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High-order Uniform Convergence Estimation of Boundary Solutions for Laplace's Equation

Dedicated to the memory of the late Professor Kôsaku Yosida

By

Kantaro HAYAKAWA* and Yuusuke ISO**

Abstract

Treated in this paper is Laplace's equation with the Neumann condition. Uniform convergence estimation of boundary element methods for this problem was done by Iso [1]. The aim of this paper is to improve the estimates given in [1] and to show high-order uniform convergence of the boundary element scheme.

§1. Introduction

Many numerical experiments have shown effectiveness of the boundary element method (=BEM). Especially for elliptic boundary value problems, we know experimentally that accurate numerical solutions are easily obtained by this method. For the Neumann problem of Laplace's equation, Iso [1] showed the uniform convergence theorems for BEM and gave a new method to construct accurate schemes. In Iso [1], the final rate of convergence is $O(h)$, although the truncation errors are of $O(h^2)$. Such loss in estimation comes from the method applied in its proof, and invention of a suitable estimation technique must enable us to clear off this loss.

The aim of this paper is to give a new technique through which the rate of convergence becomes $O(h^2)$. This technique is deeply connected with some properties of positive matrices, which are discussed in §4. In this paper, the Neumann problem for Laplace's equation is dealt with, and a rather mathematical boundary element scheme is adopted. Slight modification will be neces-

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* Department of Mathematical Science, College of Engineering, University of Osaka Prefecture, Mozu-umemachi Sakai, 591, Japan.
** Research Institute for Mathematical Sciences, Kyoto University, Sakyo, Kyoto, 606, Japan.

sary for this scheme in case of real numerical computation.

Thruoghout this paper, we use the following notations: Let Ω be a bounded domain in $\boldsymbol{E}^2$ with its boundary $\Gamma:=\partial\Omega$ smooth. And let us express a spacial point by $x=(x_1, x_2)\in\boldsymbol{E}^2$.

$|x|$; the Euclidean distance between the origin and x.

$C^s(\Omega)$; totality of s times continuously differentiable functions on Ω.

$C^s(\Gamma)$; totality of s times continuously differentiable functions on Γ.

$|||*|||_\infty$; the maximum norm introduced into $C^0(\Gamma)$;

i.e. $|||f|||_\infty:=\max_{x\in\Gamma}|f(x)|$ for $f(x)\in C^0(\Gamma)$.

$||*||_p$, $||*||_\infty$; norms introduced into $\boldsymbol{R}^N$;

i.e. $||x||_p:=(\sum_{k=1}^{N}|x_k|^p)^{1/p}$ $(p>0)$,

$||x||_\infty:=\max_{1\leq k\leq N}|x_k|$,

where $x=(x_1, x_2, \cdots, x_N)^T\in\boldsymbol{R}^N$.

§2. Boundary Integral Equations

The Neumann problem for two-dimensional Laplace's equation is dealt with. Let Ω be a bounded domain in $\boldsymbol{E}^2$, and let its boundary $\Gamma:=\partial\Omega$ be sufficiently smooth. Furthermore let us assume that the curvature of Γ is positive. Let $\vec{n}_z$ denote the unit outward normal vector to Γ at $z\in\Gamma$. Here we consider the following Neumann problem:

$$\Delta u = 0 \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\frac{\partial}{\partial n}u = q \quad \text{on} \quad \Gamma, \tag{2.2}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on Γ, and where q is a given function of $C^2(\Gamma)$ and satisfies

$$\int_\Gamma q(y)d\sigma_y = 0. \tag{2.3}$$

It is well known that the problem has, in this case, a unique solution in $C^2(\bar{\Omega})/\{\text{const.}\}$.

To apply BEM, we first derive a boundary integral equation for this problem. Let $g(x)$ be the fundamental solution of Laplace's equation;

$$g(x) = -\frac{1}{2\pi}\log|x| \qquad \text{for} \quad x\in\boldsymbol{E}^2.$$

The Dirichlet data $u(z)$ satisfies

$$\frac{1}{2}u(z)+\text{p.v.}\int_{\Gamma}\frac{\partial}{\partial n_y}g(z-y)u(y)d\sigma_y=\int_{\Gamma}g(z-y)q(y)d\sigma_y \qquad \text{for} \quad z\in\Gamma\,, \tag{2.4}$$

and this is our aimed integral equation. (See Iso [1]). Let us define a vector valued function $\vec{G}(x, y)$ by

$$\vec{G}(x, y) := \nabla_y g(x-y) \qquad \text{for} \quad x, y\in \boldsymbol{E}^2\,. \tag{2.5}$$

Hence the kernel of the integral equation is written in $\vec{G}(z, y)\cdot\vec{n}_y$ for $y, z\in\Gamma$. As is pointed out in Iso [1; §2], this kernel function can be extended as a continuous function on Γ, and 'p.v.' can be omitted in (2.4). For the sake of convenience in the following discussion, we define an operator B from $C^0(\Gamma)$ into itself and a function $r(z)$ as follows:

$$(Bu)(z) := \int_{\Gamma}\vec{G}(z, y)\cdot\vec{n}_y u(y)d\sigma_y, \qquad \text{where} \quad z\in\Gamma,\ u\in C^0(\Gamma); \tag{2.6}$$

$$r(z) := \int g(z-y)q(y)d\sigma_y \qquad \text{for} \quad z\in\Gamma\,. \tag{2.7}$$

For the operator B, we have the next proposition.

Proposition 2.1. (Kellogg [2; Chapter 11])

$$N\left(\frac{1}{2}I+B\right) = \{\text{const.}\}\,,$$

where $N\left(\frac{1}{2}I+B\right)$ *denotes the null space of the operator* $\frac{1}{2}I+B$.

Hence our problem of the boundary integral equation is to find $u\in C^0(\Gamma)$ such that

$$\left(\frac{1}{2}I+B\right)u = r\,, \tag{2.8}$$

$$\int_{\Gamma}u(y)d\sigma_y = 0\,. \tag{2.9}$$

Unique solvability for this problem is guaranteed by Kellogg [2; Chapter 11]. Furtheremore, the solution of (2.8) and (2.9) belongs to $C^2(\Gamma)$.

§3. Boundary Element Scheme

In this section, we give a numerical scheme to solve the integral equation

(2.8) and (2.9) according to the boundary element method adopted in Iso [1; §3].

Let $\{z_k\}_{k=1}^N$ be a set of nodal points on Γ, and let us define Γ_k by the closed minor arc segment of Γ cut by z_k and z_{k+1};

$$\text{i.e.}\quad \Gamma_k := \widehat{z_k z_{k+1}} \qquad \text{for} \quad 1 \leq k \leq N,$$

where z_{N+1} coincides with z_1. We call each of $\{\Gamma_k\}_{k=1}^N$ a boundary element. Here we assume that these nodal points are chosen so that they satisfy

$$|\Gamma_k| = \frac{1}{N}|\Gamma|, \tag{3.1}$$

where $|\ \ |$ denotes length of a curve. And let us define a mesh size h by

$$h := \frac{1}{N}|\Gamma|.$$

Let $\{\phi_k\}_{k=1}^N$ be a set of continuous functions on Γ which satisfy the following properties (i)—(v);

(i) $\phi_k \in C^0(\Gamma)$, $\phi_k(z_j) = \delta_{k,j}$ for $1 \leq j, k \leq N$, (3.2)

(ii) $\operatorname{supp}(\phi_k) = \Gamma_{k-1} \cup \Gamma_k$ for $1 \leq k \leq N$, where $\Gamma_0 = \Gamma_N$, (3.3)

(iii) $\phi_k|_{\Gamma_j} \in C^2(\Gamma_j)$ for $1 \leq j, k \leq N$, (3.4)

(iv) $\sum_{k=1}^N \phi_k = 1$, (3.5)

(v) $\int_\Gamma \phi_k(y)d\sigma_y = \int_\Gamma \phi_j(y)d\sigma_y$ for $1 \leq j, k \leq N$. (3.6)

Then we define an N-dimensional linear subspace V_h of $C^0(\Gamma)$ by

$$V_h := \text{linear hull} \langle \phi_1, \cdots, \phi_N \rangle.$$

Let us define a collocation operator P_h from $C^0(\Gamma)$ into V_h by

$$\begin{array}{rcl} P_h:\ C^0(\Gamma) & \longrightarrow & V_h \\ \Cup & & \Cup \\ u(x) & \longrightarrow & \sum_{1\leq k\leq N} u(z_k)\phi_k \end{array} \tag{3.7}$$

Under above preparations, we give a discretization of the integral operator B defined by (2.6) Let us define an operator B_h from V_h into itself by

$$D(B_h) = V_h, \quad B_h := P_h B,$$

where $D(B_h)$ denotes the definition domain of the operator B_h. Furthermore let $\{b_{i,j}\}_{\substack{1\leq i\leq N\\1\leq j\leq N}}$ be defined by

$$b_{i,j} := (P_h B\phi_j)(z_i) \qquad \text{for} \quad 1 \leq i, j \leq N .$$

Since V_h is an N-dimensional vector space over $\boldsymbol{R}$, we can identify the operator B_h with the following $N \times N$ matrix

$$B_h = \begin{pmatrix} b_{1,1} & \cdots & b_{1,N} \\ \vdots & & \vdots \\ b_{N,1} & \cdots & b_{N,N} \end{pmatrix}. \tag{3.8}$$

And, by this identification rule, we note that an element of V_h

$$u_h = \sum_{k=1}^{N} u_h^k \phi_k \tag{3.9}$$

can be identified with an element $(u_h^1, \cdots, u_h^N)^T \in \boldsymbol{R}^N$. Therefore, the same notations are used for the cases both of V_h and of $\boldsymbol{R}^N$ in this paper. For the matrix B_h defined by (3.8), we have the following two lemmas. (See Iso [1; §3, §4].)

Lemma 3.1. *There exist positive constants C_1 and C_2, which are independent of h, such that*

(i) $$-C_1 h \leq b_{i,j} \leq -C_2 h \qquad \textit{for} \quad 1 \leq i, j \leq N . \tag{3.10}$$

(ii) $$\sum_{j=1}^{N} b_{i,j} = -\frac{1}{2} \qquad \textit{for} \quad 1 \leq i \leq N . \tag{3.11}$$

Lemma 3.2. *Let I_h be the $N \times N$ unit matrix, then we have*

$$\operatorname{rank}\left(\frac{1}{2} I_h + B_h\right) = N - 1 .$$

From these lemmas, it follows that the matrix $\frac{1}{2} I_h + B_h$ has the eigenvalue 0 with its eigenvector $(1, \cdots, 1)^T$. Hence we must pay a little attention to give a discretization of the function $r(z)$, which is the right-hand side of the boundary integral equation.

Let $r_h \in V_h$ denote a discretization of $r(z)$;

$$\text{i.e.} \quad r_h = \sum_{k=1}^{N} r_h^k \phi_k .$$

Here let us assume that we give r_h so that it satisfies

$$\max_{1 \leq k \leq N} |r(z_k) - r_h^k| \leq Ch^2 , \tag{3.12}$$

$$r_h \in R\left(\frac{1}{2} I_h + B_h\right), \tag{3.13}$$

where C is a positive constant which is independent of h. And $R\left(\frac{1}{2}I_h+B_h\right)$ denotes the range space of the operator $\frac{1}{2}I_h+B_h$. We remark that $r_h:=P_h r$ does not always satisfy the assumption (3.13). In order to attain these assumptions, we give a discretization of r, for example, by the following manner. Let $\nu_h=(\nu_h^1, \cdots, \nu_h^N)\in \boldsymbol{R}^N$ be a solution of

$$\left(\frac{1}{2}I_h+B_h\right)^T \nu_h = 0. \tag{3.14}$$

Since

$$\operatorname{rank}\left(\frac{1}{2}I_h+B_h\right)^T = N-1\,,$$

we can choose ν_h, from the Perron-Frobenius theory (e.g. Minc [3; Chapter 1]), so that all the components of ν_h are positive. Moreover, ν_h is uniquely determined by normalization of $||\nu_h||_1=1$. And it can be proved, from Lemma 3.1, that there exists a positive constant M, which is independent of h, such that

$$(\max_k \nu_h^k)/(\min_k \nu_h^k)<M\,.$$

Here let us set

$$r_h^k := r(z_j)-\frac{1}{N}\cdot\frac{d}{\nu_h^k} \qquad \text{for} \quad 1\leq k\leq N, \tag{3.15}$$

where d denotes the defect defined by

$$d := \sum_{j=1}^{N} r(z_j)\nu_h^j\,. \tag{3.16}$$

We know, from error estimation in Iso [1, §4], that

$$\left|\frac{d}{\nu_h^k}\right| = O(h) \qquad \text{for} \quad 1\leq k\leq N\,. \tag{3.17}$$

Hence we can see that the discretization by (3.15) attains the assumptions (3.12) and (3.13).

Our aimed discretized problem is to find $u_h=(u_h^1, \cdots, u_h^N)^T\in \boldsymbol{R}^N$ such that

$$\left(\frac{1}{2}I_h+B_h\right)u_h = r_h\,, \tag{3.18}$$

$$\sum_{k=1}^{N} u_h^k = 0\,. \tag{3.19}$$

These equations correspond to a discretization of the boundary integral equations (2.8) and (2.9), and it is trivial that this system of linear algebraic equations has a unique solution.

§4. Properties of Positive Matrices

Prior to convergence estimation, some properties of positive matrices are given by Theorem 4.1 and by Theorem 4.2. These properties, especially Theorem 4.2, play an essential role to improve the error analysis. Here we state them in more general situations.

Let $\{A_N=(a^N_{i,j})_{i\downarrow,j\rightarrow}\}$ be a family of $N\times N$ real matrices, and let the following assumptions (i) and (ii) be satisfied:

(i) $$\sum_{j=1}^{N} a^N_{i,j} = 1 \qquad \text{for} \quad 1\leq i\leq N, \tag{4.1}$$

(ii) there exist positive constants p and $\tilde{C}$, which are independent of N, such that

$$\tilde{C}\cdot\frac{1}{N}\leq a^N_{i,j}\leq p\tilde{C}\cdot\frac{1}{N} \qquad \text{for} \quad 1\leq i, j\leq N. \tag{4.2}$$

Then, from the Perron-Frobenius theory (e.g. Minc [3; Chapter 1]), there exists $\mu^N=(\mu^N_1, \cdots, \mu^N_N)^T\in\boldsymbol{R}^N$ such that

$$\mu^N_j>0 \qquad \text{for} \quad 1\leq j\leq N, \tag{4.3}$$

$$(A_N)^T\mu^N = \mu^N, \tag{4.4}$$

$$||\mu^N||_1 = 1. \tag{4.5}$$

Let $e^N\in\boldsymbol{R}^N$ be $e^N=(1, 1, \cdots, 1)^T$, and let I_N be the $N\times N$ unit matrix. Furthermore, define linear subspaces $W^{(N)}_0$ and $W^{(N)}_1$ of $\boldsymbol{R}^N$ by

$$W^{(N)}_0 := \{w^N\in\boldsymbol{R}^N\,|\,(w^N, \mu^N) = 0\}, \tag{4.6}$$

$$W^{(N)}_1 := \{w^N\in\boldsymbol{R}^N\,|\,(w^N, e^N) = 0\}, \tag{4.7}$$

where (u, v) denotes the inner product of $u=(u_1, \cdots, u_N)^T$ and $v=(v_1, \cdots, v_N)^T \in\boldsymbol{R}^N$;

$$\text{i.e.} \quad (u, v) = u_1v_1+u_2v_2+\cdots+u_Nv_N.$$

Then, we have the following theorem.

Theorem 4.1. *Let* A_N *satisfy* (4.1) *and* (4.2), *then the estimates* (i)-(iii) *hold;*

(i) $$||A_Nu^N||_\infty\leq\left(1-\frac{1}{p}\right)||u^N||_\infty \qquad \textit{for} \quad {}^\forall u^N\in W^{(N)}_0, \tag{4.8}$$

(ii) $$||(I_N-A_N)u^N||_\infty \geq \frac{1}{p}||u^N||_\infty \quad \textit{for} \quad {}^\forall u^N \in W_0^{(N)}, \tag{4.9}$$

(iii) $$||(I_N-A_N)u^N||_\infty \geq \frac{1}{2p}||u^N||_\infty \quad \textit{for} \quad {}^\forall u^N \in W_1^{(N)}. \tag{4.10}$$

Proof. We first remark that μ^N, defined by (4.3)-(4.5), satisfies the following estimate;

$$\tilde{C}\cdot\frac{1}{N} \leq \mu_j^N \leq p\tilde{C}\cdot\frac{1}{N} \qquad \text{for} \quad 1\leq j\leq N. \tag{4.11}$$

Let e_j^N be an element of $\boldsymbol{R}^N$ with its j-th component 1 and otherwise 0;

$$\text{i.e.} \quad e_j^N = (0, \cdots 0, \overset{\overset{j}{\vee}}{1}, 0, \cdots, 0)^T \qquad \text{for} \quad 1\leq j\leq N.$$

Then,

$$||A_N u^N||_\infty = \max_{1\leq j\leq N} |(A_N u^N, e_j^N)|. \tag{4.12}$$

On the other hand, for an arbitrary real number λ and $u^N \in W_0^{(N)}$, we have

$$\begin{aligned} |(A_N u^N, e_j^N)| &= |(A_N u^N, e_j^N - \lambda\mu^N)| \\ &\leq ||u^N||_\infty \cdot ||(A_N)^T e_j^N - \lambda\mu^N||_1. \end{aligned} \tag{4.13}$$

Therefore, from (4.12) and (4.13), we have

$$\begin{aligned} ||A_N u^N||_\infty &\leq ||u^N||_\infty \max_{1\leq j\leq N} \{\min_\lambda ||(A_N)^T e_j^N - \lambda\mu^N||_1\} \\ &= ||u^N||_\infty \max_{1\leq j\leq N} \{\min_\lambda (\sum_{k=1}^N |a_{j,k}^N - \lambda\mu_k^N|)\}. \end{aligned}$$

Set $\lambda_0 := \frac{1}{p}$, then we have $0 \leq a_{j,k}^N - \lambda_0\mu_k^N = a_{j,k}^N - \frac{1}{p}\mu_k^N$ from (4.11). Hence, from (4.1) and (4.5), we have

$$||A_N u^N||_\infty \leq \left(1-\frac{1}{p}\right)||u^N||_\infty \qquad \text{for} \quad {}^\forall u^N \in W_0^{(N)},$$

and (4.8) has just been proved.

Next for $u^N \in W_0^{(N)}$, we have

$$\begin{aligned} ||(I_N-A_N)u^N||_\infty &\geq |\,||u^N||_\infty - ||A_N u^N||_\infty| \\ &\geq ||u^N||_\infty - \left(1-\frac{1}{p}\right)||u^N||_\infty \quad (\because (4.8)) \\ &= \frac{1}{p}||u^N||_\infty. \end{aligned}$$

Thus we have shown (4.9):

$$\|(I_N-A_N)u^N\|_\infty \geq \frac{1}{p}\|u^N\|_\infty \qquad \text{for} \quad {}^{\forall}u^N \in W_0^{(N)}.$$

Finally, let v^N be defined by

$$v^N := u^N - (u^N, \mu^N)e^N \qquad \text{for} \quad u^N \in W_1^{(N)},$$

then $v^N \in W_0^{(N)}$ and $(I_N-A_N)v^N=(I_N-A_N)u^N$. Hence we have

$$\|(I_N-A_N)v^N\|_\infty = \|(I_N-A_N)u^N\|_\infty. \tag{4.14}$$

On the other hand, since $u^N=v^N-\frac{1}{N}(v^N, e^N)e^N$, we have

$$\|u^N\|_\infty \leq 2\|v^N\|_\infty. \tag{4.15}$$

Applying (4.9) to $v^N \in W_0^{(N)}$, we have

$$\|(I_N-A_N)v^N\|_\infty \geq \frac{1}{p}\|v^N\|_\infty. \tag{4.16}$$

Therefore, from (4.14)–(4.16), we have

$$\begin{aligned}\|(I_N-A_N)u^N\|_\infty &\geq \frac{1}{p}\|v^N\|_\infty \\ &\geq \frac{1}{2p}\|u^N\|_\infty \qquad \text{for} \quad {}^{\forall}u^N \in W_1^{(N)}.\end{aligned}$$

Q.E.D.

Immediately from this theorem, we have the next estimate.

Theorem 4.2. *Let $e_h=(e_h^1, \cdots, e_h^N)^T \in \boldsymbol{R}^N$ be the solution of the following system;*

$$\left(\frac{1}{2}I_h+B_h\right)e_h = f_h, \tag{4.17}$$

$$\sum_{k=1}^{N} e_h^k = 0, \tag{4.18}$$

where $f_h=(f_h^1, \cdots, f_h^N)^T \in \boldsymbol{R}^N$ is given. Then we have

$$\|e_h\|_\infty \leq 4(C_1/C_2)\|f_h\|_\infty,$$

where C_1 and C_2 are the same constants appeared in (3.10).

Proof. It is clear that the equations (4.17) and (4.18) have the unique solution. Set $A_N := -2B_h$ and put $p := C_1/C_2$, then A_N satisfies the both assumptions (4.1) and (4.2). Since the solution e_h satisfies $e_h \in W_1^{(N)}$, we get, from

Theorem 4.1. (iii),

$$||(I_N-A_N)e_h||_\infty \geq \frac{1}{2p}||e_h||_\infty .$$

Hence we have

$$||e_h||_\infty \leq 4p||\left(\frac{1}{2}I_N-\frac{1}{2}A_N\right)e_h||_\infty = 4p||f_h||_\infty .$$

Q.E.D.

In convergence estimation in the next section, e_h and f_h in (4.17) and (4.18) are to mean the discretization errors and the truncation errors respectively.

§5. Uniform Convergence Estimation

Let u be the solution of (2.8) and (2.9), and let u_h be the solution of (3.18) and (3.19). We use these notations throughout this section. Our purpose is to estimate $|||u-u_h|||_\infty$ in $O(h^2)$. On the other hand, we have

$$|||u-u_h|||_\infty \leq |||u-P_hu|||_\infty + |||P_hu-u_h|||_\infty ,$$

and the estimate of $|||u-P_hu|||_\infty$ is of $O(h^2)$. For this reason, we are sufficient to show

$$|||P_hu-u_h|||_\infty = O(h^2) \quad \text{i.e.} \quad ||P_hu-u_h||_\infty = O(h^2) .$$

Before starting error estimation, we remark two estimates from Iso [1; §4], which are derived from the Taylor expansion.

Lemma 5.1. (Iso [1; §4]) *There exists a positive constant C, which is independent of h, such that*

$$||\, P_h\left(\frac{1}{2}I+B\right)(P_hu-u)\,||_\infty \leq Ch^2 , \tag{5.1}$$

$$\left|\int_\Gamma (P_hu-u_h)\,d\sigma\right| \leq Ch^2 . \tag{5.2}$$

Let us start error analysis. Set $\varepsilon_h := P_hu-u_h$, then we have, from (5.2),

$$\left|\sum_{k=1}^{N} \varepsilon_h^k\right| \leq Ch .$$

Define $e_h = \sum_{k=1}^N e_h^k \phi_k \in V_h$ by

$$e_h^j := \varepsilon_h^j - \frac{1}{N}\sum_{k=1}^{N} \varepsilon_h^k \qquad \text{for} \quad 1 \leq j \leq N ,$$

then it satisfies

$$\sum_{k=1}^{N} e_h^k = 0\,, \tag{5.3}$$

$$\|e_h - \varepsilon_h\|_\infty \le Ch^2\,. \tag{5.4}$$

On the other hand, we have

$$\left(\frac{1}{2}I_h + B_h\right)(P_h u - u_h) = P_h\left(\frac{1}{2}I + B\right)(P_h u - u) + (P_h r - r_h)\,.$$

Therefore we have

$$\left(\frac{1}{2}I_h + B_h\right)e_h = P_h\left(\frac{1}{2}I + B\right)(P_h u - u) + (P_h r - r_h) + \left(\frac{1}{2}I_h + B_h\right)(e_h - \varepsilon_h). \tag{5.5}$$

For the first term of the right-hand side of (5.5), we get, from (5.1),

$$\left\| P_h\left(\frac{1}{2}I + B\right)(P_h u - u) \right\|_\infty \le Ch^2\,, \tag{5.6}$$

and for the second term, we get, from (3.12),

$$\|P_h r - r_h\|_\infty \le Ch^2\,, \tag{5.7}$$

and finally for the last term, we get, from (3.11) and (5.4),

$$\left\| \left(\frac{1}{2}I_h + B_h\right)(e_h - \varepsilon_h) \right\|_\infty \le Ch^2\,. \tag{5.8}$$

Set $f_h := P_h\left(\frac{1}{2}I + B\right)(P_h u - u) + (P_h r - r_h) + \left(\frac{1}{2}I_h + B_h\right)(e_h - \varepsilon_h)$, then we get, from (5.6)-(5.8),

$$\|f_h\|_\infty \le Ch^2\,. \tag{5.9}$$

Furthermore, from the definition of f_h, we have

$$\left(\frac{1}{2}I_h + B_h\right)e_h = f_h\,,$$

$$\sum_{k=1}^{N} e_h^k = 0\,.$$

Hence, immediately from Theorem 4.2 and (5.9), we obtain

$$\begin{aligned}\|e_h\|_\infty &\le 4(C_1/C_2)\|f_h\|_\infty \\ &\le Ch^2\,.\end{aligned}$$

We have just come to our conclusion.

Theorem 5.1. *Assume that* (3.12) *and* (3.13) *are satisfied, then the system of linear equations* (3.18) *and* (3.19) *has a unique solution* $u_h \in \mathbf{R}^N$. *And there exists a positive constant* C, *which is independent of* h, *such that*

$$||P_h u - u_h||_\infty \leq Ch^2 . \tag{5.10}$$

Unique solvability of (3.18) and (3.19) is mentioned in §3.

Since we identify V_h with $\mathbf{R}^N$, we immediately have the following corollary from this.

Corollary 5.1. *There exists a positive constant* C, *which is independent of* h, *such that*

$$|||u - u_h|||_\infty \leq Ch^2 . \tag{5.11}$$

§6. Concluding Remarks

In this paper, we showed uniform convergence of $O(h^2)$ for the boundary element scheme. (See Theorem 5.1 and Corollary 5.1). The final results are improved beside those of Iso [1]. But we should, here, remark that the scheme adopted is rather mathematical and that the assumption (3.12) and (3.13) are the keys in our arguments. We give, in §3, a method to clear up these assumptions, but it requires to solve a system of linear equations (3.14). From practical viewpoints, such a procedure consumes more computing time, and an easier method to attain the assumptions should be proposed.

The assumptions (3.1) and (3.6) seem to be strong, and they can be replaced by weaker conditions. But such replacement yields not essential but rather complicated changes in error estimation. For weaker conditions, details are mentioned in Iso [1; §5].

The properties of positive matrices, stated in Theorem 4.1, will be of use not only for error analysis but for some applications in the game theories.

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