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Cyclic Representations of $U_q(\mathfrak{sl}(n+1, \boldsymbol{C}))$ at $q^N=1$

By

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§1. Introduction

In this article we deal with the q-analog of the universal enveloping algebra $U_q(\mathfrak{sl}(n+1, \boldsymbol{C}))$ when $q=\omega$ is a primitive N-th root of 1 with odd N. We shall give an explicit construction of finite-dimensional irreducible representations having $n(n+2)$ continuous parameters.

Our motivation in this problem originates in the chiral Potts model [AMPT], [BPA]. This is a solvable lattice model built upon solutions to the Yang-Baxter equation whose spectral parameters live on certain algebraic curves of genus greater than 1. Bazhanov and Stroganov [BS] showed in effect that these solutions can be derived as intertwiners between tensor products of the representations of $U_q(\widehat{\mathfrak{sl}}(2, \boldsymbol{C}))$ with q a root of 1 (see [BS], [DJMM] for details). Attempts for extending their construction to the case of $U_q(\mathfrak{sl}(n+1, \boldsymbol{C}))$ have been initiated in [BK], [DJMM] for the case $n=2$.

Representations of $U_q(\mathfrak{g})$ at roots of 1 have been studied recently by De Concini and Kac [DK] for an arbitrary finite dimensional simple Lie algebra $\mathfrak{g}$. They showed that the irreducible representations of $U_q(\mathfrak{g})$ are generically parametrized by dim $\mathfrak{g}$ number of continuous parameters which are the values of certain central elements. Our aim here is to write down such representations in the case $\mathfrak{g}=\mathfrak{sl}(n+1, \boldsymbol{C})$.

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Here is the outline of the paper. In section 2 we consider a Weyl algebra $\mathcal{W}$ with generators x_{jk}, z_{jk} $(1 \leq j \leq k \leq n)$ such that $z_{jk}x_{jk} = qx_{jk}z_{jk}$ and all others pairwise commute. We shall construct (for generic q) an algebra map $\rho_{r,s}$: $U_q(\mathfrak{sl}(n+1, \boldsymbol{C})) \to \mathcal{W}$ depending on arbitrary non-zero complex numbers r_i s_i $(1 \leq i \leq n)$. Explicitly it reads

$$\rho_{r,s}(e_i) = \sum_{k=i}^{n} \{r_i z_{ik} z_{i\,k-1} z_{i-1\,k-1}^{-1} z_{i+1\,k}^{-1}\} x_{ik} x_{i\,k+1} \cdots x_{in} ,$$

$$\rho_{r,s}(f_i) = \sum_{k=1}^{i} \{s_i z_{i+1-k\,n-k} z_{i+1-k\,n+1-k}^{-1} z_{i-k\,n+1-k} z_{i-k\,n-k}^{-1}\} x_{i+1-k\,n+1-k}^{-1} \cdots x_{in}^{-1} ,$$

$$\rho_{r,s}(t_i) = \frac{r_i}{s_i} z_{in}^2 z_{i-1\,n}^{-1} z_{i+1\,n}^{-1} .$$

When $q=\omega$, $\mathcal{W}$ admits an N^m-dimensional irreducible representation σ_{gh}: $\mathcal{W} \to \mathrm{End}((\mathbf{C}^N)^{\otimes m})$ with $m = n(n+1)/2$. Let X, Z be $N \times N$ matrices given by

$$Xu_i = u_{i+1} \ (u_N = u_0) , \quad Zu_i = \omega^i u_i ,$$

where $\{u_i\}_{0 \leq i \leq N-1}$ denotes the standard basis of $\boldsymbol{C}^N$. Let $X_{jk}, Z_{jk} \in \mathrm{End}((\mathbf{C}^N)^{\otimes m})$ denote the matrices acting as X, Z on the (j, k)-component and as identity on the other components. Then we have

$$\sigma_{gh}(x_{jk}) = g_{jk} X_{jk} , \quad \sigma_{gh}(z_{jk}) = h_{jk} Z_{jk} .$$

Here again g_{jk}, h_{jk} $(1 \leq j \leq k \leq n)$ are arbitrary non-zero complex numbers. Composing σ_{gh} with $\rho_{r,s}$ above, we obtain in section 3 a representation of $U_q(\mathfrak{sl}(n+1, \boldsymbol{C}))$ at $q=\omega$. The parameters r_i, s_i, g_{jk} and h_{jk} are not mutually independent, and there are altogether $n(n+2)$ continuous parameters. We show next that the central elements take values in an open set of $\boldsymbol{C}^{n(n+2)}$. From the results of De Concini-Kac [DK] we then conclude that these representations are generically irreducible.

§2. Algebra Homomorphism $U_q(\mathfrak{sl}(n+1, \boldsymbol{C})) \to \mathcal{W}$

Let $\boldsymbol{C}(q)$ denote the field of rational functions in an indeterminate q. In this section we construct a $\boldsymbol{C}(q)$ algebra $\mathcal{W}$ and an algebra homomorphism $\rho_{r,s}$: $U_q(\mathfrak{sl}(n+1, \boldsymbol{C})) \to \mathcal{W}$.

We use the following notations:

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} , \quad [k]! = [k][k-1] \cdots [1] , \quad \begin{bmatrix} N \\ k \end{bmatrix} = \frac{[N]!}{[k]![N-k]!} ,$$

$$\{x\} = \frac{x-x^{-1}}{q-q^{-1}}, \quad x^{(k)} = \frac{x^k}{[k]!}.$$

Suppose that L is a logical expression. We define

$$\theta(L) = 1 \quad \text{if } L \text{ is true}$$
$$= 0 \quad \text{if } L \text{ is false}.$$

The algebra $\mathcal{W}$ is generated by x_{jk}, z_{jk} $(1 \leq j \leq k \leq n)$ and the inverses x_{jk}^{-1}, z_{jk}^{-1} satisfying

$$[x_{jk}, x_{j'k'}] = [x_{jk}, z_{j'k'}] = [z_{jk}, z_{j'k'}] = 0 \quad \text{if} \quad (j, k) \neq (j', k'), \tag{2.1a}$$
$$z_{jk} x_{jk} = q x_{jk} z_{jk}. \tag{2.1b}$$

We define a $\boldsymbol{C}(q)$ linear involution $*$ by

$$x_{jk}^* = x_{k+1-j\,k}^{-1}, \quad z_{jk}^* = z_{k+1-j\,k}^{-1}.$$

We also define a $\boldsymbol{C}$ linear involution $\wedge$ by

$$\hat{q} = q^{-1}, \quad \hat{x}_{jk} = x_{jk}, \quad \hat{z}_{jk} = z_{jk}^{-1}.$$

Let (a_{ij}) be the Cartan matrix of type A_n. By definition the algebra $U_q(\mathfrak{sl}(n+1, \boldsymbol{C}))$ is a $\boldsymbol{C}(q)$ algebra generated by e_i, f_i, t_i $(1 \leq i \leq n)$ and the inverses t_i^{-1} satisfying

$$[t_i, t_j] = 0, \tag{2.2a}$$
$$t_i e_j t_i^{-1} = q^{a_{ij}} e_j, \tag{2.2b}$$
$$t_i f_j t_i^{-1} = q^{-a_{ij}} f_j, \tag{2.2c}$$
$$[e_i, f_j] = \delta_{ij} \{t_i\}, \tag{2.2d}$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{ij}-k)} = 0 \quad \text{if} \quad i \neq j, \tag{2.2e}$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)} = 0 \quad \text{if} \quad i \neq j. \tag{2.2f}$$

We denote the following $\boldsymbol{C}(q)$ linear involution of $U_q(\mathfrak{sl}(n+1, \boldsymbol{C}))$ by $*$.

$$e_i^* = f_{n+1-i}, \quad f_i^* = e_{n+1-i}, \quad t_i^* = t_{n+1-i}^{-1}.$$

We also denote the following $\boldsymbol{C}$ linear involution by $\wedge$.

$$\hat{q} = q^{-1}, \quad \hat{e}_i = e_i, \quad \hat{f}_i = f_i, \quad \hat{t}_i = t_i^{-1}.$$

We define the root vectors e_{ij} $(1 \leq i \neq j \leq n+1)$ inductively as follows.

$$e_{i\,i+1} = e_i, \tag{2.3a}$$

$$e_{ij} = e_{ik}e_{kj} - qe_{kj}e_{ik} \qquad \text{if} \quad i<k<j, \tag{2.3b}$$

$$e_{ij} = (e_{n+2-i\, n+2-j})^{*\wedge} \qquad \text{if} \quad i>j. \tag{2.3c}$$

In particular

$$e_{i+1\,i} = f_i . \tag{2.3d}$$

The consistency of this definition follows from (2.2e) for $i \neq j \pm 1$. Among the commutation relations of e_{ij}(cf. [Y]) we shall need the following.

$$e_{ik}^{(2)} e_{kj} - e_{ik}e_{kj}e_{ik} + e_{kj}e_{ik}^{(2)} = 0 \qquad i<k<j, \tag{2.4a}$$

$$e_{kj}^{(2)} e_{ik} - e_{kj}e_{ik}e_{kj} + e_{ik}e_{kj}^{(2)} = 0 \qquad i<k<j. \tag{2.4b}$$

For $r=(r_1, \cdots, r_n) \in (\boldsymbol{C}^\times)^n$ we define

$$r^* = (r_n, \cdots, r_1),$$
$$\hat{r} = (r_1^{-1}, \cdots, r_n^{-1}).$$

We construct a family of $\boldsymbol{C}(q)$ algebra homomorphisms

$$\rho_{r,s}\colon U_q(\mathfrak{sl}(n+1, \boldsymbol{C})) \to \mathcal{W}$$

depending on r, $s \in (\boldsymbol{C}^\times)^n$. Fix r, s and define

$$\xi_{ik} = x_{ik}x_{i\,k+1} \cdots x_{in},$$
$$\zeta_{ik} = r_i z_{ik} z_{i\,k-1} z_{i-1\,k-1}^{-1} z_{i+1\,k}^{-1},$$

where $z_{ik}=1$ unless $1 \leq i \leq k \leq n$. We use the following abbreviations.

$$\zeta_{i\,k_1 \cdots k_l} = \zeta_{ik_1}\zeta_{i+1\,k_2} \cdots \zeta_{i+l-1\,k_l},$$
$$\xi_{i\,k_1 \cdots k_l} = \xi_{ik_1}\xi_{i+1\,k_2} \cdots \xi_{i+l-1\,k_l}.$$

It is easy to check the following commutation relations.

$$\begin{aligned} \xi_{ik}\zeta_{ik'} &= q^{-2}\zeta_{ik'}\xi_{ik} && \text{if} \quad k<k' \\ &= q^{-1}\zeta_{ik'}\xi_{ik} && \text{if} \quad k=k' \\ &= \zeta_{ik'}\xi_{ik} && \text{if} \quad k>k', \end{aligned} \tag{2.5a}$$

$$\begin{aligned} \xi_{i+1\,k}\zeta_{ik'} &= q\zeta_{ik'}\xi_{i+1\,k} && \text{if} \quad k \leq k' \\ &= \zeta_{ik'}\xi_{i+1\,k} && \text{if} \quad k>k', \end{aligned} \tag{2.5b}$$

$$\begin{aligned} \xi_{i-1\,k}\zeta_{ik'} &= q\zeta_{ik'}\xi_{i-1\,k} && \text{if} \quad k<k' \\ &= \zeta_{ik'}\xi_{i-1\,k} && \text{if} \quad k \geq k', \end{aligned} \tag{2.5c}$$

$$\xi_{ik}\zeta_{i'k'} = \zeta_{i'k'}\xi_{ik} \qquad \text{if} \quad i \neq i',\ i' \pm 1. \tag{2.5d}$$

We define

$$\rho_{r,s}(e_i) = \sum_{k=i}^{n} \{\zeta_{ik}\} \xi_{ik}\,, \tag{2.6a}$$

$$\rho_{r,s}(f_i) = \rho_{s^*,r^*}(e_{n+1-i})^*, \tag{2.6b}$$

$$\rho_{r,s}(t_i) = \frac{r_i}{s_i} z_{in}^2 z_{i-1\,n}^{-1} z_{i+1\,n}^{-1}\,. \tag{2.6c}$$

Note that (2.6*b*) means

$$\rho_{r,s}(f_i) = \sum_{k=1}^{i} \{s_i z_{i+1-k\,n-k} z_{i+1-k\,n+1-k}^{-1} z_{i-k\,n+1-k} z_{i-k\,n-k}^{-1}\} x_{i+1-k\,n+1-k}^{-1} \cdots x_{in}^{-1}.$$

Proposition 2.1. *For* $a=e_i, f_i, t_i$ *we have*

$$\rho_{r,s}(a)^* = \rho_{s^*,r^*}(a^*)\,, \quad \rho_{r,s}(a)^\wedge = \rho_{\hat{r},\hat{s}}(\hat{a})\,.$$

Proof. Straightforward. □

Theorem 2.2. $\rho_{r,s}$ *defines a* $\boldsymbol{C}(q)$ *algebra homomorphism.*

Proof. We shall check the relations (2.2). The relation (2.2*a*) is obvious. By using Proposition 2.1, (2.2*c*) follows from (2.2*b*) and (2.2*f*) follows from (2.2*e*).

It is easy to see that

$$\rho_{r,s}(t_i)\zeta_{jk}\rho_{r,s}(t_i)^{-1} = \zeta_{jk}\,,$$
$$\rho_{r,s}(t_i)\xi_{jk}\rho_{r,s}(t_i)^{-1} = q^{a_{ij}}\xi_{jk}\,.$$

From this follows (2.2*b*).

Let us show (2.2*d*). Set

$$(i, k) = \{r_i z_{ik} z_{i-1\,k-1}^{-1} z_{i\,k-1} z_{i+1\,k}^{-1}\} x_{ik} \cdots x_{in}\,,$$
$$(j, l)' = \{s_j z_{j+1-l\,n-l} z_{j+1-l\,n+1-l}^{-1} z_{j-l\,n+1-l} z_{j-l\,n-l}^{-1}\} x_{j+1-l\,n+1-l}^{-1} \cdots x_{jn}^{-1}.$$

Then we have

$$\rho_{r,s}(e_i) = \sum_{k=i}^{n} (i, k)\,, \quad \rho_{r,s}(f_j) = \sum_{l=1}^{j} (j, l)'\,.$$

Consider the product $(i, k)\,(j, l)'$. Using the commutation relations (2.1) we can move all the x_{ab}'s in (i, k) to the right of z_{ab}'s in $(j, l)'$. This procedure picks up nonzero power of q in the following two cases.

Case 1: $l=j-i+1,\ k+l=n+1$,

$$x_{ik} \cdots x_{in}(z_{j+1-l\,n-l} z_{j+1-l\,n+1-l}^{-1}) = q(z_{j+1-l\,n-l} z_{j+1-l\,n+1-l}^{-1}) x_{ik} \cdots x_{in}\,.$$

Case 2: $l=j-i,\ k+l=n+1$,

$$x_{ik} \cdots x_{in}(z_{j-l\,n+1-l} z_{j-l\,n-l}^{-1}) = q^{-1}(z_{j-l\,n+1-l} z_{j-l\,n-l}^{-1}) x_{ik} \cdots x_{in}\,.$$

The situation is the same for the product $(j, l)'$ (i, k).

Case 1: $l=j-i+1$, $k+l=n+1$,

$$x_{j+1-l\,n+1-l}^{-1}\cdots x_{jn}^{-1}(z_{ik}z_{i-1\,k-1}^{-1}) = q(z_{ik}z_{i-1\,k-1}^{-1})x_{j+1-l\,n+1-l}^{-1}\cdots x_{jn}^{-1}.$$

Case 2: $l=j-i$, $k+l=n+1$,

$$x_{j+1-l\,n+1-l}^{-1}\cdots x_{jn}^{-1}(z_{i\,k-1}z_{i+1\,k}^{-1}) = q^{-1}(z_{i-1\,k}z_{i+1\,k}^{-1})x_{j+1-l\,n+1-l}^{-1}\cdots x_{jn}^{-1}.$$

If $i>j$, neither Case 1 nor Case 2 occurs. If $i=j$, Case 1 occurs but Case 2 does not. If $i<j$, both Case 1 and Case 2 occur. Therefore, if $i>j$ then

$$[\rho_{r,s}(e_i), \rho_{r,s}(f_j)] = 0\,. \tag{2.7}$$

When $i\leq j$, we use the following formulas for commutative x, y.

$$\begin{aligned}\{x\}\{qy\}-\{qx\}\{y\} &= \{xy^{-1}\}\,,\\ \{x\}\{q^{-1}y\}-\{q^{-1}x\}\{y\} &= \{x^{-1}y\}\,,\\ \{x\}+\{x^{-1}\} &= 0\,.\end{aligned}$$

If $i<j$ (2.7) follows from these identities. Finally, for $i=j$ we have

$$\begin{aligned}[\rho_{r,s}(e_i), \rho_{r,s}(f_i)] &= \left\{\frac{r_i}{s_i} z_{in}^2 z_{i-1\,n}^{-1} z_{i+1\,n}^{-1}\right\}\\ &= \{\rho_{r,s}(t_i)\}\,.\end{aligned}$$

From (2.5*d*) and (2.6*a*) we have $[e_i, e_j]=0$ if $i\neq j\pm1$. This is (2.2*e*) for $i\neq j\pm1$. In order to finish the proof we must show (2.2*e*) for $i=j\pm1$. Before proceeding to that proof we determine the image of e_{ij}. Since we have shown (2.2*e*) for $i\neq j\pm1$, the map $\rho_{r,s}$ is well-defined on e_{ij}.

Proposition 2.3. *For an integer l such that $1\leq l\leq n+1-i$ we have*

$$\rho_{r,s}(e_{i\,i+l}) = \sum_{\substack{k_1\geq i\\ \cdots\\ k_l\geq i+l-1}} \sum_{j=1}^{l} (-q)^{j-1}\theta(k_1\geq\cdots\geq k_j<\cdots<k_l) \times\zeta_{i\,k_1\cdots k_{j-1}}\{\zeta_{i+j-1\,k_j}\}(\zeta_{i+j\,k_{j+1}\cdots k_l})^{-1}\xi_{i\,k_1\cdots k_l}\,. \tag{2.8}$$

Proof. We use induction on l. If $l=1$ this is (2.6*a*). Suppose that (2.8) is shown for l. By the definition we have

$$e_{i\,i+l+1} = e_{i\,i+l}e_{i+l\,i+l+1}-qe_{i+l\,i+l+1}e_{i\,i+l}\,.$$

Note that

$$\rho_{r,s}(e_{i+l\,i+l+1}) = \sum_{k_{l+1}=i+l}^{n}\{\zeta_{i+l\,k_{l+1}}\}\xi_{i+l\,k_{l+1}}\,.$$

Consider the product $\rho_{r,s}(e_{i\,i+l})\rho_{r,s}(e_{i+l\,i+l+1})$. When we move $\xi_{i\,k_1\cdots k_l}$ to the right of $\rho_{r,s}(e_{i+l\,i+l+1})$, it may pick up some power of q from $\zeta_{i+l\,k_{l+1}}$. In fact, we have

$$\xi_{i\,k_1\cdots k_l}\zeta_{i+l\,k_{l+1}} = q^{\theta(k_l<k_{l+1})}\zeta_{i+l\,k_{l+1}}\xi_{i\,k_1\cdots k_l}.$$

Similarly, for the product $\rho_{r,s}(e_{i+l\,i+l+1})\rho_{r,s}(e_{i\,i+l})$, $\xi_{i+l\,k_{l+1}}$ may pick up some power of q from $\zeta_{i+l-1\,k_l}$;

$$\xi_{i+l\,k_{l+1}}\zeta_{i+l-1\,k_l} = q^{\theta(k_l\geq k_{l+1})}\zeta_{i+l-1\,k_l}\xi_{i+l\,k_{l+1}}.$$

We use the following formulas for commutative x, y.

$$\begin{aligned}
x^{-1}\{qy\}-qx^{-1}\{y\} &= x^{-1}y^{-1},\\
\{x\}\{qy\}-q\{x\}\{y\} &= \{x\}y^{-1},\\
x^{-1}\{y\}-q(qx)^{-1}\{y\} &= 0,\\
\{x\}\{y\}-q\{qx\}\{y\} &= -qx\{y\}.
\end{aligned}$$

From these formulas follows (2.8). □

Proof of Theorem 1 (*continued*). If $i=j-1$, (2.2*e*) is equivalent to

$$e_{i\,i+2}e_i = qe_ie_{i\,i+2} \tag{2.9a}$$

and if $i=j+1$ it is equivalent to

$$e_{j+1}e_{j\,j+2} = qe_{j\,j+2}e_{j+1}. \tag{2.9b}$$

We shall prove (2.9*b*). The proof of (2.9*a*) is similar. Note the following formulas.

$$\begin{aligned}
e_{j+1} &= \sum_{k\geq j+1}\{\zeta_{j+1\,k}\}\xi_{j+1\,k},\\
e_{j\,j+2} &= \sum_{\substack{k_1\geq j\\ k_2\geq j+1}}\left(\theta(k_1<k_2)\{\zeta_{jk_1}\}\zeta^{-1}_{j+1\,k_2}\xi_{jk_1}\xi_{j+1\,k_2}-q\theta(k_1\geq k_2)\zeta_{jk_1}\{\zeta_{j+1\,k_2}\}\xi_{jk_1}\xi_{j+1\,k_2}\right).
\end{aligned}$$

Consider the term in (2.9*b*) corresponding to the summation indices k, k_1, k_2. There are 9 cases.

(1) $k\leq k_1<k_2$,
(2) $k_1<k<k_2$,
(3) $k_1<k=k_2$,
(4) $k_1<k_2<k$,
(5) $k<k_2\leq k_1$,
(6) $k=k_2<k_1$,
(7) $k_2<k\leq k_1$,

(8) $k_2 \leq k_1 < k$,

(9) $k = k_2 = k_1$.

Using (2.5) we have

$$(1)+(8) = (2)+(4) = (3) = (5)+(7) = (6) = (9) = 0. \qquad \square$$

For $\lambda = (\lambda_{jk}) \in \boldsymbol{C}(^\times)^m$, let S_λ, T_λ denote the automorphisms of $\mathcal{W}$

$$S_\lambda(x_{jk}) = \lambda_{jk} x_{jk}, \quad S_\lambda(z_{jk}) = z_{jk} ,$$
$$T_\lambda(x_{jk}) = x_{jk} , \quad T_\lambda(z_{jk}) = \lambda_{jk} z_{jk} .$$

Proposition 2.4. *For $r, s, \tilde{r}, \tilde{s} \in \boldsymbol{C}(^\times)^n$, there exists a $\lambda \in (\boldsymbol{C}^\times)^m$ such that*

$$T_\lambda \circ \rho_{r,s} = \rho_{\tilde{r},\tilde{s}} \tag{2.10}$$

if and only if $r_i s_{n+1-i} = \tilde{r}_i \tilde{s}_{n+1-i}$ $(1 \leq i \leq n)$.

Before the proof we prepare

Lemma 2.5. *Given $d_j \in \boldsymbol{C}^\times$ $(1 \leq j \leq l)$, consider the equations for the unknowns λ_{jk} $(1 \leq j \leq k \leq l)$*

$$d_j = \lambda_{jk} \lambda_{j\,k-1} (\lambda_{j-1\,k-1} \lambda_{j+1\,k})^{-1} \qquad (1 \leq j \leq k \leq l) \tag{2.11}$$

where we set $\lambda_{jk} = 1$ unless $1 \leq j \leq k \leq l$. Define $d_{jk} = d_j d_{j+1} \cdots d_k$ for $1 \leq j \leq k \leq l$. Then (2.11) *has a unique solution given by*

$$\lambda_{jk} = d_{jk} d_{j-1\,k-1} \cdots d_{1\,k-j+1} . \tag{2.12}$$

Setting $\lambda^*_{jk} = \lambda^{-1}_{k+1-j\,k}$ we have

$$\lambda_{jk} \lambda^*_{jk} = 1 . \tag{2.13}$$

Proof. Set

$$\mu_{jk} = \lambda_{jk} / \lambda_{j-1\,k-1} . \tag{2.14}$$

The equation reads as

$$d_j = \mu_{jk} / \mu_{j+1\,k} .$$

This equation has a unique solution given by

$$\mu_{jk} = d_{jk} .$$

Therefore, from (2.14) we have

$$\lambda_{jk} = d_{jk} \lambda_{j-1\,k-1} = d_{jk} d_{j-1\,k-1} \cdots d_{1\,k-j+1} .$$

This is (2.12). Substituting the solution we obtain

$$\begin{aligned}\frac{\lambda_{jk}}{\lambda_{jk-1}} &= \frac{d_{jk}d_{j-1\,k-1}\cdots d_{1\,k-j+1}}{d_{jk-1}d_{j-1\,k-2}\cdots d_{1\,k-j}} \\ &= d_{k+1-j\,k} \\ &= \frac{\lambda_{k+1-j\,k}}{\lambda_{k-j\,k-1}} \\ &= \frac{\lambda^*_{j\,k-1}}{\lambda^*_{jk}}\,.\end{aligned}$$

From this follows (2.13). □

Proof of Proposition 2.4. The condition (2.10) is equivalent to the following equations.

$$\tilde{r}_i = r_i\lambda_{ik}\lambda_{i\,k-1}\lambda_{i-1\,k-1}^{-1}\lambda_{i+1\,k}^{-1}\,, \qquad (1\le i\le k\le n) \tag{2.15a}$$

$$\tilde{s}_i^* = s_i^*\lambda_{ik}^*\lambda_{i\,k-1}^*{\lambda_{i-1\,k-1}^*}^{-1}{\lambda_{i+1\,k}^*}^{-1}\,, \qquad (1\le i\le k\le n) \tag{2.15b}$$

$$\frac{\tilde{r}_i}{\tilde{s}_i} = \frac{r_i}{s_i}\lambda_{in}^2\lambda_{i-1\,n}^{-1}\lambda_{i+1\,n}^{-1}\,. \qquad (1\le i\le n)\,. \tag{2.15c}$$

Set $d_i=\tilde{r}_i/r_i$, $d_i'=\tilde{s}_i^*/s_i^*$ and $l=n$ in Lemma 2.5. The equation (2.15*a*) is solved by

$$\lambda_{ik} = d_{ik}d_{i-1\,k-1}\cdots d_{1\,k-i+1}\,,$$

and the equation (2.15*b*) is solved by

$$\lambda_{ik}^* = d_{ik}'d_{i-1\,k-1}'\cdots d_{i\,k-i+1}'\,.$$

Since $\lambda_{ik}=\lambda_{k+1-i\,k}$, these two solutions are consistent with the definition $\lambda_{ik}^*=\lambda_{k-i+1\,k}^{-1}$ if and only if $d_id_i'=1$. This is equivalent to $r_is_{n+1-i}=\tilde{r}_i\tilde{s}_{n+1-i}$. Finally, (2.15*c*) follows from (2.15*a*) and (2.15*b*) with $k=n$. □

§3. Finite-dimensional Representations

Fix a positive odd integer $N\ge 3$. Let ω be a primitive N-th root of unity, and let $\Phi_N(q)$ denote the N-th cyclotomic polynomial so that $\Phi_N(\omega)=0$. We set

$$\mathcal{A} = \{f\in \boldsymbol{C}(q)\,|\, f \text{ is regular at } \Phi_N(q)=0\}\,.$$

Let $U_{\mathcal{A}}$ denote the $\mathcal{A}$-subalgebra of U_q generated by e_i, f_i, t_i $(1\le i\le n)$. Let further $U_\omega=U_{\mathcal{A}}\otimes_{\mathcal{A}}\boldsymbol{C}_\omega$, where $\boldsymbol{C}_\omega$ denotes the $\mathcal{A}$-algebra $\boldsymbol{C}$ on which q acts as ω. We define $\mathcal{W}_{\mathcal{A}}$, $\mathcal{W}_\omega$ analogously.

Consider an N-dimensional vector space with fixed basis u_i $(0\le i\le N-1)$.

$$V^1 = \otimes_{i=0}^{N-1} Cu_i\,.$$

We define the following representation σ of the Weyl algebra $\mathcal{W}^1_\omega$ with generators x, z:

$$\sigma\colon \mathcal{W}^1_\omega \to \mathrm{End}(V^1_\omega)\,,$$
$$\sigma(x)u_i = u_{i+1}\ (u_N = u_0)\,,\quad \sigma(z)u_i = \omega^i u_i\,.$$

Set $m=n(n+1)/2$ and $V=(V^1_\omega)^{\otimes m}$. Then we have a representation $\sigma^{\otimes m}\colon \mathcal{W}_\omega \cong (\mathcal{W}^1_\omega)^{\otimes m} \to \mathrm{End}(V)$ by letting the generators x_{jk}, z_{jk} act on the (j, k)-component of V as $\sigma(x)$, $\sigma(z)$ and as identity on the other components. Composition with $\rho_{r,s}\colon U_\omega \to \mathcal{W}_\omega$ and the automorphisms S_g, T_h gives rise to a representation

$$\pi\colon U_\omega \xrightarrow{\rho_{r,s}} \mathcal{W} \xrightarrow{S_g\circ T_h} \mathcal{W} \xrightarrow{\sigma^{\otimes m}} \mathrm{End}(V)\,. \tag{3.1}$$

Besides r, $s\in(\boldsymbol{C}^\times)^n$, π contains $n(n+1)$ arbitrary parameters $g=(g_{jk})$, $h=(h_{jk})\in(\boldsymbol{C}^\times)^m$. In view of Proposition 2.4, these parameters are not all independent, and we can set e.g. $s_i=1$ $(1\le i\le n)$ without loss of generality.

The goal of this section is to show that (3.1) is irreducible for generic choice of the parameters r_i, g_{jk} and h_{jk}. For this purpose we prepare some lemmas. In what follows we set

$$\left\{\begin{matrix} x;\ \mu \\ \nu \end{matrix}\right\} = \frac{\{xq^\mu\}\{xq^{\mu-1}\}\cdots\{xq^{\mu-\nu+1}\}}{[\nu]!}\,.$$

Lemma 3.1. *For any positive integer l we have in $\mathcal{W}$*

$$\rho_{r,s}(e_i^{(l)}) = \sum_{p=1}^{l} \sum_{\substack{n\ge k_1>\cdots>k_p\ge i \\ l=\nu_1>\cdots>\nu_p\ge 1}} \prod_{r=1}^{p} \left\{\begin{matrix} \zeta_{ik_r};\ -\nu_{r+1} \\ \nu_r-\nu_{r+1} \end{matrix}\right\} \prod_{r=1}^{p} \xi_{ik_r}^{\nu_r-\nu_{r+1}}\,,$$

where $\nu_{p+1}=0$.

Proof. We use the induction on l. The case $l=1$ follows from the definition.

Assuming the Lemma for l, we can calculate $\rho_{r,s}(e_i^{l+1})/[l]!$ as

$$\sum_{k\ge i}\sum_{p=1}^{l} \sum_{\substack{n>k_1>\cdots>k_p\ge i \\ l=\nu_1>\cdots>\nu_p\ge 1}} \{\zeta_{ik}\}\xi_{ik} \prod_{r=1}^{p} \left\{\begin{matrix} \zeta_{ik_r};\ -\nu_{r+1} \\ \nu_r-\nu_{r+1} \end{matrix}\right\} \prod_{r=1}^{p} \xi_{ik_r}^{\nu_r-\nu_{r+1}}\,.$$

Since $\xi_{ik}\zeta_{il}=\zeta_{il}\xi_{ik}q^{-\theta(l\ge k)-\theta(l>k)}$. the summand becomes

$$\{\zeta_{ik}\} \prod_{r=1}^{p} \left\{\begin{matrix} \zeta_{ik_r};\ -\nu_{r+1}-\theta(k_r\ge k)-\theta(k_r>k) \\ \nu_r-\nu_{r+1} \end{matrix}\right\}\cdot\xi_{ik} \prod_{r=1}^{p} \xi_{ik_r}^{\nu_r-\nu_{r+1}}\,. \tag{3.2}$$

We divide the sum into two parts according to whether (i) $k=k_s$ with some s, or (ii) $k_{s-1}>k>k_s$ with some s. In the first case let $k'_j=k_j$ $(1\leq j\leq p)$, $\nu'_j=\nu_j+\theta(j\leq s)$. In the second case replace p by $p-1$, and set $k'_j=k_j$, $\nu'_j=\nu_j+1$ $(j<s)$, $k'_s=k$, $\nu'_s=\nu_s+1$, $k'_j=k_{j-1}$, $\nu'_j=\nu_{j-1}$ $(s<j\leq p)$. Then $l+1\geq\nu'_1>\cdots>\nu'_p\geq 1$, and the cases (i), (ii) correspond to $\nu'_s-\nu'_{s+1}>1$ or $=1$, respectively. Dropping primes and rewriting the summand, we get in both cases

$$\prod_{r<s}\begin{Bmatrix}\zeta_{ik_r};\ -\nu_{r+1}-1\\ \nu_r-\nu_{r+1}\end{Bmatrix}\cdot\begin{Bmatrix}\zeta_{ik_s};\ -\nu_{s+1}-1\\ \nu_s-\nu_{s+1}-1\end{Bmatrix}\{\zeta_{ik_s}\}\cdot\prod_{r>s}\begin{Bmatrix}\zeta_{ik_r};\ -\nu_{r+1}\\ \nu_r-\nu_{r+1}\end{Bmatrix}\prod_{r=1}^{p}\xi_{ik_r}^{\nu_r-\nu_{r+1}}.$$

It remains to show that

$$\sum_{s=1}^{p}\prod_{r<s}\begin{Bmatrix}\zeta_{ik_r};\ -\nu_{r+1}-1\\ \nu_r-\nu_{r+1}\end{Bmatrix}\cdot\begin{Bmatrix}\zeta_{ik_s};\ -\nu_{s+1}-1\\ \nu_s-\nu_{s+1}-1\end{Bmatrix}\{\zeta_{ik_s}\}\cdot\prod_{r>s}\begin{Bmatrix}\zeta_{ik_r};\ -\nu_{r+1}\\ \nu_r-\nu_{r+1}\end{Bmatrix}$$
$$=[l+1]\prod_{r=1}^{p}\begin{Bmatrix}\zeta_{ik_r};\ -\nu_{r+1}\\ \nu_r-\nu_{r+1}\end{Bmatrix}.$$

The sum for $s=p-1$, p reads

$$\prod_{r<p-1}\begin{Bmatrix}\zeta_{ik_r};\ -\nu_{r+1}-1\\ \nu_r-\nu_{r+1}\end{Bmatrix}\left(\begin{Bmatrix}\zeta_{ik_{p-1}};\ -\nu_p-1\\ \nu_{p-1}-\nu_p-1\end{Bmatrix}\{\zeta_{ik_{p-1}}\}\begin{Bmatrix}\zeta_{ik_p};\ 0\\ \nu_p\end{Bmatrix}\right.$$
$$\left.+\begin{Bmatrix}\zeta_{ik_{p-1}};\ -\nu_p-1\\ \nu_{p-1}-\nu_p\end{Bmatrix}\begin{Bmatrix}\zeta_{ik_p};\ 0\\ \nu_p\end{Bmatrix}[\nu_p]\right)$$
$$=\prod_{r<p-1}\begin{Bmatrix}\zeta_{ik_r};\ -\nu_{r+1}-1\\ \nu_r-\nu_{r+1}\end{Bmatrix}\begin{Bmatrix}\zeta_{ik_{p-1}};\ -\nu_p\\ \nu_{p-1}-\nu_p\end{Bmatrix}[\nu_{p-1}]\begin{Bmatrix}\zeta_{ik_p};\ -\nu_{p+1}\\ \nu_p-\nu_{p+1}\end{Bmatrix}.$$

The assertion follows by repeating this procedure. □

Lemma 3.2. *Let A, B be elements in an associative algebra over $\boldsymbol{C}(q)$, satisfying the relations*

$$A^{(2)}B-ABA+BA^{(2)}=0\,,$$
$$B^{(2)}A-BAB+AB^{(2)}=0\,.$$

Set $C=AB-qBA$. Then for any positive integers k, l we have

(i) $C^{(l)}=\sum_{j=0}^{l}(-q)^{l-j}A^{(j)}B^{(l)}A^{(l-j)}$,

(ii) $B^{(k)}A^{(l)}=\sum_{0\leq j\leq k,l}(-1)^j q^{-j-(k-j)(l-j)}A^{(l-j)}C^{(j)}B^{(k-j)}$,

(iii) $AB^{(k)}A^{(l)}=B^{(k-1)}A^{(l+1)}B+[l-k+1]B^{(k)}A^{(l+1)}$.

Proof. Formulas (i) and (ii) are proved in [Lu]. Formula (ii) with $k=1$ together with $AC=q^{-1}CA$ gives

$$BA^{(l)}=q^{-l}(A^{(l)}B-CA^{(l-1)})\,.$$

Multiplying by A from the left and using the definition of C, we get

$$ABA^{(l)} = A^{(l+1)}B+[l]BA^{(l+1)}\,. \tag{3.3}$$

Similarly we have

$$B^{(l)}AB = AB^{(l+1)}+[l]B^{(l+1)}A\,.$$

Then multiplying eq.(3.3) by $B^{(k-1)}$ from the left and using the above identity we obtain (iii). □

Now consider the following 'monomials' in $\mathcal{W}$:

$$\prod z_{jk}^{\alpha_{jk}} \prod x_{jk}^{\beta_{jk}} \qquad \alpha_{jk},\ \beta_{jk}\in \boldsymbol{Z}\,.$$

For convenience we shall call an element $f\in\mathcal{W}$ of *type I* if it is an $\mathcal{A}$-linear combination of monomials with α_{jk}, $\beta_{jk}\in N\boldsymbol{Z}$, and of *type II* if $f\in\Phi_N(q)\mathcal{W}_{\mathcal{A}}$.

Lemma 3.3. *For any $i\neq j$, $\rho_{r,s}(e_{ij}^N)\in\mathcal{W}$ is a sum of elements of type I and of type II.*

Proof. Assuming $i<j$ we prove the Lemma by induction on $j-i$. The case $i>j$ follows by applying the involutions $*\circ\wedge$.

Let $j-i=1$. From Lemma 3.1 we have

$$\rho_{r,s}(e_{i\,i+1}^N) = \sum_{i\leq k\leq n} \prod_{\nu=0}^{N-1} \{\zeta_{ik}q^{-\nu}\}\xi_{ik}^N + (\text{elements of type II})\,.$$

In view of the formula

$$\prod_{j=0}^{N-1}(xq^{-j}-x^{-1}q^j) = x^N q^{-N(N-1)/2} - x^{-N} q^{N(N-1)/2} + \sum_{0<\nu<N}\begin{bmatrix} N \\ \nu \end{bmatrix}(-1)^\nu q^{-N(N-1)/2+(N-1)\nu} x^{N-2\nu}$$

we find that the assertion is true in this case.

Next let us prove the case $j-i>1$. Taking

$$A = e_{ik}\,,\quad B = e_{kj}\,,\quad C = e_{ij} \qquad (i<k<j)$$

we can apply Lemma 3.2 to obtain

$$\begin{aligned} e_{ij}^N &= \frac{e_{i\,i+1}^N e_{i+1\,j}^N - q^N e_{i+1\,j}^N e_{i\,i+1}^N}{[N]!} + \sum_{0<k<N} (-q)^{N-k} \frac{e_{i\,i+1}^k}{[k]!} e_{i+1\,j}^N \frac{e_{i\,i+1}^{N-k}}{[N-k]!} \\ &= \frac{[e_{i\,i+1}^N,\, e_{i+1\,j}^N]}{[N]!} + \frac{(1-q^N)}{[N]!} e_{i+1\,j}^N e_{i\,i+1}^N \\ &\quad + \sum_{0<k<N} (-q)^{N-k} \frac{e_{i+1\,j}^N e_{i\,i+1}^N + [e_{i\,i+1}^k,\, e_{i+1\,j}^N] e_{i\,i+1}^{N-k}}{[k]![N-k]!}\,. \end{aligned}$$

From the induction hypothesis it suffices to show the following.

(i) If f, g are of type I, then so are fg and $[f, g]/[N]!$,

(ii) If f is of type I and g is of type II, then fg, gf and $[f, g]/[N]!$ are of type II,

(iii) If f is either of type I or type II and g is a monomial, then $[f, g]$ is of type II.

To show (i)–(iii), let $f=f_0 \prod z_{jk}^{\alpha_{jk}} \prod x_{jk}^{\beta_{jk}}$, $g=g_0 \prod z_{jk}^{\alpha'_{jk}} \prod x_{jk}^{\beta'_{jk}}$, $f_0, g_0 \in \mathcal{A}$. Then we have

$$fg = f_0 g_0 \prod q^{-\alpha'_{jk}\beta_{jk}} z_{jk}^{\alpha_{jk}+\alpha'_{jk}} \prod x_{jk}^{\beta_{jk}+\beta'_{jk}}$$

$$\frac{[f, g]}{[N]!} = \frac{q^{-\sum \alpha'_{jk}\beta_{jk}} - q^{-\sum \alpha_{jk}\beta'_{jk}}}{[N]!} f_0 g_0 \prod z_{jk}^{\alpha_{jk}+\alpha'_{jk}} \prod x_{jk}^{\beta_{jk}+\beta'_{jk}}.$$

The assertions (i)-(iii) are clear from these. □

We now specialize q to ω. It is clear that the image under $\sigma^{\otimes m}$ of elements of type I are scalar, while those of type II vanish. From this we have the following.

Proposition 3.4. *We have*

$$\pi(t_i^N) = Y_{ii}\mathrm{id}. \qquad (1 \leq i \leq n),$$
$$\pi(e_{ij}^N) = (\omega - \omega^{-1})^{-N} Y_{ij}\mathrm{id}. \qquad (1 \leq i \neq j \leq n+1),$$

where $Y_{ij} \in \mathbf{C}$ are given as follows.

$$Y_{ii} = (r_i s_i^{-1} h_{i\,n}^2 h_{i-1\,n}^{-1} h_{i+1\,n}^{-1})^N \tag{3.4a}$$

$$Y_{i\,i+l} = \sum_{k_1 \geq i, \cdots, k_l \geq i+l-1} \sum_{j=1}^{l} (-1)^{j-1} \theta(k_1 \geq \cdots \geq k_j < \cdots < k_l)$$
$$\eta_{i\,k_1 \cdots k_{j-1}} (\eta_{i+j-1\,k_j} - \eta_{i+j-1\,k_j}^{-1}) \eta_{i+j\,k_{j+1} \cdots k_l}^{-1} \theta_{i\,k_1 \cdots k_l} \tag{3.4b}$$

$$Y_{n-i+2\;n-i+2-l} = -\sum_{k_1 \geq i, \cdots, k_l \geq i+l-1} \sum_{j=1}^{l} (-1)^{j-1} \theta(k_1 \geq \cdots \geq k_j < \cdots < k_l)$$
$$\bar{\eta}_{i\,k_1 \cdots k_{j-1}} (\bar{\eta}_{i+j-1\,k_j} - \bar{\eta}_{i+j-1\,k_j}^{-1}) \bar{\eta}_{i+j\,k_{j+1} \cdots k_l}^{-1} \bar{\theta}_{i\,k_1 \cdots k_l} \tag{3.4c}$$

where we set $\bar{g}_{jk} = g_{k+1-j\,k}^{-1}$, $\bar{h}_{jk} = h_{k+1-j\,k}$, $\bar{r}_i = s_{n+1-i}^{-1}$,

$$\eta_{ik} = (r_i h_{ik} h_{ik-1} h_{i-1k-1}^{-1} h_{i+1k}^{-1})^N$$
$$\bar{\eta}_{ik} = (s_i \bar{h}_{ik} \bar{h}_{i\,k-1} \bar{h}_{i-1\,k-1}^{-1} \bar{h}_{i+1\,k}^{-1})^N$$
$$\theta_{ik} = \prod_{l=k}^{n} g_{il}^N, \quad \bar{\theta}_{ik} = \prod_{l=k}^{n} (\bar{g}_{il})^N$$

and $\phi_{i\,k_1 \cdots k_l} = \phi_{ik_1} \cdots \phi_{i+l-1\,k_l}$ for $\phi = \eta, \bar{\eta}, \theta, \bar{\theta}$.

Let us now examine the irreducibility of the above representaitons of U_ω.

Hereafter we assume $s_i=1$ without loss of generality. We denote by Y the map $\boldsymbol{C}^{2m+n}\to\boldsymbol{C}^{2m+n}$, $(g_{ij}, h_{ij}, r_i)\mapsto(Y_{ij})$ defined in Proposition 3.4.

Let Z_0 denote the subring of the center of U_ω generated by $t_i^{\pm N}$ and e_{jk}^N, and let Rep be the set of equivalence classes of irreducible representations of U_ω. De Concini and Kac show [DK] that

(i) Z_0 is a polynomial ring in $n(n+2)$ indeterminates,

(ii) the natural map X': Rep $\to$ Spec $Z_0\cong\boldsymbol{C}^{n(n+2)}$ has finite fibre,

(iii) $\dim R\leq N^m$ for all $R\in$ Rep, the equality being true for generic $X'(R)$.

To show the irreducibility of our representations π (for generic values of parameters), it is therefore sufficient to show that the image of Y contains an open set, since our representations are N^m dimensional. We shall show that the Jacobian determinant of Y at the point P: $g_{ij}=h_{ij}=1$ $(1\leq i\leq j\leq n)$, $r_i=t$ $(1\leq i\leq n)$ does not vanish (for generic t)

It is easy to see that the Jacobian matrix at P is of triangular form.

$$\left.\frac{\partial((Y_{ij})_{1\leq i\leq j\leq n+1})}{\partial((h_{ij})_{1\leq i\leq j\leq n},(r_i)_{1\leq i\leq n},(g_{ij})_{1\leq i\leq j\leq n}))}\right|_P$$

$$= \begin{array}{c} (h_{ij})_{1\leq i\leq j\leq n} \\ (r_i)_{1\leq i\leq n} \\ (g_{ij})_{1\leq i\leq j\leq n}\end{array}\overset{\begin{array}{ccc}(Y_{ij})_{1\leq j<i\leq n+1} & (Y_{ii})_{1\leq i\leq n} & (Y_{ij})_{1\leq i<j\leq n+1}\end{array}}{\begin{pmatrix} H & * & * \\ 0 & R & * \\ 0 & 0 & G \end{pmatrix}}.$$

We shall show that H, R, G are non-singular. From (3.4a) it follows that R is $Nt^{N-1}\mathrm{id}_N(\mathrm{id}_N$: $N\times N$ identity matrix).

Proposition 3.5. *The Jacobian matrix H is non-singular.*

Proof. The change of variables $(\bar{h}_{ij})_{1\leq i\leq j\leq n}\to(\bar{h}_{ij})_{1\leq i\leq j\leq n}$ is invertible at $h_{ij}=1$ $(1\leq i\leq j\leq n)$. The change of variables $(\bar{h}_{ij})_{1\leq i\leq j\leq n}\to(\bar{h}_{ij})_{1\leq i\leq j\leq n}$ where $\bar{\eta}_{ij}=\bar{h}_{ij}\bar{h}_{i\,j-1}\bar{h}_{i-1\,j-1}^{-1}\bar{h}_{i+1\,j}^{-1}$ is also invertible. In fact, the inverse map is given inductively by

$$\bar{h}_{ij}=\bar{\eta}_{ij}\bar{\eta}_{i+1j}\cdots\bar{\eta}_{jj}\bar{h}_{i-1j-1}.$$

Therefore, it is sufficient to show that the Jacobian matrix

$$\left.\frac{\partial((Y_{n+2-i,n+1-j})_{1\leq i\leq j\leq n})}{\partial((\bar{\eta}_{ij})_{1\leq i\leq j\leq n})}\right|_P$$

is non-singular. We have

$$\left.\frac{\partial Y_{n+2-i,n+1-j}}{\partial\bar{\eta}_{kl}}\right|_P = -2\sum_{\substack{k_1\geq i\\ \cdots\\ k_{j-i+1}\geq j}}\sum_{s=1}^{j-i+1}(-1)^{s-1}\delta_{k\,i+s-1}\delta_{lk_s}\theta(k_1\geq\cdots\geq k_s<\cdots<k_{j-i+1})$$

$$= -2(-1)^{k-i}\binom{n-l+k-i}{k-i}\binom{n-l}{j-k}.$$

Here we have adopted the following convention for the usual binomial coefficients

$$\binom{j}{k}=0 \qquad \text{if} \quad k<0 \text{ or } j<k\,.$$

If $j<k$ then $\partial Y_{n+2-i,n+1-j}/\partial\bar{\eta}_{kl}=0$ at P. Therefore, it is sufficient to show that the diagonal blocks $\binom{n-l}{j-k}_{\substack{k\leq j\leq n\\ k\leq l\leq n}}$ $(1\leq k\leq n)$ are non singular. Note that

$$\binom{n-l}{j-k}=0 \qquad \text{if} \quad l+j>n+k$$

$$=1 \qquad \text{if} \quad l+j=n+k\,.$$

The assertion follows immediately. □

Proposition 3.8. *The Jacobian matrix G is non-singular.*

Proof. We have

$$\frac{\partial Y_{i,i+l}}{\partial g_{r,s}}=\sum_{k_i\geq i,\cdots,k_l\geq i+l-1}\sum_{j=1}^{l}(-1)^{j-1}\theta(k_1\geq\cdots\geq k_j<\cdots<k_l)$$

$$\times t^{(2j-l-1)N}(t^N-t^{-N})\frac{\partial\theta_{ik_1\cdots k_l}}{\partial g_{r,s}},$$

$$\frac{\partial\theta_{ik}}{\partial g_{r,s}}=N\delta_{ir}\theta(s\geq k)\,.$$

In the limit $t\to 0$, the leading terms of the right hand side come from $j=1$. Setting $l=j+1-i$ and $r'=r-i+1$ we get

$$\frac{\partial Y_{i,j+1}}{\partial g_{r,s}}=-Nt^{-(j-i+1)N}(D_{ij,rs}+O(t))$$

where

$$D_{ij,rs}=\sum_{i\leq k_1<\cdots<k_{j-i+1}}\theta(s\geq k_{r'})$$

$$=\sum_{r\leq k\leq s}\ \sum_{i\leq k_1<\cdots<k_{r'-1}<k}\ \sum_{k<\cdots<k_{j-i+1}\leq n}1$$

$$=\sum_{r\leq k\leq s}\binom{k-i}{r-i}\binom{n-k}{j-r}. \tag{3.5}$$

It remains to show that $\det(D_{ij,rs})_{\substack{1\le i\le j\le n\\1\le r\le s\le n}}\neq 0$. Since the summand of (3.5) is independent of s, it is easy to see that $\det(D_{ij,rs})=\det(D'_{ij,rs})$ where $D'_{ij,rs}=\binom{s-i}{r-i}\binom{n-s}{j-r}$. This matrix is block triangular since $D'_{ij,rs}=0$ if $r<i$. Further if $r=i$ we have $\det\,(D'_{ij,is})_{i\le j,s\le n}=1$. This completes the proof. □

If we specialize the parameters, we get invariant subspaces. Let l be an integer such that $0\le l\le n-1$ and let i be an integer such that $0\le i\le N-1$. Set

$$u_{i,l}=\underbrace{u_i\otimes\cdots\otimes u_i}_{(j,\,k)\text{-component},\,1\le j\le k\le l},$$

and

$$V_{i,l}=u_{i,l}\otimes\underbrace{V^1\otimes\cdots\otimes V^1}_{(j,k)\text{-component},\,1\le j\le k,\,l<k\le n}.$$

Fix the parameters r_j $(1\le j\le l)$ and $h_{j\,k}$ $(1\le j\le k\le l)$ in such a way that

$$r_j=\omega^{-2}, \tag{3.6a}$$
$$h_{jk}h_{j\,k-1}h^{-1}_{j-1\,k-1}h^{-1}_{j+1\,k}=\omega^{1-i\delta_{j,1}} \qquad \text{for}\quad 1\le j\le k\le l, \tag{3.6b}$$
$$h^*_{jk}h^*_{j\,k-1}h^{*-1}_{j-1\,k-1}h^{*-1}_{j+1\,k}=\omega^{-1+i\delta_{j,1}} \qquad \text{for}\quad 1\le j\le k\le l, \tag{3.6c}$$

where $h^*_{jk}=h^{-1}_{k+1-j\,k}$.

From Lemma 2.5, (3.6*b*) and (3.6*c*) are consistent and have a unique solution. The parameters r_i $(l<i\le n)$, g_{ik} $(1\le i\le k\le n)$ and h_{ik} $(1\le j\le k,\ l<k\le n)$ are free.

Proposition 3.9. *The subspace $V_{i,l}$ is U_ω invariant.*

Proof. It is easy to see that $V_{i,l}$ is invariant by e_j, f_{n+1-j} with $l<j\le n$ and t_i for all i. If $1\le j\le l$, we have

$$\rho_{r,s}(e_j)=\sum_{k\ge j}\xi_{jk}\{\omega^{-1}z_{jk}z_{j\,k-1}z^{-1}_{j-1\,k-1}z^{-1}_{j+1\,k}\},$$
$$\rho_{r,s}(f_{n+1-j})=\sum_{k\ge j}\xi^*_{jk}\{\omega z^*_{jk}z^*_{j\,k-1}z^{*-1}_{j-1\,k-1}z^{*-1}_{j+1\,k}\}.$$

If $1\le j\le k\le l$, then

$$z_{jk}z_{j\,k-1}z^{-1}_{j-1\,k-1}z^{-1}_{j+1\,k}u_{i,l}=\omega^{i\delta_{j1}}u_{i,l},$$
$$z^*_{jk}z^*_{j\,k-1}z^{*-1}_{j-1\,k-1}z^{*-1}_{j+1\,k}u_{j,l}=\omega^{-i\delta_{j1}}u_{i,l},$$

and therefore, by using (3.6*b*), (3.6*c*) and $\{1\}=0$,

$$\sigma^{\otimes m}\circ S_g\circ T_h(\{\omega^{-1}z_{jk}z_{j\,k-1}z^{-1}_{j-1\,k-1}z^{-1}_{j+1\,k}\})u_{i,l}=0,$$
$$\sigma^{\otimes m}\circ S_g\circ T_h(\{\omega z^*_{jk}z^*_{j\,k-1}z^{*-1}_{j-1\,k-1}z^{*-1}_{j+1\,k}\})u_{i,l}=0.$$

Thus we have shown that $V_{i,l}$ is invariant by the action of e_j and f_j. □

In [DJMM], we have constructed an N^n dimensional irreducible representation. We shall show that it is isomorphic to the representation on $V_{0,n-1}$. First we recall the construction in [DJMM]. Let $W=(V^1)^{\otimes(n+1)}$ and let Z_j and X_j $(1\leq j\leq n+1)$ be the operators acting as $\sigma(z)$ and $\sigma(x)$ on the j-th component of W. Set $W^{(0)}=\{u\in W\mid \prod_{j=1}^{n+1} Z_j u=u\}$. Define π': $U_\omega\to \mathrm{End}_C(W^{(0)})$ by

$$\pi'(e_j)=b_j\{a_jZ_j\}X_jX_{j+1}^{-1},$$
$$\pi'(f_j)=b_j^{-1}\{a_{j+1}Z_{j+1}\}X_j^{-1}X_{j+1},$$
$$\pi'(t_j)=\frac{a_j}{a_{j+1}}Z_jZ_{j+1}^{-1}.$$

Now consider $V_{0,n-1}\cong(V^1)^{\otimes n}$. Let C_j and Ω_j $(1\leq j\leq n)$ be the operators acting as $\sigma(z)$ and $\sigma(x)$ on the j-th component of $(V^1)^{\otimes n}$. Then the subrepresentation of π on $V_{0,n-1}$, which we denote also by π, reads as

$$\pi(e_j)=g_{jn}\{r_jh_{jn}h_{j\,n-1}h_{j-1\,n-1}^{-1}h_{j+1\,n}^{-1}C_jC_{j+1}^{-1}\}\Omega_j,$$
$$\pi(f_j)=g_{jn}^{-1}\{h_{jn}^{-1}h_{j-1\,n-1}^{-1}h_{j\,n-1}h_{j-1\,n}C_j^{-1}C_{j-1}\}\Omega_j^{-1},$$
$$\pi(t_j)=r_jh_{jn}^2h_{j-1\,n}^{-1}h_{j+1\,n}^{-1}C_j^2C_{j-1}^{-1}C_{j+1}^{-1}.$$

Choose d_j $(1\leq j\leq n)$ in such a way that

$$\frac{d_j^N}{d_{j+1}^N}\,\frac{a_j^N-a_j^{-N}}{a_{j+1}^N-a_{j+1}^{-N}}=1. \tag{3.7}$$

For $1\leq j\leq n$, define mutually commuting operators

$$\begin{aligned}\Phi_j&=(f_j(Z_j)X_j)(f_{j+1}(Z_{j+1})X_{j+1})^{-1}\\&=\frac{f_j(Z_j)}{f_{j+1}(\omega Z_{j+1})}X_jX_{j+1}^{-1}=\left(\frac{f_{j+1}(Z_{j+1})}{f_j(\omega Z_j)}X_{j+1}X_j^{-1}\right)^{-1},\end{aligned}$$

where

$$f_j(Z_j)=d_j\{a_jZ_j\}.$$

The equation (3.7) implies $\Phi_j^N=1$.

Define a linear isomorphism

$$\psi\colon (V^1)^{\otimes n}\to W^{(0)}$$

by

$$\psi(u_{p_1}\otimes\cdots\otimes u_{p_n})=\prod_{j=1}^{n}\Phi_j^{p_j}u_0^{\otimes(n+1)}.$$

We denote also by ψ the induced algebra homomorphism

$$\psi\colon \mathrm{End}_C((V^1)^{\otimes n}) \to \mathrm{End}_C(W^{(0)}) .$$

Then we have

$$\psi(C_j C_{j-1}^{-1}) = Z_j , \tag{3.8a}$$

$$\psi(\Omega_j) = \Phi_j . \tag{3.8b}$$

We shall show that by a suitable choice of the parameters h_{jn}, g_{jn} $(1\leq j\leq n)$ and r_n we have

$$\psi\circ\pi(e_j) = \pi'(e_j) , \tag{3.9a}$$

$$\psi\circ\pi(f_j) = \pi'(f_j) , \tag{3.9b}$$

$$\psi\circ\pi(t_j) = \pi'(t_j) . \tag{3.9c}$$

The equations (3.9a) and (3.9b) give rise to

$$b_j\{a_j Z_j\} = g_{jn}\{r_j h_{jn}\, h_{j\,n-1} h_{j-1\,n-1}^{-1} h_{j+1\,n}^{-1} Z_{j+1}^{-1}\} \frac{d_j\{a_j Z_j\}}{d_{j+1}\{\omega a_{j+1} Z_{j+1}\}} ,$$

$$b_j^{-1}\{a_{j+1} Z_{j+1}\} = g_{jn}^{-1}\{h_{jn}^{-1} h_{j-1\,n-1}^{-1} h_{j\,n-1} h_{j-1\,n} Z_j^{-1}\} \frac{d_{j+1}\{a_{j+1} Z_{j+1}\}}{d_j\{\omega a_j Z_j\}} .$$

These are satisfied if we choose

$$g_{jn} = -b_j d_{j+1}/d_j ,$$
$$h_{jn} h_{j-1\,n}^{-1} = \omega a_j h_{j\,n-1} h_{j-1\,n-1}^{-1} ,$$
$$r_n h_{nn} = \omega^{-1} a_{n+1}^{-1} h_{n-1\,n-1} .$$

Finally, (3.9c) is checked easily.

In closing we shall explain how we get the above representations.

For $\lambda=(\lambda_1, \cdots, \lambda_n)\in C^n$, let $M_\lambda=U_q/I$ denote the Verma module where $I=\sum_i U_q e_i+\sum_i U_q(t_i-q^{\lambda_i})$. Let $v_\lambda=1 \bmod I$ be the highest weight vector. Let further $F^k(j_{1k} \cdots j_{kk})=f_1^{(j_{1k})} \cdots f_k^{(j_{kk})}$ $(j_{rk}\in Z_{\geq 0},\ 1\leq r\leq k\leq n)$. We define

$$v_\lambda[j] = F^n(j_{1n}, \cdots, j_{nn}) \cdots F^1(j_{11})v_\lambda , \quad j = (j_{rs})_{1\leq r\leq s\leq n}\in Z^m_{\geq 0} .$$

From now on, $j_{rs}=0$ unless $1\leq r\leq s\leq n$, and by convention $F^k(j_{1k}, \cdots, j_{kk})=0$, $v_\lambda[j]=0$ if a negative value of some j_{rs} appears in the expressions. Then we have

Proposition 3.10. *Set* $\varepsilon_{jk}=(\delta_{jr}\delta_{ks})_{1\leq r\leq s\leq n}$. *The action of* e_i, f_i *and* t_i *on vectors* $v_\lambda[j]$ *is given by*

$$e_i v_\lambda[j] = \sum_{k=i}^{n} [1+\lambda_i - j_{ik} - 2\sum_{l=i}^{k-1} j_{il} + \sum_{l=i-1}^{k-1} j_{i-1\,l} + \sum_{l=i+1}^{k} j_{i+1\,l}]\, v_\lambda[j-\varepsilon_{ik}]\,,$$

$$f_i v_\lambda[j] = \sum_{k=n+1-i}^{n} [1+j_{i+k-n\,k} - j_{i+k-n-1\,k}]\, v_\lambda[j+\sum_{l\geq k} \varepsilon_{l+i-n\,l} - \sum_{l\geq k+1} \varepsilon_{l+i-n-1\,l}]\,,$$

$$t_i v_\lambda[j] = q^{\lambda_i - 2\sum_{l=i}^{n} j_{il} + \sum_{l=i-1}^{n} j_{i-1\,l} + \sum_{l=i+1}^{n} j_{i+1\,l}}\, v_\lambda[j]\,.$$

Proof. From $[e_i, f_j^{(l)}] = \delta_{ij} f_i^{(l-1)}\{q^{1-l} t_i\}$ and Lemma 3.2 with $A=f_i$ and $B=f_{i-1}$, we have

$$[e_i, F^k(j_{1k}, \cdots, j_{kk})] = \theta(k\geq i)$$
$$\times F^k(j_{1k}, \cdots, j_{i-1\,k}, j_{ik}-1, j_{i+1\,k}, \cdots, j_{kk})\{q^{1-j_{ik}+j_{i+1\,k}} t_i\}\,,$$

and for $1\leq i\leq k$

$$f_i F^k(j_{1k}, \cdots, j_{kk}) = F^k(j_{1k}, \cdots, j_{i-2\,k}, j_{i-1\,k}-1, j_{ik}+1, j_{i+1\,k}, \cdots, j_{kk}) f_{i-1}$$
$$+[j_{ik} - j_{i-1\,k}+1] F^k(j_{1k}, \cdots, j_{i-1\,k}, j_{ik}+1, j_{i+1\,k}, \cdots, j_{kk})\,.$$

Here f_0 is understood to be zero. From these the propoistion follows. □

Returning to the Weyl algebra $\mathcal{W}$, let us consider the following representation of $\mathcal{W}$ on the linear span of the symbols $V[J]$, $J\in \boldsymbol{Z}^m$:

$$x_{jk} V[J] = V[J-\varepsilon_{jk}]\,, \quad z_{jk} V[J] = q^{-J_{jk}} V[J]\,.$$

This induces a representation of U_q via $\rho_{r,s}$ with $r_i = q^{\lambda_i}$, $s_i=1$. Explicitly it reads

$$e_i V[J] = \sum_{k=i}^{n} [1+\lambda_i - J_{ik} + J_{i+1\,k} + J_{i-1\,k-1} - J_{i\,k-1}] V[J - \sum_{l\geq k} \varepsilon_{il}]\,,$$

$$f_i V[J] = \sum_{k=n+1-i}^{n} [1+J_{i+k-n\,k} - J_{i+k-n-1\,k} - J_{i+k-n\,k-1} + J_{i+k-n-1\,k-1}]$$
$$\times V[J+\sum_{l\geq k} \varepsilon_{l+i-n\,l}]\,,$$

$$t_i V[J] = q^{\lambda_i - 2J_{in} + J_{i-1\,n} + J_{i+1\,n}} V[J]\,.$$

If we formally identify $V[J] = v_\lambda[j]$ with $J_{rs} = \sum_{t=r}^{s} j_{rt}$, we see that this representation has the same form as the one for the Verma module described above. (Note that in the latter case the vectors $v_\lambda[j]$ are not linearly independent in general.)

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