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A Generalization of Eichler Integrals and Certain Local Systems over Spin Riemann Surfaces

In the memory of Professor Michio Kuga

By

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Abstract

Let $(X, K_X^{1/2})$ be a spin Riemann surface*) of genus ≥ 2. By using infinite dimensional representations of the fundamental group of X, we obtain many local systems on X, which taken together define *a resolution of the halfcanonical ring of X* and indicate a *non-abelian theory of abelian integrals on X*. The work has a root in a study of the complex structure on the Fricke moduli space [14].

*) a Riemann surface together with a halfcanonical bundle (4.1).

Summary of Results

Let $(X, K_X^{1/2})$ be a spin Riemann surface of genus ≥ 2. For each $n \geq 0$, we construct an increasing sequence $F_{X,i}^n$ $i \in \mathbf{Z}_{\geq 0}$ of finite dimensional $\mathbf{C}$-vector subsheaves contained in the $\mathcal{O}_X(K_X^{-n/2})$, regarded as local systems over X (cf. (4.3) and (5.3)). They are characterized by the following properties.

a) The initial local system $F_{X,0}^n$ for $i = 0$ is isomorphic to the one induced from n th symmetric tensor product $Sym^n(\mathbf{C}^2)$ of the vector representation space $\mathbf{C}^2$ of $SL(2, \mathbf{R})$ (i.e. $F_{X,0}^n \simeq (\mathbf{H} \times Sym^n(\mathbf{C}^2))/\pi_1(X)$).

b) Set $G_{X,i}^n := F_{X,i+1}^n / F_{X,i}^n$ for $n, i \geq 0$. Then each graded piece $G_{X,i}^n$ is a trivial local system and the initial one $G_{X,0}^n$ is isomorphic to $H^0(X, \mathcal{O}(K_X^{n/2+1}))$ for $i = 0$ (5.4.5).

c) The cohomology of the $F_{X,i}^n$ for $n, i \geq 0$ is as follows ((2.2.5), (3.5.1) and (4.4) Lemma):

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$$H^*(X, F^n_{X,i}) \simeq H^*(X, \mathcal{O}_X(K_X^{-n/2})) \oplus G^n_{X,i}[1] \oplus G^n_{X,i-1}[2].$$

d) The sum $\mathcal{G}_X := \bigoplus_i \bigoplus_n G^n_{X,i}$ forms a (infinitely generated) graded ring over (a part of) the halfcanonical ring $\mathcal{G}_{X,0} := \bigoplus_{n=0}^{\infty} G^n_{X,0}$. Each graded piece $\mathcal{G}_{X,i} = \bigoplus_n G^n_{X,i}$ for $i \geq 0$ is a finite $\mathcal{G}_{X,0}$-module (§6 (6.1) and (6.2)).

e) There are derivation maps $\delta^0 : \mathcal{G}_X \to \mathcal{G}_X \otimes_{\mathbf{Z}} H^1(X, \mathbf{Z})$ and $\delta^1 : \mathcal{G}_X \otimes_{\mathbf{Z}} H^1(X, \mathbf{Z}) \to (\mathcal{G}_X \oplus F^0_{X,0}) \otimes_{\mathbf{Z}} H^2(X, \mathbf{Z})$ (3.6) such that the following sequence (6.3.1) gives a resolution of the halfcanonical ring $\mathcal{G}_{X,0}$:

$$0 \longrightarrow \mathcal{G}_{X,0} \longrightarrow \mathcal{G}_X \xrightarrow{\delta^0} \mathcal{G}_X \otimes_{\mathbf{Z}} H^1(X) \xrightarrow{\delta^1 \oplus \iota} (\mathcal{G}_X \oplus F^0_{X,0}) \otimes_{\mathbf{Z}} H^2(X) \longrightarrow 0.$$

where the map ι to the factor $F^0_{X,0} \simeq \mathbf{C}$ is given by abelian integrals (6.3.2):

$$f \in \mathrm{Hom}_{\mathbf{Z}}(\Gamma_g, G^0_{X,0}) \longmapsto \sum_{i=1}^{g} \left(\int_{b_i} f(a_i) - \int_{a_i} f(b_i) \right) \in \mathbf{C}$$

f) Statement e) implies the following generating formula for the classes $[G^n_{X,i}]$ in the Grothendieck-group of mapping class group equivariant vector spaces ((3.7) Corollary and (4.4) Lemma).

$$\sum_{i=0}^{\infty} [G^0_{X,i}] t^i = \frac{[H^0(X, \mathcal{O}_X(K_X))] - [H^2(X, \mathbf{C}_X)] t}{1 - [H^1(X, \mathbf{C}_X)] t + [H^2(X, \mathbf{C}_X)] t^2} \quad \text{for} \quad n = 0,$$

$$\sum_{i=0}^{\infty} [G^n_{X,i}] t^i = \frac{[H^0(X, \mathcal{O}_X(K_X^{n/2+1}))]}{1 - [H^1(X, \mathbf{C}_X)] t + [H^2(X, \mathbf{C}_X)] t^2} \quad \text{for} \quad n > 0.$$

Contents

Notation. Let A be an algebra with 1 and Γ a group. An *A-Γ module* is a module with commuting left A and right Γ actions. An *A-Γ homomorphism* is a map that commutes with the A-Γ actions. A Γ-module M is *reduced* if $M^\Gamma := \{m \in M : m \cdot \gamma = m \text{ for any } \gamma \in \Gamma\}$ is equal to 0. The i th cohomology group of Γ with coefficients in an A-Γ module M is denoted by $H^i(\Gamma, M)$, which is of course an A-module.

By $\mathbf{C}$, we mean the complex number field and $\mathbf{H} := \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$ is the complex upper half plane, whose coordinate is denoted by z.

By $\mathcal{O}_X$ and K_X we denote the sheaf of germs of holomorphic functions (resp. the canonical line bundle) of a complex manifold X.

§1. Introduction

(1.1) The Teichmüller space as a real manifold is well known to carry a complex structure (Weil, Ahlfors, Bers [2]). The complex structure can be recovered by several different approaches: Earle & Eells, Beilinson, Manin & Schechtman [1], Hitchin [8], Pekornen [13], Tromba [18], Wolpert [20]. Although the approaches are different, each construction uses some analysis, or to put it better, uses some infinite-dimensional spaces. In contrast, the symplectic structure on the Teichmüller space is algebraic (cf. [7]).

(1.2) We recently showed [14] that the complex structure on the Teichmüller space can be recovered in terms of representations of Fuchsian groups into an infinite-dimensional space. The starting point of the present paper is an attempt to "approximate" this infinite-dimensional space by finite-dimensional subspaces. We first explain this in (1.2).

Let Γ_g be the surface group of genus $g \geq 2$ and $\rho: \Gamma_g \to SL(2, \mathbf{R})$ a faithful discrete representation. The ρ determines a Riemann surface $\rho(\Gamma_g)\backslash\mathbf{H}$ and conjugacy class $[\rho]$ of ρ determines a point in the moduli space of spin Riemann surfaces of genus g (cf. §4). The tangent space of the moduli at $[\rho]$ is identified with the cohomology group $H^1(\Gamma_g, \mathfrak{g})$ (Weil [19]), where Γ_g acts on $\mathfrak{g} := sl(2, \mathbf{R})$ by the adjoint action of $SL(2, \mathbf{R})$ composed with ρ. The space acquires a complex structure via the Eichler-Shimura isomorphism ([5, 15]):

$$(*) \qquad \iota_*: H^1(\Gamma_g, \mathfrak{g}) \simeq H^1(\Gamma_g, \Gamma(\mathbf{H}, \Theta)).$$

Here $\Gamma(\mathbf{H}, \Theta)$ is the complex vector space of holomorphic vector fields on the upper half-plane $\mathbf{H}$, Γ_g acts on $\Gamma(\mathbf{H}, \Theta)$ by pulling back vector fields and ι_* is induced by the infinitesimal action $\iota: \mathfrak{g} \to \Gamma(\mathbf{H}, \Theta)$, $x \mapsto (-1, z) X \begin{pmatrix} z \\ 1 \end{pmatrix} \frac{d}{dz}$. The integrability of this almost complex structure is proved directly in [14] (cf. [10]).

The Γ_g-action on $\Gamma(\mathbf{H}, \Theta)$ is neither unitary nor completely reducible. Hence one is led to look for a "small" $\mathbf{C}$-Γ_g module $\mathcal{F}$ with $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} \subset \mathcal{F} \subset \Gamma(\mathbf{H}, \Theta)$ such that the inclusion maps induce isomorphisms: $H^1(\Gamma_g, \mathfrak{g}) \simeq H^1(\Gamma_g, \mathcal{F}) \simeq H^1(\Gamma_g, \Gamma(\mathbf{H}, \Theta))$. We will see that there exists a smallest such $\mathcal{F}_\rho$ depending on the representation ρ. Furthermore, there exists an increasing sequence $(F_{\rho,i})_{i \geq 0}$ of $\mathbf{C}$-Γ_g modules of finite rank such that $\mathcal{F}_\rho = \bigcup_{i=0}^{\infty} F_{\rho,i}$ and $F_{\rho,i+1}/F_{\rho,i}$ for $i \geq 0$ are trivial Γ_g modules. We call $\mathcal{F}_\rho$ the *minimal module* attached to the Eichler-Shimura isomorphism $(*)$.

(1.3) The construction of (1.2) can be carried out in the same way for the representations of Γ_g on the infinite-dimensional spaces $\Gamma(\mathbf{H}, \mathcal{O}(K^{-n/2}))$ for $n \geq 0$ with the role of $\mathfrak{g}$ replaced by $Sym^n(\mathbf{R}^2)$, where $n = 2$ is the case discussed in (1.2) (see §4). The present paper studies minimal modules $\mathscr{F}_\rho^n$ attached to this situation. The minimal module $\mathscr{F}_\rho^n$ is again exhausted by an increasing sequence $\{F_{\rho,i}^n\}_{i\geq 0}$ of $\mathbf{C}$-Γ_g modules of finite rank such that $F_{\rho,i+1}^n/F_{\rho,i}^n$ for $i \geq 0$ are trivial Γ_g modules. The first step $F_{\rho,1}^n$ of the sequence is already known as the space of Eichler integrals and $F_{\rho,1}^n/F_{\rho,0}^n \simeq \Gamma(\mathbf{H}, \mathcal{O}(K^{n/2+1}))^{\Gamma_g}$ (cf. (4.4) *Remark* and (5.4.5)). Therefore we regard the spaces $F_{\rho,i}^n$ for $i > 0$ as generalizations of this and called them *higher Eichler integrals.*

Geometrically, the representation $\rho: \Gamma_g \to SL(2, \mathbf{R})$ determines a Riemann surface $X := \rho(\Gamma_g)\backslash\mathbf{H}$, together with halfcanonical (spin) bundle $K_X^{1/2}$ (c.f. (4.1)). Then $F_{X,i}^n := (\mathbf{H} \times F_{\rho,i}^n)/\Gamma_g$ for n, $i \geq 0$ are local systems on X contained in $\mathcal{O}_X(K^{-n/2})$, which are shown to have the properties stated in the Summary.

Although each step of the construction and proofs is carried out in terms of elementary use of group cohomology, as a whole, it seems that we are treating some new object, which may be summarized in a word: *non-abelian theory of abelian integrals on a spin Riemann surface.* To describe elements of $F_{\rho,i}^n$ as global holomorphic forms on $\mathbf{H}$ (as solutions of algebraic differential equations) seems to be an interesting but hard problem, which we do not consider in this paper.

Many of the constructions in this paper may be carried out for further classes of discrete groups, including groups acting on higher dimensional spaces. Some Lemmas are formulated in this generality. It would be interesting to clarify the meaning of the generalization of the complex $(\mathscr{G}_X \otimes_{\mathbf{Z}} H^{\cdot}(\Gamma), \delta^{\cdot})$ in the Summary e) for such a discrete group Γ.

(1.4) The construction of the paper is as follows.

§2 treats the construction of a minimal module in a general setting. §3 is devoted to a calculation of the cohomology of a surface group Γ_g by introducing the concept of i-regularity. §4 recalls the Eichler-Shimura isomorphism and shows that the associated minimal modules $\mathscr{F}_\rho^n = (F_{\rho,1}^n)_{i\geq 0}$ satisfy the regularity condition of §3. In §5, we show that the $F_{\rho,i}^n$ form local systems over the Riemann surface $\Gamma_g\backslash\mathbf{H}$, depending only on the spin of ρ. The $F_{\rho,i}^n$ are understood as solution spaces of some linear differential equations. We study the algebra structure on $\mathscr{H}_\rho := \bigoplus_{n=0}^{\infty} \mathscr{F}_\rho^n/F_{\rho,0}^n$ and on $\mathrm{gr}(\mathscr{H}_\rho) \simeq \bigoplus_{n=0}^{\infty} \mathrm{gr}(\mathscr{F}_\rho^n)$ in §6. The Appendix treats the $\mathbf{Z}$-structure on $G_{X,i}^n$ for $n > 0$, $i \geq 0$.

(1.5) The author is grateful to Prof. Y. Ihara for pointing out that (1.2) (*) is known as the Eichler-Shimura isomorphism, which is the starting point of the present work. He thanks Prof. M. Kashiwara and Prof. T. Oda for discussions and for drawing attention to a Theorem of Labute [22, 1] and its

generalization due to Kohno and Oda [23, Theorem (1.4)] (cf. Summary f) and (2.6)).

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§2. The Minimal Module $\mathscr{F} = (F_i)_{i\geq 0}$

Let Γ be a finitely generated group and R an $\mathbf{R}$-Γ module of finite rank. In (2.1) we consider a setting, in which the cohomology group $H^1(\Gamma, R)$ acquires a $\mathbf{C}$-structure. Then Lemma (2.2) associates with this situation a construction of a filtered $\mathbf{C}$-Γ module $\mathscr{F} = (F_i)_{i\geq 0}$, which we shall call the *minimal module.*

(2.1) **Definition.** A pair $(\mathscr{S}, \iota)$ consisting of a $\mathbf{C}$-Γ module $\mathscr{S}$ and an $\mathbf{R}$-Γ homomorphism $\iota\colon R \to \mathscr{S}$ will be called a *complexification* of R, if ι induces

i) an injective $\mathbf{C}$-Γ homomorphism $\mathbf{C} \otimes_{\mathbf{R}} R \to \mathscr{S}$,

ii) a $\mathbf{C}$-isomorphism $\mathbf{C} \otimes_{\mathbf{R}} R^{\Gamma} \simeq \mathscr{S}^{\Gamma}$,

iii) an $\mathbf{R}$-isomorphism $\iota_*\colon H^1(\Gamma, R) \simeq H^1(\Gamma, \mathscr{S})$.

(2.2) **Lemma.** *Let R be an $\mathbf{R}$-Γ module of finite rank and $(\mathscr{S}, \iota)$ a complexification of R.*

1. *There exists a $\mathbf{C}$-Γ submodule $\mathscr{F}$ of $\mathscr{S}$ having an increasing filtration $(F_i)_{i=0}^{\infty}$ by $\mathbf{C}$-Γ modules of finite rank such that $\mathscr{F} = \bigcup_{i=0}^{\infty} F_i$ and*

i) *ι induces an isomorphism*:

$$\mathbf{C} \otimes_{\mathbf{R}} \iota\colon \mathbf{C} \otimes_{\mathbf{R}} R \simeq F_0. \tag{2.2.1}$$

ii) *For each $i \in \mathbf{Z}_{\geq 0}$, the module F_{i+1} is the largest $\mathbf{C}$-Γ subspace of $\mathscr{F}$ containing F_i such that F_{i+1}/F_i is a Γ-trivial module.*

iii) *The inclusion map $\iota\colon R \to \mathscr{F}$ induces an $\mathbf{R}$-isomorphism*

$$H^1(\Gamma, R) \simeq H^1(\Gamma, \mathscr{F}). \tag{2.2.2}$$

2. *The pair $(\mathscr{F}, \iota)$ with the properties* i)-iii) *is unique and rigid in the following two senses.*

i) *Let $\mathscr{H}$ be any $\mathbf{C}$-Γ submodule of $\mathscr{S}$ containing R such that the natural maps $H^1(\Gamma, R) \to H^1(\Gamma, \mathscr{H}) \to H^1(\Gamma, \mathscr{S})$ are isomorphisms. Then $\mathscr{F} \subset \mathscr{H}$.*

ii) *If $\varphi\colon \mathscr{F} \to \mathscr{F}$ is a $\mathbf{C}$-Γ homomorphism which commutes with ι, then φ preserves the filtration on $\mathscr{F}$ and $\mathrm{gr}(\varphi)\colon \mathrm{gr}(\mathscr{F}) \to \mathrm{gr}(\mathscr{F})$ is the identity. If $\mathscr{S}$ is reduced, then φ itself is the identity.*

Proof **1.** Define the sequence F_i by induction on $i = 0, 1, \ldots$.

$$F_0 := \mathbf{C}\iota(R) = \iota(R) \oplus \sqrt{-1}\,\iota(R)$$

$$F_{i+1} := \{f \in \mathscr{S} : f\cdot\gamma - f \in F_i \quad \text{for all } \gamma \in \Gamma\} \qquad \text{for} \quad i \geq 0.$$

By construction, the F_i are $\mathbb{C}$-Γ modules, and induced action of Γ on F_{i+1}/F_i is trivial. Thus the properties i) and ii) are shown.

Recall that the first cohomology group $H^1(\Gamma, \mathscr{S})$ is given by $Z^1(\Gamma, \mathscr{S})/\delta\mathscr{S}$, where

$$Z^1(\Gamma, \mathscr{S}) := \{c: \Gamma \to \mathscr{S} \mid c(\gamma\delta) = c(\gamma)\delta + c(\delta)\}$$

and $\delta(f)(\gamma) := f\cdot\gamma - f$ (for $f \in \mathscr{S}$ and $\gamma \in \Gamma$). Hence $\ker(\delta) = \mathscr{S}^\Gamma$.

Using this notation, the definition of the F_i can be rewritten as

$$F_{i+1} := \delta^{-1}(Z^1(\Gamma, F_i)) \quad i \geq 0. \tag{2.2.3}$$

Thus δ induces bijection: $F_{i+1}/\mathscr{S}^\Gamma \xrightarrow{\simeq} Z^1(\Gamma, F_i) \cap \delta(\mathscr{S})$ for $i \geq 0$. If F_i is of finite rank, then so is $Z^1(\Gamma, F_i)$ since Γ is finitely generated, and hence so is also $F_{i+1}/\mathscr{S}^\Gamma$. Since $\mathscr{S}^\Gamma \subset R \otimes \mathbb{C}$ is of finite rank, one concludes F_{i+1} is of finite rank.

The inclusions $R \subset F_i \subset \mathscr{S}$ induce maps $H^1(\Gamma, R) \to H^1(\Gamma, F_i) \to H^1(\Gamma, \mathscr{S})$, whose compose is an $\mathbb{R}$-isomorphism by assumption. This implies that $Z^1(\Gamma, F_i)$ maps surjectively to $H^1(\Gamma, \mathscr{S})$; the kernel is by definition $F_{i+1}/\mathscr{S}^\Gamma$. Hence one has an exact sequence:

$$0 \longrightarrow F_{i+1}/\mathscr{S}^\Gamma \xrightarrow{\ \delta\ } Z^1(\Gamma, F_i) \longrightarrow H^1(\Gamma, \mathscr{S}) \longrightarrow 0, \quad \text{for } i \geq 0, \tag{2.2.4}$$

and a splitting as $\mathbb{C}$-vector spaces:

$$H^1(\Gamma, F_i) \simeq H^1(\Gamma, \mathscr{S}) \oplus F_{i+1}/F_i \tag{2.2.5}$$

for $i \geq 1$, where the projection $H^1(\Gamma, F_i) \to H^1(\Gamma, \mathscr{S})$ is induced by $F_i \subset \mathscr{S}$ and the inclusion $H^1(\Gamma, \mathscr{S}) \subset H^1(\Gamma, F_i)$ by $F_{i-1} \to F_i$. (A decomposition of the form (2.2.5) for $i = 0$ will be treated in (2.4).)

We define $\mathscr{F}$ as the union $\mathscr{F} := \bigcup_{i=0}^{\infty} F_i$. The inductive limit of (2.2.5) gives an isomorphism $H^1(\Gamma, \mathscr{F}) \simeq H^1(\Gamma, \mathscr{S})$ and hence (2.2.2).

2. i) Let $\mathscr{H}$ be a $\mathbb{C}$-Γ submodule of $\mathscr{S}$ containing R such that the inclusion $\mathscr{H} \subset \mathscr{S}$ induces an isomorphism: $H^1(\Gamma, \mathscr{H}) \simeq H^1(\Gamma, \mathscr{S})$. We prove the inclusion $F_i \subset \mathscr{H}$ by induction. The case $i = 0$ is the assumption. The inclusion maps $F_i \subset \mathscr{H} \subset \mathscr{S}$ induce maps: $H^1(\Gamma, F_i) \to H^1(\Gamma, \mathscr{H}) \simeq H^1(\Gamma, \mathscr{S})$. Recalling (2.2.4), one obtains that $\delta(F_{i+1}) = \delta(\mathscr{H}) \cap Z^1(\Gamma, F_i)$. This implies that $\delta(F_{i+1}) \subset \delta(\mathscr{H})$ and hence $F_{i+1}/\mathscr{S}^\Gamma \subset \mathscr{H}/(\mathscr{H} \cap \mathscr{S}^\Gamma)$. Since $\mathscr{S}^\Gamma \subset F_0 \subset \mathscr{H}$, one has $F_{i+1} \subset \mathscr{H}$.

ii) Let $\varphi: \mathscr{F} \to \mathscr{F}$ be a $\mathbb{C}$-Γ homomorphism commuting with ι. We show by induction on i that the restriction $\varphi \mid F_i$ (for $i \geq 0$) is of the form $id + \phi_i$, where ϕ_i is a map from F_i to F_{i-1}. When $i = 0$, we have $\phi_0 = 0$ by the assumption on φ. For $i \geq 0$, apply the induction hypothesis to the element $f\cdot\gamma$

$-f \in F_i$ for $f \in F_{i+1}$ and $\gamma \in \Gamma$. For all $\gamma \in \Gamma$, one gets a relation $\varphi(f)\cdot\gamma - \varphi(f) = \varphi(f\cdot\gamma - f) = f\cdot\gamma - f + \phi_i\ (f\cdot\gamma - f)$, which we rewrite as

$$(f-\varphi(f)) - ((f-\varphi(f)))\cdot\gamma = \begin{cases} \phi_i(f\cdot\gamma - f) \in F_{i-1} & \text{if } i > 0 \\ 0 & \text{if } i = 0. \end{cases}$$

Thus $f - \varphi(f) \in F_i$ and hence $\varphi|_{F_{i+1}} \equiv id \pmod{F_i}$. If $\mathscr{S}$ is reduced, then $\phi_i = 0$ by induction. This completes the proof. □

Remark 1. The condition 1. ii) of the Lemma is necessary for the unicity of the filtration $(F_i)_{i \in \mathbf{N}}$. This is equivalent to the exactness of the sequence:

(2.2.6) $$0 \longrightarrow F_{i+1}/F_i \xrightarrow{\ \delta\ } H^1(\Gamma, F_i) \longrightarrow H^1(\Gamma, \mathscr{F}) \longrightarrow 0 \quad \text{for } i \geq 0.$$

2. The complex structure J on $H^1(\Gamma, R)$ induced from $H^1(\Gamma, \mathscr{S})$ is not enough to determine the $\mathbf{C}$-Γ module $\mathscr{F} = (F_i)_{i \geq 0}$, since the short exact sequence (2.2.4) depends not only on J but on the data of the projection: $Z^1(\Gamma, F_i) \to H^1(\Gamma, \mathscr{S}) (\simeq H^1(\Gamma, R))$.

(2.3) **Definition.** The filtered $\mathbf{C}$-Γ module $(\mathscr{F}, \iota)$ introduced in Lemma (2.2) is called the *minimal module* relative to $(\mathscr{S}, \iota)$. The induced complex structure on $H^1(\Gamma, R)$ will be denoted by J.

(2.4) The complex structure J on $H^1(\Gamma, R)$ determines the first term F_1/F_0 as follows. Let the setting be as in Lemma (2.2).

Lemma bis. *The image of* F_1/F_0 *in* $\mathbf{C} \otimes_{\mathbf{R}} H^1(\Gamma, R)$ *under the map* δ (2.2.6) *is the eigenspace of* J *for the eigenvalue* $-\sqrt{-1}$. *Hence one has a direct sum decomposition*:

(2.4.1) $$F_1/F_0 \oplus \overline{F_1/F_0} \overset{\delta \oplus \bar{\delta}}{\simeq} \mathbf{C} \otimes_{\mathbf{R}} H^1(\Gamma, R),$$

and an isomorphism:

(2.4.2) $$\overline{F_1/F_0} \simeq H^1(\Gamma, \mathscr{S}),$$

where $\bar{A}$ *denotes the complex conjugate of a* $\mathbf{C}$*-vector space* A.

Proof. The definitions of $F_0 := \mathbf{C} \otimes_{\mathbf{R}} R$ and F_1 in (2.2) give rise to the following commutative diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathbf{C}\otimes_{\mathbf{R}} R/R^{\Gamma} & \longrightarrow & F_1/F^{\Gamma} \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\
\mathbf{C}\otimes_{\mathbf{R}} Z^1(\Gamma, R) & \xrightarrow{\simeq} & Z^1(\Gamma, F_0) \\
\downarrow & & \downarrow \\
\mathbf{C}\otimes_{\mathbf{R}} H^1(\Gamma, R) & \longrightarrow & Z^1(\Gamma, F_0)/F_1 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

with exact columns. Then one has the exact sequence:

$$0 \longrightarrow \ker(\mathbf{C}\otimes_{\mathbf{R}} H^1(\Gamma, R) \longrightarrow Z^1(\Gamma, F_0)/F_1) \longrightarrow \operatorname{coker}(\mathbf{C}\otimes_{\mathbf{R}} R \to F_1) \longrightarrow 0.$$

The $Z^1(\Gamma, F_0)/F_1$ in the first parenthesis is isomorphic to $H^1(\Gamma, \mathscr{F})$ as a $\mathbf{C}$-vector space ((2.2.4)). The image of $\mathbf{C}\otimes_{\mathbf{R}} R$ in the second parenthesis is F_0/F^{Γ} ((2.2.1)). These imply the isomorphism $F_1/F_0 \simeq \ker(\mathbf{C}\otimes_{\mathbf{R}} H^1(\Gamma, R) \to H^1(\Gamma, \mathscr{F}))$. Hence one obtains the exact sequence:

$$0 \longrightarrow F_1/F_0 \longrightarrow H^1(\Gamma, R)\otimes\mathbf{C} \longrightarrow H^1(\Gamma, \mathscr{F}) \longrightarrow 0$$

For an element $u := x + \sqrt{-1}\, y \in H^1(\Gamma, R) \oplus \sqrt{-1}\, H^1(\Gamma, R)$, we have $\iota(u) = 0 \iff \iota(x) + \sqrt{-1}\,\iota(y) = \iota(x + Jy) = 0 \iff x + Jy = 0 \iff u = x + \sqrt{-1}\, Jx \iff Ju = -\sqrt{-1}\, u$. $\square$

(2.5) The coboundary map δ (cf. (3.6)).

Let the setting be as in (2.2) and let $\mathscr{F} = (F_i)_{i\geq 0}$ be a minimal module with its filtration. For $i \geq 0$ we define a map:

$$F_{i+1}/F_i \xrightarrow{\delta} F_i/F_{i-1} \otimes_{\mathbf{Z}} H^1(\Gamma, \mathbf{Z}), \tag{2.5.1}$$

called the coboundary map. In view of $\operatorname{Hom}(\Gamma, \mathbf{Z}) \cong H^1(\Gamma, \mathbf{Z})$, we identify the target space with $\operatorname{Hom}(\Gamma, F_i/F_{i-1})$. Set

$$\delta([f])(\gamma) :\equiv f\cdot\gamma - f \qquad \operatorname{mod}(F_{i-1})$$

for $\gamma \in \Gamma$ and $f \in F_{i+1}$ representing $[f] \in F_{i+1}/F_i$. Here we write

$$F_{-1} := \sum_{\gamma\in\Gamma} F_0\cdot(\gamma - 1). \tag{2.5.2}$$

$\delta([f])$ is a homomorphism, since

$$\delta([f])(\gamma_1\cdot\gamma_2) \equiv f\cdot\gamma_1\cdot\gamma_2 - f$$

$$\equiv (f\cdot\gamma_1 - f) + (f\cdot\gamma_2 - f) + ((f\cdot\gamma_1 - f)\cdot\gamma_2 - (f\cdot\gamma_1 - f))$$
$$\equiv \delta([f])(\gamma_1) + \delta([f])(\gamma_2).$$

The map δ (for i > 0)*) is part of a long exact sequence:

(2.5.3) $$0 \longrightarrow F_{i+1}/F_i \xrightarrow{\delta} \mathrm{Hom}_{\mathbf{Z}}(\Gamma, F_i/F_{i-1}) \longrightarrow H^2(F_{i-1}) \longrightarrow H^2(F_i) \longrightarrow H^2(F_i/F_{i-1})$$

Proof. For brevity, we drop Γ in the notation for cohomology. Consider the long exact sequence associated to the short exact sequence $0 \to F_{i-1} \to F_i \to F_i/F_{i-1} \to 0$ (for $i > 0$):

$$0 \longrightarrow F_i/F_{i-1} \to H^1(F_{i-1}) \longrightarrow H^1(F_i) \longrightarrow H^1(F_i/F_{i-1}) \longrightarrow H^2(F_{i-1}) \longrightarrow \cdots.$$

Since F_i/F_{i-1} is Γ-trivial, one has $H^1(F_i/F_{i-1}) = \mathrm{Hom}_{\mathbf{Z}}(\Gamma, F_i/F_{i-1})$. Applying (2.2.6) and (2.2.5) for $H^1(F_i)$, one eliminates the first two terms, so that one obtains a sequence of the form (2.5.3) for some map δ. Since the isomorphism of (2.2.5) on the second factor is induced by a coboundary map δ, one gets the description of δ given in (2.5.1). □

*) Note that (2.5.3) does not hold for $i = 0$.

(2.6) *Remark.* Let $\mathbf{Z}[\Gamma]$ be the group algebra for a group Γ and let $\mathscr{I}$ be its ideal generated by $\gamma - 1$ for all $\gamma \in \Gamma$. Let $\mathscr{F} = \bigcup_{i=0}^{\infty} F_i$ be a minimal module (2.3). Then for any integers $i \geq 0$ and $j \geq 1$, the action of $\mathbf{Z}[\Gamma]$ on the quotient module F_{i+j}/F_i is factored through an action of $\mathbf{Z}[\Gamma]/\mathscr{I}^j$. Therefore, the action of Γ on F_{i+j}/F_i is factored through Γ/Γ_j, where Γ_j is the j th lower central series of Γ. That is: $\Gamma_1 := \Gamma$ and $\Gamma_{j+1} := [\Gamma, \Gamma_j]$ for $j \geq 1$.

Compare the formula f) in the Summary of the present paper with a Theorem of Labute [22, 1] and its generalization due to Kohno and Oda [23, Theorem (1.4)].

§3. The Case of Surface Group

We calculate the cohomology of the surface group of genus $g > 1$

$$\Gamma_g := \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{j=1}^{g} [a_j, b_j] = 1 \rangle$$

using surface topology (3.1)-(3.3). The key result of this section is the heredity property of regularity (3.4) and its consequences. (For the cohomology of a group with a single relation, see [11].)

(3.1) Let $X = \{0\} \coprod \coprod_{k=1}^{g} (a_k \coprod b_k) \coprod Y$ be a canonical dissection of a compact

Riemann surface X of genus $g \geq 2$; that is i) $O \in X$ is a point; ii) a_k and b_k (for $k = 1, \dots, g$) are simple closed curves on X with base point O disjoint outside O with intersection number $\langle a_i, b_j \rangle = -\langle b_j, a_i \rangle = \delta_{ij}$ and $\langle a_i, a_j \rangle = \langle b_i, b_j \rangle = 0$ (for $1 \leq i, j \leq g$); and iii) Y is an open $4g$-polygon. We identify a_k and b_k with the elements of $\pi_1(X, O)$ represented by them. This induces the isomorphism: $\Gamma_g \simeq \pi_1(X, O)$.

(3.2) Let $\mathbf{H} \to X$ be the holomorphic universal covering of X with $\mathbf{H}$ the complex upperhalf plane. The pull-back of the dissection on X induces a cell decomposition of $\mathbf{H}$. Let C_i be the free abelian group generated by the i-cells for $i = 0, 1, 2$. Since Γ_g acts freely on the set of i-cells, C_i is a $\mathbf{Z}\Gamma_g$-free module. Since $\mathbf{H}$ is contractible, the coboundary maps ∂ define a free $\mathbf{Z}\Gamma_g$-resolution of $\mathbf{Z}$:

(3.2.1) $$0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0.$$

Let us fix a base point $\tilde{O} \in \mathbf{H}$ over $O \in X$. The 1-cell of $\mathbf{H}$ from $\gamma\tilde{O}$ to $\delta\tilde{O}$ (if it exists) is denoted by $[\gamma\tilde{O}, \delta\tilde{O}] = -[\delta\tilde{O}, \gamma\tilde{O}]$. We denote by Z the 2-cell of $\mathbf{H}$ surrounded by the 1-cells $[R_{i-1}\tilde{O}, R_i\tilde{O}]$ $(1 \leq i \leq 4g)$, where $R_i := \gamma_1\gamma_2 \cdots \gamma_i$ and $\gamma_1, \gamma_2, \dots, \gamma_{4g}$ is the sequence

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}.$$

Using this notation, the complex (C_*, ∂) (3.2.1) is given explicitly as:

$$C_0 = \mathbf{Z}\Gamma_g \cdot \tilde{O},\ C_1 = \bigoplus_{i=1}^{g} \mathbf{Z}\Gamma_g \cdot [\tilde{O}, a_i^{-1}\tilde{O}] \oplus \bigoplus_{i=1}^{g} \mathbf{Z}\Gamma_g \cdot [\tilde{O}, b_i^{-1}\tilde{O}],\ C_2 = \mathbf{Z}\Gamma_g \cdot Z.$$

$$\partial_1[\tilde{O}, a_i^{-1}\tilde{O}] = (a_i^{-1} - 1)\cdot\tilde{O},\ \partial_1[\tilde{O}, b_i^{-1}\tilde{O}] = (b_i^{-1} - 1)\cdot\tilde{O},\ \partial_2 Z = \sum_{i=1}^{4g} [R_{i-1}\tilde{O},$$

$$R_i\tilde{O}] = \sum_{k=1}^{g} R_{4(k-1)} a_k b_k ((a_k^{-1} - 1)\cdot[\tilde{O}, b_k^{-1}\tilde{O}] - (b_k^{-1} - 1)\cdot[\tilde{O}, a_k^{-1}\tilde{O}]).$$

It is a direct calculation to check that the following f_*(3.2.2) gives a homotopy equivalence between the complex (C_*, ∂) (3.2.1) and the standard resolution F_* of $\mathbf{Z}$ (see [4, Chap. I.5]).

(3.2.2) $$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\varepsilon} & \mathbf{Z} \longrightarrow 0 \\ & & & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \\ \cdots & \longrightarrow & F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \xrightarrow{\varepsilon} & \mathbf{Z} \longrightarrow 0 \end{array}$$

where $f_0(\tilde{O}) := [\,]$,

$f_1([\tilde{O}, a_i^{-1}\tilde{O}]) := [a_i^{-1}]$ and $f_1([\tilde{O}, b_i^{-1}\tilde{O}]) := [b_i^{-1}]$ for $i = 1, \dots, g$,

$$f_2(Z) := \sum_{k=1}^{g} ([R_{4(k-1)} a_k b_k (a_k^{-1} - 1) \mid b_k^{-1}] - [R_{4(k-1)} a_k b_k (b_k^{-1} - 1) \mid a_k^{-1}]).$$

(3.3) For a right Γ_g-module F, the cohomology group is defined as the homology $H^{\cdot}(\Gamma_g, F) \simeq H^{\cdot}((\mathrm{Hom}_{\mathbf{Z}\Gamma_g}(C_*, F), \delta^*))$ of the following complex:

(3.3.1)

$$\mathrm{Hom}_{\mathbf{Z}\Gamma_g}(C_*, F) := \{c \in \mathrm{Hom}_{\mathbf{Z}}(C_*, F) \colon c(\gamma^{-1} \cdot z) = c(z) \cdot \gamma \quad {}^{\forall}\gamma \in \Gamma_g, {}^{\forall}z \in C_*\}$$

$$\simeq \begin{cases} F \text{ (resp. } F^{2g}, F) & \text{if } * = 0 \text{ (resp. } * = 1, * = 2), \\ \{0\} & \text{otherwise.} \end{cases}$$

$$\delta^0(m) := (m \cdot (a_i - 1), m \cdot (b_i - 1))_{i=1}^{g},$$

$$\delta^1((c(a_i), c(b_i))_{i=1}^{g}) := \sum_{k=1}^{g} (c(b_k) \cdot (a_k - 1) - c(a_k) \cdot (b_k - 1))(R_{4(k-1)} a_k b_k)^{-1}$$

$$= \sum_{k=1}^{g} (c(b_k) \cdot (a_k - 1) - c(a_k) \cdot (b_k - 1)) \cdot a_k^{-1} b_k^{-1} \prod_{j=k+1}^{g} (a_j b_j a_j^{-1} b_j^{-1}).$$

Remark 1. In view of the isomorphisms $Z^1(\Gamma_g, F) \simeq Z^1(\mathrm{Hom}_{\mathbf{Z}\Gamma}(C_*, F))$ given by $c \mapsto \{c(a_i), c(b_i)\}_{i=1}^{g}$, we identify 1-cocycles for the standard complex and for the complex (3.3.1).

2. It is obvious from (3.3.1) that $H^j(\Gamma_g, F) = 0$ for $j \geq 3$.

3. Let $F = \mathbf{Z}$ be the infinite cyclic group with trivial Γ_g action and denote $H^k(\Gamma_g, \mathbf{Z})$ by $H^k(\Gamma_g)$ (or by H^k). The cup product is a skew symmetric form on $H^1(\Gamma_g)$ ($\simeq \mathbf{Z}^{2g}$) with values in $H^2(\Gamma_g) \simeq \mathbf{Z}$ (see [4]).

(3.4) Let $(\mathscr{S}, \iota)$ be the complexification (2.1) of an $\mathbf{R}$-Γ_g module R for the surface group Γ_g (3.1) and $\mathscr{F} = (F_i)_{i \geq 0}$ its minimal module (2.3). To calculate the cohomology of F_i, we introduce a new concept: *regularity* of $\mathscr{F}$.

Definition 1. Let i be an integer with $i \geq 1$. We say that $\mathscr{F}$ is *i-regular* if the multiplication by $\gamma - 1$ on F_i is a surjection onto F_{i-1} for all $\gamma \in \{a_1, b_1, \ldots, a_g, b_g\}$.

2. $\mathscr{F}$ is *0-regular* if the following two conditions are satisfied.

i) $\mathrm{rank}_{\mathbf{R}}(\ker(\gamma - 1 \colon R \to R)) \leq 1$ for all $\gamma \in \{a_1, b_1, \ldots, a_g, b_g\}$,

ii) $R = R \cdot (a_i - 1) + R \cdot (b_i - 1)$ for $i = 1, \ldots, 2g$.

The following heredity of regularity is a key fact throughout the rest of the paper.

Lemma. *If $\mathscr{F}$ is i-regular, then it is $i + 1$-regular for $i \geq 0$.*

Proof. We prove this only for $\gamma = a_1$. We proceed in 3 steps.

Assertion 1. *Assume that $\mathscr{F}$ is i-regular. Then the following map*

$$\pi \colon Z^1(\Gamma_g, F_i) \longrightarrow F_i, \text{ defined by } \pi(c) := c(a_1)$$

is surjective.

Proof. We understand $Z^1(\Gamma_g, F_i)$ as the space of the 1-cocycles for the complex (3.3.1) in view of (3.3) Remark 1. One has to show that for any $p_1 \in F_i$ one can find $c = \{p_j, q_j\}_{j=1}^{g} \in F_i^{2g}$ satisfying the cocycle condition $\delta^1(c) = 0$ (3.3.1). This can be solved as follows.

Case $i > 0$. Put $c = (p_1, q_1, 0, \dots, 0)$, where $q_1 \in F_i$ is chosen such that $q_1 \cdot (a_1 - 1) = p_1 \cdot (b_1 - 1) \in F_{i-1}$. This is possible by i-regularity assumption.

Case $i = 0$. Put $c = (p_1, 0, p_2, q_2, \dots, 0)$, where $p_2, q_2 \in F_0$ are chosen such that $p_1 \cdot (b_1 - 1) a_1^{-1} b_1^{-1} a_2 b_2 = q_2 \cdot (a_2 - 1) - p_2 \cdot (b_2 - 1) \in F_0$. □

Assertion 2. *Put* $Z_\pi^1(\Gamma_g, F_i) := \ker(\pi)$ *where* π *is defined in Assertion* 1. *If* $\mathscr{F}$ *is i-regular, then the natural inclusion* $F_i \subset \mathscr{S}$ *induces a surjection* $Z_\pi^1(\Gamma_g, F_i) \to H^1(\Gamma_g, \mathscr{S})$.

Proof. Case $i = 0$. It is enough to show that $\operatorname{Im}(Z_\pi^1(\Gamma_g, R))$ in $H^1(\Gamma_g, R) \simeq H^1(\Gamma_g, \mathscr{S})$ has $\mathbf{R}$-codimension ≤ 1, since then $\operatorname{Im}(Z_\pi^1(\Gamma_g, F_0))$ is a $\mathbf{C}$-module containing $\operatorname{Im}(Z_\pi^1(\Gamma_g, R))$ and hence its $\mathbf{C}$-codimension in $H^1(\Gamma_g, \mathscr{S})$ can be at most $[1/2] = 0$.

For an element $c \in Z^1(\Gamma_g, R)$, if $c(a_1) \in R \cdot (a_1 - 1)$, say $c(a_1) = r \cdot (a_1 - 1)$ for an $r \in R$, then $c - \delta^0(r) \in Z_\pi^1(\Gamma_g, R)$. Since $\operatorname{codim}_{\mathbf{R}} R \cdot (a_1 - 1) \le 1$, this implies that $\delta^0(R) + Z_\pi^1(\Gamma_g, R)$ has $\mathbf{R}$-codimension at most 1 in $Z^1(\Gamma_g, R)$. Hence the same holds for the image of $Z_\pi^1(\Gamma_g, R)$ in $Z^1(\Gamma_g, R)/\delta^0(R)$.

Case $i > 0$. It is enough to show that $Z_\pi^1(\Gamma_g, F_i) + \delta^0 F_i$ is surjective onto $H^1(\Gamma_g, \mathscr{S})$. Let c be any cocycle in $Z^1(\Gamma_g, F_{i-1})$. By the hypothesis, there exists an element $f \in F_i$ such that $f \cdot (a_1 - 1) = \pi(c) := c(a_1) \in F_{i-1}$. Then by definition $c - \delta^0(f) \in Z_\pi^1(\Gamma_g, F_i)$. This means $Z^1(\Gamma_g, F_{i-1}) \subset Z_\pi^1(\Gamma_g, F_i) + \delta^0 F_i$. Since $Z^1(\Gamma_g, F_{i-1})$ surjects to $H^1(\Gamma_g, \mathscr{S})$, this completes the proof of Assertion 2.

Assertion 3. *Surjectivity of the two maps* $\pi: Z^1(\Gamma_g, F_i) \to F_i$ *and* $Z_\pi^1(\Gamma_g, F_i) \to H^1(\Gamma_g, \mathscr{S})$ *in Assertions* 1 *and* 2 *imply the surjectivity of the map*: $F_{i+1} \times (a_1 - 1) \to F_i$ *for* $i \ge 0$.

Proof. Recalling (2.2.4), we obtain the following diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(a_1 - 1)/\mathscr{S}^{\Gamma} & \longrightarrow & Z^1_{\pi}(\Gamma_g, F_i) & \longrightarrow & H^1(\Gamma_g, \mathscr{S}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F_{i+1}/\mathscr{S}^{\Gamma} & \xrightarrow{\delta} & Z^1(\Gamma_g, F_i) & \longrightarrow & H^1(\Gamma_g, \mathscr{S}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_1 - 1} & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
0 & \longrightarrow & F_{i+1} \cdot (a_i - 1) & \longrightarrow & F_i & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Here all columns and rows except for the last row are exact by the assumptions. Then the last row is also exact. □

This completes the proof of the Lemma. □

(3.5) An immediate consequence of Lemma (3.4) is the following.

Corollary. *Assume that $\mathscr{F}$ is i-regular. Then*

(3.5.1) $$H^2(\Gamma_g, F_j) \simeq F_j/F_{j-1} \qquad \textit{for} \quad j \geq i.$$

(3.5.2) $$H^2(\Gamma_g, \mathscr{F}) = 0.$$

(*Note that $F_{-1} = F_0$ in case of* 0*-regularity.* (*cf.* (2.5.2)).

Proof. Recall the complex (3.3.1) so that $H^2(\Gamma_g, F_j) \simeq F_j/\delta^1 F_j^{2g}$. If $\mathscr{F}$ is i-regular for $i > 0$, then the Lemma implies $\delta^1 F_j^{2g} = F_{j-1}$ for $j \geq i$ so that (3.5.1) holds. Also 0-regularity implies $\delta^1 F_0^{2g} = F_0$ and hence $H^2(\Gamma_g, F_0) = 0$. Taking the inductive limit of (3.5.1), (3.5.2) follows. □

(3.6) The coboundary map δ^k (cf. (2.5)).

We show that there is an exact sequence (3.6.4) (cf. (6.3.1)) for a complex defined on the graded pieces of an i-regular minimal module $\mathscr{F}$, where the coboundary maps δ^k are defined below. First we put

(3.6.1) $$G_i := F_{i+1}/F_i$$

for $i \geq -1$. Now introduce the coboundary maps

(3.6.2) $$\delta^k \colon G_i \otimes_{\mathbf{Z}} H^k(\Gamma_g, \mathbf{Z}) \longrightarrow G_{i-1} \otimes_{\mathbf{Z}} H^{k+1}(\Gamma_g, \mathbf{Z}) \qquad \text{for} \quad k = 0, 1$$

as follows. Put $\delta^0 := \delta$ for the coboundary map $\delta \colon G_i \to G_{i-1} \otimes_{\mathbf{Z}} H^1(\Gamma_g)$ introduced in (2.5.1). If δ^k is defined already, then δ^{k+1} is defined as the composite:

$$G_i \otimes H^k(\Gamma_g) \xrightarrow{\delta \times id} G_{i-1} \otimes H^1(\Gamma_g) \otimes H^k(\Gamma_g) \xrightarrow{\text{cup product}} G_{i-1} \otimes H^{k+1}(\Gamma_g).$$

In our Fuchsian group case, let us give an explicit formula:

$$\delta^1(\varphi) := \sum_{k=1}^{g} (\hat{\varphi}(b_k)\cdot(a_k - 1) - \hat{\varphi}(a_k)\cdot(b_k - 1)) \mod F_{i-1}, \tag{3.6.3}$$

where $\hat{\varphi}(\gamma) \in F_{i+1}$ *is an element representing* $\varphi(\gamma) \in G_i := F_{i+1}/F_i$.

Proof of (3.6.3). Recalling the definition of δ (2.5.1), we have

$$(\delta \times id)(\varphi)(u, v) \equiv \hat{\varphi}(u)\cdot(v-1) \mod F_{i-1}$$

for u, $v \in H_1(\Gamma_g) \simeq \Gamma_g/[\Gamma_g, \Gamma_g]$. Let $e_1, \dots, e_{2g} \in H_1(\Gamma_g)$ be a basis and $e^1, \dots, e^{2g} \in \mathrm{Hom}_{\mathbf{Z}}(\Gamma_g, \mathbf{Z})$ the dual basis. Then one has an expression

$$\delta^1(\varphi) = \sum_{k,l=1}^{2g} ((\delta \times id)(\varphi)(e_k, e_l))(e^k \vee e^l).$$

By taking the generators a_k, $b_k \in \Gamma_g (1 \le k \le g)$ as for representatives of a symplectic basis of $\Gamma_g/[\Gamma_g, \Gamma_g]$, we obtain the formula. □

Lemma. *Assume that* $\mathscr{F}$ *is i-regular. Denote* $\mathrm{H}^k(\Gamma_g) := \mathrm{H}^k(\Gamma_g, \mathbf{Z})$. *Then the following sequences are exact*;

$$0 \longrightarrow G_j \xrightarrow{\delta^0} G_{j-1} \otimes_{\mathbf{Z}} H^1(\Gamma_g) \xrightarrow{\delta^1} G_{j-2} \otimes_{\mathbf{Z}} H^2(\Gamma_g) \longrightarrow 0 \tag{3.6.4}$$

for $j > i$. *If* $i > 0$ *then the following is also exact*:

$$0 \longrightarrow G_i \xrightarrow{\delta^0} G_{i-1} \otimes_{\mathbf{Z}} H^1(\Gamma_g) \longrightarrow H^2(F_{i-1}) \longrightarrow 0. \tag{3.6.5}$$

Proof. Suppose $i > 0$. For $j \ge i$ the natural map $H^2(\Gamma_g, F_j) \to H^2(\Gamma_g, F_j/F_{j-1}) \simeq F_j/F_{j-1} \otimes_{\mathbf{Z}} H^2(\Gamma_g)$ is bijective, since both modules are isomorphic to F_j/F_{j-1} in a natural way (cf. (3.5.1), (3.3.1)). We apply this fact to the sequence (2.5.3), and obtain for $j \ge i$

$$0 \longrightarrow F_{j+1}/F_j \xrightarrow{\delta^0} \mathrm{Hom}_{\mathbf{Z}}(\Gamma_g, F_j/F_{j-1}) \longrightarrow H^2(\Gamma_g, F_{j-1}) \longrightarrow 0.$$

Let us show that the map $\mathrm{Hom}_{\mathbf{Z}}(\Gamma_g, F_j/F_{j-1}) \to H^2(\Gamma_g, F_{j-1})$ in this sequence is naturally identified with the map δ^1 (3.6.2) for $j > i$. This can be checked by comparing the explicit descriptions of the coboundary map δ^1 in (3.3.1) and (3.6.3).

If $i = 0$, then $\mathscr{F}$ is also 1-regular. So we apply (3.6.4) for $i = 1$. Since $H^2(F_0) \simeq F_0/\delta(F_0^{2g}) \simeq 0$ by 0-regularity, (3.6.5) becomes

$$0 \longrightarrow G_1 \xrightarrow{\delta^0} G_0 \otimes_{\mathbf{Z}} H^1(\Gamma_g) \longrightarrow 0. \tag{3.6.6}$$

Since $G_{-1} := F_0/F_{-1} = 0$ in the 0-regular case, (3.6.6) is the initial case of (3.6.4) for $i = 0$.

(3.7) The class $[G_i]$ in $K^0\mathbf{C}$ (= the K-group of $\mathbf{C}$-vector spaces, see the Remark below) is determined recursively by the use of the sequences (3.6.4) and (3.6.5).

Corollary. *Let the setting be as in* (3.4). *Assume that* $\mathscr{F}$ *is i-regular. Then the generating function for the sequence* $[G_j]$ (*for* $j \in \mathbf{Z}_{\geq 0}$) *is a rational function of the form*

$$\sum_{j=0}^{\infty} [G_j]\, t^j = \frac{P(t)}{1 - [H^1(\Gamma_g, \mathbf{C})]\cdot t + [H^2(\Gamma_g, \mathbf{C})]\cdot t^2} \tag{3.7.1}$$

where $P(t)$ *is a polynomial of* deg $\leq i$. *In particular if* $\mathscr{F}$ *is* 0-*regular,*

$$\sum_{i=0}^{\infty} [G_i]\, t^i = \frac{[\overline{H^1(\Gamma_g, \mathscr{S})}]}{1 - [H^1(\Gamma_g, \mathbf{C})]\cdot t + [H^2(\Gamma_g, \mathbf{C})]\cdot t^2} \tag{3.7.2}$$

Proof. In the formal power series ring, put

$$\begin{aligned} P(t) &:= \Big(\sum_{j=0}^{\infty} [G_j]t^j\Big)(1 - [H^1]\cdot t + [H^2]\cdot t^2) \\ &= [G_0] + ([G_1] - [H^1] \otimes_{\mathbf{C}} [G_0])t \\ &\qquad + \sum_{j=2}^{\infty} ([G_j] - [H^1] \otimes_{\mathbf{C}} [G_{j-1}] + [H^2] \otimes [G_{j-2}])\cdot t^j \end{aligned}$$

where $H^k := H^k(\Gamma_g, \mathbf{C})$ for $k = 1, 2$. If $i > 0$, then (3.6.4) gives the recursion relations: $[G_j] - [H^1] \otimes_{\mathbf{C}} [G_{j-1}] + [H^2] \otimes_{\mathbf{C}} [G_{j-2}] = 0$ for $j > i$, in the K-group, so that $P(t)$ is a polynomial of degree $\leq i$. If $i = 0$, then the recursion relations, including (3.6.6), imply that $P(t)$ is a constant, which is the class of $G_0 := F_1/F_0 = \overline{H^1(\Gamma, \mathscr{S})}$ (2.4.2). □

Remark. There is an isomorphism $K^0\mathbf{C} \simeq \mathbf{Z}$ by the correspondence $[G] \mapsto \dim_{\mathbf{C}} G$. Here in the (3.7.1) and (3.7.2), we used $[G]$ rather than $\dim_{\mathbf{C}} G$ for the following reason. In the next paragraph, we study minimal modules associated to Eichler-Shimura isomorphism. The graded pieces $G^n_{\rho,i}$ for them depends analytically on the representation ρ of the surface group into $SL(2, \mathbf{R})$. That is, they form vector bundles over the Teichmüller space T_g equivariant with the mapping class group action. Then the formula (3.7.1) and (3.7.2) are the formula for the class in the K-group (the Grothendieck group) for such vector bundles over T_g.

§ 4. The Eichler-Shimura Isomorphism

We recall the Eichler-Shimura isomorphism (see [4], [15, 16]). The minimal modules relative to the isomorphisms are shown to be 0 (or 1-) regular in the sense of (3.4).

(4.1) Let $Sym^n(\mathbf{R}^2)$ be the n th symmetric tensor product space over $\mathbf{R}$ of the vector representation space $\mathbf{R}^2$ of $SL(2, \mathbf{R})$ for $n \in \mathbf{Z}_{\geq 0}$. The space can be identified with the space of all real polynomials in one variable, say z, of degreee $\leq n$. An element $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbf{R})$ acts on $\varphi(z) \in Sym^n(\mathbf{R}^2)$ on the right:

$$\varphi(z) \cdot ad^{n/2}(A) := \varphi\left(\frac{az+b}{cz+d}\right)(cz+d)^n. \tag{4.1.1}$$

(We use the notation $ad^{n/2}$ for the action, since in case of $n = 2$ the action is identified with the adjoint action of $SL(2, \mathbf{R})$ on $sl(2, \mathbf{R})$.)

Let $\mathcal{O}_{\mathbf{H}}(K_{\mathbf{H}}^{-n/2}) := \mathcal{O}_{\mathbf{H}}\left(\frac{d}{dz}\right)^{n/2}$ be the invertible sheaf on the upper half plane $\mathbf{H} := \{z \in \mathbf{C} : \operatorname{Im}(z) > 0\}$ with the base $\left(\frac{d}{dz}\right)^{n/2}$. The element $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbf{R})$ acts on the sheaf from the right, denoted again by $ad^{n/2}$,

$$\left(\varphi(z)\left(\frac{d}{dz}\right)^{n/2}\right) \cdot ad^{n/2}(A) := \varphi\left(\frac{az+b}{cz+d}\right)(cz+d)^n \left(\frac{d}{dz}\right)^{n/2}. \tag{4.1.2}$$

By definition, the $\mathbf{R}$-linear embedding

$$\iota_n : Sym^n(\mathbf{R}^2) \longrightarrow \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}(K_{\mathbf{H}}^{-n/2})), \quad \varphi(z) \longmapsto \varphi(z)\left(\frac{d}{dz}\right)^{n/2}, \tag{4.1.3}$$

for $n \geq 0$ is equivariant with respect to the $ad^{n/2}$ actions of $SL(2, \mathbf{R})$.

Let Γ_g be the surface group of genus $g \geq 2$ (3.1.1) and let

$$\rho : \Gamma_g \longrightarrow SL(2, \mathbf{R}) \tag{4.1.4}$$

be a faithful discrete and cocompact representation. Γ_g acts on $Sym^n(\mathbf{R}^2)$ and $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}(K_{\mathbf{H}}^{-n/2}))$ by the composite $ad_\rho^{n/2} := ad^{n/2} \circ \rho$, which is equivariant with respect to ι_n.

Remark. 1. ρ induces a representation $\hat{\rho} : \Gamma \to PSL(2, \mathbf{R})$ and defines a Riemann surface $X := \hat{\rho}(\Gamma)\backslash\mathbf{H}$ of genus g. Conversely, for a given $\hat{\rho}$ there exists a lifting ρ (4.1.4), determined up to the choice of spin structure $\in \mathbf{Z}_2^{2g}$ (cf. [17], [2]).

2. The right action of $ad_\rho^{n/2}(\gamma)$ on $\mathcal{O}_{\mathbf{H}}(K_{\mathbf{H}}^{-n/2})$ is equivariant with the left action of $\hat{\rho}(\gamma)^{-1}$ on $\mathbf{H}$ (for $\gamma \in \Gamma_g$), so the quotient $\mathcal{O}_{\mathbf{H}}(K_{\mathbf{H}}^{-n/2})/\Gamma_g$ is an invertible sheaf on the surface X, denoted by $\mathcal{O}_X(K_X^{-n/2})$. *The pair* $(X, \mathcal{O}_X(K_X^{1/2}))$ *is called a spin Riemann surface.* Two representations ρ and ρ^* give isomorphic spin Riemann surface if and only if they are conjugate in $SL(2, \mathbf{R})$ up to $Aut(\Gamma_g)$.

(4.2) The Eichler-Shimura isomorphism is formulated as follows.

Theorem. *The map ι_n induces an isomorphism of* $\mathbf{R}$*-vector spaces*

$$H_\rho^1(\Gamma_g, Sym^n(\mathbf{R}^2)) \simeq H_\rho^1(\Gamma_g, \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}(K_{\mathbf{N}}^{-n/2})) \tag{4.2.1}$$

of the 1 *st cohomology group of* Γ_g *w.r.t. the action* $ad_\rho^{n/2}$.

(*The lower script* ρ *at the notation is added to indicate the dependence of the cohomology group on* ρ (4.1.4).)

An outline of the proof [15]. Serre duality implies that the right-hand side of (4.2.1) is dual to $H_\rho^0(\Gamma_g, \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}(K^{n/2+1})))$ = the space of automorphic forms of weight $n+2$. Then the $\mathbf{R}$-ranks of the both sides of (4.2.1) are shown to be equal in Eichler [4]. The injectivity is shown by Shimura [15] using the Peterson inner product on the automorphic forms.

The isomorphism is generalized to higher-dimensional cases. (See Murakami [12] and the references given there.)

(4.3) The pair $(\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}(K^{-n/2}), \iota_n)$ is a complexification of $Sym^n(\mathbf{R}^2)$ in the sense of (2.1) in view of the isomorphism (4.2.1).

The minimal module relative to the pair will be denoted by

$$\mathscr{F}_\rho^n = (F_{\rho,i}^n)_{i\geq 0},$$

which is a filtered $\mathbf{C}$-Γ_g *submodule of* $\Gamma(\mathbf{H}, \mathcal{O}(K_{\mathbf{H}}^{-n/2}))$ *depending on* ρ, n *and* i (cf. (2.3)). *Of course, one has the isomorphism*: $Sym^n(\mathbf{C}^2) \simeq F_{\rho,0}^n$.

(4.4) The following is a key fact in all what follows.

Lemma. i) *The minimal modules* $\mathscr{F}_\rho^0$ *for any* ρ *are* 1*-regular.*
ii) *The minimal modules* $\mathscr{F}_\rho^n$ *for any* ρ *and* $n>0$ *are* 0*-regular.*
(*For the definition of i-regularity, recall* (3.4).)

Proof. i) For $f \in \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}})$,

$$\begin{aligned} f \in F_{\rho,1}^0 \quad &\langle = \rangle\ f(\rho(\gamma)z) - f(z) = \text{const} \quad \text{for any } \gamma \in \Gamma_g \\ &\langle = \rangle\ d\rho(\gamma)^*(df) - df = 0 \quad \text{for any } \gamma \in \Gamma_g \\ &\langle = \rangle\ \omega := df \in \Gamma(\mathbf{H}, \Omega_{\mathbf{H}}^1)^{\rho(\Gamma)}. \end{aligned}$$

That is, any $f \in F_{\rho,1}^0$ can be expressed as an indefinite integral $\int^z \omega$ for an abelian differential ω of the first kind on the Riemann surface $X := \rho(\Gamma_g)\backslash\mathbf{H}$ in such a way that that $f\cdot\gamma - f = \oint_\gamma \omega$. The space $\Gamma(\mathbf{H}, \Omega_{\mathbf{H}}^1)^{\rho(\Gamma)} \simeq \Gamma(X, \Omega^1)$ is of rank g over $\mathbf{C}$ and hence $2g$ over $\mathbf{R}$, and is dual to $H_1(X, \mathbf{R})$. Hence for a generator a_1 of Γ_g one can find an abelian differential ω s.t. $\oint_{a_1} \omega = 1$.

ii) It is well known that the image $\rho(\gamma) \in SL(2, \mathbf{R})$ of an element $\gamma \neq 1$ of a surface group Γ_g is hyperbolic, and hence it is conjugate to $A_r := \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}$ for some $r \in \mathbf{R}$ with $|r| > 1$. The action of A_r on $Sym^n(\mathbf{R}^2)$ is given by $z^k \cdot ad^{n/2}(A_r) = r^{2k-n} z^k$ for $k = 0,.,n$. Hence it is semisimple with real eigenvalues r^{2k-n} (for $0 \leq k \leq n$). In particular $ad^{n/2}(A) - 1$ has non-trivial kernel if and only if n is even, and then its rank is equal to 1. For even n, let us show that

(*) *For any two hyperbolic elements A and B of $SL(2, \mathbf{R})$ the following three conditions* i) $\sim$ iii) *are equivalent.*

i) *A and B commute*,

ii) $\ker(\mathrm{ad}^{n/2}(A) - 1) = \ker(\mathrm{ad}^{n/2}(B) - 1)$,

iii) $\mathrm{im}(\mathrm{ad}^{n/2}(A) - 1) = \mathrm{im}(\mathrm{ad}^{n/2}(B) - 1)$.

Proof. We may assume $A = A_r$ and $B = A_s \cdot Ad\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that i) is equivalent to "either $a = d = 0$ or $b = c = 0$", since an element of $GL(2)$ commutes with A_r if and only if it is diagonal and $A_s \cdot Ad\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A_s^{\pm 1}$ for $a = d = 0$ or $b = c = 0$. Since the fixed points for A (resp. B) is spanned by $z^{n/2}$ (resp. $((az + b)(cz + d))^{n/2}$), ii) is possible only when either $b = c = 0$ or $a = d = 0$. The image for A (resp. B) is spanned by z^k (resp. $(az + b)^k (cz + d)^{n-k}$) for $k = 0, \ldots, n$, $k \neq n/2$. Hence iii) implies in particular that $(az + b)^n$ and $(cz + d)^n$ do not contain the monomial $z^{n/2}$. This implies again either $a = d = 0$ or $b = c = 0$. Hence *) is proved.

(*) implies conditions i) and ii) of (3.4), Lemma 2. □

Remark 1. As seen in the proof, the space $F_{\rho,1}^0$ is identified with the space of abelian integrals of the first kind. Generally for $n \geq 0$, an element $\varphi(z) \left(\frac{d}{dz}\right)^{n/2} \in \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}(K^{-n/2}))$ belongs to $F_{\rho,1}^n$ if and only if $\varphi^{(n)}\left(\frac{az + b}{cz + d}\right)(\mathrm{c}z + d)^{n+2} - \varphi^{(n)}(z) = 0$ (5.4.5). Such $\varphi(z)\left(\frac{d}{dz}\right)^{n/2}$ is called the Eichler integral of weight n [4]. For this reason, we regard elements of $F_{\rho,i}^n$ for $i \in \mathbf{N}$ as generalizations of Eichler integrals. In fact $F_{\rho,i}^n$ may be regarded as the null space for certain linear differential operators on X (cf. (5.4)).

2. $H^2(\Gamma_g, Sym^0(\mathbf{R}^2)) = \mathbf{R}$ and $H^2(\Gamma_g, Sym^n(\mathbf{R}^2)) = 0$ for $n > 0$. (by (3.3.1) and the Lemma.)

(4.5) As consequences of Lemma (4.3), we can apply several result of § 3. For the sake of completeness, we recall and summarize them.

Assertion 1. *Put*

$$G^n_{\rho,i} := F^n_{\rho,i+1}/F^n_{\rho,i}$$

for $n \geq 0$ *and* $i \geq -1$. *Then one has the following exact sequences*

$$0 \longrightarrow G^n_{\rho,j} \xrightarrow{\delta^0} G^n_{\rho,j-1} \otimes_{\mathbf{Z}} H^1(\Gamma_g) \xrightarrow{\delta^1} G^n_{\rho,j-2} \otimes_{\mathbf{Z}} H^2(\Gamma_g) \longrightarrow 0$$

for $n \geq 0$ *and* $j > 1$. *For* $j = 1$, *we have separate cases.*

$$0 \longrightarrow G^0_{\rho,1} \xrightarrow{\delta^0} G^0_{\rho,0} \otimes_{\mathbf{Z}} H^1(\Gamma_g) \longrightarrow F^0_{X,0} \otimes_{\mathbf{Z}} H^2(\Gamma_g) \longrightarrow 0$$

and

$$0 \longrightarrow G^n_{\rho,1} \xrightarrow{\delta^0} G^n_{\rho,0} \otimes_{\mathbf{Z}} H^1(\Gamma_g) \longrightarrow 0 \qquad \text{for } n > 0.$$

Here we notice that $F^0_{X,0} := \mathbf{C} \otimes_{\mathbf{R}} R^0 \simeq \mathbf{C}$.

2. *The formula for the cohomology of* $F^n_{\rho,i}$ *in Summary* c) *can be proven by the formula* (2.2.5) *and Corollary* (3.5).

3. *The formula for the generating functions* $\sum_{i=0} [G^n_{\rho,i}] \cdot t^i$ *in Summary* f) *can be proven by Corollary* (3.7).

§5. Local Systems over Spin Riemann Surfaces

We describe the transformation rules of the filters $F^n_{\rho,i}$ of the minimal modules $\mathscr{F}^n_\rho$ for a representation ρ (4.1.4) by the actions of $PSL(2, \mathbf{R})$ and $Aut(\Gamma_g)$. This leads to local systems $F^n_{X,i}$ over a spin Riemann surface $(X, K_X^{1/2})$ depending only on the spin class of ρ.

(5.1) For $A \in PSL(2, \mathbf{R})$ and a representation ρ, we denote by $\rho \cdot Ad(A)$ the representation given by $\gamma \in \Gamma_g \mapsto A^{-1}\rho(\gamma)A \in SL(2, \mathbf{R})$.

Assertion. *For given* ρ *and* n, *the right action* $ad^{n/2}(A)$ *of* $A \in SL(2, \mathbf{R})$ *on* $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}(K_{\mathbf{H}}^{-n/2}))$ *induces bijections*: $\mathscr{F}^n_\rho \xrightarrow{\simeq} \mathscr{F}^n_{\rho\cdot Ad(A)}$ *and* $F^n_{\rho,1} \xrightarrow{\simeq} F^n_{\rho\cdot Ad(A),i}$ *for* $i \in \mathbf{Z}_{\geq 0}$. *The bijections are equivariant with respect to the action of* $\gamma \in \Gamma_g$. *Hence one has commutative diagrams*:

$$\begin{array}{ccc} F^n_{\rho,i} & \xrightarrow{ad^{n/2}(A)} & F^n_{\rho\cdot Ad(A),i} \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\ F^n_{\rho,i} & \xrightarrow{ad^{n/2}(A)} & F^n_{\rho\cdot Ad(A),i} \end{array} \quad \text{and} \quad \begin{array}{ccc} F^n_{\rho,i+1} & \xrightarrow{ad^{n/2}(A)} & F^n_{\rho\cdot Ad(A),i+1} \\ \downarrow{\scriptstyle\delta_\rho} & & \downarrow{\scriptstyle\delta_{\rho\cdot Ad(A)}} \\ Z^1(\Gamma, F^n_{\rho,i}) & \xrightarrow{ad^{n/2}(A)} & Z^1(\Gamma, F^n_{\rho\cdot Ad(A),i}) \end{array}$$

Proof. Recall that the right action of $\gamma \in \Gamma_g$ on $\Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}(K_{\mathbf{H}}^{-n/2}))$ (or on $F^n_{\rho,i}$) is defined by the composition $ad^{n/2}_\rho(\gamma) := ad^{n/2}(\rho(\gamma))$ (4.1). We verify the following commutativity of the actions on $\Gamma(\mathbf{H}, \Theta)$:

$$ad_\rho^{n/2}(\gamma)\cdot ad^{n/2}(A) = ad^{n/2}(A)\cdot ad^{n/2}_{\rho\cdot Ad(A)}(\gamma).$$

(Since $ad^{n/2}_{\rho\cdot Ad(A)}(\gamma) = ad^{n/2}((\rho\cdot Ad(A))(\gamma)) = ad^{n/2}(A^{-1}\rho(\gamma)A) = ad^{n/2}(A)^{-1}\cdot ad^{n/2}(\rho(\gamma))\cdot ad^{n/2}(A) \doteq ad^{n/2}(A)^{-1}\cdot ad_\rho^{n/2}(\gamma)\cdot ad^{n/2}(A)$.)

By subtracting $ad^{n/2}(A)$ from both sides of the equality,

$$(ad_\rho^{n/2}(\gamma)-1)\cdot ad^{n/2}(A) = ad^{n/2}(A)\cdot(ad^{n/2}_{\rho\cdot Ad(A)}(\gamma)-1).$$

This implies that $ad^{n/2}(A)$ maps $F^n_{\rho,i}$ into $F^n_{\rho\cdot Ad(A),i}$ by induction on i, where the case $i=0$ is trivial: $F^n_{\rho,0} = \mathbf{C}\otimes\iota(Sym^n(\mathbf{R}^2))$ is invariant under $SL(2,\mathbf{R})$. Since $ad^{n/2}(A^{-1}) = (ad^{n/2}(A))^{-1}$, the maps are bijective. The commutativity of diagrams follows from the same relation. □

(5.2) An element $\alpha\in Aut(\Gamma_g)$ acts on a representation ρ on the left by $(\alpha\cdot\rho)(\gamma) := \rho(\alpha^{-1}(\gamma))$. In view of $\rho(\Gamma_g) = (\alpha\cdot\rho)(\Gamma_g)$, we have $\mathscr{F}^n_\rho = \mathscr{F}^n_{\alpha\cdot\rho}$ and $F^n_{\rho,i} = \mathscr{F}^n_{\alpha\cdot\rho,i}$ for $i,\ n\geq 0$ as subsets of $\Gamma(\mathbf{H}, \mathcal{O}_\mathbf{H}(K_\mathbf{H}^{-n/2}))$.

Assertion. *One has the commutative diagrams:*

$$\begin{array}{ccc} F^n_{\rho,i} & = & F^n_{\alpha\cdot\rho,i} \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha(\gamma)} \\ F^n_{\rho,i} & = & F^n_{\alpha\cdot\rho,i} \end{array} \quad \textit{and} \quad \begin{array}{ccc} F^n_{\rho,i+1} & = & F^n_{\alpha\cdot\rho,i+1} \\ \downarrow{\scriptstyle\delta_\rho} & & \downarrow{\scriptstyle\delta_{\alpha\cdot\rho}} \\ Z^1(\Gamma_g, F^n_{\rho,i}) & \xrightarrow{\alpha*} & Z^1(\Gamma_g, F^n_{\alpha\cdot\rho,i}) \end{array}$$

where α_* is the bijection defined by $\alpha_*(c)(\gamma) := c(\alpha^{-1}(\gamma))$.

Proof. This follows from the equality

$$ad^{n/2}_{\alpha\cdot\rho}(\gamma) = ad^{n/2}(\rho(\alpha^{-1}(\gamma))) = ad^{n/2}_\rho(\alpha^{-1}(\gamma)). \qquad \square$$

(5.3) Define the local system over the Riemann surface $X_{\hat\rho} := \hat\rho(\Gamma_g)\backslash\mathbf{H}$,

$$F^n_{\rho,i} := \Gamma_g\backslash(\mathbf{H}\times F^n_{\rho,i})$$

by the diagonal action of $\gamma\in\Gamma$: $\gamma\cdot(z,f) := (\hat\rho(\gamma)z,\ f\cdot ad_\rho^{n/2}(\gamma^{-1}))$.

Lemma. *The local system $F^n_{\rho,i}$ ($n,\ i\in\mathbf{Z}_{\geq 0}$) depends only on the isomorphism class of the spin Riemann surface $X = (X_{\hat\rho}, K_X^{1/2})$.*

Proof. Recall that two representations ρ and ρ^* give the same spin Riemann surface if and only if there exist $A\in PSL(2,\mathbf{R})$ and $\alpha\in Aut(\Gamma_g)$ with $\rho^* = \alpha\cdot\rho\cdot Ad(A)$. Then the isomorphism is given by

$$\begin{aligned} \psi_{\alpha,A}: X_{\hat\rho} := \hat\rho(\Gamma_g)\backslash\mathbf{H} &\simeq X_{\alpha\cdot\hat\rho\cdot Ad(A)} := (\alpha\cdot\hat\rho\cdot Ad(A))(\Gamma_g)\backslash\mathbf{H} \\ \hat\rho(\Gamma_g)\backslash z &\longmapsto (\alpha\cdot\hat\rho\cdot Ad(A))(\Gamma_g)\backslash A^{-1}(z) \\ ad^{n/2}_{\alpha,A}: \mathcal{O}(K_\mathbf{H}^{-n/2})/\rho(\Gamma_g) &\simeq \mathcal{O}(K_\mathbf{H}^{-n/2})/\alpha\cdot\rho\cdot Ad(A)(\Gamma_g) \\ (z,f) &\longmapsto (A^{-1}(z),\ f\cdot ad^{n/2}(A)). \end{aligned}$$

It is enough to show that $ad^{n/2}_{\alpha,A}$ induces a right $\psi_{\alpha,A}$-isomorphism of the local systems: *) $ad^{n/2}_{\alpha,A}: F^n_{\rho,i} \xrightarrow{\simeq} F^n_{\alpha\cdot\rho\cdot Ad(A),i}$ with the property: **) $ad^{n/2}_{\beta\cdot\alpha,A\cdot B} = ad^{n/2}_{\alpha,A}\cdot ad^{n/2}_{\beta,B}$. *) follows from the facts (5.1) and (5.2). **) is also a straightforward calculation and is omitted. □

The isomorphism class of local system $F^n_{\rho,i}$ over X is denoted

(5.3.1) $$F^n_{X,i} \qquad \text{for} \quad i, n \in \mathbf{Z}_{\geq 0}$$

$$\mathscr{F}^n_X := \bigcup_{i=0}^{\infty} F^n_{X,i} \qquad \text{for} \quad n \in \mathbf{Z}_{\geq 0}.$$

By construction, the local systems are embedded into the sheaf $\mathcal{O}_X(K^{-n/2})$ over X as multi-valued global sections. The graded pieces

(5.3.2) $$G^n_{X,i} := F^n_{X,i+1}/F^n_{X,i} \qquad (i, n \in \mathbf{Z}_{\geq 0})$$

are trivial local systems over X, whose ranks (which depend only on n, i and the genus g of X) are denoted by $g^n_{g,i} := \operatorname{rank}_{\mathbf{C}} G^n_{X,i}$. By (4.5) and (3.7), we have

(5.3.3) $$\sum_{i=0}^{\infty} g^n_{g,i}t^i = \frac{(g-1)(n+1)}{1-2gt+t^2} \qquad \text{for} \quad n > 0,$$

$$= \frac{g-t}{1-2gt+t^2} \qquad \text{for} \quad n = 0.$$

(5.4) We give an interpretation of the local systems $F^n_{X,i}$ as sheaves of solutions of linear differential equations on X.

Assertion. *There exists a sequence of holomorphic linear differential operators*

(5.4.1) $$D^n_{X,i}: \mathcal{O}_X(K_X^{d^n_{g,i}/2}) \longrightarrow \mathcal{O}_X(K_X^{d^n_{g,i+1}/2})$$

of degree $g^n_{g,i}$ for $i = 0, 1, 2, \ldots$, such that

i) *The local system $F^n_{X,i}$ is characterized as the solution of the equation: $D^n_{X,i}\circ D^n_{X,i-1}\circ\cdots\circ D^n_{X,0}(\phi) = 0$. That is,*

(5.4.2) $$\ker(D^n_{X,i}\circ D^n_{X,i-1}\circ\cdots\circ D^n_{X,0}) = F^n_{X,i}.$$

ii) *The map $D^n_{X,i}\circ D^n_{X,i-1}\circ\cdots\circ D^n_{X,0}$ induces an injection:*

(5.4.3) $$D^n_{X,i}\circ D^n_{X,i-1}\circ\cdots\circ D^n_{X,0}: G^n_{X,i} \subset \Gamma(X, \mathcal{O}_X(K_X^{d^n_{g,i+1}/2})).$$

Here $d^n_{g,i}$ $(i \geq 1)$ is given inductively by the formula:

(5.4.4) $$d^n_{g,1} = n+2 \quad \textit{and} \quad d^n_{g,i+1} = (d^n_{g,i} + g^n_{g,i})(g^n_{g,i}+1).$$

Proof. The Lemma is proved using the Wronskian as follows. At the start, for $i=0$, put

$$D^n_{X,0}\left(\varphi(z)\left(\frac{d}{dz}\right)^{n/2}\right):=\left(\left(\frac{d}{dz}\right)^{n+1}\varphi(z)\right)(dz)^{(n+2)/2}.$$

The following transformation rule can be checked directly:

$$\left(\frac{d}{dz}\right)^{n+1}\left(\varphi\left(\frac{az+b}{cz+d}\right)(cz+d)^n\right)=\left(\left(\frac{d}{dz}\right)^{n+1}\varphi\right)\left(\frac{az+b}{cz+d}\right)(cz+d)^{-n-2}.$$

This was studied by G. Bol [3], Peterson and Eichler [4]. i) and ii) for $i=0$ follows obviously from this. Particularly, $D^n_{X,0}$ induces an isomorphism:

(5.4.5) $$D^n_{X,0}: G^n_{X,0}\simeq \Gamma(X,\mathcal{O}_X(K_X^{n/2+1}))$$

Suppose the operators $D^n_{X,0},\dots,D^n_{X,i-1}$ for $i>0$ are constructed. Let $\varphi_j(z)(dz)^{d_i/2}\in D^n_{X,i-1}\circ D^n_{X,i-2}\circ\cdots\circ D^n_{X,0}(F^n_{\rho,i})$ (for $1\le j\le g_i:=g^n_{g,i}$), be a $\mathbb{C}$-basis for the image of $G^n_{\rho,i}$. For $\varphi(z)(dz)^{d_i/2}\in\mathcal{O}_X(K_X^{d_i/2})$, put

$$D^n_{X,i}(\varphi(z)(dz)^{d_i/2}):=\begin{vmatrix}\varphi_1(z), & \varphi_2(z),\dots, & \varphi_{g_i}(z), & \varphi(z)\\ \varphi'_1(z), & \varphi'_2(z),\dots, & \varphi'_{g_i}(z), & \varphi'(z)\\ \varphi^{(2)}_1(z), & \varphi^{(2)}_2(z),\dots, & \varphi^{(2)}_{g_i}(z), & \varphi^{(2)}(z)\\ \cdots\cdots, & \cdots\cdots,\ \cdots, & \cdots\cdots, & \cdots\cdots\\ \varphi^{(g_i)}_1(z), & \varphi^{(g_i)}_2(z),\ \cdots, & \varphi^{(g_i)}_{g_i}(z), & \varphi^{(g_i)}(z)\end{vmatrix}(dz)^{d_{i+1}/2}$$

(Here in this definition, there is an ambiguity of a choice of the basis of $G^n_{X,i}$. See the following Remark 1.)

$D^n_{X,i}$ commutes with the adjoint action of the Γ_g. To see this, for $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\in\rho(\Gamma_g)$, substitute $\varphi_1(A(z))(cz+d)^{-d_i},\dots,\varphi_{g_i}(A(z))(cz+d)^{-d_i}$, $\varphi(A(z))(cz+d)^{-d_i}$ for $\varphi_1(z),\dots,\varphi_{g_i}(z),\varphi(z)$ in the definition of $D^n_{X,i}$. Apply the fact that $\left(\frac{d}{dz}\right)^k(\varphi(A(z))(cz+d)^{-d})=\varphi^{(k)}(A(z))(cz+d)^{-d-2k}+\sum_{j=1}^{k}c_j\varphi^{(k-j)}(A(z))(cz+d)^{-d-2k+j}$ for some constants c_j, so that the right hand is equal to

$$\begin{vmatrix}(cz+d)^{-d_i} & & \\ & (cz+d)^{-d_i-2} & 0\\ & \cdots\cdots & \\ 0 & & (cz+d)^{-d_i-2g_i}\end{vmatrix}\begin{vmatrix}\varphi_1(A(z)),\dots, & \varphi(A(z))\\ \varphi^{(2)}_1(A(z)),\dots, & \varphi^{(2)}(A(z))\\ \cdots\cdots,\ \cdots, & \cdots\\ \varphi^{(g_i)}_1(A(z)),\dots, & \varphi^{(g_i)}(A(z))\end{vmatrix}(dz)^{d_{i+1}/2}$$

$=D^n_{X,i}(\varphi(z)(dz)^{d_i/2})\cdot ad^{-d_{i+1}/2}(A)$ for $\sum_{j=0}^{g_i}(d_i+2j)=d_{i+1}$. On the other hand,

$\varphi_j(z)(dz)^{d_i/2} \in \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}(K^{d_i/2}))^{\Gamma_g}$ means $\varphi_j(A(z))(cz+d)^{-d_i} = \varphi_j(z)$ for $1 \le j \le g_i$. This implies that

$$D^n_{X,i}(f \cdot ad_\rho^{-d_i/2}(\gamma)) = D^n_{X,i}(f) \cdot ad_\rho^{-d_i+1/2}(\gamma)$$

for $f \in \mathcal{O}_{\mathbf{H}}(K_{\mathbf{H}}^{-d_i/2})$ and $\gamma \in \Gamma_g$ as a generalization of Bol's relation. So the operator is well defined. By properties of the Wronskian, i) is verified. To prove ii), observe that $f \in \Gamma(\mathbf{H}, \mathcal{O}(K_{\mathbf{H}}^{-n/2}))$ belongs to $F^n_{\rho,i+1}$ if and only if $f - f \cdot ad_\rho^{n/2}(\gamma) \in F^n_{\rho,i}$. Applying $D^n_{X,i} \circ \cdots \circ D^n_{X,0}$ to this relation, we obtain

$$D^n_{X,i} \circ \cdots \circ D^n_{X,0}(f) - D^n_{X,i} \circ \cdots \circ D^n_{X,0}(f) \cdot ad_\rho^{-d_i+1/2} = 0. \qquad \square$$

Remark 1. To be more precise, the operator $D^n_{X,i}$ must be normalized as $\frac{1}{\det(G^n_{X,i})} D^n_{X,i}$ to kill the ambiguity arising from the choice of basis of $G^n_{X,i}$. For the purpose, one needs to study the vector bundles $\bigcup_{[x] \in \mathcal{T}_g} (G^n_{X,i})$ over the moduli space $\mathcal{T}_g$ of curves of genus g. This is beyond the treatment in this paper.
2. The isomorphisms $\overline{F^n_{\rho,1}/F^n_{\rho,0}} \simeq H^1(X, \mathcal{O}_X(K_X^{-n/2})$ (2.4.2) and $D^n_{X,0}: F^n_{\rho,1}/F^n_{\rho,0} \simeq \Gamma(X, \mathcal{O}_X(K_X^{n/2+1}))$ (5.4.5), and Serre duality between $H^1(X, \mathcal{O}_X(K_X^{-n/2}))$ and $\Gamma(X, \mathcal{O}_X(K_X^{n/2+1}))$ implies that the space $G^n_{\rho,1} := F^n_{\rho,1}/F^n_{\rho,0}$ and its conjugate are $\mathbf{C}$ dual of each other. This is nothing but the Weil-Petersson metric on the space.

§6. Algebra Structure on $\mathcal{H}_X$

We introduce an algebra structure on $\mathcal{H}_X := \bigoplus_{n=0}^{\infty} \mathcal{F}^n_X / F^n_{X,0}$, whose associated graded ring $\mathcal{G}_X := gr(\mathcal{H}_X)$ admits derivations by the elements of Γ_g.

(6.1) Recall Bol's map $D^n_{X,0} := \partial^{n+1}: \Gamma(\mathbf{H}, \mathcal{O}(K^{-n/2})) \to \Gamma(\mathbf{H}, \mathcal{O}(K^{n/2+1}))$ (5.4) on global sections. Obviously ∂^{n+1} is surjective and equivariant with respect to the action of Γ_g. Its kernel is F^n_0.

Assertion. *The product $K_{\mathbf{H}}^{n/2+1} \times K_{\mathbf{H}}^{m/2+1} \to K_{\mathbf{H}}^{(n+m)/2+2}$ on the tensors of the halfcanonical bundle $K_{\mathbf{H}}^{1/2}$ induces a Γ_g-equivariant and filter preserving product map*:

$$\text{(6.1.1)} \qquad \partial^{n+1}(\mathcal{F}^n_X) \times \partial^{m+1}(\mathcal{F}^m_X) \longrightarrow \partial^{n+m+3}(\mathcal{F}^{n+m+2}_X).$$

$$\text{(6.1.1)*} \qquad \partial^{n+1}(F^n_{X,i+1}) \times \partial^{m+1}(F^m_{X,j+1}) \longrightarrow \partial^{n+m+3}(F^{n+m+2}_{X,i+j+1})$$

for $i, j \ge -1$.

Proof. We have only to show that the image of the filters by the product map belongs to the filter described in (6.1.1)*. We proceed by induction on i

and j. First we remark that:

for an element $\varphi \in \Gamma(\mathbf{H}, \mathcal{O}(K^{n/2+1}))$ *and for* $i \in \mathbf{Z}_{\geq 0}$, *we have* $\varphi \in \partial^{n+1}(F^n_{X,i+1})$ *if and only if* $\varphi \cdot ad_\rho^{-n/2-1}(\gamma) - \varphi \in \partial^{n+1} F^n_{X,i}$ *for all* $\gamma \in \Gamma_g$.

If $i = -1$ (resp. $j = -1$), then $\partial^{n+1}(F^n_{X,0})$ (resp. $\partial^{m+1}(F^m_{X,0})$) is 0 so that the image of the map is $0 \subset \partial(F^{n+m+2}_{X,0})$. Now suppose $i, j \geq 0$ and take elements $\varphi \in \partial^{n+1}(F^n_{X,i+1})$ and $\phi \in \partial^{m+1}(F^m_{X,j+1})$). Then for $\gamma \in \Gamma_g$,

$$*) \quad (\varphi \cdot \phi)\, ad_\rho(\gamma) - \varphi \cdot \phi$$
$$= (\varphi \cdot ad_\rho(\gamma) - \varphi) \cdot \phi + \varphi \cdot (\phi \cdot ad_\rho(\gamma) - \phi) + (\varphi \cdot ad_\rho(\gamma) - \varphi) \cdot (\phi \cdot ad_\rho(\gamma) - \phi),$$

which belongs to $\partial^{n+m+3}(F^{n+m+2}_{X,i+j})$ by hypothesis. By the above remark, $\varphi \cdot \phi$ belongs to $\partial^{n+m+3}(F^{n+m+2}_{X,i+j+1})$. □

Corollary. *The product* (6.1.1) *induces a product* $G^n_{X,i} \times G^m_{X,j} \to G^{n+m+2}_{X,i+j}$ *for* $i, j, n, m \geq 0$. *For* $\gamma \in \Gamma_g$, *let* δ_γ *be the coboundary map defined in* (2.5.1). *Then*

$$\delta_\gamma(\varphi \cdot \phi) = \delta_\gamma(\varphi) \cdot \phi + \varphi \cdot \delta_\gamma(\phi) \tag{6.1.2}$$

for $\varphi \in G^n_{X,i}$, $\phi \in G^m_{X,j}$ *and* $\gamma \in \Gamma_g$. *Here* $G^n_{X,i} := F^n_{X,i+1}/F^n_{X,i}$ cf. (3.6.1).

(6.2) Inspired by the calculations in (6.1), we introduce:

$$\mathscr{H}_X := \bigoplus_{n=0}^{\infty} \mathscr{F}^n_X / F^n_{X,0} \simeq \bigoplus_{n=0}^{\infty} \partial^{n+1} \mathscr{F}^n_X. \tag{6.2.1}$$

Each summand will be denoted as

$$\mathscr{H}^n_X := \mathscr{F}^n_X / F^n_{X,0} \simeq \partial^{n+1} \mathscr{F}^n_X; \tag{6.2.2}$$

this carries a filtration

$$H^n_{X,i} := F^n_{X,i+1}/F^n_{X,0} \simeq \partial^{n+1} F^n_{X,i+1} \quad i = 0, 1, 2, \ldots \tag{6.2.3}$$

induced by that on $\mathscr{F}^n_X$. Then (6.1.1)* can be rewritten as a product:

$$H^n_{X,i} \times H^m_{X,j} \longrightarrow H^{n+m+2}_{X,i+j}$$

and so $\mathscr{H}_X$ is a graded algebra with increasing filtration

$$\mathscr{H}_{X,i} := \bigoplus_{n=0}^{\infty} H^n_{X,i}, \qquad i = 0, 1, 2, \ldots. \tag{6.2.4}$$

The first filter: $\mathscr{H}_{X,0} \simeq \bigoplus_{n=0}^{\infty} F^n_{X,1}/F^n_{X,0} \simeq \bigoplus_{n=0}^{\infty} \Gamma(\mathbf{H}, \mathcal{O}_{\mathbf{H}}(K_{\mathbf{H}}^{n/2+1}))^{\Gamma_g}$ is (a part of) the half canonical ring for the spin Riemann surface $(X, K_X^{1/2})$ associated to the representation ρ. It is well known that $\mathscr{H}_{X,0}$ is a finitely generated algebra over $\mathbf{C}$ and is noetherian.

In the remaining of this paragraph, we prove the following:

Lemma. *$\mathscr{H}_X$ is an integral domain over $\mathscr{H}_{X,0}$ such that each filter $\mathscr{H}_{X,i}$ is a finite module over $\mathscr{H}_{X,0}$.*

By generalities in commutative algebra, to prove the Lemma, it is enough to prove the same statement (Lemma* below) for the graded algebra $gr(\mathscr{H}_X) := \bigoplus_{i=0}^{\infty} \mathscr{H}_{X,i}/\mathscr{H}_{X,i-1}$ (Here $\mathscr{H}_{X,-1} := 0$). The associated graded algebra of $\mathscr{H}_X$, which we denote by $\mathscr{G}_X$, is given by

$$\mathscr{G}_X := gr(\mathscr{H}_X) = \bigoplus_{i=0}^{\infty} \mathscr{G}_{X,i}, \tag{6.2.5}$$

where $\mathscr{G}_{X,i} := \mathscr{H}_{X,i}/\mathscr{H}_{X,i-1} = \bigoplus_{n=0}^{\infty} F^n_{X,i+1}/F^n_{X,i} = \bigoplus_{n=0}^{\infty} G^n_{X,i} (i \geq 0)$. Of course $\mathscr{G}_X$ is a bigraded algebra, as $G^n_{X,i} \times G^m_{X,j} \to G^{n+m+2}_{X,i+j}$ is induced from (6.1.1). The 0-graded part $\mathscr{G}_{X,0}$ w.r.t. the index i is the same as the $\mathscr{H}_{X,0}$. Therefore we prove the following:

Lemma* i) *$\mathscr{G}_X$ is an integral domain over $\mathscr{G}_{X,0}$.*
ii) *For $i \geq 0$, $\mathscr{G}_{X,i}$ is a $\mathscr{G}_{X,0}$-finite module.*

(6.3) To prove the Lemma*, we summarize some of the previous results in a Theorem, which describes a resolution of the half canonical ring $\mathscr{G}_{X,0}$; this can be regarded as the main result of the present paper.

Theorem. *Let $\delta^k: \mathscr{G}_{X,i} \otimes H^k(\Gamma_g, \mathbf{Z}) \to \mathscr{G}_{X,i-1} \otimes H^{k+1}(\Gamma_g, \mathbf{Z})$ be the coboundary map introduced in (3.6.2) for $i \geq 0$ and $k = 0, 1$. Then they are $\mathscr{G}_{X,0}$-module homomorphisms, and they define the following resolution of the halfcanonical ring $\mathscr{G}_{X,0}$:*

(6.3.1)

$$0 \longrightarrow \mathscr{G}_{X,0} \longrightarrow \mathscr{G}_X \xrightarrow{\delta^0} \mathscr{G}_X \otimes_{\mathbf{Z}} H^1(\Gamma_g) \xrightarrow{\delta^1 \otimes \iota} (\mathscr{G}_X \oplus F^0_{X,0}) \otimes_{\mathbf{Z}} H^2(\Gamma_g) \longrightarrow 0.$$

Here $F^0_{X,0} \simeq \mathbf{C}$ as a $\mathscr{G}_{X,0}$-module is annihilated by any element of $\mathscr{G}_{X,0}$. The map ι to the factor $F^0_{X,0}$ is given by abelian integrals:

$$f \in Hom_{\mathbf{Z}}(\Gamma_g, G^0_{X,0}) \longrightarrow \sum_{i=1}^{g} \int_{b_i} f(a_i) - \int_{a_i} f(b_i) \in F^0_{X,0} \simeq \mathbf{C}. \tag{6.3.2}$$

Proof. The exactness of (6.3.1) follows from Lemma (4.3) and Lemma (3.6). The fact that δ^0 and δ^1 are commutative with the $\mathscr{G}_{X,0}$-module structure follows from (6.1.2). The map ι comes from the exact sequence (3.6.5) for $i = 1$ and $n = 0$:

$$0 \longrightarrow G^0_{X,1} \longrightarrow G^0_{X,0} \otimes_{\mathbf{Z}} H^1(\Gamma_g, \mathbf{Z}) \xrightarrow{\iota} F^0_{X,0} \otimes_{\mathbf{Z}} H^2(\Gamma_g, \mathbf{Z}) \longrightarrow 0 \tag{6.3.3}$$

To obtain an explicit formula (6.3.2), we refer to (3.6.3) and its proof. □

Remark. The exactness of (6.3.1) can be reformulated in terms of the cohomology of the complex $(\mathscr{G}_X \otimes_{\mathbf{Z}} H^{\cdot}, \delta^{\cdot})$ defined by

$$H^k(\mathscr{G}_X \otimes_{\mathbf{Z}} H^{\cdot}(\Gamma_g), \delta^{\cdot}) = \begin{cases} \mathscr{G}_{X,0} & k = 0 \\ F^0_{X,0} & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(6.4) Recall the coboundary map $\delta_\gamma \colon G^n_{X,i+1} \to G^n_{X,i}$ for $\gamma \in \Gamma_g$ (2.5). The relation (6.1.2) implies that δ_γ is a derivation of the algebra $\mathscr{G}_X$ over $\mathscr{G}_{X,0}$. By a use of derivations, one can recover the filtration on $\mathscr{G}_X$ as follows:

for an element $f \in \mathscr{G}_X$, the following are equivalent

i) $f \in \bigoplus_{i=0}^{d} \mathscr{G}_{X,i}$, ii) $\delta_1 \cdots \delta_{d+1} f = 0$ *for all* $\delta_1, \dots, \delta_{d+1} \in Der(\mathscr{G}_X/\mathscr{G}_{X,0})$.

Proof. ii) implies i) by the injectivity of δ^0 in (3.6.4). i) implies ii) because the derivations have degree -1 by definition. □

Proof of Lemma i).* It is enough to show that the product:

$$\mathscr{G}_{X,i} \times \mathscr{G}_{X,j} \longrightarrow \mathscr{G}_{X,i+j}$$

does not have zero-divisors. Let us prove this by induction on i and j. The start of the induction: when $i = j = 0$, the above product is just the product in the half canonical ring $\mathscr{G}_{X,0}$ which is obviously integral.

Take $f \in \mathscr{G}_{X,i}$ for $i \geq 1$, and $g \in \mathscr{G}_{X,0}$ and assume $fg = 0$. For any derivation δ, one has $0 = \delta(fg) = \delta(f)g + f\delta(g) = \delta(f)g$. Hence either $g = 0$ or $\delta(f) = 0$ for any δ by induction on i. Hence either $g = 0$ or $f = 0$.

Let $i, j \geq 1$ and take $f \in \mathscr{G}_{X,i}$ and $g \in \mathscr{G}_{X,j}$ with $f \neq 0$, $g \neq 0$. Suppose $fg = 0$. Let δ be any derivation. If $\delta^i f \neq 0$, then let us show $\delta^{j-k} g = 0$ for $k = -1, 0, 1, \dots, j$ by induction on k, where the case of $k = -1$ is clear. Assume true for $k-1$. Then the relation $0 = \delta^{i+j-k}(fg) = \binom{i+j-k}{i}(\delta^i f)(\delta^{j-k} g)$ implies $\delta^{j-k} g = 0$ by induction. Hence $g = 0$ for $k = j$ contradicts the assumption on g. Thus $\delta^i f = 0$, and similarly $\delta^j g = 0$. If $i - 1 > 0$ and $\delta^{i-1} f \neq 0$, then again a similar argument shows that $\delta^{j-1-k} g = 0$ for $k = -1, 0, \dots, j-1$, which gives a contradiction. This implies again that $\delta^{i-1} f = 0$ and also $\delta^{j-1} g = 0$ as far as $i - 1 > 0$ and $j - 1 > 0$. Repeating a similar argument for p inductively, we obtain $\delta^{i-p} f = 0$ and $\delta^{j-p} g = 0$ for $0 \leq p < \min(i, j)$. By assuming $i \leq j$, this implies that $\delta f = 0$ for any derivation δ of degree -1 and hence $f = 0$.

ii) The exact sequence (6.3.1) implies the exact sequences:

$$0 \longrightarrow \mathscr{G}_{X,1} \xrightarrow{\delta^0} \mathscr{G}_{X,0} \otimes_{\mathbf{Z}} H^1 \xrightarrow{\delta^1} G^0_{X,-1} \otimes_{\mathbf{Z}} H^2 \longrightarrow 0$$

and

$$0 \longrightarrow \mathscr{G}_{X,i+1} \xrightarrow{\delta^0} \mathscr{G}_{X,i} \otimes_{\mathbf{Z}} H^1 \xrightarrow{\delta^1} \mathscr{G}_{X,i-1} \otimes_{\mathbf{Z}} H^2 \longrightarrow 0 \qquad (i > 0)$$

as $\mathscr{G}_{X,0}$-module. Since $\mathscr{G}_{X,0}$ is noetherian, by induction on i, this implies the finiteness of $\mathscr{G}_{X,i}$ as $\mathscr{G}_{X,0}$-module for $i \geq 0$. □

Remark. The ring $\mathscr{G}_X$ is neither noetherian nor of finite Krull dimension. (The denominator $1 - 2gt + t^2$ of the dimension formula (5.3.3) has a real root $t_0 = g - \sqrt{g^2 - 1}$ such that $0 < t_0 < 1$. This implies the exponential growth $g^n_{g,i} \sim O(t_0^{-i})$ of the ranks of $G^n_{X,i}$ in i so that the graded ring can not be noetherian. □)

Appendix

In the Appendix, for a 0-regular minimal module $\mathscr{F} = (F_i)_{i \geq 0}$ for a surface group Γ_g, we give a $\mathbf{Z}$-structure on the $\mathbf{C}$-vector spaces $G_i := F_{i+1}/F_i$ (for $i \geq 0$). This is done by an explicit lattice description of the G_i's (see (A.5)).

We introduce a sequence of lattices L_n and maps $\delta^0_n\colon L_{n+1} \to Hom_{\mathbf{Z}}(\Gamma_g, L_n) \simeq H \otimes_{\mathbf{Z}} L_n$ (for $n \geq 0$) by induction on n. (Here $H := Hom_{\mathbf{Z}}(\Gamma_g, \mathbf{Z})$ is the symplectic lattice together with the symplectic form I.) *The induction starts with* $L_0 := \mathbf{Z}$, $L_1 := H$ and $\delta^0_0\colon L_1 \to H \otimes_{\mathbf{Z}} \mathbf{Z}$ the natural isomorphism. Suppose that L_n and δ^0_{n-1} are already defined for some $n > 0$; define the map $\delta^1_n\colon H \otimes_{\mathbf{Z}} L_n \to L_{n-1}$ as the composite $H \otimes_{\mathbf{Z}} L_n \xrightarrow{id \times \delta^0} H \otimes_{\mathbf{Z}} H \otimes_{\mathbf{Z}} L_{n-1} \xrightarrow{I \times id} L_{n-1}$. Then we put

(A.1) $\quad L_{n+1} := \ker(\delta^1_n)$,

$\delta^0_n :=$ the canonical inclusion map of L_{n+1} into $H \otimes_{\mathbf{Z}} L_n$.

Thus one has the following exact sequence:

(A.2) $$0 \longrightarrow L_{n+1} \xrightarrow{\delta^0_n} H \otimes_{\mathbf{Z}} L_n \xrightarrow{\delta^1_n} L_{n-1} \qquad (n \geq 0).$$

(*Here by convention, we set* $L_{-1} = 0$.)

Lemma 1. *For* $p, q \geq 0$, *one can define a contraction map* $\iota_{pq}\colon L_p \otimes_{\mathbf{Z}} L_q \to L_{p-1} \otimes L_{q-1}$ *in term of the cup product* I, *giving rise to an exact sequence*:

(A.3) $$0 \longrightarrow L_{p+q} \longrightarrow L_p \otimes_{\mathbf{Z}} L_q \xrightarrow{\iota_{pq}} L_{p-1} \otimes_{\mathbf{Z}} L_{q-1} \longrightarrow 0.$$

2. *There exists an integral bilinear form* I_n *on* L_n $(n \geq 0)$, *symmetric or skew-symmetric according as n is even or odd, and compatible with the inclusions* L_{p+q}

$\to L_p \otimes_{\mathbf{Z}} L_q$ *for* $p, q \geq 0$.

Proof. Rewriting the definition of L_n inductively, one has the following description of L_n for $n \geq 1$.

(A.4) $$L_n \simeq \{\varphi \in H^{\otimes n}: I_j(\varphi) = 0 \quad \text{for} \quad j = 1, \dots, n-1.\}$$

where $H^{\otimes n}$ denotes the tensor product of n copies of H over $\mathbf{Z}$ and I_j is the contraction map from $H^{\otimes n}$ to $H^{\otimes (n-2)}$ defined by the cup product I of j th and $(j+1)$ th components for $1 \leq j \leq n-1$.

1. The restriction of the linear form I_p on $H^{\otimes (p+q)}$ induces the map ι_{pq} on $L_p \otimes_{\mathbf{Z}} L_q$ whose range is $L_{p-1} \otimes_{\mathbf{Z}} L_{q-1} \subset H^{\otimes (p+q-2)}$ and whose kernel is L_{p+q}. The surjectivity of ι_{pq} will be shown by induction on p. If $p = 0$, this is trivial since $L_{-1} = 0$ and $L_0 = \mathbf{Z}$. For $p = 1$, we put $\iota_q := \iota_{1q}$ and prove the surjectivity of ι_q $(q \geq 0)$ as follows.

Asserion. *Let* e_1, e_2, f_1, f_2 *be elements of* H *with the properties*: $I(e_1, e_2) = I(e_2, e_1) = 0$ *and* $I(e_1, f_1) = I(e_2, f_2) = 1$. *Then the restriction of* ι_q: $e_i \otimes L_q \to L_{q-1}$ *is surjective for* $i = 1, 2$ *and* $q \in \mathbf{N}$.

Proof. This is shown by induction on q. The case of $q = 1$ follows from the fact that $I(e_i, \mathbf{Z}f_i) = \mathbf{Z}$ $(i = 1, 2)$. Assume that $\iota_q: e_2 \otimes L_q \to L_{q-1}$ is surjective. Then for any $x \in L_q$ there exists $y \in L_q$ such that $\iota_q(f_1 \otimes x) = \iota_q(e_2 \otimes y)$. This means that $z := f_1 \otimes x - e_2 \otimes y$ belongs to L_{q+1} by definition. Since $\iota_{q+1}(e_1 \otimes z) = I(e_1, f_1)x - I(e_1, e_2)y = x$, this implies the surjectivity of $e_1 \otimes L_{q+1} \to L_q$. □

In a symplectic lattice H, one can choose $e_1 = e_2 = a$ and $f_1 = f_2 = b$ for the symplectic pair a and b. Thus ι_n is surjective for $n \geq 0$.

Return to the proof of the surjectivity of ι_{pq} for $p > 1$. Consider the following natural commutative diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{p+q+1} & \longrightarrow & L_{p+1} \otimes_{\mathbf{Z}} L_q & \xrightarrow{\iota_{p+1,q}} & L_p \otimes_{\mathbf{Z}} L_{q-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H \otimes_{\mathbf{Z}} L_{p+q} & \longrightarrow & H \otimes_{\mathbf{Z}} L_p \otimes_{\mathbf{Z}} L_q & \xrightarrow{\iota_{p,q}} & H \otimes_{\mathbf{Z}} L_{q-1} \otimes_{\mathbf{Z}} L_{q-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_{p+q}} & & \downarrow{\scriptstyle \iota_p} & & \downarrow{\scriptstyle \iota_{p-1}} & & \\
0 & \longrightarrow & L_{p+q-1} & \longrightarrow & L_{p-1} \otimes_{\mathbf{Z}} L_q & \xrightarrow{\iota_{p-1,q}} & L_{p-2} \otimes_{\mathbf{Z}} L_{q-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

By assumption, all columns are exact. By the induction hypothesis the bottom and the middle rows are exact. Hence the first row is exact. This completes the induction on p and the proof of the Lemma 1.

2. In view of the description of (A.4), the bilinear form $I^{\otimes n}$ on $H^{\otimes n}$ induces a bilinear form I_n on L_n, which is symmetric or skew symmetric according as n is even or odd. The fact that I_n are compatible with the inclusion maps is obvious.

□

Let us consider the formal power series: $f(t) := \sum_{n=0}^{\infty} [L_n]\, t^n$. Then (A.3) implies: $0 \to \frac{d}{dt}(tf(t)) \to f(t)^2 \to t^2 f(t)^2 \to 0$, which can be solved to give $f(t) = (1 - [H]\cdot t + t^2)^{-1}$. In particular we have

$$\sum_{n=0}^{\infty} \operatorname{rank}(L_n)\, t^n = (1 - 2g\cdot t + t^2)^{-1},$$

and hence an explicit formula:

$$\operatorname{rank}(L_n) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2g)^{n-2k}.$$

Remark 1. The construction of the sequence L_n can be started from any lattice H. The condition of the Assertion is satisfied by a wide class of lattices (e.g. one containing a unimodular sublattice of rank ≥ 2), and the exactness (A.3) holds also for them.

2. Let us denote by $P_n(g)$ the polynomial of the right-hand side of the formula for $\operatorname{rank}(L_n)$. Then $P_n(g) = 0$ has n distinct real roots in g, which separates the roots of $P_{n+1}(g) = 0$.

Now, we are able to describe graded pieces of a 0-regular minimal module in terms of integral lattices L_n.

Lemma. *Let the setting be as in* (3.4). *Assume that $\mathscr{F}$ is* 0-*regular. Then there is a canonical isomorphism*

$$\text{(A.5)} \qquad F_{i+1}/F_i \simeq L_i \otimes_{\mathbf{Z}} \overline{H^1(\Gamma_g, \mathscr{F})}.$$

Proof. This is proved by an induction on i, where $i = -1$ is trivial by convention that both sides are zero. The case $i = 0$ is proved in Lemma (2.4.2).

The exact sequence (A.2) induces a corresponding exact sequence for $L_i \otimes_{\mathbf{Z}} \overline{H^1(\Gamma_g, \mathscr{F})}$. A comparison of this exact sequence with that of (3.6.4) implies (A.5). □

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