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Entropy for Canonical Shifts. II

By

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Abstract

Let $N \subset M$ be a pair of factors. Associated with a conditional expectation E from M onto N with finite index, we introduce the canonical shift Γ on the von Neumann algebra A, with the canonical state ϕ, generated by the tower of relative commutants for the basic constructions iterated from E. Related with the minimum index $[M:N]_0$, we investigate the entropy $h_\phi(\Gamma)$ of Γ and the entropy $H_\phi(A|\Gamma(A))$ of A relative to the subalgebra $\Gamma(A)$. The inequalities $h_\phi(\Gamma) \le \log[M:N]_0$ and $\frac{1}{2}H_\phi(A|\Gamma(A)) \le \log[M:N]_0$ hold in general. Furthermore when E has the minimum index and $N \subset M$ has finite depth, we establish $h_\phi(\Gamma) = \frac{1}{2}H_\phi(A|\Gamma(A)) = \log[M:N]_0$.

Introduction

Based on Connes' spatial theory [9] and Haagerup's theory on operator valued weights [15], Kosaki [24] extended Jones' index theory [22] for type II_1 factors to that for conditional expectations between arbitrary factors. For a pair of factors $N \subset M$, let $\mathscr{E}(M, N)$ be the set of all faithful normal conditional expectations from M onto N. Although Kosaki's index $\operatorname{Index} E$ varies depending on $E \in \mathscr{E}(M, N)$, it was shown in [19] (also by Longo [30]) that if $\operatorname{Index} E < \infty$ for some $E \in \mathscr{E}(M, N)$, then there exists a unique $E_0 \in \mathscr{E}(M, N)$ which minimizes $\operatorname{Index} E$ for $E \in \mathscr{E}(M, N)$. Then the minimum index $[M:N]_0$ for $N \subset M$ is defined as $\operatorname{Index} E_0$.

Pimsner and Popa [33] extensively developed the entropy $H(M|N)$ for type II_1 factors $N \subset M$ in connection with Jones' index $[M:N]$. Among other things, they showed the inequality $H(M|N) \le \log[M:N]$ and obtained several characterizations for the equality. Also it was noted in [19] that $H(M|N) = \log[M:N]$ is equivalent to $[M:N] = [M:N]_0$. The entropy $H(M|N)$ was first used in Connes and Stǿrmer [12] to study the entropy of Kolmogorov-Sinai

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type for automorphisms of finite von Neumann algebras. Furthermore the notion of entropy for automorphisms was extended by Connes [10] and Connes, Narnhofer and Thirring [11] to the general setup of C^*-algebras or von Neumann algebras.

For von Neumann algebras $N \subset M$ and a faithful normal state φ on M, the entropy $H_\varphi(M|N)$ is defined as in [10], which coincides with the above $H(M|N)$ when φ is a trace. General properties of $H_\varphi(M|N)$ were given in [21]. When there exists $E \in \mathscr{E}(M, N)$ with $\varphi \circ E = \varphi$, another entropy $K_\varphi(M|N)$ was defined in [20] (also [23]) by taking account of Pimsner and Popa's estimate of $H(M|N)$. Given factors $N \subset M$ and $E \in \mathscr{E}(M, N)$, the relation between the entropy $K_E(M|N)$ and the minimum index $[M: N]_0$ was established in [20] in a way analogous to [33]. Here $K_E(M|N) = K_\varphi(M|N)$ independently of φ with $\varphi \circ E = \varphi$, and $K_E(M|N) = H(M|N)$ when $N \subset M$ are type II_1 factors and E is the conditional expectation with respect to the trace.

The entropy $H(\sigma)$ for a $*$-endomorphism σ of a finite von Neumann algebra A was investigated in [6, 7] in connection with the entropy $H(A|\sigma(A))$ and the generalized index $\lambda(A, \sigma(A))$ introduced in [33]. Here the entropy for $*$-endomorphisms can be defined in the same way as that for automorphims. For an inclusion $N \subset M$ of type II_1 factors with finite index, Ocneanu [31] introduced a special kind of $*$-endormorphism Γ, called the canonical shift, on the tower of relative commutants induced by the tower of basic constructions. The canonical shift Γ is extended on the von Neumann algebra A generated by the tower of relative commutants, which becomes a typical example of 2-shifts. Under a certain assumption (equivalent to the equality $H(M|N) = \log[M: N]$), the following relations were obtained in [7]:

$$H(A|\Gamma(A)) \leq 2H(\Gamma) \leq \log\lambda(A, \Gamma(A))^{-1} = 2\log[M: N].$$

These numbers are all identical particularly when $N \subset M$ has finite depth. The aim of this paper is to extend the results in [7] to the canonical shift defined for a pair of arbitrary factors.

In §1 of this paper, for the reader's convenience, we list definitions and preliminaries on the index theory and the entropy theory. In particular, we note that the main results for automorphisms in [10, 11] remain valid also for $*$-endomorphisms. Now let $N \subset M$ be factors and $E \in \mathscr{E}(M, N)$ with $\operatorname{Index} E < \infty$. Then we obtain the basic construction for E following [24], which consists of a factor $M_1 \supset M$, a projection $e \in M_1$ with $M_1 = \langle M, e\rangle$, and $E_1 \in \mathscr{E}(M_1, M)$. In §2, we obtain an algebraic (up to isomorphisms) characterization of the basic construction (M_1, e, E_1) for E. A similar characterization was given in [17].

Iterating the basic constructions from E, we obtain the tower of factors $N \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots$ with projections $e_{n-1} \in M_n$ and $E_n \in \mathscr{E}(M_n, M_{n-1})$ for $n \geq 1$. In §3, on the tower of relative commutants $M' \cap M_1 \subset M' \cap M_2 \subset \cdots$, we define the mirrorings γ_n and the canonical shift Γ. To do so, we adopt a powerful idea of Longo's canonical endomorphism [28–30]. Here it is worth noting that the canonical shift Γ on $\bigcup_n (M' \cap M_n)$ is independent (up to isomorphisms) of the choice of $E \in \mathscr{E}(M, N)$. But taking the GNS representation associated with the state ϕ canonically determined by $\{E_n\}$, we extend Γ (denoted by the same Γ) to a $*$-endomorphism of the von Neumann algebra A generated by $\bigcup_n (M' \cap M_n)$. We call (A, ϕ, Γ) the canonical shift associated with E. In particular, when $E = E_0$ (i.e. $\operatorname{Index} E = [M : N]_0$), ϕ is a trace and Popa's analysis [36] on sequences of commuting squares can be applied to the tower of relative commutants in our setup. Thus A is a type II_1 factor when $E = E_0$ and $N \subset M$ has finite depth.

In §§4 and 5, we establish relations among the entropy $h_\phi(\Gamma)$ of Γ, the entropy $H_\phi(A \mid \Gamma(A))$ relative to $\Gamma(A)$, and the minimum index $[M : N]_0$. The following inequalities hold in general:

$$h_\phi(\Gamma) \leq \frac{1}{2}\{K_E(M \mid N) + K_{E_1}(M_1 \mid M)\} \leq \log [M : N]_0,$$

$$H_\phi(A \mid \Gamma(A)) \leq 2 \log [M : N]_0.$$

We show that $E = E_0$ if $h_\phi(\Gamma) = \log [M : N]_0$ or if $H_\phi(A \mid \Gamma(A)) = 2 \log [M : N]_0$. The inequality $H_\phi(A \mid \Gamma(A)) \leq 2 h_\phi(\Gamma)$ holds when $E = E_0$. Furthermore when $E = E_0$ and $N \subset M$ has finite depth, we obtain

$$H_\phi(A \mid \Gamma(A)) = 2 h_\phi(\Gamma) = \log \lambda_\phi(A, \Gamma(A))^{-1} = 2 K_E(M \mid N) = 2 \log [M : N]_0.$$

Finally in §6, we give two typical examples to illustrate our main results.

§1. Preliminaries on Index and Entropy

In this paper, all von Neumann algebras are assumed to be σ-finite. Let M be a von Neumann algebra. The set of all faithful normal states on M is denoted by $\mathscr{S}(M)$. Given a von Neumann subalgebra N of M, we denote by $\mathscr{E}(M, N)$ the set of all faithful normal conditional expectations from M onto N. In this section, we collect definitions and preliminaries on index and entropy for the reader's convenience.

(1.1) Index for conditional expectations

Let $N \subset M$ be von Neumann algebras on a Hilbert space $\mathscr{H}$. For each $E \in \mathscr{E}(M, N)$, there corresponds uniquely a faithful normal semifinite operator valued weight E^{-1} from N' to M' by the equation $d(\varphi \circ E)/d\psi = d\varphi / d(\psi \circ E^{-1})$ of spatial derivatives where φ and ψ are any faithful normal semifinite weights on N and M', respectively (see [15], [37, 12.11]). When $N \subset M$ is a pair of factors, Kosaki [24] defined the index of E by $\operatorname{Index} E = E^{-1}(1)$. Kosaki's index extends Jones' index [22] in the sense that if M is a finite factor and E_N is the conditional expectation [39] onto a subfactor N with respect to the trace, then $\operatorname{Index} E_N$ coincides with Jones' index $[M: N]$.

The following formula (the best constant for Pimsner and Popa's inequality) serves as another definition of $\operatorname{Index} E$. This formula for Jones' index is due to [33, Theorem 2.2], while for infinite factors it was obtained in several ways in [3, 16, 27, 30]. The proof when $\operatorname{Index} E < \infty$ is not difficult as shown in [27]. A nice proof of full generality is given in [26].

Theorem 1.1. *Let $N \subset M$ be factors where M is not finite dimensional. Then for every $E \in \mathscr{E}(M, N)$,*

$$(\operatorname{Index} E)^{-1} = \max\{\lambda \geq 0 \colon E(x) \geq \lambda x,\ x \in M_+\}.$$

Moreover $(\operatorname{Index} E)^{-1}$ *is always the best constant for the complete positivity of $E - \lambda\, \mathrm{id}_M$ including the case M is finite dimensional.*

(1.2) Minimum index

Given a pair of factors $N \subset M$, the value of $\operatorname{Index} E$ depends on the choice of $E \in \mathscr{E}(M, N)$. But if $\operatorname{Index} E < \infty$ for some $E \in \mathscr{E}(M, N)$, then the relative commutant $N' \cap M$ is finite dimensional and $\operatorname{Index} E < \infty$ for all $E \in \mathscr{E}(M, N)$ as noted in [19]. In this case, it was proved in [19] that there exists a unique $E_0 \in \mathscr{E}(M, N)$ such that

$$\operatorname{Index} E_0 = \min\{\operatorname{Index} E \colon E \in \mathscr{E}(M, N)\},$$

which is characterized by the condition

$$E_0^{-1} | N' \cap M = (\operatorname{Index} E_0) E_0 | N' \cap M.$$

In fact, $E_0 | N' \cap M$ becomes a trace on $N' \cap M$. We define the minimum index $[M: N]_0$ for a pair $N \subset M$ by $[M: N]_0 = \operatorname{Index} E_0$. Also let $[M: N]_0 = \infty$ if $\mathscr{E}(M, N) = \emptyset$ or $\operatorname{Index} E = \infty$ for all $E \in \mathscr{E}(M, N)$. Note [24, Theorem 4.4] that if $\operatorname{Index} E < 4$ for some $E \in \mathscr{E}(M, N)$, then $N' \cap M = \mathbf{C}$ and hence $\mathscr{E}(M, N) = \{E\}$. See [18, 20, 30] for properties of the minimum index.

(1.3) Commuting squares

Consider a square

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ B & \subset & A \end{array}$$

of von Neumann algebras and let $\varphi \in \mathscr{E}(M)$. Then the next proposition can be proved as [14, 4.2.1].

Proposition 1.2. *Assume that there exist conditional expectations $E \colon M \to N$,*

$F: M \to A$ and $G: M \to B$ with respect to φ (i.e. $\varphi \circ E = \varphi \circ F = \varphi \circ G = \varphi$). Then the following conditions are equivalent:

(i) $E(A) \subset B$;

(ii) $E \circ F = G$;

(iii) $E \circ F = F \circ E$ *and* $A \cap N = B$;

(iv) $E|A = G|A$.

We say that

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ B & \subset & A \end{array}$$

is a commuting square with respect to φ if there exist the conditional expectations E, F and G as above and the equivalent conditions (i)–(iv) hold.

(1.4) Entropies $H_\varphi(M|N)$ and $K_\varphi(M|N)$

Let $N \subset M$ be von Neumann algebras with $\varphi \in \mathscr{E}(M)$. Following Connes [10], we define the entropy $H_\varphi(M|N)$ of M relative to N and φ by

$$H_\varphi(M|N) = \sup_{(\psi_i)} \sum_i \{S(\varphi, \psi_i) - S(\varphi|N, \psi_i|N)\},$$

where the supremum is taken over all finite families $(\psi_1, \dots, \psi_n)$ of $\psi_i \in M_*^+$ with $\sum_i \psi_i = \varphi$. Here $S(\varphi, \psi)$ denotes the relative entropy of φ, $\psi \in M_*^+$, which was first introduced by Umegaki [40] in the semifinite case and extended by Araki [1, 2] to the general case. Particularly if M is finite with a faithful normal trace τ, $\tau(1) = 1$, then $H(M|N) = H_\tau(M|N)$ is given by

$$H(M|N) = \sup_{(x_i)} \sum_i \{\tau(\eta E_N(x_i)) - \tau(\eta x_i)\},$$

where $\eta(t) = -t \log t$ on $[0, \infty)$, $E_N: M \to N$ is the conditional expectation with respect to τ, and the supremum is taken over all finite families $(x_1, \dots, x_n)$ of $x_i \in M_+$ with $\sum_i x_i = 1$. See [21] for general properties of $H_\varphi(M|N)$.

When there exists $E \in \mathscr{E}(M, N)$ with $\varphi \circ E = \varphi$, another entropy $K_\varphi(M|N)$ of M relative to N and φ was defined in [20] (also [23]) by

$$K_\varphi(M|N) = -S(\hat{\omega}, \omega)$$

where $\omega = \varphi|N' \cap M$ and $\hat{\omega} = \varphi \circ (E^{-1}|N' \cap M)$. Here unless $\hat{\omega}$ is bounded, the relative entropy $S(\hat{\omega}, \omega)$ is given by the infimum of $S(\omega', \omega)$ for $\omega' \in (N' \cap M)_*^+$ with $\omega' \le \hat{\omega}$. If N is a factor, then $K_\varphi(M|N)$ is independent of the choice of $\varphi \in \mathscr{E}(M)$ with $\varphi \circ E = \varphi$, so that we write $K_\varphi(M|N) = K_E(M|N)$. Although the entropies $H_\varphi(M|N)$ and $K_\varphi(M|N)$ are not generally identical, we have $H(M|N) = K_\tau(M|N)$ when M is a type II_1 factor.

(1.5) Relation between entropy and index

Pimsner and Popa [33] established the relation between the entropy $H(M|N)$ and Jones' index $[M: N]$ for a pair of type II_1 factors $N \subset M$. The entropy $K_\varphi(M|N)$ was investigated in [20] in connection with Kosaki's index and the minimum index for a pair of general factors. In the following, we state the main results in [20] restricting to the case of factors $N \subset M$. Here note that the centralizer $(N' \cap M)_E$ of $E \in \mathscr{E}(M, N)$ is atomic whenever so is $N' \cap M$. Also for each nonzero projection q in $N' \cap M$, $E(q)$ is a scalar and $E_q \in \mathscr{E}(M_q, N_q)$ is defined by $E_q(x) = E(q)^{-1}E(x)q$, $x \in M_q$.

Theorem 1.3. *Let $E \in \mathscr{E}(M, N)$.*

(1) *If $N' \cap M$ has a nonatomic part, then $K_E(M|N) = \infty$.*

(2) *If $N' \cap M$ is atomic and $\{q_i\}$ is a set of atoms in $(N' \cap M)_E$ with $\sum_i q_i = 1$, then*

$$K_E(M|N) = \sum_i E(q_i) \log \frac{\operatorname{Index} E_{q_i}}{E(q_i)^2}.$$

Theorem 1.4. (1) $K_E(M|N) \le \log [M: N]_0$ *for every $E \in \mathscr{E}(M, N)$.*

(2) *If $[M: N]_0 < \infty$, then the following conditions for $E \in \mathscr{E}(M, N)$ are equivalent*:

(i) $\operatorname{Index} E = [M: N]_0$, *i.e.* $E = E_0$;

(ii) $K_E(M|N) = \log [M: N]_0$;

(iii) $K_E(M|N) = \log \operatorname{Index} E$;

(iv) $\operatorname{Index} E_q = E(q)^2 \operatorname{Index} E$ *for every nonzero projection $q \in N' \cap M$.*

(1.6) Entropy for $*$-endomorphisms

After Connes and Størmer [12] developed the entropy of Kolmogorov-Sinai type for automorphisms of finite von Neumann algebras, Connes [10] and Connes, Narnhofer and Thirring [11] extended it to the general setup of C^*-algebras or von Neumann algebras. As in [6], to fix the notations, we briefly survey its definition extending to the case for $*$-endomorphisms of general von Neumann algebras.

Let A be a von Neumann algebra with $\phi \in \mathscr{E}(A)$ and σ a $*$-endomorphism of A with $\phi \circ \sigma = \phi$. Then σ is unital, injective and weakly continuous. For each $n \in \mathbf{N}$, we denote by $\mathscr{P}_n$ the set of all families $\Psi = (\psi_{i_1,\dots,i_n})_{i_1,\dots,i_n \in \mathbf{N}}$ of $\psi_{i_1,\dots,i_n} \in A_*^+$ such that $\sum_{i_1,\dots,i_n} \psi_{i_1,\dots,i_n} = \phi$ and $\psi_{i_1,\dots,i_n} = 0$ except for a finite number of indices. For $\Psi \in \mathscr{P}_n$, $k \in \{1,\dots,n\}$ and $j \in \mathbf{N}$, let

$$\Psi_j^k = \sum_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_n} \psi_{i_1,\dots,i_{k-1},j,i_{k+1},\dots,i_n}.$$

Given finite dimensional subalgebras $B_1,\dots,B_n$ of A, Connes [10] defined

$$H_\phi(B_1,\dots,B_n) = \sup_{\Psi\in\mathcal{P}_n} \{ \sum_{i_1,\dots,i_n} \eta(\psi_{i_1,\dots,i_n}(1)) + \sum_{k=1}^{n} \sum_j S(\phi|B_k, \Psi_j^k|B_k)\}.$$

Then the following is easy to check as in [6, Lemma 2]:

$$H_\phi(\sigma(B_1),\dots,\sigma(B_n)) \le H_\phi(B_1,\dots,B_n).$$

By this and [10, Théorème 5], the following limit exists for each finite dimensional subalgebra B of A:

$$h_{\phi,\sigma}(B) = \lim_{n\to\infty} \frac{1}{n} H_\phi(B, \sigma(B),\dots,\sigma^{n-1}(B)).$$

Now the entropy $h_\phi(\sigma)$ of σ relative to ϕ is defined by the supremum of $h_{\phi,\sigma}(B)$ for all finite dimensional subalgebras B of A.

The arguments in [11, § VII] remain true also for $*$-endomorphism, so that we have:

Theorem 1.5. *If $\{B_k\}$ is an increasing sequence of finite dimensional subalgebras with $A = (\bigcup_k B_k)''$ (so A is approximately finite dimensional), then*

$$h_\phi(\sigma) = \lim_{k\to\infty} h_{\phi,\sigma}(B_k).$$

Proposition 1.6. (1) $h_\phi(\sigma) = h_{\phi\circ\theta}(\theta^{-1}\sigma\theta)$ *for every automorphism θ of A.*

(2) $h_\phi(\sigma^n) \le nh_\phi(\sigma)$ *for all $n\in\mathbf{N}$, and the equality holds if A is approximately finite dimensional.*

§ 2. Basic Construction and Algebraic Basic Construction

The concept of the basic construction invented by Jones is a core in the index theory [22, 24]. In this section, we present some preliminary results on the (algebraic) basic construction, which will be useful in the next section.

First let us recall the procedure of the basic construction [24]. Let $N \subset M$ be factors and $E\in\mathcal{E}(M, N)$ with Index $E < \infty$. Choosing $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E = \varphi$, we represent M standardly on $\mathcal{H} = \mathcal{H}_\varphi$ equipped with the natural cone $\mathcal{H}^+$ and the associated modular conjugation J. Define the factor $M_1 = JN'J$ and the projection e by $e(x\xi) = E(x)\xi$, $x\in M$, where $\varphi = \omega_\xi$, $\xi\in\mathcal{H}^+$. Then

$$M_1 = \langle M, e\rangle (=(M\cup\{e\})'').$$

Also $E_1\in\mathcal{E}(M_1, M)$ is defined by

$$E_1(x) = (\text{Index}\, E)^{-1} JE^{-1}(JxJ)J, \qquad x\in M_1.$$

The construction of (M_1, e, E_1) is called the basic construction for E. We write this procedure as follows:

$$N \underset{E}{\subset} M \underset{E_1}{\subset}{}^{e} M_1.$$

The next proposition is a restatement of [25, Appendix I], which shows that the basic construction is canonical up to spatial isomorphisms.

Proposition 2.1. *Let $\hat{N} \subset \hat{M}$ be factors with $\hat{E} \in \mathscr{E}(\hat{M}, \hat{N})$. Let $\theta: M \to \hat{M}$ be an isomorphism such that $\theta(N) = \hat{N}$ and $\theta \circ E = \hat{E} \circ \theta$ (hence* Index $\hat{E}$ = Index E $< \infty$). *Let*

$$\hat{N} \underset{\hat{E}}{\subset} \hat{M} \underset{\hat{E}_1}{\subset}{}^{\hat{e}} \hat{M}_1$$

be the basic construction for $\hat{E}$ defined on a Hibert space $\hat{\mathscr{H}}$ with the conjugation $\hat{J}$. Then there exists a unitary $u: \mathscr{H} \to \hat{\mathscr{H}}$ such that

(i) $uxu^* = \theta(x)$, $x \in M$,

(ii) $uJu^* = \hat{J}$,

(iii) $ueu^* = \hat{e}$,

(iv) $uM_1u^* = \hat{M}_1$,

(v) $\mathrm{Ad}(u) \circ E_1 = \hat{E}_1 \circ \mathrm{Ad}(u)$ *on* M_1.

Now let us introduce the notion of algebraic basic constructions. Let $N \subset M \subset M_1$ be factors, $e \in M_1$ a projection, $E \in \mathscr{E}(M, N)$ and $E_1 \in \mathscr{E}(M_1, M)$. We call (M_1, e, E_1) an algebraic basic construction for E if

(1) $M_1 = \langle M, e\rangle$,

(2) $E_1(e) = \lambda 1$ $(\lambda > 0)$,

(3) $exe = E(x)e$, $x \in M$.

In this case, we have Index $E < \infty$ by Theorem 1.1 because for $x \in M_+$

$$\lambda E(x) = E_1(E(x)e) = E_1(exe) \geq E_1(x^{1/2}e)^* E_1(x^{1/2}e) = \lambda^2 x.$$

Note [24, Lemma 3.2 and (8)] that the basic construction for E is an algebraic one. The next proposition shows the uniqueness of algebraic basic constructions for E up to isomorphisms. In other words, the above (1)–(3) algebraically characterize the basic construction for E (see [34, Proposition 1.2] for more elegant characterizations in the type II_1 case). A similar result was obtained by Hamachi and Kosaki [17, Theorem 8].

Proposition 2.2. *Besides (M_1, e, E_1) for $E \in \mathscr{E}(M, N)$ as above, let $\tilde{N} \subset \tilde{M} \subset \tilde{M}_1$ be factors, $\tilde{e} \in \tilde{M}_1$ a projection, $\tilde{E} \in \mathscr{E}(\tilde{M}, \tilde{N})$ and $\tilde{E}_1 \in \mathscr{E}(\tilde{M}_1, \tilde{M})$ such that $(\tilde{M}_1, \tilde{e}, \tilde{E}_1)$ is an algebraic basic construction for $\tilde{E}$. If $\theta: M \to \tilde{M}$ is an isomorphism such that $\theta(N) = \tilde{N}$ and $\theta \circ E = \tilde{E} \circ \theta$, then there exists a unique isomorphism $\theta_1: M_1 \to \tilde{M}_1$ such that*

(i) $\theta_1(x) = \theta(x)$, $x \in M$,

(ii) $\theta_1(e) = \tilde{e}$,

(iii) $\theta_1 \circ E_1 = \tilde{E}_1 \circ \theta_1$.

Proof. Since $\operatorname{Index} E < \infty$ as remarked above, we can take the basic construction for E. So we may assume that (M_1, e, E_1) is the basic construction (in the spatial sense) for E. Let $\tilde{E}_1(\tilde{e}) = \tilde{\lambda}1$ ($\tilde{\lambda} > 0$), while $E_1(e) = \lambda 1$ with $\lambda = (\operatorname{Index} E)^{-1}$. Let $\varphi \in \mathscr{E}(M)$ and define $\psi \in \mathscr{E}(M_1)$, $\tilde{\psi} \in \mathscr{E}(\tilde{M}_1)$ by $\psi = \varphi \circ E_1$, $\tilde{\psi} = \varphi \circ \theta^{-1} \circ \tilde{E}_1$. According to [33, Proposition 1.3] and [41, 2.5.3], there exists a basis $\{m_1, \ldots, m_n\}$ in M for E which satisfies $\sum_i m_i e m_i^* = 1$. The existence of such a basis shows that M_1 is the linear span of $\{aeb\colon a, b \in M\}$ (see the proof of [33, Proposition 1.5]). For any $a_i, b_i \in M$, $1 \le i \le k$, we have

$$\psi(\sum_i a_i e b_i) = \lambda\varphi(\sum_i a_i b_i) = \mu\tilde{\psi}(\sum_i \theta(a_i)\tilde{e}\theta(b_i))$$

where $\mu = \lambda\tilde{\lambda}^{-1}$. Therefore a map $\theta_1\colon M_1 \to \tilde{M}_1$ is well defined by

$$\theta_1(\sum_i a_i e b_i) = \sum_i \theta(a_i)\tilde{e}\theta(b_i), \qquad a_i, b_i \in M.$$

Then (ii) holds and θ_1 is a $*$-homomorphism. Since $\psi = \mu\tilde{\psi} \circ \theta_1$, θ_1 is normal and hence injective. We get

$$\theta_1(1)\tilde{e} = \theta_1(e) = \tilde{e}\theta_1(1),$$

and since $\theta_1(1) = \sum_i \theta(m_i)\tilde{e}\theta(m_i^*)$,

$$\theta_1(1)\theta(x) = \theta_1(x) = \theta(x)\theta_1(1), \qquad x \in M,$$

so that $\theta_1(1) = 1$, showing (i). Moreover the linear span of $\{\tilde{a}\tilde{e}\tilde{b}\colon \tilde{a}, \tilde{b} \in \tilde{M}\}$ is the $*$-algebra generated by $\tilde{M} \cup \{\tilde{e}\}$, so that θ_1 is surjective. Since

$$(\theta_1 \circ E_1)(aeb) = \lambda\theta(ab) = \mu(\tilde{E}_1 \circ \theta_1)(aeb), \qquad a, b \in M,$$

we get $\mu = 1$ and (iii) holds. It is immediate that (i) and (ii) uniquely determine θ_1. ■

Proposition 2.2 shows that algebraic properties of the basic construction automatically become those of an algebraic basic construction (M_1, e, E_1) for E. So we have for instance

(4) $\operatorname{Index} E = \operatorname{Index} E_1 = \lambda^{-1}$ for λ in (2),

(5) $N = M \cap \{e\}'$,

(6) $e \in (N' \cap M_1)_{E \circ E_1}$, the centralizer of $E \circ E_1$.

Furthermore we have:

Proposition 2.3. *In any representation of* $N \subset M \subset M_1$,

(7) $E^{-1}(e) = 1$,

(8) $exe = \lambda E_1^{-1}(x)e$, $x \in M'$.

Proof. Since (7) and (8) hold for the basic construction, it suffices by

Proposition 2.2 to show that the validity of (7) and (8) is preserved under isomorphisms, that is, if $\alpha: M_1 \to \tilde{M}_1$ is an isomorphism and if $\tilde{M} = \alpha(M)$, $\tilde{N} = \alpha(N)$, $\tilde{E} = \alpha_0 \circ E \circ \alpha_0^{-1}$ ($\alpha_0 = \alpha | M$), $\tilde{E}_1 = \alpha \circ E_1 \circ \alpha^{-1}$ and $\tilde{e} = \alpha(e)$, then (7) and (8) imply

(7′) $\tilde{E}^{-1}(\tilde{e}) = 1$,

(8′) $\tilde{e}x\tilde{e} = \lambda \tilde{E}_1^{-1}(x)\tilde{e}$, $x \in \tilde{M}'$.

We may separately consider an amplification, an induction and a spatial isomorphism. The first two cases are easily shown by [20, Propositions 1.7 and 1.5]. The final case is obvious. ■

From now on, let $N \subset M$ be always a pair of factors with $[M : N]_0 < \infty$. By iterating the basic construction from $E \in \mathscr{E}(M, N)$, we obtain the tower

$$N \underset{\overleftarrow{E}}{\subset} M = M_0 \underset{\overleftarrow{E_1}}{\subset} M_1 \underset{\overleftarrow{E_2}}{\subset} M_2 \subset \cdots$$

of factors M_n together with conditional expectations $E_n \in \mathscr{E}(M_n, M_{n-1})$. Then we have the commuting square property as follows.

Proposition 2.4. *Let $\varphi_0 \in \mathscr{E}(M)$ with $\varphi_0 \circ E = \varphi_0$, and define $\varphi_n \in \mathscr{E}(M_n)$ by*

$$\varphi_n = \varphi_0 \circ E_1 \circ \cdots \circ E_n, \qquad n \geq 1.$$

Then for every $0 \leq j \leq k \leq n$

$$\begin{array}{ccc} M_k & \subset & M_n \\ \cup & & \cup \\ M_j' \cap M_k & \subset & M_j' \cap M_n \end{array}$$

is a commuting square with respect to φ_n.

Proof. Let $E_{k,n} = E_{k+1} \circ \cdots \circ E_n$ which is the conditional expectation $M_n \to M_k$ with respect to φ_n. Hence by [38], we have

$$\sigma_t^{\varphi_n}(M_j' \cap M_n) = M_j' \cap M_n, \quad \sigma_t^{\varphi_n}(M_j' \cap M_k) = M_j' \cap M_k, \qquad t \in \mathbf{R},$$

so that there exist the conditional expectations $M_n \to M_j' \cap M_n$ and $M_n \to M_j' \cap M_k$ with respect to φ_n. Now it is easy to check that $E_{k,n}(M_j' \cap M_n) = M_j' \cap M_k$, implying the conclusion by Proposition 1.2. ■

Let $E_0 \in \mathscr{E}(M, N)$ be such that $\operatorname{Index} E_0 = [M : N]_0$. Then the characterization of E_0 in (1.2) immediately shows the following:

Proposition 2.5. $E = E_0$ *if and only if* $E(x) = E_1(Jx^*J)$ *for all* $x \in N' \cap M$.

The next important result on the minimum index for the tower is proved by Kosaki and Longo [26].

Theorem 2.6. *Suppose $E = E_0$. Then for every $0 \leq k < n$, $E_{k+1} \circ \cdots \circ E_n \in \mathscr{E}(M_n, M_k)$ gives the minimum index for $M_k \subset M_n$; equivalently $[M_n : M_k]_0 = [M : N]_0^{n-k}$. In particular, $E_{k+1} \circ \cdots \circ E_n | M'_k \cap M_n$ is a trace.*

§ 3. Definitions of Mirrorings and Canonical Shift

Ocneanu [31] introduced the important concepts of the mirrorings and the canonical shift on the tower of relative commutants for the inclusion of type II_1 factors with finite index. The aim of this section is to present precise definitions of the mirrorings and the canonical shift which are available to the inclusion $N \subset M$ of general factors.

Given $E \in \mathscr{E}(M, N)$ where $\operatorname{Index} E < \infty$ by assumption, let

$$N \underset{\overleftarrow{E}}{\subset} M = M_0 \underset{\overleftarrow{E_1}}{\subset}{}^{e_0} M_1 \underset{\overleftarrow{E_2}}{\subset}{}^{e_1} M_2 \underset{\overleftarrow{E_3}}{\subset}{}^{e_2} M_3 \subset \cdots$$

be the tower of basic constructions iterated from E. Here for each $n \geq 1$, M_n is standardly represented on a Hilbert space $\mathscr{H}_n$ with the modular conjugation J_n, so that

$$M_{n+1} = J_n M'_{n-1} J_n = \langle M_n, e_n \rangle,$$
$$E_{n+1}(x) = \lambda J_n E_n^{-1}(J_n x J_n) J_n, \qquad x \in M_{n+1},$$

where $\lambda = (\operatorname{Index} E)^{-1}$. Then we obtain the tower

$$M' \cap M_1 \subset M' \cap M_2 \subset M' \cap M_3 \subset \cdots$$

of relative commutant algebras, which is an increasing sequence of finite dimensional algebras.

For each fixed $n \geq 1$, let us define

$$\tilde{M}_k = J_n M'_{2n-k} J_n, \qquad n+1 \leq k \leq 2n,$$
$$\tilde{e}_k = J_n e_{2n-k} J_n, \qquad n+1 \leq k \leq 2n-1,$$
$$\tilde{E}_k(x) = \lambda J_n E_{2n-k+1}^{-1}(J_n x J_n) J_n, \qquad x \in \tilde{M}_k,\ n+2 \leq k \leq 2n,$$

where M'_{2n-k} and E_{2n-k+1}^{-1} are defined for $M_{2n-k} \subset M_{2n-k+1}$ represented on $\mathscr{H}_n$. Then

$$M_n \subset M_{n+1} = \tilde{M}_{n+1} \subset \tilde{M}_{n+2} \subset \cdots \subset \tilde{M}_{2n},$$
$$\tilde{e}_{k-1} \in \tilde{M}_k,\ \tilde{E}_k \in \mathscr{E}(\tilde{M}_k, \tilde{M}_{k-1}), \qquad n+2 \leq k \leq 2n,$$

and we have:

Lemma 3.1. *For every $n+1 \leq k \leq 2n-1$, $(\tilde{M}_{k+1}, \tilde{e}_k, \tilde{E}_{k+1})$ is an algebraic basic construction for $\tilde{E}_k$ where $\tilde{E}_{n+1} = E_{n+1}$.*

Proof. Let us check three conditions for the algebraic basic construction. Since $M_{2n-k-1} = M_{2n-k} \cap \{e_{2n-k}\}'$, $\widetilde{M}_{k+1} = \langle \widetilde{M}_k, \tilde{e}_k \rangle$. Since $E_{2n-k}^{-1}(e_{2n-k}) = 1$ by Proposition 2.3 (7),

$$\widetilde{E}_{k+1}(\tilde{e}_k) = \lambda J_n E_{2n-k}^{-1}(e_{2n-k}) J_n = \lambda 1.$$

Since by Proposition 2.3 (8)

$$e_{2n-k} x e_{2n-k} = \lambda E_{2n-k+1}^{-1}(x) e_{2n-k}, \qquad x \in M'_{2n-k},$$

we get for $x \in \widetilde{M}_k$

$$\tilde{e}_k x \tilde{e}_k = \lambda J_n E_{2n-k+1}^{-1}(J_n x J_n) e_{2n-k} J_n = \widetilde{E}_k(x) \tilde{e}_k. \quad ∎$$

Lemma 3.1 enables us to apply Proposition 2.2 recursively, so that we have the following:

Proposition 3.2. *For every $n \geq 1$, there exists a unique isomorphism $\theta_n: M_{2n} \to \widetilde{M}_{2n}$ such that*

(1) $\theta_n(x) = x$, $x \in M_{n+1}$,
(2) $\theta_n(M_k) = \widetilde{M}_k$, $n+2 \leq k \leq 2n$,
(3) $\theta_n(e_k) = \tilde{e}_k$, $n+1 \leq k \leq 2n-1$,
(4) $\theta_n \circ E_k = \widetilde{E}_k \circ \theta_n$ *on* M_k, $n+2 \leq k \leq 2n$.

Using θ_n in Proposition 3.2, we now define antiautomorphisms γ_n of $M' \cap M_{2n}$, $n \geq 1$, by

$$\gamma_n(x) = \theta_n^{-1}(J_n \theta_n(x^*) J_n), \qquad x \in M' \cap M_{2n}.$$

(Also let $\gamma_0 = \mathrm{id}$ on $M' \cap M = \mathbf{C}$.) We obviously have $\gamma_n \circ \gamma_n = \mathrm{id}_{M' \cap M_{2n}}$. These γ_n are called the mirrorings on the tower of relative commutants, which extend those in [31]. The next proposition shows that the sequence $\{\gamma_n\}$ of mirrorings is determined (up to isomorphisms) independently of the choice of the tower.

Proposition 3.3. *Besides the tower and $\{\gamma_n\}$ described above, let*

$$N \underset{E}{\subset} M = \hat{M}_0 \underset{\hat{E}_1}{\subset}^{\hat{e}_0} \hat{M}_1 \underset{\hat{E}_2}{\subset}^{\hat{e}_1} \hat{M}_2 \subset \cdots$$

be another tower of basic constructions from E, and $\{\hat{\gamma}_n\}$ be the associated sequence of mirrorings. Then there exists an isomorphism $\Theta: \bigcup_n M_n \to \bigcup_n \hat{M}_n$ such that for every $n \geq 1$

(1) $\Theta(x) = x$, $x \in M$,
(2) $\Theta(M_n) = \hat{M}_n$,
(3) $\Theta \circ E_n = \hat{E}_n \circ \Theta$ *on* M_n,
(4) $\Theta \circ \gamma_n = \hat{\gamma}_n \circ \Theta$ *on* $M' \cap M_{2n}$.

Proof. Let $\hat{M}_n$ be standardly represented on a Hilbert space $\hat{\mathscr{H}}_n$ with the conjugation $\hat{J}_n$. Applying Proposition 2.1 recursively, we have a sequence $\{u_n : n \geq 0\}$ of unitaries $u_n : \mathscr{H}_n \to \hat{\mathscr{H}}_n$ such that for every $n \geq 0$

(i) $u_n x u_n^* = u_{n-1} x u_{n-1}^*$, $x \in M_n$ (with convention $u_{-1} = 1$),

(ii) $u_n J_n u_n^* = \hat{J}_n$,

(iii) $u_n e_n u_n^* = \hat{e}_n$,

(iv) $u_n M_{n+1} u_n^* = \hat{M}_{n+1}$,

(v) $\mathrm{Ad}(u_n) \circ E_{n+1} = \hat{E}_{n+1} \circ \mathrm{Ad}(u_n)$ on M_{n+1}.

Thus an isomorphism $\Theta : \bigcup_n M_n \to \bigcup_n \hat{M}_n$ can be defined by $\Theta(x) = u_n x u_n^*$, $x \in M_n$, $n \geq 0$. Then (1)–(3) are immediate. Let us prove (4). Besides $\theta_n : M_{2n} \to \tilde{M}_{2n}(= J_n M' J_n)$, let $\hat{\theta}_n : \hat{M}_{2n} \to \hat{J}_n M' \hat{J}_n$, $n \geq 1$, be isomorphisms as in Proposition 3.2 associated with the second tower. For each $n \geq 1$, we can define the isomorphism $\Theta_{2n} : M_{2n} \to \hat{M}_{2n}$ by $\Theta_{2n} = \hat{\theta}_n^{-1} \circ \mathrm{Ad}(u_n) \circ \theta_n$ because $u_n(J_n M' J_n) u_n^* = \hat{J}_n M' \hat{J}_n$. Then by Proposition 3.2 together with (i)–(v), we have the following:

(1°) $\Theta_{2n} = \mathrm{Ad}(u_n)$ on M_{n+1},

(2°) $\Theta_{2n}(M_k) = \hat{M}_k$, $n+2 \leq k \leq 2n$,

(3°) $\Theta_{2n}(e_k) = \hat{e}_k$, $n+1 \leq k \leq 2n-1$,

(4°) $\Theta_{2n} \circ E_k = \hat{E}_k \circ \Theta_{2n}$, $n+2 \leq k \leq 2n$.

Since the above (1°)–(4°) are the conditions which uniquely determine the isomorphism $\Theta | M_{2n}$, it follows that $\Theta_{2n} = \Theta | M_{2n}$. Therefore we get for $x \in M' \cap M_{2n}$

$$(\Theta \circ \gamma_n)(x) = \hat{\theta}_n^{-1}(u_n J_n \theta_n(x^*) J_n u_n^*) = \hat{\theta}_n^{-1}(\hat{J}_n \hat{\theta}_n(\Theta(x)^*) \hat{J}_n) = (\hat{\gamma}_n \circ \Theta)(x),$$

as desired. ■

In the proof of the next proposition, we adopt a key idea of Longo's canonical endomorphism investigated in [28–30].

Proposition 3.4. *For every $n \geq 1$, $\gamma_{n+1} \circ \gamma_n = \gamma_n \circ \gamma_{n-1}$ on $M' \cap M_{2n-2}$.*

Proof. In view of Proposition 3.3, we may show in the particular choice of the tower. First assume that M is infinite (hence so is N). Since the assumption of finite index implies that N' is σ-finite on the standard Hilbert space $\mathscr{H}$ for M, we can choose $\xi_0 \in \mathscr{H}$ which is cyclic and separating for both M and N (see [13]). Let J_M and J_N be the modular conjugations associated with ξ_0 for M and N, respectively, and let $W = J_M J_N$. We define the basic construction (M_1, e_0, E_1) for E via the natural cone associated with ξ_0 for M. Since

$$M_1 = J_M N' J_M = W N W^*,$$

M_1 is standard on $\mathscr{H}$ with the conjugation $J_1 = W J_N W^*$ associated with ξ_0. Hence the basic construction for E_1 can be defined via the natural cone associated with ξ_0 for M_1, so that we have two steps of the tower

$$N \underset{E}{\overleftarrow{\subset}} M \underset{E_1}{\overleftarrow{\subset}}{}^{e_0} M_1 \underset{E_2}{\overleftarrow{\subset}}{}^{e_1} M_2$$

on the same $\mathscr{H}$. Here

$$M_2 = WJ_N W^* M' WJ_N W^* = WMW^*.$$

Now define for $n \geq 1$

$$M_{2n} = W^n M W^{*n},\ M_{2n+1} = W^{n+1} N W^{*n+1} (= W^n M_1 W^{*n}),$$
$$J_{2n} = W^n J_M W^{*n},\ J_{2n+1} = W^{n+1} J_N W^{*n+1} (= W^n J_1 W^{*n}),$$
$$e_{2n} = W^n e_0 W^{*n},\ e_{2n+1} = W^n e_1 W^{*n},$$
$$E_{2n} = W^n E(W^{*n} \cdot W^n) W^{*n},\ E_{2n+1} = W^n E_1 (W^{*n} \cdot W^n) W^{*n}.$$

Then it is easy to see that the tower

$$N \underset{E}{\overleftarrow{\subset}} M \underset{E_1}{\overleftarrow{\subset}}{}^{e_0} M_1 \underset{E_2}{\overleftarrow{\subset}}{}^{e_1} M_2 \underset{E_3}{\overleftarrow{\subset}}{}^{e_2} M_3 \subset \cdots$$

is a realization of the tower of basic constructions from E. For this tower, the isomorphisms $\theta_n: M_{2n} \to \tilde{M}_{2n}$ in Proposition 3.2 are simply $\theta_n = \mathrm{id}_{M_{2n}}$ ($\tilde{M}_{2n} = M_{2n}$). Therefore

$$\gamma_n(x) = J_n x^* J_n, \quad x \in M' \cap M_{2n}, \quad n \geq 1.$$

This shows that for $n \geq 1$

$$\gamma_n(\gamma_{n-1}(x)) = WxW^*, \qquad x \in M' \cap M_{2n-2},$$

and particularly $\gamma_{n+1} \circ \gamma_n = \gamma_n \circ \gamma_{n-1}$ on $M' \cap M_{2n-2}$.

Next assume that M is finite. Taking the tensor product of the tower from E with any infinite factor P, we obtain

$$N \otimes P \underset{E \otimes \mathrm{id}_P}{\overleftarrow{\subset}} M \otimes P \underset{E_1 \otimes \mathrm{id}_P}{\overleftarrow{\subset}}{}^{e_0 \otimes 1} M_1 \otimes P \underset{E_2 \otimes \mathrm{id}_P}{\overleftarrow{\subset}}{}^{e_1 \otimes 1} M_2 \otimes P \subset \cdots,$$

which is really the tower of basic constructions from $E \otimes \mathrm{id}_P$. Since the isomorphisms in Proposition 3.2 for the tensored tower are $\theta_n \otimes \mathrm{id}_P: M_{2n} \otimes P \to \tilde{M}_{2n} \otimes P$ and $M' \cap M_{2n} = (M \otimes P)' \cap (M_{2n} \otimes P)$, the mirrorings for the tensored tower coincide with those for the original tower. Thus the desired equality follows from the infinite case. ■

The next proposition is a partial extension of [34, Theorem 2.6] (also [31]).

Proposition 3.5. *For every* $0 \leq k < n$, $E_{n+1} \circ \cdots \circ E_{2n-k}$ *is the basic construction of* $E_{k+1} \circ \cdots \circ E_n$; *more precisely*

$$(E_{n+1}\circ\cdots\circ E_{2n-k})(x)=\lambda^{n-k}J_n(E_{k+1}\circ\cdots\circ E_n)^{-1}(J_n\theta_n(x)J_n)J_n$$

for all $x\in M_{2n-k}$ *with* θ_n *in Proposition 3.2.*

Proof. By the proof of Proposition 3.3, it suffices to show in the particular choice of the tower. When M is infinite, we take the tower specified in the proof of Proposition 3.4 where $\theta_n=\mathrm{id}_{M_{2n}}$. The case $k=n-1$ is just the definition of the basic construction. Suppose the equality holds for some $0<k<n$. Then for every $x\in M_{2n-k+1}$, we have

$$\begin{aligned}(E_{n+1}\circ\cdots\circ E_{2n-k+1})(x)&=\lambda^{n-k+1}J_n(E_{k+1}\circ\cdots\circ E_n)^{-1}(E_k^{-1}(J_nxJ_n))J_n\\&=\lambda^{n-k+1}J_n(E_k\circ\cdots\circ E_n)^{-1}(J_nxJ_n)J_n,\end{aligned}$$

because

$$E_{2n-k+1}(x)=\lambda J_nE_k^{-1}(J_nxJ_n)J_n$$

as easily checked. Hence the conclusion follows by induction. When M is finite, we can do as in the proof of Proposition 3.4. ■

Given the tower from E, Proposition 3.4 enables us to define a $*$-endomorphism Γ of $\bigcup_n(M'\cap M_n)$ by

$$\Gamma(x)=\gamma_{n+1}(\gamma_n(x)),\qquad x\in M'\cap M_{2n},$$

which is called the canonical shift on the tower of relative commutants. In view of the proof of Proposition 3.4, we know that the canonical shift Γ as well as the mirrorings γ_n can be constructed apart from the choice of $E\in\mathscr{E}(M,N)$. In this sense, Γ is canonical for the inclusion $N\subset M$ rather than for E. Now the faithful state ϕ on $\bigcup_n(M'\cap M_n)$ is defined by

$$\phi|M'\cap M_n=E_1\circ\cdots\circ E_n|M'\cap M_n,\qquad n\geq 1.$$

Then we have:

Proposition 3.6. (1) $\gamma_n(M_j'\cap M_k)=M_{2n-k}'\cap M_{2n-j}$, $0\leq j\leq k\leq 2n$.
(2) $\Gamma(M_k'\cap M_n)=M_{k+2}'\cap M_{n+2}$, $0\leq k\leq n$.
(3) $\Gamma^k\circ\gamma_n=\gamma_{n+k}$ *on* $M'\cap M_{2n}$, $k,n\geq 0$.
(4) $\phi\circ\Gamma=\phi$ *on* $\bigcup_n(M'\cap M_n)$.
(5) *If* $E=E_0$, *then* $\phi\circ\gamma_n=\phi$ *on* $M'\cap M_{2n}$, $n\geq 0$.

Proof. The case of M being finite follows from the infinite case by taking the tensor product with an infinite factor. So let M be infinite. It suffices as before to show for the tower specified in the proof of Proposition 3.4. Then (1)–(3) are directly checked for $\Gamma(x)=WxW^*$ and $\gamma_n(x)=J_nx^*J_n$, $x\in M'\cap M_{2n}$. (In fact, $W^kJ_n=J_nW^{*k}=J_{n+k}$.)

(4) Proposition 3.5 implies that

$$(E_{k+1} \circ \cdots \circ E_n)(x) = \lambda^{n-k} J_n(E_{n+1} \circ \cdots \circ E_{2n-k})^{-1}(J_n x J_n) J_n$$

for all $x \in M_n$, $n > k \geq 0$. Hence for every $x \in M' \cap M_n$, we get

$$\begin{aligned}\phi(x) &= (E_1 \circ \cdots \circ E_{n+1})(x) \\ &= \lambda^{n+1} J_{n+1}(E_{n+2} \circ \cdots \circ E_{2n+2})^{-1}(J_{n+1} x J_{n+1}) J_{n+1}.\end{aligned}$$

Since $J_{n+1} x J_{n+1} \in M'_{n+2} \cap M_{2n+2}$, we get $E_{n+2}^{-1}(J_{n+1} x J_{n+1}) = \lambda^{-1} J_{n+1} x J_{n+1}$. Therefore

$$\begin{aligned}\phi(x) &= \lambda^n J_{n+1}(E_{n+3} \circ \cdots \circ E_{2n+2})^{-1}(J_{n+1} x J_{n+1}) J_{n+1} \\ &= J_{n+1} J_{n+2}(E_3 \circ \cdots \circ E_{n+2})(J_{n+2} J_{n+1} x J_{n+1} J_{n+2}) J_{n+2} J_{n+1} \\ &= (E_3 \circ \cdots \circ E_{n+2})(W x W^*) = \phi(\Gamma(x)).\end{aligned}$$

(5) Let $E = E_0$ and $x \in M' \cap M_{2n}$. Combining Proposition 2.5, Theorem 2.6 and Proposition 3.5, we have

$$\begin{aligned}\phi(x) = (E_1 \circ \cdots \circ E_{2n})(x) &= (E_{2n+1} \circ \cdots \circ E_{4n})(J_{2n} x^* J_{2n}) \\ &= \phi(\gamma_{2n}(x)) = \phi(\Gamma^n(\gamma_n(x))) = \phi(\gamma_n(x))\end{aligned}$$

by (3) and (4). ■

Let us extend Γ to a $*$-endomorphism of the von Neumann algebra generated by $\bigcup_n (M' \cap M_n)$. So define

$$A = \pi_\phi\Big(\bigcup_n (M' \cap M_n)\Big)''$$

where π_ϕ is the GNS representation of $\bigcup_n (M' \cap M_n)$ associated with ϕ. Further let $\bar{\phi}$ be the normal extension of ϕ on A, so that $\bar{\phi}(\pi_\phi(x)) = \phi(x)$, $x \in \bigcup_n (M' \cap M_n)$.

The inclusion $N \subset M$ is said to have finite depth if

$$\sup_n \dim Z(M' \cap M_n) < \infty$$

where $Z(M' \cap M_n)$ denotes the center of $M' \cap M_n$ (this condition does not depend on the choice of E).

Proposition 3.7. (1) *$\bar{\phi}$ is a faithful normal state on A.*

(2) *There exists a unique $*$-endomorphism $\bar{\Gamma}$ of A such that $\bar{\phi} \circ \bar{\Gamma} = \bar{\phi}$ and $\bar{\Gamma}(\pi_\phi(x)) = \pi_\phi(\Gamma(x))$, $x \in \bigcup_n (M' \cap M_n)$.*

(3) *If $E = E_0$, then $\bar{\phi}$ is a faithful normal trace on A.*

(4) *If $E = E_0$ and $N \subset M$ has finite depth, then A is a type II_1 factor.*

Proof. (1) Let A^0 be the C^*-completion of $\bigcup_n (M' \cap M_n)$ with the extension ϕ^0 of ϕ. Then π_ϕ is nothing but the GNS representation of A^0 associated with

ϕ^0. Letting $\phi_n = \phi | M' \cap M_n$, since $\sigma_t^{\phi_{n+1}} | M' \cap M_n = \sigma_t^{\phi_n}$ for all $n \geq 1$, we obtain a one-parameter automorphism group σ_t^0 of A^0 such that $\sigma_t^0 | M' \cap M_n = \sigma_t^{\phi_n}$, $n \geq 1$. Hence it follows (see [4, 5.3.9] for instance) that the normal extension $\bar{\phi}$ of ϕ^0 is faithful.

(2) follows from Proposition 3.6 (4).

(3) follows from Theorem 2.6.

(4) Suppose $E = E_0$. Then Popa's arguments in [36, §2] work in our setup as well. In fact, the results [36, Proposition 2.1, Corollaries 2.2 and 2.3] (also [14, 4.6.3]) hold for $\lambda = [M : N]_0^{-1}$ and $\{B_n = M' \cap M_n : n \geq 0\}$, when we consider the dimension vector and the trace vector of B_n with respect to the trace ϕ together with the inclusion matrix of $B_n \subset B_{n+1}$. Thus the same proof as [36, Corollary 2.5] implies the desired conclusion under the finite depth assumption. ■

Since π_ϕ faithfully imbeds $\bigcup_n (M' \cap M_n)$ in A, we consider $\bigcup_n (M' \cap M_n)$ as a subalgebra of A and denote $\bar{\Gamma}$, $\bar{\phi}$ by Γ, ϕ again. We call the $*$-endomorphism Γ extended on A, or more precisely (A, ϕ, Γ), the canonical shift associated with E. In particular, let $N \subset M$ be type II_1 factors and (A, ϕ, Γ) the canonical shift associated with the conditional expectation $E_N : M \to N$ with respect to the trace. Then ϕ is a trace whether $E_N = E_0$ or not. This Γ is the canonical shift for $N \subset M$ investigated in [7].

On the lines of [29, Theorem 5.1], we have the ergodic property of Γ extending [7, Proposition 21].

Proposition 3.8. $\bigcap_{k=1}^{\infty} \Gamma^k(A) = \mathbf{C}$.

Proof. Let $\|\cdot\|_\phi$ be the norm on A induced by ϕ, i.e. $\|x\|_\phi = \phi(x^* x)^{1/2}$. Let $x \in \bigcap_k \Gamma^k(A)$. For any $\varepsilon > 0$, there exist k and $y \in M' \cap M_{2k}$ such that $\|x - y\|_\phi < \varepsilon$. For every $n \geq 2k$, Proposition 2.4 shows that

$$\begin{array}{ccc} M' \cap M_{2k} & \subset & M' \cap M_n \\ \cup & & \cup \\ \mathbf{C} & \subset & M_{2k}' \cap M_n \end{array}$$

is a commuting square with respect to $\phi | M' \cap M_n$. By Proposition 3.6 (2), $\Gamma^k(A)$ is generated by $\bigcup_n (M_{2k}' \cap M_n)$. Hence we see that

$$\begin{array}{ccc} M' \cap M_{2k} & \subset & A \\ \cup & & \cup \\ \mathbf{C} & \subset & \Gamma^k(A) \end{array}$$

is a commuting square with respect to ϕ. So there exists the conditional expectation $F : A \to \Gamma^k(A)$ with respect to ϕ, which satisfies $F(M' \cap M_{2k}) = \mathbf{C}$. Since $F(x) = x$ and $F(y) = \phi(y)$, we get

$$\|x - \phi(x)\|_\phi \le \|F(x-y)\|_\phi + |\phi(y-x)| \le 2\|x-y\|_\phi < 2\varepsilon,$$

which implies $x \in \mathbf{C}$. ■

§4. Entropy $h_\phi(\Gamma)$

Let (A, ϕ, Γ) be the canonical shift associated with $E \in \mathscr{E}(M, N)$ defined in the previous section. Let $B_n = M' \cap M_n$ and $\phi_n = \phi | B_n$ for $n \ge 0$. Then $\{B_n\}$ is an increasing sequence of finite dimensional subalgebras of A with $A = (\bigcup_n B_n)''$. The aim of this section is to establish the relation between the entropy $h_\phi(\Gamma)$ and the minimum index $[M: N]_0$.

Lemma 4.1. (1) *For every $n, m \ge 0$, $(\bigcup_{j=0}^m \Gamma^j(B_n))''$ is included in B_{n+2m}.*

(2) *Let $k_n = \left[\dfrac{n+1}{2}\right]$. Then for every $n, m \ge 0$, $\Gamma^{(m+1)k_n}(B_n)$ commutes with $(\bigcup_{j=0}^{mk_n} \Gamma^j(B_n))''$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x \in (\bigcup_{j=0}^{mk_n} \Gamma^j(B_n))''$ and $y \in \Gamma^{(m+1)k_n}(B_n)$.*

(3) *The conditional expectation $A \to \Gamma^j(B_n)$ with respect to ϕ exists for every $n, j \ge 0$, and*

$$\begin{array}{ccc} B_n & \subset & A \\ \cup & & \cup \\ \Gamma(B_{n-2}) & \subset & \Gamma(B_n) \end{array}$$

is a commuting square with respect to ϕ for every $n \ge 2$.

(4) *$\Gamma(B_{2n}) = \gamma_{n+1}(B_{2n})$ for all $n \ge 0$.*

Moreover if $E = E_0$, then Γ is a 2-shift on the tower $\{B_n\}$ in the sense of [7].

Proof. (1) and (2) follow from Proposition 3.6 (2).

(3) By the proof of Proposition 3.7 (1) and [4, 5.3.4], the conditional expectation $A \to B_n$ with respect to ϕ exists for every $n \ge 0$. Then Proposition 2.4 shows the desired conclusions (see the proof of Proposition 3.8).

(4) is obvious from Proposition 3.6 (3).

By Propositions 3.6 (5) and 3.7 (3), the above (1)–(4) show the last statement. ■

Proposition 4.2. *$h_\phi(\Gamma) = \lim_{n\to\infty} \dfrac{1}{n} H_\phi(B_{2n}) = \lim_{n\to\infty} \dfrac{1}{n} S(\phi_{2n})$ where $S(\phi_n)$ is the entropy of ϕ_n.*

Proof. For each $n, m \ge 1$, let $B = (\bigcup_{j=0}^{m-1} \Gamma^{jk_n}(B_n))''$ where $k_n = \left[\dfrac{n+1}{2}\right]$, and $\{q_i: 1 \le i \le l\}$ be a set of atoms in the centralizer of ϕ_n with $\sum_i q_i = 1$. Furthermore let $q_i^j = \Gamma^{(j-1)k_n}(q_i)$ for $1 \le i \le l$ and $1 \le j \le m$. Then by Proposition 3.6 (4) and Lemma 4.1 (2), $\{q_{i_1}^1 \cdots q_{i_m}^m: 1 \le i_1, \dots, i_m \le l\}$ is a set of atoms in the centralizer of $\phi | B$ such that $\sum_{i_1,\dots,i_m} q_{i_1}^1 \cdots q_{i_m}^m = 1$ and $\phi(q_{i_1}^1 \cdots q_{i_m}^m)$

$= \phi(q_{i_1}^1) \cdots \phi(q_{i_m}^m)$. Hence by Lemma 4.1 and [10, Théorème 5(E)] (also [11, Corollary VIII.8]), we get

$$H_\phi(B_n, \Gamma^{k_n}(B_n), \ldots, \Gamma^{(m-1)k_n}(B_n)) = S(\phi | B) = \sum_{i_1, \ldots, i_m} \eta(\phi(q_{i_1}^1 \cdots q_{i_m}^m))$$
$$= mS(\phi_n) = mH_\phi(B_n).$$

Now the proof is the same as [7, Theorem 1] in view of Theorem 1.5 and Proposition 1.6 (2). ■

Theorem 4.3. (1) $h_\phi(\Gamma) \leq \frac{1}{2}\{K_E(M|N) + K_{E_1}(M_1|M)\} \leq \log [M : N]_0$.

(2) *If* $h_\phi(\Gamma) = \log [M : N]_0$, *then* $E = E_0$.

(3) *Suppose* $N \subset M$ *has finite depth. Then the following conditions are equivalent*:

(i) $E = E_0$;
(ii) $h_\phi(\Gamma) = \log [M : N]_0$;
(iii) $h_\phi(\Gamma) = \log \operatorname{Index} E$.

Proof. (1) For each $n \geq 1$, choose a set $\{q_i^{(2n)}\}$ of atoms in the centralizer of ϕ_{2n} with $\sum_i q_i^{(2n)} = 1$. Since the centralizer of $E_1 \circ \cdots \circ E_{2n}$ is nothing but that of ϕ_{2n} and

$$\operatorname{Index}(E_1 \circ \cdots \circ E_{2n})_{q_i^{(2n)}} \geq 1,$$

Theorem 1.3 (2) implies that

$$K_{E_1 \circ \cdots \circ E_{2n}}(M_{2n}|M) \geq 2 \sum_i \eta(\phi_{2n}(q_i^{(2n)})) = 2S(\phi_{2n}).$$

Furthermore by [20, Theorem 5.1 (1)] and [21, Proposition 8.1], we have

$$K_{E_1 \circ \cdots \circ E_{2n}}(M_{2n}|M) \leq \sum_{j=1}^{2n} K_{E_j}(M_j|M_{j-1}) = n\{K_E(M|N) + K_{E_1}(M_1|M)\}.$$

Thus Proposition 4.2 implies the first inequality. Also we get the second inequality by Theorem 1.4 (1).

(2) Suppose $h_\phi(\Gamma) = \log [M : N]_0$. Then $K_E(M|N) = \log [M : N]_0$ holds, which is equivalent to $E = E_0$ by Theorem 1.4 (2).

(3) In view of (2), it suffices to show that (i) implies (ii) under the finite depth condition. So suppose $N \subset M$ has finite depth and $E = E_0$. Then there exists n_0 such that $\dim Z(B_{2n_0+2}) = \dim Z(B_{2n_0})$. Since [36, Corollary 2.3] holds for $\{B_n\}$ in our setup as noted in the proof of Proposition 3.7 (4), the trace vector of B_{2n} with respect to the trace ϕ is given by $(\lambda^{n-n_0} s_k)_k$ for any $n \geq n_0$, where $\lambda = [M : N]_0^{-1}$ and (s_k) is the trace vector of B_{2n_0} with respect to ϕ. Let $s = \max_k s_k$ and $n \geq n_0$. Since $\phi_{2n}(q_i^{(2n)}) \leq \lambda^{n-n_0} s$, we have by Theorems 2.6 and

1.4 (2)

$$\operatorname{Index}(E_1\circ\cdots\circ E_{2n})_{q_i^{(2n)}}=\phi_{2n}(q_i^{(2n)})^2\ \operatorname{Index}(E_1\circ\cdots\circ E_{2n})$$
$$\le(\lambda^{n-n_0}s)^2\lambda^{-2n}=(\lambda^{-n_0}s)^2$$

for all i. Therefore by Theorems 2.6, 1.4 (2) and 1.3 (2), we have

$$2n\log[M:N]_0=\log[M_{2n}:M]_0=K_{E_1\circ\cdots\circ E_{2n}}(M_{2n}|M)$$
$$\le\sum_i\phi_{2n}(q_i^{(2n)})\log\frac{(\lambda^{-n_0}s)^2}{\phi_{2n}(q_i^{(2n)})^2}=2S(\phi_{2n})+2\log(\lambda^{-n_0}s),$$

so that $h_\phi(\Gamma)\ge\log[M:N]_0$ by Proposition 4.2. ∎

Specializing Theorem 4.3 to the type II_1 case, we have:

Corollary 4.4. *Let $N\subset M$ be type II_1 factors and $H(\Gamma)$ the entropy of the canonical shift Γ for $N\subset M$. Then*:

(1) $H(\Gamma)\le\frac{1}{2}\{H(M|N)+H(M_1|M)\}\le\log[M:N]_0\le\log[M:N]$.

(2) *If* $H(\Gamma)=\log[M:N]_0$, *then* $[M:N]=[M:N]_0$ *and* $E_{M'\cap M_1}(e_0)=[M:N]^{-1}1$ *where $E_{M'\cap M_1}$ is the conditional expectation $M_1\to M'\cap M_1$ with respect to the trace.*

(3) *If $N\subset M$ has finite depth, then*

$$H(\Gamma)=H(M|N)=\log[M:N]_0=\log[M:N].$$

Proof. Let E_N be the conditional expectation $M\to N$ with respect to the trace. We know by [33, Corollary 4.5] and [19] that $E_N=E_0$ (i.e. $[M:N]=[M:N]_0$) if and only if $E_{M'\cap M_1}(e_0)=[M:N]^{-1}1$. According to [36, Corollary 3.7], if $N\subset M$ has finite depth, then $E_{M'\cap M_1}(e_0)=[M:N]^{-1}1$ automatically holds. Thus the corollary is the specialization of Theorem 4.3. ∎

§5. Entropy $H_\phi(A|\Gamma(A))$

Let (A,ϕ,Γ) be the canonical shift associated with $E\in\mathscr{E}(M,N)$. Let B_n and ϕ_n be as in §4, and $C_n=M_2'\cap M_n$ $(=\Gamma(B_{n-2}))$, $n\ge 2$. In this section, we investigate the entropy $H_\phi(A|\Gamma(A))$ in connection with $[M:N]_0$ and $h_\phi(\Gamma)$.

The entropy $H_\phi(B_n|C_n)$ is given in [10] by

$$H_\phi(B_n|C_n)=\sup_{(\psi_k)}\sum_k\{S(\phi|B_n,\psi_k|B_n)-S(\phi|C_n,\psi_k|C_n)\},$$

where the supremum is taken over all finite families (ψ_k) of $\psi_k\in A_*^+$ with $\sum_k\psi_k=\phi$. But we have $H_\phi(B_n|C_n)=H_{\phi_n}(B_n|C_n)$ by Lemma 4.1 (3). Proposition 2.4 and [21, Proposition 2.12 (1)] show the following:

Proposition 5.1. *$H_\phi(A|\Gamma(A)) = \lim_{n\to\infty} H_\phi(B_n|C_n)$ increasingly.*

Proposition 5.2. *$h_\phi(\Gamma) \le H_\phi(A|\Gamma(A))$.*

Proof. By [21, Proposition 2.2 (1)] and Proposition 3.6, we get for $n \ge 1$

$$H_\phi(B_{2n}) \le H_\phi(B_{2n}|C_{2n}) + H_\phi(C_{2n}) = H_\phi(B_{2n}|C_{2n}) + H_\phi(B_{2n-2}).$$

This implies that

$$\frac{1}{n} H_\phi(B_{2n}) \le \frac{1}{n}\sum_{j=1}^{n} H_\phi(B_{2j}|C_{2j}).$$

Hence the desired inequality follows from Propositions 4.2 and 5.1. ∎

Theorem 5.3. (1) *$H_\phi(A|\Gamma(A)) \le 2\log[M:N]_0$.*

(2) *If $H_\phi(A|\Gamma(A)) = 2\log[M:N]_0$, then $E = E_0$.*

(3) *Suppose $N \subset M$ has finite depth. Then $E = E_0$ if and only if $H_\phi(A|\Gamma(A)) = 2\log[M:N]_0$.*

Proof. (1) Let

$$\underset{E_0}{\overleftarrow{N \subset}} \underset{E_{0,1}}{\overleftarrow{M \subset}}{}^{e_0} \underset{E_{0,2}}{\overleftarrow{M_1 \subset}}{}^{e_1} M_2 \subset \cdots$$

be the tower of basic constructions iterated from E_0. Here we can assume as remarked before Proposition 3.6 that the factors M_n are the same as those in the tower iterated from E. For $n \ge 1$, let $\tau_n = E_{0,1} \circ \cdots \circ E_{0,n}|B_n$ which is a trace by Theorem 2.6, and let $h_n = d(E_n|M'_{n-1} \cap M_n)/d(E_{0,n}|M'_{n-1} \cap M_n)$. Since $E_n = h_n^{1/2} E_{0,n} h_n^{1/2} (= E_{0,n}(h_n^{1/2} \cdot h_n^{1/2}))$ by [8, Théorème 5.3], we get

$$E_1 \circ \cdots \circ E_n = (h_1 \cdots h_n)^{1/2} (E_{0,1} \circ \cdots \circ E_{0,n}) (h_1 \cdots h_n)^{1/2},$$

so that $d\phi_n/d\tau_n = h_1 \cdots h_n$. For each fixed $n \ge 2$, we denote by F and F_0 the conditional expectations $B_n \to C_n$ with respect to ϕ_n and τ_n, respectively. For $1 \le k \le n$, let us define $E'_{0,k} \in \mathscr{E}(M'_{k-1}, M'_k)$ by $E'_{0,k} = \lambda E_{0,k}^{-1}$ where $\lambda = [M:N]_0^{-1}$ and $E_{0,k}^{-1}$ is defined on the standard Hilbert space for M_n. Then it follows that

$$\underset{E'_{0,n}}{\overleftarrow{M'_n \subset}} \underset{E'_{0,n-1}}{\overleftarrow{M'_{n-1} \subset}}{}^{e_{n-1}} M'_{n-2} \subset \cdots \subset^{e_2} \underset{E'_{0,1}}{\overleftarrow{M'_1 \subset}}{}^{e_1} M'$$

is n steps of algebraic basic constructions. Since by Theorem 2.6

$$E'_{0,n} \circ \cdots \circ E'_{0,1}|B_n = \lambda^n (E_{0,1} \circ \cdots \circ E_{0,n})^{-1}|B_n = \tau_n,$$

we have $F_0 = E'_{0,2} \circ E'_{0,1}|B_n$, so that

$$\frac{d(\phi_n|C_n)}{d(\tau_n|C_n)} = F_0(h_1 \cdots h_n) = h_3 \cdots h_n.$$

Hence the cocycle derivative $[DF: DF_0]_t$ of F and F_0 is computed as follows (see [8, 15]):

$$\begin{aligned}[DF: DF_0]_t &= [D(\tau_n \circ F): D(\tau_n \circ F_0)]_t \\ &= [D(\phi_n \circ F): D(\tau_n \circ F)]_t^* [D(\phi_n \circ F): D(\tau_n \circ F_0)]_t \\ &= [D(\phi_n | C_n): D(\tau_n | C_n)]_t^* [D\phi_n: D\tau_n]_t \\ &= (h_3 \cdots h_n)^{-it}(h_1 \cdots h_n)^{it} = (h_1 h_2)^{it}.\end{aligned}$$

Now let $\psi_1, \ldots, \psi_m \in (B_n)_*^+$ be faithful with $\sum_k \psi_k = \phi_n$. Then

$$\begin{aligned}[D(\psi_k \circ F): D\psi_k]_t &= [D(\psi_k \circ F): D(\psi_k \circ F_0)]_t [D(\psi_k \circ F_0): D\psi_k]_t \\ &= [DF: DF_0]_t [D(\psi_k \circ F_0): D\psi_k]_t \\ &= (h_1 h_2)^{it} [D(\psi_k \circ F_0): D\psi_k]_t.\end{aligned}$$

Hence by [32, Theorem 4], we get

$$\begin{aligned}S(\psi_k \circ F, \psi_k) &= i \lim_{t \to +0} \frac{1}{t} \psi_k([D(\psi_k \circ F): D\psi_k]_t - 1) \\ &= S(\psi_k \circ F_0, \psi_k) + \psi_k(\log h_1 h_2).\end{aligned}$$

Since by Theorem 1.1

$$F_0(x) \geq (\mathrm{Index}(E'_{0,2} \circ E'_{0,1}))^{-1} x = \lambda^2 x, \qquad x \in (B_n)_+,$$

we have $\psi_k \circ F_0 \geq \lambda^2 \psi_k$, so that

$$\sum_k S(\psi_k \circ F_0, \psi_k) \leq \sum_k \psi_k(1) \log \lambda^{-2} = 2 \log [M: N]_0.$$

Therefore

$$\sum_k S(\psi_k \circ F, \psi_k) \leq 2 \log [M: N]_0 + \tau_2(\eta(h_1 h_2)).$$

By the lower semicontinuity of the relative entropy ([2, Theorem 3.7]), the above inequality holds for any $\psi_1, \ldots, \psi_m \in (B_n)_*^+$ with $\sum_k \psi_k = \phi_n$. This implies by [21, Lemma 2.6] that

$$H_\phi(B_n | C_n) \leq 2 \log [M: N]_0 + \tau_2(\eta(h_1 h_2)).$$

Since $\tau_2(\eta(h_1 h_2)) \leq \eta(\tau_2(h_1 h_2)) = 0$, the desired inequality follows from Proposition 5.1.

(2) Suppose $H_\phi(A | \Gamma(A)) = 2 \log [M: N]_0$. Then $\tau_2(\eta(h_1 h_2)) = 0$ and hence $h_1 h_2 = 1$ by the strict concavity of η. This implies $E_1 \circ E_2 = E_{0,1} \circ E_{0,2}$, so that $E_1 = E_{0,1}$, equivalently $E = E_0$.

(3) Suppose $N \subset M$ has finite depth and $E = E_0$. Since $\gamma_n(C_{2n}) = B_{2n-2}$,

it follows from Proposition 3.6 (5) that $H_\phi(B_{2n}|C_{2n}) = H_\phi(B_{2n}|B_{2n-2})$ for all $n \geq 1$. Now let us proceed as in the proof of [36, Corollary 2.4]. Choose n_0 such that $\dim Z(B_{2n_0+2}) = \dim Z(B_{2n_0})$. Let $\vec{d}$ be the dimension vector of B_{2n_0}, Λ the inclusion matrix of $B_{2n_0} \subset B_{2n_0+1}$, and (s_k) the trace vector of B_{2n_0} with respect to the trace ϕ. Then according to [36, Corollary 2.3], $(\Lambda\Lambda^t)^n \vec{d}$ is the dimension vector of B_{2n_0+2n} and $(\lambda^n s_k)_k$ is the trace vector of B_{2n_0+2n} with respect to ϕ for any $n \geq 0$ where $\lambda = [M : N]_0^{-1}$. Hence by [33, Theorem 6.2] (also [35]), we have for every $n \geq 1$

$$H_\phi(B_{2n_0+2n}|B_{2n_0+2n-2}) = \sum_{k,l} (\Lambda\Lambda^t)_{kl}((\Lambda\Lambda^t)^{n-1}\vec{d})_k(\lambda^n s_l) \log \frac{((\Lambda\Lambda^t)^n \vec{d})_l(\lambda^{n-1}s_k)}{((\Lambda\Lambda^t)^{n-1}\vec{d})_k(\lambda^n s_l)}.$$

Since (s_k) is the Perron-Frobenius eigenvector of $\Lambda\Lambda^t$ with the eigenvalue λ, we have

$$\lim_{n\to\infty} \frac{((\Lambda\Lambda^t)^n \vec{d})_l(\lambda^{n-1}s_k)}{((\Lambda\Lambda^t)^{n-1}\vec{d})_k(\lambda^n s_l)} = \lambda^{-2}$$

for all k, l. Therefore

$$H_\phi(A|\Gamma(A)) = \lim_{n\to\infty} H_\phi(B_{2n_0+2n}|B_{2n_0+2n-2}) = \log \lambda^{-2},$$

as desired. ∎

Following [33], we define the number $\lambda_\phi(A, \Gamma(A))$ by

$$\lambda_\phi(A, \Gamma(A)) = \max\{\lambda \geq 0 : E_{\Gamma(A)}(x) \geq \lambda x,\ x \in A_+\},$$

where $E_{\Gamma(A)}$ is the conditional expectation $A \to \Gamma(A)$ with respect to ϕ. In view of Theorem 1.1, we can consider $\lambda_\phi(A, \Gamma(A))^{-1}$ as a generalized index of $E_{\Gamma(A)}$ when A is not necessarily a factor.

Proposition 5.4. $\lambda_\phi(A, \Gamma(A))^{-1} \leq (\operatorname{Index} E)^2$.

Proof. Let $\lambda_\phi(B_n, C_n)$ be defined for the conditional expectation $B_n \to C_n$ with respect to ϕ_n. Then as [33, Proposition 2.6], we have $\lambda_\phi(A, \Gamma(A)) = \lim_{n\to\infty} \lambda_\phi(B_n, C_n)$ decreasingly by Proposition 2.4. For each fixed $n \geq 1$, let us use the notations in the proof of Theorem 5.3 (1). Since $[DF : DF_0]_t = (h_1 h_2)^{it}$, $F = (h_1 h_2)^{1/2} F_0 (h_1 h_2)^{1/2}$ follows from [8, Proposition 4.11]. Define $E' = (h_1 h_2)^{1/2}(E'_{0,2} \circ E'_{0,1})(h_1 h_2)^{1/2}$. Then $E' \in \mathscr{E}(M', M'_2)$ because for $x \in M'_2$

$$E'(x) = E'_{0,1}(h_1)E'_{0,2}(h_2)x = E_{0,1}(h_1)E_{0,2}(h_2)x = x.$$

Moreover it follows (see [19]) that

$$\operatorname{Index} E' = (E'_{0,2} \circ E'_{0,1})^{-1}((h_1 h_2)^{-1}) = \lambda^{-2} E_{0,1}(h_1^{-1}) E_{0,2}(h_2^{-1}) = (\operatorname{Index} E)^2.$$

Since $F_0 = E'_{0,2} \circ E'_{0,1} | B_n$, we have by Theorem 1.1

$$F(x) = E'(x) \geq (\operatorname{Index} E)^{-2} x, \qquad x \in (B_n)_+,$$

so that $\lambda_\phi(B_n, C_n) \geq (\operatorname{Index} E)^{-2}$, implying the desired inequality. ■

The next theorem is an extended version of [7, Theorem 14].

Theorem 5.5. *Suppose* $E = E_0$. *Then*:

(1) $H_\phi(A|\Gamma(A)) \leq 2h_\phi(\Gamma) \leq \log \lambda_\phi(A, \Gamma(A))^{-1} = 2K_E(M|N) = 2\log[M: N]_0$.

(2) *If* $\lim_{n\to\infty} \frac{1}{n} \log k_n = 0$ (*this is the case if* $\sup_n \frac{1}{n} k_n < \infty$) *where* k_n *is the number of simple summands of* B_n, *then* $H_\phi(A|\Gamma(A)) = 2h_\phi(\Gamma)$.

(3) *If* $N \subset M$ *has finite depth* (*in particular, if* $\operatorname{Index} E < 4$), *then the numbers in* (1) *are all identical together with* $\log[A: \Gamma(A)]$.

Proof. (1) By Lemma 4.1, Γ is a 2-shift on the tower $\{B_n\}$. The results [36, Proposition 2.1 and Corollary 2.2] hold for $\{B_n\}$ and $\lambda = [M: N]_0^{-1}$. Furthermore we have by Theorems 1.3, 1.4 and 2.6

$$H_\phi(B_{2n}) \leq \frac{1}{2} K_{E_1 \circ \cdots \circ E_{2n}}(M_{2n}|M) = \frac{1}{2} \log[M_{2n}, M]_0 = -n \log \lambda.$$

Thus we conclude that $\{B_n\}$ is a locally standard tower for λ^2 with period 4 in the sense of [7, Definition 3]. Hence [7, Theorem 8] implies that

$$H_\phi(A|\Gamma(A)) \leq 2h_\phi(\Gamma) \leq 2\log[M: N]_0 \leq \log \lambda_\phi(A, \Gamma(A))^{-1}.$$

Since $\lambda_\phi(A, \Gamma(A))^{-1} \leq [M: N]_0^2$ by Proposition 5.4, we obtain the conclusion.

(2) For $n \geq 0$, let K_n be the set of simple summands of B_{2n}. Then by assumption, $\lim_{n\to\infty} \frac{1}{n} \log |K_n| = 0$ where $|\cdot|$ denotes the cardinal number. We denote by $(d_k^{(n)})_{k \in K_n}$ the dimension vector of B_{2n} and by $(t_k^{(n)})_{k \in K_n}$ the trace vector of B_{2n} with respect to the trace ϕ. Moreover let $(a_{kl}^{(n)})_{k \in K_n, l \in K_{n+1}}$ be the inclusion matrix of $B_{2n} \subset B_{2n+2}$ and let $L_n = \{(k, l) \in K_n \times K_{n+1} : a_{kl}^{(n)} > d_k^{(n)}\}$. To simplify the notation, we define as in [6, 7]

$$I_\phi(B_{2n}) = \sum_{k \in K_n} d_k^{(n)} t_k^{(n)} \log \frac{d_k^{(n)}}{t_k^{(n)}}$$

and analogously $I_\phi(C_{2n})$. For each $n \geq 1$, since the mirroring γ_n maps $B_{2n-2} \subset B_{2n}$ to $C_{2n} \subset B_{2n}$, the inclusion matrix of $C_{2n} \subset B_{2n}$ coincides with $(a_{kl}^{(n-1)})_{k \in K_{n-1}, l \in K_n}$ and the dimension vector of C_{2n} coincides with $(d_k^{(n-1)})_{k \in K_{n-1}}$

under the identification of respective simple summands via γ_n. Also let $(\tilde{t}_k^{(n)})_{k\in K_n}$ be the trace vector of B_{2n} corresponding to $C_{2n} \subset B_{2n}$, which is a permutation of $(t_k^{(n)})$ via γ_n. Then according to [33, Theorem 6.2], we have

$$H_\phi(B_{2n}|C_{2n}) = I_\phi(B_{2n}) - I_\phi(B_{2n-2}) + \sum_{(k,l)\in L_{n-1}} d_k^{(n-1)} a_{kl}^{(n-1)} \tilde{t}_l^{(n)} \log \frac{d_k^{(n-1)}}{a_{kl}^{(n-1)}},$$

because $I_\phi(C_{2n}) = I_\phi(B_{2n-2})$ by Proposition 3.6. Now let $\tilde{\lambda} = \max\{\lambda, 1-\lambda\}$. Then $0 < \tilde{\lambda} < 1$ except the trivial case $N = M$. Since B_{2n} contains mutually commuting projections $e_1, e_3, \ldots, e_{2n-1}$, and since

$$\phi(f_1 f_3 \cdots f_{2n-1}) = \phi(f_1)\phi(f_3)\cdots\phi(f_{2n-1}) \le \tilde{\lambda}^n$$

for $f_{2i-1} = e_{2i-1}$ or $f_{2i-1} = 1 - e_{2i-1}$, $1 \le i \le n$, we get $t_k^{(n)} \le \tilde{\lambda}^n$ for all n, k. Furthermore according to [36, Corollary 2.2], we get $a_{kl}^{(n)} \le [M:N]_0$ for all n, k, l. These imply that

$$\begin{aligned} 0 &\le -\sum_{(k,l)\in L_{n-1}} d_k^{(n-1)} a_{kl}^{(n-1)} \tilde{t}_l^{(n)} \log \frac{d_k^{(n-1)}}{a_{kl}^{(n-1)}} \\ &\le \sum_{(k,l)\in L_{n-1}} (a_{kl}^{(n-1)})^2 \tilde{t}_l^{(n)} \log a_{kl}^{(n-1)} \\ &\le |K_{n-1}|\,|K_n| \tilde{\lambda}^n [M:N]_0^2 \log [M:N]_0, \end{aligned}$$

which tends to 0 as $n \to \infty$. On the other hand, it follows (see [6, Proposition 16], [7, Proposition 4]) that

$$0 \le 2H_\phi(B_{2n}) - I_\phi(B_{2n}) \le \log |K_n|.$$

Therefore by Propositions 5.1 and 4.2

$$H_\phi(A|\Gamma(A)) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} H_\phi(B_{2j}|C_{2j}) = \lim_{n\to\infty} \frac{1}{n} I_\phi(B_{2n}) = 2h_\phi(\Gamma).$$

(3) As [36, Corollary 6.7] (also [14, 4.6.6]), we see that if Index $E < 4$, then $N \subset M$ has finite depth. Thus the desired assertion is immediate from (1) and Theorem 5.3 (3) together with Proposition 3.7 (4). ■

Finally in the type II_1 case, we have the next proposition (without the assumption $E_N = E_0$) in view of the proof of Theorem 5.5 (2).

Proposition 5.6. *Let $N \subset M$ be type II_1 factors and $H(\Gamma)$ the entropy of the canonical shift Γ for $N \subset M$. Then*:

(1) $H(A|\Gamma(A)) \le 2H(\Gamma)$.

(2) *If* $\lim_{n\to\infty} \frac{1}{n} \log k_n = 0$ *where k_n is the number of simple summands of B_n, then* $H(A|\Gamma(A)) = 2H(\Gamma)$.

§6. Examples

In this section, we present two simple examples to illustrate the results in §§4 and 5. In the following, we use the same notations as before.

Example 6.1. Let $M = R$ be the hyperfinite type II_1 factor, $N = R_\lambda$ Jones' subfactor [22] of M with $[M: N] = \lambda^{-1}$, and Γ the canonical shift for $N \subset M$. It follows from Proposition 3.2 that $\gamma_n(e_k) = e_{2n-k}$ for every $n \geq 1$ and $1 \leq k \leq 2n-1$. Hence

$$\Gamma(e_n) = \gamma_{n+1}(\gamma_n(e_n)) = \gamma_{n+1}(e_n) = e_{n+2}, \qquad n \geq 1.$$

Suppose $\lambda \geq 1/4$. Then it is known (see [14, 4.7.b]) that $M' \cap M_n = \langle 1, e_1, \ldots, e_{n-1} \rangle$. Hence $A = \{e_n: n \geq 1\}'' \ (\simeq R)$, so that Γ coincides with σ_λ^2 where σ_λ is a special case of the shifts discussed in [5, 6]. We have by [6, Example 2]

$$\frac{1}{2}H(A|\Gamma(A)) = H(\Gamma) = 2H(\sigma_\lambda) = H(M|N) = \log \lambda^{-1}.$$

Next suppose $\lambda < 1/4$ and $t(1-t) = \lambda$, $t > 0$. We get

$$\begin{aligned} H(\Gamma) &= \lim_{n\to\infty} \frac{1}{n} H(B_{2n}) \geq \lim_{n\to\infty} \frac{1}{n} H(\langle 1, e_1, \ldots, e_{2n-1} \rangle) \\ &= 2H(\sigma_\lambda) = 2\eta(t) + 2\eta(1-t) = H(M|N) \end{aligned}$$

by [7, Theorem 1], [6, Example 2] and [33, Corollary 5.3]. On the other hand,

$$H(\Gamma) \leq \frac{1}{2}\{H(M|N) + H(M_1|M)\} = H(M|N)$$

by Corollary 4.4 (1) and [21, Proposition 8.4]. Hence $H(\Gamma) = H(M|N)$. Moreover since the Bratteli diagram for the tower $B_0 \subset B_1 \subset B_2 \subset \cdots$ is Pascal's triangle (see [14, p. 231]), we have $H(A|\Gamma(A)) = 2H(\Gamma)$ by Proposition 5.6 (2). Therefore $\frac{1}{2}H(A|\Gamma(A)) = H(\Gamma) = H(M|N)$ for any λ.

Example 6.2. Let us consider $M = N \otimes B \supset N = N \otimes \mathbf{C}$ where N is any factor and $B = M_m(\mathbf{C})$. Let $\varphi_0 \in \mathscr{E}(B)$ and $h = d\varphi_0/d\tau$ where τ is the normalized trace on B. Define $E \in \mathscr{E}(M, N)$ by $E = \mathrm{id}_N \otimes \varphi_0$ and $\varphi_1 \in \mathscr{E}(B)$ by $d\varphi_1/d\tau = h^{-1}/\tau(h^{-1})$. Then it follows (see [21, Example 8.3]) that the basic constructions $E_n \in \mathscr{E}(M_n, M_{n-1})$, $n \geq 1$, iterated from E are given as follows:

$$M_n = M \otimes B^{(n)} = M_{n-1} \otimes B,$$

$$E_{2n-1} = \mathrm{id}_{M_{2n-2}} \otimes \varphi_1, \quad E_{2n} = \mathrm{id}_{M_{2n-1}} \otimes \varphi_0,$$

where $B^{(n)} = \bigotimes_1^n B$. Moreover it is easy to see that the mirrorings γ_n on $M' \cap M_{2n} = B^{(2n)}$ are given by

$$\gamma_n(a_1 \otimes a_2 \otimes \cdots \otimes a_{2n-1} \otimes a_{2n}) = a_{2n}^t \otimes a_{2n-1}^t \otimes \cdots \otimes a_2^t \otimes a_1^t,$$

where a^t denotes the transpose of a. Therefore

$$\begin{aligned}\Gamma(a_1 \otimes a_2 \otimes \cdots \otimes a_{2n-1} \otimes a_{2n}) &= \gamma_{n+1}(a_{2n}^t \otimes a_{2n-1}^t \otimes \cdots \otimes a_2^t \otimes a_1^t \otimes 1 \otimes 1)\\ &= 1 \otimes 1 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{2n-1} \otimes a_{2n},\end{aligned}$$

so that Γ is the unilateral shift on $(A, \phi) = \bigotimes_1^\infty (B^{(2)}, \varphi_1 \otimes \varphi_0)$. This example clarifies that ϕ is not generally invariant for γ_n but Γ preserves ϕ. By [10, Corollaire 10], we have

$$h_\phi(\Gamma) = S(\varphi_1 \otimes \varphi_0) = S(\varphi_0) + S(\varphi_1) = \frac{1}{2}\{K_E(M|N) + K_{E_1}(M_1|M)\}.$$

When $\varphi_0 = \tau$ and hence $\varphi_1 \otimes \varphi_0$ is the trace on $B^{(2)}$, we get by [6, Example 1]

$$\frac{1}{2}H_\phi(A|\Gamma(A)) = h_\phi(\Gamma) = 2\log m = \log[M: N]_0.$$

Also when $\varphi_0 \neq \tau$ (hence A is a type III factor), we get (see [21, Theorem 6.6], [20, Proposition 3.6 and Example 4.6])

$$\frac{1}{2}H_\phi(A|\Gamma(A)) \leq S(\varphi_0) + S(\varphi_1) = h_\phi(\Gamma).$$

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