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Supplement to "Hodge Spectral Sequence on Compact Kähler Spaces"

By

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1. The purpose of this short note is to supplement the author's previous article [1] by exhibiting a rigorous proof of the dual L^2 Poincaré lemma as well as its $\bar{\partial}$-analogue (cf. Proposition 4.1 (7), p. 272 in [1]). We have stated there that up to a slight change its proof is similar to that of the ordinary one, for which we had given a detailed proof. However it turned out that the proof requires a different technique which does not seem to be in the literature. Therefore the author would like to present it here.

2. Let $V \subset \mathbf{C}^N$ be an irreducible complex analytic set of dimension n containing the origin as an isolated singular point, let $z = (z_1, \ldots, z_N)$ be the coordinate of $\mathbf{C}^N$, and let $\|z\|^2 = \sum_{i=1}^{N} |z_i|^2$. We fix $c \in (0, 1)$ so that the spheres $S_c \subset \mathbf{C}^N$ of radius c centered at 0 intersect with V transversally for all $c' \in (0, c]$. We set $U = \{z \in V;\ \|z\| < c\}$ and $U' = U \setminus \{0\}$. Let us denote by $C_0^r(U')$ (resp. $C_0^{p,q}(U')$) the set of (**C**-valued) compactly supported C^∞ r-forms (resp. (p, q)-forms) on U' and set $C_0(U') = \bigoplus_{r=0}^{2n} C_0^r(U')$. Let ds^2 be the restriction of the euclidean metric to U' and let dv be the volume element with respect to ds^2. For all $u \in C_0(U')$, $|u|$ will denote the length of u with respect to ds^2, and the L^2 norm $\|u\|$ of u will be defined as the square root of the integral of $|u|^2 dv$ over U'. Let $d: C_0(U') \hookleftarrow$ be the exterior derivative and let $\bar{\partial}$ (resp. ∂) be its (0, 1)-component (resp. (1, 0)-component). The maximal closed extensions of these operators to the completion $\overline{C_0(U')}$ with respect to $\|\ \|$ will be denoted by the same symbols. We put $\Phi = \{K \subset U';\ \bar{K} \subset U\}$, $S^r = \{u \in \overline{C_0^r(U')};\ \operatorname{supp} u \in \Phi\}$ and $S^{p,q} = \{u \in \overline{C_0^{p,q}(U')};\ \operatorname{supp} u \in \Phi\}$. Then our goal is to prove the following.

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Lemma Under the above situation, $\mathrm{Ker}\, d \cap S^0 = \mathrm{Ker}\, \bar{\partial} \cap S^{p,0} = \{0\}$ for all p,

$$\mathrm{Ker}\, d \cap S^r = \{du;\ u \in \mathscr{D}om\, d \cap S^{r-1}\} \tag{1}$$

and

$$\mathrm{Ker}\, \bar{\partial} \cap S^r = \{\bar{\partial} u;\ u \in \mathscr{D}om\, \bar{\partial} \cap S^{r-1}\} \tag{2}$$

if $0 < r < n$.

Proof. That $\mathrm{Ker}\, d \cap S^0 = \mathrm{Ker}\, \bar{\partial} \cap S^{p,0} = \{0\}$ is trivial. Let $0 < r < n$, $f \in \mathrm{Ker}\, d \cap S^r$, and let $f_{p,q}$ be the (p, q)-component of f. First of all we note that $\bar{\partial} f_{0,r} = 0$, $\partial f_{p-1,r-p+1} + \bar{\partial} f_{p,r-p} = 0$ for $1 \leq p \leq r$ and $\partial f_{r,0} = 0$. Let us fix $b \in (0, c)$ so that $\mathrm{supp} f \subset U_b' := \{z \in U';\ \|z\| < b\}$. Then we put

$$F_{r,k} = -\log(b^2 - \|z\|^2) + (n-r)\log\|z\|^2 - k \log\log\|z\|^{-1}$$

for $k \geq 0$. Then $F_{r,k}$ is a strictly plurisubharmonic function on U_b'. Let $ds_{r,k}^2$ be the restriction of the complex Hessian of $F_{r,k}$ to U_b', and let $\| \ \|_{r,k}$ be the L^2 norm with respect to $ds_{r,k}^2$ and the weight function $e^{F_{r,k}}$. Namely we put

$$\|u\|_{r,k}^2 = \int_{U'} e^{F_{r,k}} |u|_{r,k}^2 \, dv_{r,k},$$

where $|\ |_{r,k}$ (resp. $dv_{r,k}$) denotes the length (resp. the volume form) with respect to $ds_{r,k}^2$. Since $f \in S^r$, $\|f\|_{r,k} < \infty$ if $k \geq 2$. Note that $(U_b', ds_{r,k}^2)$ is a complete Kähler manifold for any $k > 0$. Therfore by Andreotti-Vesentini's theorem (cf. [2] p. 31 Theorem 1.3), one can find a measurable $(0, r-1)$-form say $g_{0,r-1}$ on U_b' satisfying $f_{0,r} = \bar{\partial} g_{0,r-1}$, $\|g_{0,r-1}\|_{r,k} < \infty$ and that $g_{0,r-1}$ is orthogonal to $\mathrm{Ker}\, \bar{\partial}$ with respect to the inner product associated to $\| \ \|_{r,k}$. Then, by Bochner-Nakano formula for the complex Laplacian with respect to $\| \ \|_{r,k}$ we obtain

$$\|(\partial + \partial F_{r,k}) g_{0,r-1}\|_{r,k} \leq \|f_{0,r}\|_{r,k} < \infty.$$

Hence we obtain $\|g_{0,r-1}\|_{r,k+2} < \infty$. Next we solve the $\bar{\partial}$-equation $\bar{\partial} g_{1,r-2} = f_{1,r-1} - \partial g_{0,r-1}$ similarly as above with respect to the L^2 norm $\| \ \|_{r,k+2}$. Repeating this process and noting that $f_{r,0} - \partial g_{r-1,0} = 0$ since $\bar{\partial}(f_{r,0} - \partial g_{r-1,0}) = 0$ and $\|f_{r,0} - \partial g_{r-1,0}\|_{r,k+2r} < \infty$, we finally obtain an $(r-1)$-form, say g such that $\|g\|_{r,k+2r} < \infty$ and $dg = f$. Since g is an $(r-1)$ form, finiteness of $\|g\|_{r,k+2r}$ implies that of $\|g\|_{n,0}$. Thus the trivial extension $\tilde{g}$ of g to U' satisfies that $\tilde{g} \in S^{r-1}$ and $d\tilde{g} = f$ (cf. Lemma 2.4 of [1]). This completes the proof of (1). Clearly the above proof contains that of (2).

References

[1] Ohsawa, T., Hodge spectral sequence on compact Kähler spaces, *publ. RIMS, Kyoto Univ.* **23** (1987), 613–625.

[2] Vesentini, E., Lectures on Levi convexity of complex manifolds and cohomology vanishing theorems, Tata Inst. Bombay, 1967.