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Inverse Image for the Functor μhom

By

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§ 0. Introduction

Let $f: Y \to X$ be a morphism of real C^∞ manifolds and let F, K be sheaves on X (more precisely objects of the derived category $D^b(X)$).

In this paper we study the microlocal inverse images of sheaves.

In particular we recall the construction of the functors $f_{\mu,p}^{-1}$, $f_{\mu,p}^{!}$ of [K-S 4] (which makes use of the categories of ind-objects and pro-objects on the microlocalization of $D^b(X)$) and study some of their properties.

Then we give a theorem, namely Theorem 2.2.3 below, which asserts that the natural morphism:

$$\mu\hom(f_{\mu,p}^{-1} K, f_{\mu,p}^{!} F \otimes \omega_Y^{\otimes -1})_{p_Y} \longrightarrow \mu\hom(K, F \otimes \omega_X^{\otimes -1})_{p_X}, \tag{0.1}$$

is an isomorphism as soon as a very natural hypothesis, similar to that of "microhyperbolicity" for microdifferential systems, is satisfied (here ω_X denotes the dualizing complex on X and μhom the microlocalization bifunctor of [K-S 4]).

In fact, one could say that this theorem is a statement of the microlocal well poseness for the Cauchy problem.

As an application, we then state and prove a theorem, namely Theorem 3.1.1, on the well poseness for the Cauchy problem, in a sheaf theoretical frame.

This theorem generalizes what was obtained in [D'A-S] and will allow us not only to recover the classical results on the ramified Cauchy problem (cf. [H-L-W], [K-S 1], [Sc]), but also the result of [K-S 2] on the hyperbolic Cauchy problem.

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§1. Review on Sheaves

In this chapter we collect the notations that will be used throughout this paper.

We also give some basic results on ind-objects and pro-objects that are necessary for the proof of the main theorem.

The frame is that of the microlocal study of sheaves as developped in [K-S 3] and [K-S 4].

Until chapter 4 all manifolds and morphisms of manifolds will be real and of class C^∞.

§1.1. Geometry

To a manifold X one associates its tangent and cotangent bundles noted $\tau_X\colon TX \to X$ and $\pi_X\colon T^*X \to X$ respectively. One notes $\dot{T}^*X$ the cotangent bundle with the zero-section removed and denotes by $\dot{\pi}_X$ the projection $\dot{T}^*X \to X$.

If M is a closed submanifold of X, one denotes by T_M^*X the conormal bundle to M in X. If A is a subset of X, one denotes by $N^*(A)$ the strict conormal cone to A, a closed, proper, convex conic subset of T^*X.

If $f\colon Y \to X$ is a morphism of manifolds, one denotes by ${}^tf'$ and f_π the natural mappings associated to f:

$$T^*Y \xleftarrow{{}^tf'} Y \times_X T^*X \xrightarrow{f_\pi} T^*X.$$

One sets: $T_Y^*X = {}^tf'^{-1}(T_Y^*Y)$.

If N (resp. M) is a closed submanifold of Y (resp. X) with $f(N) \subset M$, one denotes by ${}^tf'_N$ and $f_{N\pi}$ the natural mappings associated to f:

$$T_N^*Y \xleftarrow{{}^tf'_N} N \times_M T_M^*X \xrightarrow{f_{N\pi}} T_M^*X.$$

If A is a closed conic subset of T^*X, one says that f is *non-characteristic* for A iff ${}^tf'^{-1}(T_Y^*Y) \cap f_\pi^{-1}(A) \subset Y \times_X T_X^*X$. If V is a subset of T^*Y, we refer to [K-S 3] for the definition of f being non-characteristic for A on V.

§1.2. The Category $\mathrm{D}^b(X)$

We fix a commutative ring A with finite global dimension (e.g. $A = \mathbb{Z}$).

Let X be a manifold. One denotes by $\mathrm{D}^b(X)$ the derived category of the category of bounded complexes of sheaves of A-modules on X.

If $F \in \mathrm{Ob}(\mathrm{D}^b(X))$, one notes by $\mathrm{SS}(F)$ the *micro-support* of F (cf. [K-S 3]). This is a closed conic involutive subset of T^*X that describes the directions of non propagation for the cohomology of F.

If M is a closed submanifold of X, one denotes by $\mu_M(F)$ the Sato's microlocalization of F along M, an object of $D^b(T^*_M X)$. If G is another object of $D^b(X)$, following [K-S 3], one defines the microlocalization of F along G by:

$$\mu\text{hom}(G, F) = \mu_\Delta \mathrm{R}\mathscr{H}om(q_2^{-1} G, q_1^! F),$$

where Δ is the diagonal of $X \times X$ and q_1, q_2 denote the projections from $X \times X$ to X. This is an object of $D^b(T^*X)$ with the following properties:

(1.2.1) $$\mathrm{R}\pi_{X*}\mu\text{hom}(G, F) = \mathrm{R}\mathscr{H}om(G, F),$$

(1.2.2) $$\mu\text{hom}(A_M, F) = \mu_M(F),$$

(1.2.3) $$\text{supp}\,\mu\text{hom}(G, F) \subset \text{SS}(G) \cap \text{SS}(F).$$

Here, as general notation on sheaves, for Z a locally closed subset of X, one denotes by A_Z the sheaf which is 0 on $X \setminus Z$ and the constant sheaf with stalk A on Z.

If Y is another manifold and $F \in \text{Ob}(D^b(X))$, $G \in \text{Ob}(D^b(Y))$, one defines the *external product* of F and G by:

$$F \overset{\mathrm{L}}{\boxtimes} G = q_1^{-1} F \overset{\mathrm{L}}{\otimes} q_2^{-1} G,$$

where q_1 (resp. q_2) is the projection from $X \times Y$ to X (resp. Y). This is an object of $D^b(X \times Y)$.

Let $f: Y \to X$ be a morphism of manifolds. One denotes by $\omega_{Y/X}$ the relative dualizing complex defined by $\omega_{Y/X} = f^! A_X$. One sets $\omega_X = a_X^! A$, where $a_X: X \to \{pt\}$. If or_X is the orientation sheaf, one has an isomorphism $\omega_X \cong \text{or}_X[\dim X]$, and hence, for local problems, ω_X plays essentially the role of a shift.

If F is an object of $D^b(X)$, one says that f is non-characteristic for F if f is non-characteristic for $\text{SS}(F)$.

§1.3. The Category $D^b(X; p_X)$

Let X be a manifold and let Ω be a subset of T^*X. One denotes by $D^b(X; \Omega)$ the localized category $D^b(X)/D^b{}_\Omega(X)$, where $D^b{}_\Omega(X)$ is the null system: $D^b{}_\Omega(X) = \{F \in \text{Ob}(D^b(X)); \text{SS}(F) \cap \Omega = \emptyset\}$. Recall that the objects of $D^b(X; \Omega)$ are the same as those of $D^b(X)$ and that a morphism $u: F \to G$ in $D^b(X)$ becomes an isomorphism in $D^b(X; \Omega)$ if $\Omega \cap \text{SS}(H) = \emptyset$, H being the third term of a distinguished triangle: $F \xrightarrow{u} G \longrightarrow H \xrightarrow{+1}$. Such an u is called an isomorphism on Ω. If $p_X \in T^*X$ one writes $D^b(X; p_X)$ instead of $D^b(X; \{p_X\})$.

A question naturally arising is whether a functor, acting on derived categories of sheaves, still has a "microlocal" meaning, i.e. if it is well defined as a functor acting on these localized categories. In this section we will mainly be concerned in giving an answer to this problem for several well known functors.

Let Y be another manifold and denote by q_1 (resp. q_2) the projections from

$X \times Y$ to X (resp. Y). Let M be a closed submanifold of X. Take a point $p_X \in T^*X$ (resp. $p_Y \in T^*Y$) and set $p_{X\times Y} = (p_X, p_Y) \in T^*(X \times Y)$.

Proposition 1.3.1. *The functors*:

$$\cdot \overset{\mathrm{L}}{\boxtimes} \cdot : \mathrm{D}^b(X) \times \mathrm{D}^b(Y) \longrightarrow \mathrm{D}^b(X \times Y),$$

$$\mathrm{R}\mathcal{H}om(q_2^{-1}(\cdot), q_1^!(\cdot)) : \mathrm{D}^b(Y)^\circ \times \mathrm{D}^b(X) \longrightarrow \mathrm{D}^b(X \times Y),$$

$$\mu_M(\cdot) : \mathrm{D}^b(X) \longrightarrow \mathrm{D}^b(T_M^* X),$$

$$\mu\mathrm{hom}(\cdot,\cdot) : \mathrm{D}^b(X)^\circ \times \mathrm{D}^b(X) \longrightarrow \mathrm{D}^b(T^* X),$$

are microlocally well defined, i.e. extend naturally as functors (*that we denote by the same names*):

$$\cdot \overset{\mathrm{L}}{\boxtimes} \cdot : \mathrm{D}^b(X; p_X) \times \mathrm{D}^b(Y; p_Y) \longrightarrow \mathrm{D}^b(X \times Y; p_{X\times Y}),$$

$$\mathrm{R}\mathcal{H}om(q_2^{-1}(\cdot), q_1^!(\cdot)) :$$

$$\mathrm{D}^b(Y; p_Y)^\circ \times \mathrm{D}^b(X; p_X) \longrightarrow \mathrm{D}^b(X \times Y; p_{X\times Y}),$$

$$\mu_M(\cdot) : \mathrm{D}^b(X; p_X) \longrightarrow \mathrm{D}^b(T_M^* X; p_X), \ (p_X \in T_M^* X)$$

$$\mu\mathrm{hom}(\cdot,\cdot) : \mathrm{D}^b(X; p_X)^\circ \times \mathrm{D}^b(X; p_X) \longrightarrow \mathrm{D}^b(T^* X; p_X).$$

Here $\mathrm{D}^b(Y)^\circ$ denotes the opposite category to $\mathrm{D}^b(Y)$, i.e. the category whose objects are the same as those of $\mathrm{D}^b(Y)$ and whose morphisms are reversed.

Proof. Let $F, G \in \mathrm{Ob}(\mathrm{D}^b(X))$ and $H \in \mathrm{Ob}(\mathrm{D}^b(Y))$. Recall the following estimates of the micro-support (cf[K-S 3, Proposition 4.2.1, 4.2.2, Theorem 5.2.1]):

$$\mathrm{SS}(F \overset{\mathrm{L}}{\boxtimes} H) \subset \mathrm{SS}(F) \times \mathrm{SS}(H),$$

$$\mathrm{SS}(\mathrm{R}\mathcal{H}om(q_2^{-1}(H), q_1^!(F))) \subset \mathrm{SS}(F) \times \mathrm{SS}(H)^a,$$

$$\mathrm{SS}(\mu_M(F)) \subset C_{T_M^* X}(\mathrm{SS}(F)),$$

$$\mathrm{SS}(\mu\mathrm{hom}(G, F)) \subset C(\mathrm{SS}(F), \mathrm{SS}(G)),$$

where ^{a} denotes the antipodal and C the Whithey normal cone.

Since the proofs are similar we will treat only the first functor. The hypothesis $F \in \mathrm{D}^b{}_{\{p_X\}}(X)$ or $H \in \mathrm{D}^b{}_{\{p_Y\}}(X)$ means that $p_X \notin \mathrm{SS}(F)$ or $p_Y \notin \mathrm{SS}(H)$. Then it follows from the first estimate that $p_{X\times Y} \notin \mathrm{SS}(F \overset{\mathrm{L}}{\boxtimes} H)$. Q.E.D.

§ 1.4. Complements on Ind-objects and Pro-objects

Let $f: Y \to X$ be a morphism of manifolds. Take a point $p \in Y \times_X T^*X$ and set $p_X = f_\pi(p)$, $p_Y = {}^t f'(p)$. Contrarily to the case of the functors treated in Proposition 1.3.1, the functors Rf_*, $Rf_!$ (resp. f^{-1}, $f^!$) are not microlocal, i.e. are not well defined as functors from $D^b(Y; p_Y)$ (resp. $D^b(X; p_X)$) to $D^b(X; p_X)$ (resp. $D^b(Y; p_Y)$). To give a microlocal meaning to these functors one must enlarge the category $D^b(X; p_X)$ and work with ind-objects and pro-objects. In this section we recall the definition of ind-objects and pro-objects and, as a preparation for the next section, we give some of their basic properties.

Let us first recall some basic notions on ind-objects and pro-objects due to Grothendieck [G] (for an exposition e.g. cf. [K-S 4, Chapter 1, §11]).

Let $\mathscr{C}$ be a category. Denote by $\mathscr{C}^\wedge$ (resp. $\mathscr{C}^\vee$) the category of covariant (resp. contravariant) functors from $\mathscr{C}$ to the category of sets. Notice first that $\mathscr{C}$ may be considered as a full subcategory of $\mathscr{C}^\wedge$ or $\mathscr{C}^\vee$ via the fully faithfull functors:

$$\begin{aligned} h^\wedge : \mathscr{C} &\longrightarrow \mathscr{C}^\wedge & h^\vee : \mathscr{C} &\longrightarrow \mathscr{C}^\vee \\ X &\longmapsto \mathrm{Hom}_{\mathscr{C}}(X, \cdot) & X &\longmapsto \mathrm{Hom}_{\mathscr{C}}(\cdot, X) \end{aligned}$$

An object ϕ of $\mathscr{C}^\wedge$ in the image of $h^\wedge$ is called *representable*. An object X of $\mathscr{C}$ such that $\phi = h^\wedge(X)$ is called a *representative* of ϕ. Representatives are defined up to an isomorphism.

A category $\mathscr{I}$ is called *filtrant* if for $i, j \in \mathrm{Ob}(\mathscr{I})$ there exist $k \in \mathrm{Ob}(\mathscr{I})$ and morphisms $i \to k$, $j \to k$ and if for two morphisms $f, g \in \mathrm{Hom}_{\mathscr{I}}(i, j)$ there exists a morphism $h: j \to k$ such that $h \circ f = h \circ g$.

Let ϕ be a covariant functor from a filtrant category $\mathscr{I}$ to $\mathscr{C}$. Recall that the object "$\varinjlim_{\mathscr{I}}$" $\phi(i)$ of $\mathscr{C}^\vee$ is defined by "$\varinjlim_{\mathscr{I}}$" $\phi(i)(X) = \varinjlim_{\mathscr{I}} \mathrm{Hom}_{\mathscr{C}}(X, \phi(i))$ for $X \in \mathrm{Ob}(\mathscr{C})$. Here $\varinjlim_{\mathscr{I}}$ denotes the classical inductive limit in the category of sets. Similarly, if ϕ is a contravariant functor from $\mathscr{I}$ to $\mathscr{C}$, "$\varprojlim_{\mathscr{I}}$" $\phi(i)$ is the object of $\mathscr{C}^\wedge$ defined by "$\varprojlim_{\mathscr{I}}$" $\phi(i)(X) = \varinjlim_{\mathscr{I}} \mathrm{Hom}_{\mathscr{C}}(\phi(i), X)$. The category of ind-objects (resp. pro-objects) is the full subcategory of $\mathscr{C}^\vee$ (resp. $\mathscr{C}^\wedge$) consisting of those objects isomorphic to "$\varinjlim_{\mathscr{I}}$" $\phi(i)$ (resp. "$\varprojlim_{\mathscr{I}}$" $\phi(i)$) for some covariant (resp. contravariant) functor ϕ from $\mathscr{I}$ to $\mathscr{C}$.

We will give now some results on ind-objects and pro-objects which will be useful in section 2.

Let $\mathscr{I}$, $\mathscr{I}'$ be two filtrant categories and, for simplicity, assume $\mathrm{Ob}(\mathscr{I})$ and $\mathrm{Ob}(\mathscr{I}')$ being sets. One defines the filtrant category $\mathscr{I} \times \mathscr{I}'$ in the obvious way. Let $\mathscr{C}$, $\mathscr{C}'$ be two categories and let ϕ, ϕ' be two covariant functors from $\mathscr{I}$ to $\mathscr{C}$ and from $\mathscr{I}'$ to $\mathscr{C}'$ respectively. One can prove the following result as in [K-S 4, Corollary 1.11.8].

Proposition 1.4.1. *Keeping the same notations as above, let T be a bifunctor from $\mathscr{C} \times \mathscr{C}'$ to a category $\mathscr{C}''$. If "$\varinjlim_{\mathscr{I}}$" $\phi(i)$ and "$\varinjlim_{\mathscr{I}'}$" $\phi'(i')$ are representable* *then so is* " $\varinjlim_{\mathcal{I}\times\mathcal{I}'}$ " $T(\phi(i), \phi'(i'))$, *a representative being given by* $T($"$\varinjlim_{\mathcal{I}}$" $\phi(i)$, "$\varinjlim_{\mathcal{I}'}$" $\phi'(i'))$.

A similar result holds if ϕ or ϕ' or both of them are contravariant.

Let $\mathscr{C}$ be a category. Let $\mathcal{I}$ and $\mathcal{I}'$ be filtrant categories and let $\iota\colon \mathcal{I} \to \mathcal{I}'$ be a functor. Let ϕ be a covariant (resp. contravariant) functor from $\mathcal{I}'$ to $\mathscr{C}$.

Definition 1.4.2. One says that $\mathcal{I}$ and $\mathcal{I}'$ are *cofinal* with respect to ϕ by ι if the following properties hold:

(a) For any $i' \in \mathrm{Ob}(\mathcal{I}')$ there exists $i \in \mathrm{Ob}(\mathcal{I})$ and a morphism $\phi(i') \to \phi(\iota(i))$ (resp. $\phi(\iota(i)) \to \phi(i')$).

(b) For any $i \in \mathrm{Ob}(\mathcal{I})$, $i' \in \mathrm{Ob}(\mathcal{I}')$ and a morphism $f\colon \iota(i) \to i'$, there exists a morphism $g\colon i \to i_1$ in I such that $\phi(\iota(g))$ factors trough $\phi(f)$.

If $\mathcal{I}$ and $\mathcal{I}'$ are cofinal with respect to the identical functor of $\mathcal{I}'$ for ι, we will say that $\mathcal{I}$ and $\mathcal{I}'$ are cofinal (by ι). This is the classical definition (cf. [K-S 4, Exercice 1.38]). Note that if $\mathcal{I}$ and $\mathcal{I}'$ are cofinal by ι then they are cofinal with respect to any $\phi\colon \mathcal{I}' \to \mathscr{C}$ by ι.

Let us now state a proposition that extends to this more general definition a result of [K-S 4, Exercice 1.38].

Proposition 1.4.3. *With the same notations as above, if $\mathcal{I}$ and $\mathcal{I}'$ are cofinal with respect to ϕ by ι, the natural morphism*:

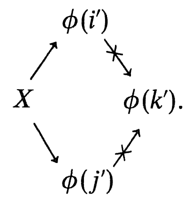
$$\text{“}\varinjlim_{\mathcal{I}}\text{”}\ \phi \circ \iota \longrightarrow \text{“}\varinjlim_{\mathcal{I}'}\text{”}\ \phi$$

(*resp.*

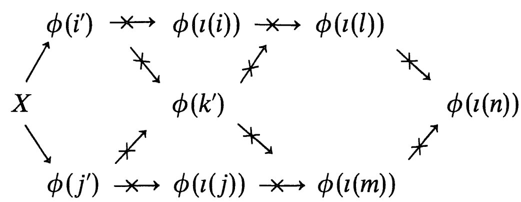
$$\text{“}\varprojlim_{\mathcal{I}'}\text{”}\ \phi \longrightarrow \text{“}\varprojlim_{\mathcal{I}}\text{”}\ \phi \circ \iota)$$

is an isomorphism.

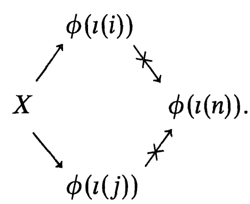
Proof. For $X \in \mathrm{Ob}(\mathscr{C})$ set $A_X = \varinjlim_{\mathcal{I}} \mathrm{Hom}_{\mathscr{C}}(X, \phi(\iota(i)))$, $B_X = \varinjlim_{\mathcal{I}'} \mathrm{Hom}_{\mathscr{C}}(X, \phi(i'))$. We have to show that $A_X \simeq B_X$ for every X. Let $[u\colon X \to \phi(i')]$ be an element of B_X (here $[u]$ denotes the equivalence class of u in B_X). Due to (*a*) of Definition 1.4.2 we can find a morphism $v\colon i' \to \iota(i)$ in $\mathcal{I}'$ with $i \in \mathrm{Ob}(\mathcal{I})$. We define a map $F\colon A_X \to B_X$ by $F([u]) = [\phi(v) \circ u]$. We then have to show that F is well defined, injective and surjective. Since the proofs of these facts are similar, we will assume that the definition of F does not depend on the choice of the representative v of $[v]$ and we will only prove that it does not depend on the choice of u either. Let $[u\colon X \to \phi(i')] = [u'\colon X \to \phi(j')]$ in B_X and let be given morphisms $i' \to \iota(i)$, $j' \to \iota(j)$ in $\mathcal{I}'$. In what follows $\twoheadrightarrow$ will denote a morphism induced by a morphism in $\mathcal{I}'$ and $\ominus\!\!\rightarrow$ a morphism induced by one of $\mathcal{I}$. $[u] = [u']$ means that there is a commutative diagram:



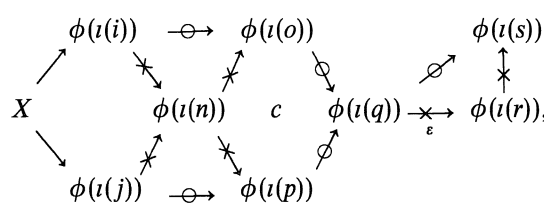
Due to (a) of Definition 1.4.2 and to the fact that $\mathscr{I}$ and $\mathscr{I}'$ are filtrant, it is then easy to get the following commutative diagram:



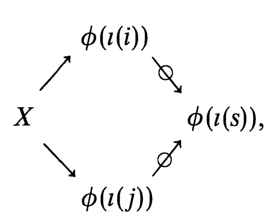
i.e. we have a commutative diagram:



Using (b) of Definition 1.4.2 one then easily get the diagram:



where all the diagrams, except c, are commutative. Nevertheless $\varepsilon \circ c$ is commutative. Hence we have the commutative diagram:



which means that $F([u]) = F([u'])$. Q.E.D.

§ 2. Microlocal Inverse Image Theorem

§ 2.1. Microhyperbolic Theorem for Sheaves

Let $f\colon Y \to X$ be a morphism of manifolds. Let M (resp. N) be a closed submanifold of X (resp. Y), with $f(N) \subset M$.

In [K-S 4] (or [K-S 3]) the main result on the comparison between inverse image and microlocalization is the following.

Theorem 2.1.1. (*cf.* [K-S 4, *Theorem* 6.7.1] *or* [K-S 3, *Theorem* 5.4.1].) *Let V be an open subset of $T_N^* Y$ and let $F \in \mathrm{Ob}(\mathrm{D}^b(X))$. Assume*:

(i) *f is non-characteristic for F on V,*
(ii) *$f_{N\pi}$ is non-characteristic for $C_{T_M^*X}(\mathrm{SS}(F))$ on ${}^tf_N'^{-1}(V)$,*
(iii) *${}^tf'^{-1}(V) \cap f_\pi^{-1}(\mathrm{SS}(F)) \subset Y \times_X T_M^* X$.*

Then the natural morphism:

$$\mu_N(f^! F)|_V \longrightarrow \mathrm{R}{}^tf'_{N*} f^!_{N\pi} \mu_M(F)|_V, \tag{2.1.1}$$

is an isomorphism.

§ 2.2. Inverse Image for μhom

In this section we aim at giving our main result, i.e. Theorem 2.2.3 below, which is a variation of Theorem 2.1.1. To this end, let us recall the definition of microlocal images.

Let $f\colon Y \to X$ be a morphism. Let $p \in Y \times_X T^* X$ and set $p_X = f_\pi(p)$, $p_Y = {}^tf'(p)$.

Definition 2.2.1. Let F be an object of $\mathrm{D}^b(X)$. We denote by $\mathcal{P}ro\jmath_F(p_X)$ (resp. $\mathcal{I}nd_F(p_X)$) the filtrant category whose objects consist of the morphisms $u\colon F' \to F$ (resp. $u\colon F \to F'$) in $\mathrm{D}^b(X)$ which are isomorphisms at p_X. A morphism $(u\colon F' \to F) \to (u'\colon F'' \to F)$ of $\mathcal{P}ro\jmath_F(p_X)$ is defined by a morphism $v\colon F'' \to F'$ in $\mathrm{D}^b(X)$ with $u' = u \circ v$ (and similarly for $\mathcal{I}nd_F(p_X)$).

Definition 2.2.2. (cf [K-S 4, Definition 6.1.7].)

(i) Let $F \in \mathrm{Ob}(\mathrm{D}^b(X; p_X))$. One denotes by $f_{\mu,p}^{-1} F$ (resp. $f_{\mu,p}^! F$) the pro-object (resp. ind-object) " $\varprojlim_{\mathcal{P}ro\jmath_F(p_X)}$ " $f^{-1}F'$ (resp. " $\varinjlim_{\mathcal{I}nd_F(p_X)}$ " $f^! F'$). Here f^{-1} is the functor from $\mathcal{P}ro\jmath_F(p_X)$ to $\mathrm{D}^b(Y; p_Y)$ which associates $f^{-1}F'$ to $F' \to F$ (and similarly for $f^!$). One calls $f_{\mu,p}^{-1} F$ the *microlocal inverse image* of F at p.
(ii) Let $G \in \mathrm{Ob}(\mathrm{D}^b(Y; p_Y))$. One denotes by $f_!^{\mu,p} G$ (resp. $f_*^{\mu,p} G$) the pro-object (resp. ind-object) " $\varprojlim_{\mathcal{P}ro\jmath_G(p_Y)}$ " $\mathrm{R}f_! G'$ (resp. " $\varinjlim_{\mathcal{I}nd_G(p_Y)}$ " $\mathrm{R}f_* G'$). Here $\mathrm{R}f_!$ is the functor from $\mathcal{P}ro\jmath_G(p_Y)$ to $\mathrm{D}^b(X; p_X)$ which associates

$Rf_! G'$ to $G' \to G$ (and similarly for Rf_*). One calls $f_*^{\mu,p} G$ the *microlocal direct image* of G at p.

From now on, for a given $p \in Y \times_X T^*X$ we will set $p_X = f_\pi(p)$ and $p_Y = {}^t f'(p)$. We shall now give a variation of Theorem 2.1.1.

For F and K objects of $D^b(X)$, there is a natural morphism:

$$\mu\text{hom}(f^{-1}K, f^!F) \longrightarrow R^t f'_* f_\pi^! \mu\text{hom}(K, F). \tag{2.2.1}$$

Theorem 2.2.3. *Let F and K be objects of $D^b(X)$ and take $p \in Y \times_X T^*X$. Let V be an open neighborhood of p_Y and assume:*

(i) $p \notin T_Y^* X$,
(ii) $f_{\mu,p}^{-1} K$ *and* $f_{\mu,p}^! F$ *are representable in* $D^b(Y; p_Y)$,
(iii) f_π *is non-characteristic for* $C(SS(F), SS(K))$ *on* ${}^t f'^{-1}(V)$.

Then the morphism (2.2.1) *induces an isomorphism*:

$$\mu\text{hom}(f_{\mu,p}^{-1} K, f_{\mu,p}^! F \otimes \omega_Y^{\otimes -1})_{p_Y} \xrightarrow{\sim} \mu\text{hom}(K, F \otimes \omega_X^{\otimes -1})_{p_X}. \tag{2.2.2}$$

In the left hand side of (2.2.2) we consider μhom acting microlocally as remarked in Proposition 1.3.1. Taking the germ at p_Y we get a bifunctor μhom: $D^b(Y; p_Y)^\circ \times D^b(Y; p_Y) \to D^b(\mathcal{M}od(A))$. Hence the isomorphism in (2.2.2) holds in $D^b(\mathcal{M}od(A))$, the derived category of the category of A-modules.

Let us explain how the morphism (2.2.2) is deduced from (2.2.1). Consider the maps:

$$\begin{array}{ccccc} T^*Y & \xleftarrow{{}^t f'} & Y \times_X T^*X & \xrightarrow{f_\pi} & T^*X \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_X} \\ Y & \xleftarrow{\sim} & Y & \xrightarrow{f} & X. \end{array}$$

By adjonction, the morphism (2.2.1) induces the morphism:

$${}^t f'^{-1} \mu\text{hom}(f^{-1}K, f^!F) \longrightarrow f_\pi^! \mu\text{hom}(K, F). \tag{2.2.3}$$

By (iii), the natural morphism: $f_\pi^{-1} \mu\text{hom}(K, F) \otimes \pi^{-1}\omega_{Y/X} \to f_\pi^! \mu\text{hom}(K, F)$ is an isomorphism on ${}^t f'^{-1}(V)$ (cf. [K-S 3, Proposition 5.3.2]). Composing (2.2.3) with the inverse of this last morphism and recalling that $\pi^{-1}\omega_{Y/X} \cong \pi^{-1}\omega_Y \otimes \pi^{-1} f^{-1} \omega_X^{\otimes -1}$ we then get the morphism:

$${}^t f'^{-1} \mu\text{hom}(f^{-1}K, f^!F \otimes \omega_Y^{\otimes -1}) \longrightarrow f_\pi^{-1} \mu\text{hom}(K, F \otimes \omega_X^{\otimes -1}).$$

Taking the fiber at p and via the natural morphisms $f_{\mu,p}^! F \to f^! F$ and $f_{\mu,p}^{-1} K \leftarrow f^{-1} K$, we obtain the morphism of (2.2.2).

In order to prove Theorem 2.2.3, we shall need Theorem 2.2.4 below.

Let M (resp. N) be a closed submanifold of X (resp. Y) such that $f(M)$

$\subset N$. For $p \in N \times_M T_M^* X$ we will denote $p_X = f_{N\pi}(p)$, $p_Y = {}^t f'_N(p)$, coherently with the previous notations. Recall that for $F \in \mathrm{Ob}(\mathrm{D}^b(X))$ there is a natural morphism corresponding to (2.2.1):

$$\mu_N(f^! F) \longrightarrow \mathrm{R}^t f'_{N*} f^!_{N\pi} \mu_M(F). \tag{2.2.4}$$

Theorem 2.2.4. *Let $F \in \mathrm{Ob}(\mathrm{D}^b(X))$ and take $p \in N \times_M T_M^* X$. Let V be an open neighborhood of p_Y in $T_N^* Y$ and assume:*

(i) $p \notin T_Y^* X$,

(ii) *$f^!_{\mu,p} F$ is representable,*

(iii) *$f_{N\pi}$ is non-characteristic for $C_{T_M^* X}(\mathrm{SS}(F))$ on ${}^t f_N'^{-1}(V)$.*

Then the morphism (2.2.4) *induces an isomorphism:*

$$\mu_N(f^!_{\mu,p} F \otimes \omega_Y^{\otimes -1})_{p_Y} \xrightarrow{\sim} \mu_M(F \otimes \omega_X^{\otimes -1})_{p_X}. \tag{2.2.5}$$

The isomorphism holds once more in $\mathrm{D}^b(\mathcal{M}od(A))$, and (2.2.5) is deduced from (2.2.4) similarly as (2.2.2) was deduced from (2.2.1).

§2.3. A Particular Case

Let us first recall some results of [K-S 4] on microlocal images. Let $f: Y \to X$ be a morphism of manifolds and take $p \in Y \times_X T^* X$. Set $p_X = f_\pi(p)$ and $p_Y = {}^t f'(p)$.

Proposition 2.3.1. (*cf.* [K-S 4, *Proposition* 6.1.8].) *Let $F \in \mathrm{Ob}(\mathrm{D}^b(X; p_X))$ and $G \in \mathrm{Ob}(\mathrm{D}^b(Y; p_Y))$. The following equalities hold:*

$$\mathrm{Hom}_{\mathrm{D}^b(X;p_X)^\wedge}(f_!^{\mu,p} G, F) = \mathrm{Hom}_{\mathrm{D}^b(Y;p_Y)^\vee}(G, f^!_{\mu,p} F), \tag{2.3.1}$$

$$\mathrm{Hom}_{\mathrm{D}^b(X;p_X)^\vee}(G, f_*^{\mu,p} F) = \mathrm{Hom}_{\mathrm{D}^b(Y;p_Y)^\wedge}(f^{-1}_{\mu,p} G, F). \tag{2.3.2}$$

Moreover there are canonical morphisms:

$$f_!^{\mu,p} G \longrightarrow f_*^{\mu,p} G, \tag{2.3.3}$$

$$f^{-1}_{\mu,p} F \otimes \omega_{Y/X} \longrightarrow f^!_{\mu,p} F. \tag{2.3.4}$$

Proposition 2.3.2. (*cf.* [K-S 4, *Proposition* 6.1.10].) *Let $G \in \mathrm{Ob}(\mathrm{D}^b(Y))$. If* $\mathrm{supp}(G)$ *is proper over X and if:*

$$f_\pi^{-1}(p_X) \cap {}^t f'^{-1}(\mathrm{SS}(G)) \subset \{p\}, \tag{2.3.5}$$

then $f_!^{\mu,p} G$ and $f_*^{\mu,p} G$ are representable and one has the isomorphisms:

$$f_!^{\mu,p} G \simeq f_*^{\mu,p} G \simeq \mathrm{R} f_* G,$$

in $\mathrm{D}^b(X; p_X)$.

We are now ready to prove a particular case of Theorem 2.2.4.

Proposition 2.3.3. *Let* $f: Y \to X$ *be a closed embedding and set* $M = f(N)$. *Take a point* $p \in N \times_M T_M^* X \cong T_M^* X$ *and set* $p_X = f_{N\pi}(p)$, $p_Y = {}^t f'_N(p)$. *Let* $F \in \mathrm{Ob}(\mathrm{D}^b(X))$ *and assume that* $f^!_{\mu,p} F$ *is representable. Then the natural morphism* (2.2.4) *induces an isomorphism*:

$$\mu_N(f^!_{\mu,p} F)_{p_Y} \xrightarrow{\sim} \mu_M(F)_{p_X}.$$

Proof. It is enough to prove the isomorphism for the cohomology groups. One has:

$$\begin{aligned} \mathscr{H}^j \mu_N(f^!_{\mu,p} F)_{p_Y} &\cong \mathrm{Hom}_{\mathrm{D}^b(Y;\, p_Y)}(A_N, f^!_{\mu,p} F[j]) \\ &\cong \mathrm{Hom}_{\mathrm{D}^b(Y;\, p_Y)^\vee}(A_N, f^!_{\mu,p} F[j]) \\ &\cong \mathrm{Hom}_{\mathrm{D}^b(X;\, p_X)^\wedge}(f_!^{\mu,p} A_N, F[j]) \\ &\cong \mathrm{Hom}_{\mathrm{D}^b(X;\, p_X)}(A_M, F[j]) \\ &\cong \mathscr{H}^j \mu_M(F)_{p_X}. \end{aligned}$$

Here the first isomorphism follows from [K-S 4, Theorem 6.1.2], the second expresses the fact that $\mathrm{D}^b(Y;\, p_Y)$ is a full subcategory of $\mathrm{D}^b(Y;\, p_Y)^\vee$, the third follows from Proposition 2.3.1 and the forth from the fact that, since f_π is injective, we can apply Proposition 2.3.2 and get: $f_!^{\mu,p} A_N = \mathrm{R}f_* A_N = A_M$.
Q.E.D.

§2.4. The Microlocal Cut-Off Lemma

First let us recall the definition of cutting functors as it has been given in [K-S 4, chapter 6].

Since we are concerned with problems of a local nature, we will assume X being a vector space. In what follows we will often identify X with T_0X.

Let γ be a (not necessarily proper) closed convex cone of T_0X. Let ω be an open neighborhood of 0 in X with smooth boundary. We shall denote by q_1 and q_2 the projections from $X \times X$ to X, by γ^0 the polar to γ and by s the map $s(x_1, x_2) = x_1 - x_2$. The following definition is a slight modification of that of [K-S 4, Proposition 6.1.4, 6.1.8].

Definition 2.4.1. Let γ and ω be as above and let F be an object of $\mathrm{D}^b(X)$. We set:

$$\Phi_X(\gamma, \omega; F) = \mathrm{R}q_{2*}(s^{-1} A_\gamma \overset{\mathrm{L}}{\otimes} q_1^{-1} F_\omega),$$

$$\Psi_X(\gamma, \omega; F) = \mathrm{R}q_{2!}\, \mathrm{R}\Gamma_{s^{-1}(\gamma^a)}(q_1^!\, \mathrm{R}\Gamma_\omega(F)).$$

Notice that for $\gamma' \subset \gamma$, $\omega' \supset \omega$, one has the following natural morphisms in $\mathrm{D}^b(X)$:

$$\Phi_X(\gamma, \omega; F) \longrightarrow \Phi_X(\gamma', \omega'; F),$$
$$\Psi_X(\gamma', \omega'; F) \longrightarrow \Psi_X(\gamma, \omega; F). \tag{2.4.1}$$

In particular, recalling the isomorphisms $\mathrm{R}q_{2*}(s^{-1} A_{\{0\}} \overset{\mathrm{L}}{\otimes} q_1^{-1} F) \tilde{\to} F$, $F \tilde{\to} \mathrm{R}q_{2!} \mathrm{R}\Gamma_{s^{-1}(0)}(q_1^! F)$, we get natural morphisms:

$$\Phi_X(\gamma, \omega; F) \longrightarrow F,$$
$$F \longrightarrow \Psi_X(\gamma, \omega; F). \tag{2.4.2}$$

One has the following result.

Proposition 2.4.2. (*cf.* [K-S 4, *Theorem* 5.2.3] *or* [K-S 3, *Proposition* 3.2.2]) *With the same notations as above*:

a) $\mathrm{SS}(F)$ *is contained in* $\bar{\omega} \times \gamma^{\circ a}$ *if and only if the morphism* $\Phi_X(\gamma, \omega; F) \to F$ *(resp.* $F \to \Psi_X(\gamma, \omega; F)$*) is an isomorphism.*
b) $\Phi_X(\gamma, \omega; F) \to F$ *(resp.* $F \to \Psi_X(\gamma, \omega; F)$*) is an isomorphism on* $\omega \times \mathrm{Int}\,\gamma^{\circ a}$.

In particular one has the following estimates:

$$\mathrm{SS}(\Phi_X(\gamma, \omega; F)) \subset \bar{\omega} \times \gamma^{\circ a},$$
$$\mathrm{SS}(\Psi_X(\gamma, \omega; F)) \subset \bar{\omega} \times \gamma^{\circ a}.$$

In order to give a sharper result on the cutting of the microsupport one should take care of the relation between γ and ω. Refining [K-S 4, Proposition 6.1.4], we give the following definition:

Definition 2.4.3. Take $\xi_0 \in \dot{T}_0^* X$. Let $\gamma \subset T_0 X$ and $\omega \subset X$ be such that:

(i) γ is a closed proper convex cone,
(ii) $\partial\gamma \setminus \{0\}$ is C^1,
(iii) $\xi_0 \in \mathrm{Int}\,\gamma^{\circ a}$,
(iv) ω is an open neighborhood of 0,
(v) $\partial\omega$ is C^1,
(vi) $\omega \subset \{x; |x| < \varepsilon\}$ for some $\varepsilon > 0$,
(vii) $\forall x \in \partial\omega \cap \partial\gamma$, $N_x^*(\omega)^a = N_x^*(\gamma)$.

We will call a pair (γ, ω) satisfying (i)–(vii) a *refined cutting pair* on X at $(0; \xi_0)$.

Note that since $\partial\omega$ and $\partial\gamma$ are smooth, condition (vii) means that $\partial\omega$ and $\partial\gamma$ are tangent at their intersection. More precisely, if $g(x) < 0$ (resp. $h(x) \leq 0$) is a local equation for ω (resp. γ) at $x \in \partial\omega \cap \partial\gamma$, this means that $-\mathrm{d}\,g(x) \in \mathbf{R}^+ \mathrm{d}\,h(x)$.

Let S be a vector space and take $p_S \in T_0^* S$. If (γ, ω) is a refined cutting pair on X at $(0; \xi_0)$, and if ω is defined by $\omega = \{x; g(x) < 0\}$ for a C^1 function g with

$\mathrm{d}g \neq 0$, we can find an open neighborhood ω_S of 0 in $X \times S$ with smooth boundary such that:

$$\text{(2.4.3)} \qquad \begin{cases} \omega_S = \{(x, s) \in X \times S;\ \langle s, p_S\rangle + g(x) < 0\} \text{ near } X \times \{0\}, \\ \omega_S \subset \{(x, s);\ |(x, s)| < \varepsilon\} \end{cases}$$

The following proposition is an extension of Proposition 6.1.4 of [K-S 4].

Proposition 2.4.4. *Let $H \in \mathrm{Ob}(\mathrm{D}^b(X \times S))$ and let (γ, ω) be a refined cutting pair on X at $(0; \xi_0)$, $\xi_0 \neq 0$. Take $p_S \in T_0^* S$ and set $H' = \Phi_{X\times S}(\gamma \times \{0\}, \omega_S; H)$ (resp. $H' = \Psi_{X\times S}(\gamma \times \{0\}, \omega_S; H)$) for ω_S defined as in* (2.4.3). *The following estimate holds*:

$$\mathrm{SS}(H') \cap (\pi_X^{-1}(0) \times \{p_S\}) \subset$$
$$(\{\xi \in \gamma^{\circ a}\setminus\{0\};\ \exists x \in \bar{\omega}\colon ((x;\xi), p_S) \in \mathrm{SS}(H)\} \cup \{0\}) \times \{p_S\}.$$

We will give a proof based on the same line as the one of [K-S 4, Proposition 6.1.4].

Proof. By Proposition 2.4.2 we know that $H \cong H'$ on $\omega_S \times \mathrm{Int}((\gamma \times \{0\})^{\circ a})$ and that $\mathrm{SS}(H') \subset \overline{\omega_S} \times (\gamma \times \{0\})^{\circ a}$. It then remains to show that

$$\text{(2.4.4)} \qquad \begin{array}{c} \xi \in \partial((\gamma \times \{0\})^{\circ a})\setminus\{0\},\ ((0;\xi), p_S) \in \mathrm{SS}(H') \\ \Downarrow \\ \exists (x;\xi);\ x \in \bar{\omega},\ ((x;\xi), p_S) \in \mathrm{SS}(H). \end{array}$$

The map $q_2\colon \mathrm{supp}(s^{-1} A_{\gamma\times\{0\}} \overset{\mathrm{L}}{\otimes} q_1^{-1} H_{\omega_S}) \to X \times S$ is proper due to (2.4.3) and (i) of Definition 2.4.3. One may then apply Propositions 5.4.4, 5.4.5 and 5.4.14 of [K-S 4] and get the estimate:

$$\text{(2.4.5)} \qquad \begin{array}{c} ((0;\xi), p_S) \in \mathrm{SS}(H') \\ \Downarrow \\ \exists x\colon ((x;\xi), p_S) \in \mathrm{SS}(A_{\gamma\times\{0\}})^a \cap \mathrm{SS}(H_{\omega_S}). \end{array}$$

Let us then prove (2.4.4) using (2.4.5). Since $\xi \neq 0$ and $((x;\xi), p_S) \in \mathrm{SS}(A_{\gamma\times\{0\}})^a$, we have $x \in \partial\gamma$.

If $x \in X\setminus\bar{\omega}$ then $\mathrm{SS}(H_{\omega_S}) \cap \pi_{X\times S}^{-1}((x, 0)) = \emptyset$.

If $x \in \omega$ then $H_{\omega_S} \cong H$ at $(x, 0)$.

If $x \in \partial\omega$, by (vii) of Definition 2.4.3 we get: $N^*_{(x,0)}(\omega_S) \cap N^*_{(x,0)}(\gamma \times \{0\})^a = \mathbf{R}_{\geq 0}(\xi, p_S)$.

Assume $((x;\xi), p_S) \notin \mathrm{SS}(H)$, then one may estimate $\mathrm{SS}(H_{\omega_S})$ as

$$\mathrm{SS}(H_{\omega_S}) \cap \pi_{X\times S}^{-1}((x, 0)) \subset -\mathbf{R}_{\geq 0}(\xi, p_S) + (\mathrm{SS}(H) \cap \pi_{X\times S}^{-1}((x, 0)))$$

which implies $((x;\xi), p_S) \in \mathrm{SS}(H)$. This is a contradiction and this completes the proof. Q.E.D.

Corollary 2.4.5. (*cf.* [K-S 4, *Proposition* 6.1.4, 6.1.8]) *Keep the same notations as above. Let K be a proper closed convex cone of $T_0^* X$ and let $U \subset K$ be an open cone. Let $F \in \mathrm{Ob}(\mathrm{D}^b(X))$ and let W be a conic neighborhood of $K \cap (\mathrm{SS}(F) \setminus \{0\})$. Then*:

a) (*Refined microlocal cut-off lemma*). *There exists $F' \in \mathrm{Ob}(\mathrm{D}^b(X))$ and a morphism $u\colon F' \to F$ satisfying*:

(1) *u is an isomorphism on U*;

(2) $\pi_X^{-1}(0) \cap \mathrm{SS}(F') \subset W \cup \{0\}$.

b) (*Dual refined microlocal cut-off lemma*). *Same as a) with $u\colon F \to F'$.*

Proof. It is not restrictive to assume $\bar{U} \subset \{0\} \cup \mathrm{Int}\, K$. Take $\xi_0 \in U$ and choose a refined cutting pair (γ, ω) on X at $(0; \xi_0)$ with $K^{\circ a} \subset \gamma \subset U^{\circ a}$. It then remains to apply Proposition 2.4.4 to the case $S = \{pt\}$. Q.E.D.

§2.5. Complements on the Microlocal Inverse Image

As a preparation to the proof of the theorems of §2.2 we need to give some results concerning microlocal inverse images.

Let $f\colon Y \to X$ be a morphism of manifolds. Take $p \in Y \times_X T^*X$ and set $p_X = f_\pi(p)$, $p_Y = {}^t f'(p)$. Assume $p_X \notin T_X^* X$. Set $x_0 = \pi_X(p_X)$, $y_0 = \pi_Y(p_Y)$. Fix a local system of coordinates $(x) \in X$ in a neighborhood of $x_0 = \pi_X(p_X)$ and let $(x; \xi)$ be the associated symplectic coordinates in T^*X. Since all statements in what follows are of a local nature, we may assume X is a vector space. Let $p_X = (x_0; \xi_0)$ and recall that we assumed $\xi_0 \neq 0$. Let $\gamma \subset T_{x_0}X$ be a cone and let $\omega \subset X$ be an open set such that:

$$\begin{cases} \xi_0 \in \mathrm{Int}\, \gamma^{\circ a}, \\ x_0 \in \omega. \end{cases} \tag{2.5.1}$$

Let $\mathscr{C}ut_X(p_X)$ be the category whose objects are the pairs (γ, ω) satisfying (2.5.1) and whose morphisms are defined as:

$$\mathrm{Hom}_{\mathscr{C}ut_X(p_X)}((\gamma, \omega), (\gamma', \omega')) = \begin{cases} \{>\} & \text{if } \gamma \supset \gamma',\ \omega \subset \omega', \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\phi_F(p_X)\colon \mathscr{C}ut_X(p_X) \to \mathscr{P}roj_F(p_X)$ be the functor associating to an object (γ, ω) of $\mathscr{C}ut_X(p_X)$ the morphism $u\colon \Phi_X(\gamma, \omega; F) \to F$ defined in (2.4.2) (note that u belongs to $\mathrm{Ob}(\mathscr{P}roj_F(p_X))$ due to Proposition 2.4.2) and to a morphism $(\gamma, \omega) > (\gamma', \omega')$ the morphism defined in (2.4.1). Similarly, let $\psi_F(p_X)\colon \mathscr{C}ut_X(p_X) \to \mathscr{I}nd_F(p_X)$ be defined by $\psi_F(p_X)((\gamma, \omega)) = (F \to \Psi_X(\gamma, \omega; F))$.

Proposition 2.5.1. *Let $F \in \mathrm{Ob}(\mathrm{D}^b(X))$ and take $p \in Y \times_X T^*X \setminus T_Y^*X$. The following isomorphisms hold*:

(i) $$f^{-1}_{\mu,p} F \cong \text{“}\varinjlim_{\mathscr{C}ut_X(p_X)}\text{”}\, f^{-1}\Phi_X(\gamma, \omega; F)$$

(ii) $$f^{!}_{\mu,p} F \cong \text{“}\varinjlim_{\mathscr{C}ut_X(p_X)}\text{”}\, f^{!}\Psi_X(\gamma, \omega; F)$$

Here $f^{-1}\Phi_X(\gamma, \omega; F)$ is the functor from $\mathscr{C}ut_X(p_X)$ to $D^b(Y; p_Y)$ which associates the object $f^{-1}\Phi_X(\gamma, \omega; F)$ to $(\gamma, \omega) \in \mathrm{Ob}(\mathscr{C}ut_X(p_X))$.

Proof. Since the proofs of (i) and (ii) are similar we will treat only the case (i). Denote by $f^{-1}F'$ the functor from $\mathscr{P}roj_F(p_X)$ to $D^b(Y; p_Y)$ which associates $f^{-1}F'$ to $F' \to F$. Due to Proposition 1.4.3 we have to show that $\mathscr{C}ut_X(p_X)$ and $\mathscr{P}roj_F(p_X)$ are cofinal with respect to $f^{-1}F'$ by $\phi_F(p_X)$. Let $u\colon F' \to F$ be an object of $\mathscr{P}roj_F(p_X)$. In order to prove that (a) of Definition 1.4.2 holds, we have to find a pair (γ, ω) satisfying (2.5.1) and a morphism $f^{-1}\Phi_X(\gamma, \omega; F) \to f^{-1}F'$ in $D^b(Y; p_Y)$. To this end, embed u in a distinguished triangle $F' \xrightarrow{u} F \longrightarrow F_0 \xrightarrow{+1}$. Since $u \in \mathscr{P}roj_F(p_X)$, we have $p_X \notin SS(F_0)$. Take a proper closed convex cone K and an open convex cone U such that $p_X \in U \subset K$ and $SS(F_0) \cap K \subset \{0\}$. Following the proof of Corollary 2.4.5 we can find a refined cutting pair (γ, ω) on X at p_X such that f is non-characteristic for $\Phi_X(\gamma, \omega; F_0)$ at x_0 and $f_\pi^{-1} SS(\Phi_X(\gamma, \omega; F_0)) \cap {}^t f'^{-1}(p_Y) = \emptyset$. Hence $p_Y \notin SS(f^{-1}\Phi_X(\gamma, \omega; F_0))$ and this means that the morphism $v\colon f^{-1}\Phi_X(\gamma, \omega; F') \to f^{-1}\Phi_X(\gamma, \omega; F)$, obtained by applying $f^{-1}\Phi_X(\gamma, \omega; \cdot)$ to u, is an isomorphism at p_Y. Composing, in $D^b(Y; p_Y)$, v^{-1} with the natural morphism $f^{-1}\Phi_X(\gamma, \omega; F') \to f^{-1}F'$, we get the desired morphism $f^{-1}\Phi_X(\gamma, \omega; F) \to f^{-1}F'$. As for (b) of Definition 1.4.2 we have to show that for any (γ, ω) as in (2.5.1), any $(F \to F') \in \mathrm{Ob}(\mathscr{P}roj_F(p_X))$ and any morphism $u\colon F' \to \Phi_X(\gamma, \omega; F)$, there exists $(\gamma', \omega') \in \mathrm{Ob}(\mathscr{C}ut_X(p_X))$ such that the natural morphism $\Phi_X(\gamma', \omega'; F) \to \Phi_X(\gamma, \omega; F)$ obtained from (2.4.1) factors as:

$$\begin{array}{ccc} f^{-1}\Phi_X(\gamma', \omega'; F) & \longrightarrow & f^{-1}\Phi_X(\gamma, \omega; F) \\ & \searrow \quad \nearrow & \\ & f^{-1}F' & \end{array}$$

Reasoning as for part (a), one can find a refined cutting pair $(\gamma', \omega') > (\gamma, \omega)$ so that the natural morphisms:

$$\begin{aligned} f^{-1}\Phi_X(\gamma', \omega'; F') &\longrightarrow f^{-1}\Phi_X(\gamma', \omega'; \Phi_X(\gamma, \omega; F)) \\ &\longrightarrow f^{-1}\Phi_X(\gamma', \omega'; F), \end{aligned}$$

are isomorphisms at p_Y. Composing, in $D^b(Y; p_Y)$, the inverse of this composite with the natural arrow $f^{-1}\Phi_X(\gamma', \omega'; F') \to f^{-1}F'$, we get the claim. Q.E.D.

Let S be another manifold and consider the map:

$$\hat{f} = f \times \mathrm{id}_S\colon Y \times S \longrightarrow X \times S.$$

We will identify $T^*(X \times S)$ with $T^*X \times T^*S$. For $p \in Y \times_X T^*X$ and $p_S \in T^*S$, set $\hat{p} = (p, p_S)$ and define $p_X = f_\pi(p)$, $p_Y = {}^t f'(p)$, $\hat{p}_X = \hat{f}_\pi(\hat{p})$ and $\hat{p}_Y = {}^t\hat{f}'(\hat{p})$. Set $x_0 = \pi_X(p_X)$, $y_0 = \pi_Y(p_Y)$, $s_0 = \pi_S(p_S)$. Fix a local system of coordinates (s) on S at s_0 and consider it as a vector space.

Proposition 2.5.2. *Let* $F \in \mathrm{Ob}(\mathrm{D}^b(X))$, $G \in \mathrm{Ob}(\mathrm{D}^b(S))$ *and take* $p \in Y \times_X T^*X \setminus T_Y^*X$, $p_S \in T^*S$. *Assume that* $f_{\mu,p}^{-1}F$ (*resp.* $f_{\mu,p}^{!}F$) *is representable. Then the following isomorphisms hold in* $\mathrm{D}^b(Y \times S; \hat{p}_Y)$:

$$\text{(i)} \qquad \hat{f}_{\mu,\hat{p}}^{-1}(F \overset{\mathrm{L}}{\boxtimes} G) \cong (f_{\mu,p}^{-1}F) \overset{\mathrm{L}}{\boxtimes} G,$$

$$\text{(ii)} \qquad \hat{f}_{\mu,\hat{p}}^{!} \mathrm{R}\mathscr{H}om(q_1^{-1}F, q_2^{!}G) \cong \mathrm{R}\mathscr{H}om(q_1^{-1}f_{\mu,p}^{-1}F, q_2^{!}G)$$

(*resp.*

$$\text{(iii)} \qquad \hat{f}_{\mu,\hat{p}}^{!} \mathrm{R}\mathscr{H}om(q_2^{-1}G, q_1^{!}F) \cong \mathrm{R}\mathscr{H}om(q_2^{-1}G, q_1^{!}f_{\mu,p}^{!}F)).$$

Here q_1 and q_2 denote the projections from $X \times S$ to X and S respectively and we remark that (i)–(iii) make sense due to Proposition 1.3.1.

Proof. Since the proofs are similar we will treat only the case (i). For a pair $(\gamma, \omega) \in \mathrm{Ob}(\mathscr{C}ut_X(p_X))$ and an open subset $\omega' \subset S$, it is easy to check that

$$f^{-1}\Phi_X(\gamma, \omega; F) \overset{\mathrm{L}}{\boxtimes} G_{\omega'} \cong \hat{f}^{-1}\Phi_{X\times S}(\gamma \times \{0\}, \omega \times \omega'; F \overset{\mathrm{L}}{\boxtimes} G).$$

We then have the isomorphisms in $\mathrm{D}^b(Y \times S; \hat{p}_Y)$:

$$\begin{aligned}(f_{\mu,p}^{-1}F) \overset{\mathrm{L}}{\boxtimes} G &\cong (\text{“}\varinjlim_{\mathscr{C}ut_X(p_X)}\text{”} f^{-1}\Phi_X(\gamma, \omega; F)) \overset{\mathrm{L}}{\boxtimes} (\text{“}\varinjlim_{\omega'}\text{”}\, G_{\omega'}) \\ &\cong \text{“}\varinjlim_{\mathscr{C}ut_X(p_X)\times\omega'}\text{”}\, (f^{-1}\Phi_X(\gamma, \omega; F) \overset{\mathrm{L}}{\boxtimes} G_{\omega'}) \\ &\cong \text{“}\varinjlim_{\mathscr{C}ut_X(p_X)\times\omega'}\text{”}\, \hat{f}^{-1}\Phi_{X\times S}(\gamma \times \{0\}, \omega \times \omega'; F \overset{\mathrm{L}}{\boxtimes} G).\end{aligned}$$

Here ω' ranges over an open neighborhood system of s_0. Notice that the first isomorphism follows from Proposition 2.5.1 and the second one from Proposition 1.4.1.

We need now a lemma.

Lemma 2.5.3. *Keeping the same notations as above and for* ω_S *as in* (2.4.3), *the following isomorphism holds for* $H \in \mathrm{Ob}(D^b(X \times S))$:

$$\hat{f}_{\mu,\hat{p}}^{-1}(H) \cong \text{“}\varinjlim_{\mathscr{C}ut_X(p_X)}\text{”}\, \hat{f}^{-1}\Phi_{X\times S}(\gamma \times \{0\}, \omega_S; H).$$

Proof. Let be given a morphism in $\mathrm{D}^b(X \times S)$ $H' \to H$ which is an isomorphism at $\hat{p}_X$. Let H_0 be the third term of a distinguished triangle: $H' \to H \to H_0 \xrightarrow{+1}$. By the same proof as in Proposition 2.5.1 it is enough to show that there exists a pair $(\gamma, \omega) \in \mathrm{Ob}(\mathscr{C}ut_X(p_X))$ such that $\hat{p}_Y \notin \mathrm{SS}(\hat{f}^{-1}\Phi_{X\times S}(\gamma \times \{0\}, \omega_S; H_0))$.

For that purpose it is enough to prove that $SS(\Phi_{X\times S}(\gamma \times \{0\}, \omega_S; H_0)) \cap \hat{f}_\pi{}^t\hat{f}'^{-1}(p_Y) \subset \{0\}$. Since $\hat{f}_\pi{}^t\hat{f}'^{-1}(\hat{p}_Y) = f_\pi{}^tf'^{-1}(p_Y) \times \{p_S\}$, this follows from Proposition 2.4.4. Q.E.D.

End of the proof of Proposition 2.5.2. The only thing which is left to prove is the isomorphism

$$\text{“}\varinjlim_{\mathscr{C}ut_X(p_X)}\text{”}\, \hat{f}^{-1}\Phi_{X\times S}(\gamma \times \{0\}, \omega_S; F \overset{L}{\boxtimes} G)$$
$$\cong \text{“}\varinjlim_{\mathscr{C}ut_X(p_X)\times\omega'}\text{”}\, \hat{f}^{-1}\Phi_{X\times S}(\gamma \times \{0\}, \omega \times \omega'; F \overset{L}{\boxtimes} G),$$

but this follows from the fact that both ω_S and $\omega \times \omega'$ describe a fundamental neighborhood system of (x_0, s_0). Q.E.D.

Let $g: Y \to X \times S$ be a morphism of manifolds. Consider the composite:

$$f: Y \xrightarrow{g} X \times S \xrightarrow{q_1} X.$$

For $p \in Y \times_X T^*X$ we will set $p_Y = {}^tf'(p)$, $p_X = f_\pi(p)$, $y_0 = \pi_Y(p_Y) \in Y$, $(x_0, s_0) = g(y_0)$, $\hat{p}_X = {}^tq_1'((x_0, s_0), p_X) \in T^*(X \times S)$ and $\hat{p} = (y_0, \hat{p}_X) \in Y \times_{(X\times S)} T^*(X \times S)$.

Proposition 2.5.4. *Let* $F \in \mathrm{Ob}(\mathrm{D}^b(X))$ *and take* $p \in Y \times_X T^*X \setminus T_Y^*X$. *The following isomorphisms hold*:

(i) $$f_{\mu,p}^{-1}F \cong g_{\mu,\hat{p}}^{-1}(q_1^{-1}F),$$

(ii) $$f_{\mu,p}^{!}F \cong g_{\mu,\hat{p}}^{!}(q_1^{!}F).$$

Proof. Since the proofs are similar we will treat only the case (i). Due to Proposition 2.5.1 we have to prove the isomorphism:

$$\text{“}\varinjlim_{\mathscr{C}ut_X(p_X)}\text{”}\, g^{-1}q_1^{-1}\Phi_X(\gamma, \omega; F) \cong \text{“}\varinjlim_{\mathscr{C}ut_{X\times S}(\hat{p}_X)}\text{”}\, g^{-1}\Phi_{X\times S}(\Gamma, \Omega; q_1^{-1}F).$$

Let $\hat{j}: \mathscr{C}ut_X(p_X) \to \mathscr{C}ut_{X\times S}(\hat{p}_X)$ be the functor of filtrant categories defined by $\hat{j}((\gamma, \omega)) = (\gamma \times \{0\}, \omega \times S)$ for $(\gamma, \omega) \in \mathrm{Ob}(\mathscr{C}ut_X(p_X))$. One has the following evident isomorphism:

$$\Phi_{X\times S}(\gamma \times \{0\}, \omega \times S; F \overset{L}{\boxtimes} A_S) \cong q_1^{-1}\Phi_X(\gamma, \omega; F),$$

and hence the proposition is proven if we show that $\mathscr{C}ut_X(p_X)$ and $\mathscr{C}ut_{X\times S}(\hat{p}_X)$ are cofinal with respect to $g^{-1}\Phi_{X\times S}(\Gamma, \Omega; q_1^{-1}F)$ by $\hat{j}$. To this end it is enough to prove that they are cofinal by $\hat{j}$. In order to prove that (a) of Definition 1.4.3 holds, for a given $(\Gamma, \Omega) \in \mathrm{Ob}(\mathscr{C}ut_{X\times S}(\hat{p}_X))$, we have to find $(\gamma, \omega) \in \mathrm{Ob}(\mathscr{C}ut_X(p_X))$ and a morphism $\Phi_{X\times S}(\gamma \times \{0\}, \omega \times S; F \overset{L}{\boxtimes} A_S) \to \Phi_{X\times S}(\Gamma, \Omega; F \overset{L}{\boxtimes} A_S)$ in $\mathrm{D}^b(X \times S; (x_0, s_0))$. It is not restrictive to assume (Γ, Ω) being a refined cutting pair on $X \times S$ at $\hat{p}_X$. Consider a distinguished triangle: $\Phi_{X\times S}(\Gamma, \Omega; F \overset{L}{\boxtimes} A_S) \to F \overset{L}{\boxtimes} A_S \to H \overset{+1}{\longrightarrow}$. Choose a refined cutting pair (γ, ω) on X at p_X such that

$-\{(x, s_0)\} \times (\gamma \times \{0\})^{\circ a} \cap SS(H) \subset \{0\}$ for $x \in \bar{\omega}$,

$-N_x^*(\omega) \subset (\gamma \times \{0\})^{\circ a}$ $\forall x \in \bar{\omega} \cap \gamma$.

Set $H' = \Phi_{X\times S}(\gamma \times \{0\}, \omega \times S; H)$. Due to [K-S 4, Proposition 5.4.8] we have the estimate: $SS(H_{\omega\times S}) \subset N^*(\omega \times S)^a + SS(H)$. Due to (vii) of Definition 2.4.3, we have: $N^*_{(x,s_0)}(\omega \times S) = N_x^*(\gamma)^a \times \{0\}$ for any $x \in \partial\gamma \cap \partial\omega$, and hence we get the estimate:

$$SS(H_{\omega\times S}) \cap (\pi_X^{-1}(x_0) \times \{s_0\}) \cap (\gamma \times \{0\})^{\circ a} \subset \{0\} \qquad \forall x \in \partial\gamma \cap \partial\omega.$$

From the estimate:

$$SS(H') \cap (\pi_X^{-1}(x_0) \times \{s_0\}) \subset SS(A_{\gamma\times\{0\}})^a \cap SS(H_{\omega\times S}),$$

we then get:

$$SS(H') \cap (\pi_X^{-1}(x_0) \times \{s_0\}) \subset \{0\},$$

and hence H' is a complex of constant sheaves. Moreover, since the stalks at (x_0, s_0) of both sides of the morphism:

$$\Phi_{X\times S}(\gamma \times \{0\}, \omega \times S; \Phi_{X\times S}(\Gamma, \Omega; F \overset{L}{\boxtimes} A_S)) \longrightarrow \tag{2.5.2}$$
$$\longrightarrow \Phi_{X\times S}(\gamma \times \{0\}, \omega \times S; F \overset{L}{\boxtimes} A_S),$$

are isomorphic to the stalk of $F \overset{L}{\boxtimes} A_S$ at (x_0, s_0), then $H' = 0$ at (x_0, s_0). This means that (2.5.2) is an isomorphism at (x_0, s_0) and to conclude it is then enough to compose the inverse of this morphism with the natural morphism $\Phi_{X\times S}(\gamma \times \{0\}, \omega \times S; \Phi_{X\times S}(\Gamma, \Omega; F \overset{L}{\boxtimes} A_S)) \to \Phi_{X\times S}(\Gamma, \Omega; F \overset{L}{\boxtimes} A_S)$.

Part (b) of Definition 1.4.2 is similarly proven. Q.E.D.

§2.6. Proof of the Theorems

We are now ready to prove the theorems stated in §2.2.

Proof of Theorem 2.2.4: Let us decompose f as:

$$\begin{array}{ccccc} Y & \overset{j}{\longrightarrow} & Y \times X & \overset{q}{\longrightarrow} & X \\ \uparrow & & \uparrow & & \uparrow \\ N & \overset{\sim}{\longrightarrow} & N & \longrightarrow & M, \end{array}$$

where j is the graph map, q denotes the second projection and we identified N and $j(N)$. We will divide the proof in several steps.

The first step will concern the map q for which we shall use Theorem 2.1.1. Remark that $f_{N\pi} = (j \times {}_M \mathrm{id}_{T^*_M X}) \circ q_{N\pi}$. Then one checks easily that the hypothesis (iii) of Theorem 2.2.4 implies the corresponding hypothesis:

(iii)' *there is an open neighborhood* W *of* (y_0, p_X) *in* $T^*_N(Y \times X)$ *such that* $q_{N\pi}$ *is non-characteristic for* $C_{T^*_M X}(\mathrm{SS}(F))$ *on* ${}^t q_N'^{-1}(W)$.

Here y_0 is the projection of p_Y on Y.

Since q is smooth the hypotheses of Theorem 2.1.1 are all satisfied. Applying this theorem we get:

$$\mu_N(q^! F)_{(y_0, p_X)} \xrightarrow{\sim} \mathrm{R}^t q'_{N*}\, q^!_{N\pi}\, \mu_M(F)_{(y_0, p_X)}.$$

Moreover, since ${}^t q'_N$ is a closed embedding one has the isomorphisms:

$$\begin{aligned} \mathrm{R}^t q'_{N*}\, q^!_{N\pi}\, \mu_M(F)_{(y_0, p_X)} &\cong (q^!_{N\pi}\, \mu_M(F))_{(y_0, p_X)} \\ &\cong \mu_M(F)_{p_X} \otimes \omega_{N/M}. \end{aligned}$$

One then gets:

$$\mu_N(q^! F)_{(y_0, p_X)} \xrightarrow{\sim} \mu_M(F)_{p_X} \otimes \omega_{N/M}. \tag{2.6.1}$$

As for the second step let us apply Proposition 2.3.3 to the closed embedding j. We get the isomorphism:

$$\mu_N(j^!_{\mu, \hat{p}}\, q^! F)_{p_Y} \xrightarrow{\sim} \mu_N(q^! F)_{(y_0, p_X)}, \tag{2.6.2}$$

where $\hat{p} = (y_0, {}^t q'((y_0, f(y_0)), p_X))$. Notice that in $Y \times_{(Y \times X)} (T^* Y \times T^* X)$, $\hat{p}$ is written as $\hat{p} = (y_0, (y_0, p_X))$. Finally remark that

$$f^!_{\mu, p} F \cong j^!_{\mu, \hat{p}}\, q^! F \tag{2.6.3}$$

due to Proposition 2.5.4. By combining (2.6.1), (2.6.2) and (2.6.3) the proof is complete. Q.E.D.

Proof of Theorem 2.2.3: Decompose the map $\tilde{f} = f \times f$ as follows:

$$\begin{array}{ccccc} Y \times Y & \xrightarrow{{}^2 f} & Y \times X & \xrightarrow{{}^1 f} & X \times X \\ \uparrow & & \uparrow & & \uparrow \\ \varDelta_Y & \longrightarrow & \varDelta & \longrightarrow & \varDelta_X, \end{array}$$

where ${}^2 f = id_Y \times f$, ${}^1 f = f \times id_X$, $\varDelta_Y$ is the diagonal of $Y \times Y$, and $\varDelta = {}^2 f(\varDelta_Y)$. One has the chain of isomorphisms:

$$\mu\mathrm{hom}\,(f^{-1}_{\mu, p} K, f^!_{\mu, p} F)_{(p_Y, p_Y)}$$

$$
\begin{aligned}
&= (\mu_{\Delta_Y} \mathrm{R}\mathscr{H}om(q_2^{-1} f_{\mu,p}^{-1} K, q_1^! f_{\mu,p}^! F))_{(p_Y, p_Y)} \\
&\cong (\mu_{\Delta_Y} {}^2 f^!_{\mu,(p_Y,p)} \mathrm{R}\mathscr{H}om(q_2^{-1} K, q_1^! f_{\mu,p}^! F))_{(p_Y, p_Y)} \\
&\cong (\mu_\Delta \mathrm{R}\mathscr{H}om(q_2^{-1} K, q_1^! f_{\mu,p}^! F))_{(p_Y, p_X)} \otimes \omega_{\Delta/\Delta_X} \\
&\cong (\mu_\Delta {}^1 f^!_{\mu,(p,p_X)} \mathrm{R}\mathscr{H}om(q_2^{-1} K, q_1^! F))_{(p_Y, p_X)} \otimes \omega_{\Delta/\Delta_X} \\
&\cong (\mu_{\Delta_X} \mathrm{R}\mathscr{H}om(q_2^{-1} K, q_1^! F))_{(p_X, p_X)} \otimes \omega_{\Delta_Y/\Delta_X} \\
&= \mu\mathrm{hom}(K, F)_{p_X} \otimes \omega_{Y/X}.
\end{aligned}
$$

Here q_1 and q_2 denote the projections from $Y \times Y$, $Y \times X$ or $X \times X$ to the corresponding factor, the meaning being clear from the context. The second and the forth isomorphisms follow from Proposition 2.5.2 applied to ${}^2f$ and ${}^1f$ respectively. The third and the fifth one follow from Theorem 2.2.4. Q.E.D.

§3. The Inverse Image Theorem for Sheaves

In [D'A-S] is given a theorem on the well poseness for the Cauchy problem in a sheaf theoretical frame that allows to recover classical results as those of [H-L-W], [K-S 1] or [Sc].

In the statement of this theorem, among the others, there are some hypotheses concerning microlocal inverse images. When dealing with micro-local images there are two ways that may be taken: to work with ind-objects and pro-objects or else to restrict the attention to a class of complexes with prescribed conditions on the micro-support. The first choice is the one of §2.2, while the second is the one of [D'A-S]. Using the results of section 2, we are then able to state here a sharper result than that of [D'A-S] that will allow us to recover also the result of [K-S 2] on the hyperbolic Cauchy problem.

§3.1. Cauchy Problem in Sheaf Theory

Let X be a manifold. We say that $K \in \mathrm{Ob}(\mathrm{D}^b(X))$ is *weakly cohomologically constructible* (w-c-c for short), if the following conditions are satisfied:

(i) For any $x \in X$, $\text{“}\varinjlim_{U \ni x}\text{”} \mathrm{R}\Gamma(U; F)$ is represented by F_x,

(ii) For any $x \in X$, $\text{“}\varprojlim_{U \ni x}\text{”} \mathrm{R}\Gamma_c(U; F)$ is represented by $\mathrm{R}\Gamma_{\{x\}} F$.

Here U ranges over an open neighborhood system of x.

In particular, weakly **R**-constructible complexes on a real analytic manifold are w-c-c (cf. [K-S 3, §8.4]).

Let $f: Y \to X$ be a morphism of manifolds. Let Z be a subset of Y (e.g. $Z = \{y\}$ for $y \in Y$).

Theorem 3.1.1. *Let F and K be objects of* $\mathrm{D}^b(X)$, *let L be an object of* $\mathrm{D}^b(Y)$. *Assume to be given a morphism* $\psi: L \to f^{-1}K$. *Let V be an open* *neighborhood of* $\dot{\pi}_Y^{-1}(Z)$. *Assume that*:

(i) f *is non-characteristic for* F *on* V,

(ii) f_π *is non-characteristic for* $C(\mathrm{SS}(F), \mathrm{SS}(K))$ *on* ${}^tf'^{-1}(V)$.

Assume that for every $p_Y \in \dot{\pi}_Y^{-1}(Z)$ *there exist* $p_1, \dots, p_r$ *in* ${}^tf'^{-1}(p_Y)$ *with*:

(iii) ${}^tf'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(F)) \subset \{p_1, \dots, p_r\}$,

(iv) $f_{\mu, p_j}^{-1} K$ *is representable for* $j = 1, \dots, r$,

(v) *the morphism induced by* ψ, $L \to f_{\mu, p_j}^{-1} K$, *is an isomorphism in* $\mathrm{D}^b(Y; p_Y)$ *for* $j = 1, \dots, r$.

Finally assume:

(vi) K *and* L *are w-c-c*,

(vii) *the morphism induced by* ψ, $\mathrm{R}\Gamma_{\{y\}}(L \otimes \omega_Y) \to \mathrm{R}\Gamma_{\{x\}}(K \otimes \omega_X)$, *is an isomorphism for every* $y \in Z$, $x = f(y)$.

Then the natural morphism induced by ψ:

$$f^{-1} \mathrm{R}\mathscr{H}om(K, F)|_Z \longrightarrow \mathrm{R}\mathscr{H}om(L, f^{-1}F)|_Z, \tag{3.1.1}$$

is an isomorphism.

The only difference between this statement and that of Theorem 2.1.1 of [D'A-S] is the hypothesis (iv) which is actually weakened.

Proof. One has a morphism induced by ψ:

$$\mathrm{R}{}^tf'_! f_\pi^{-1} \mu\mathrm{hom}(K, F) \longrightarrow \mu\mathrm{hom}(L, f^{-1}F).$$

As in [D'A-S], following an idea of [K-S 1], we consider the commutative diagram:

$$\begin{array}{ccccccc} \mathrm{R}\pi_{Y!}A & \longrightarrow & \mathrm{R}\pi_{Y*}A & \longrightarrow & \mathrm{R}\dot{\pi}_{Y*}A & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{R}\pi_{Y!}B & \longrightarrow & \mathrm{R}\pi_{Y*}B & \longrightarrow & \mathrm{R}\dot{\pi}_{Y*}B & \xrightarrow{+1} & , \end{array}$$

where $A = \mathrm{R}{}^tf'_! f_\pi^{-1} \mu\mathrm{hom}(K, F)$ and $B = \mu\mathrm{hom}(L, f^{-1}F)$.

Due to (1.2.1), we are easily reduced to prove that the first and the third vertical arrows are isomorphisms on Z.

The proof of the first vertical arrow being an isomorphism follows from hypotheses (vi) and (vii) and is given in [D'A-S].

Let us consider the third vertical arrow.

We have to prove that the natural morphism:

$$\mathrm{R}{}^tf'_! f_\pi^{-1} \mu\mathrm{hom}(K, F)_{p_Y} \longrightarrow \mu\mathrm{hom}(L, f^{-1}F)_{p_Y},$$

is an isomorphism for every $p_Y \in \dot{\pi}_Y(Z)$. Due to the assumption (iii) we can find refined cutting pairs (γ_j, ω_j) on X at $p_{X,j}$ (where $p_{X,j} = f_\pi(p_j)$) such that:

$$f_\pi^{-1}\,\mathrm{SS}(\Psi_X(\gamma_j,\omega_j;F))\cap {}^t f'^{-1}(p_Y)\subset\{p_j\}.$$

Of course, $\Psi_X(\gamma_j, \omega_j; F)$ is isomorphic to F in $\mathrm{D}^b(X; p_{X,j})$, and hence, due to Proposition 2.3.2:

$$f^!_{\mu,p_j}F=f^!\Psi_X(\gamma_j,\omega_j;F).$$

Set $F_j=\Psi_X(\gamma_j,\omega_j;F)$. One has the isomorphism $F\cong\oplus_j F_j$ in $\mathrm{D}^b(X;f_\pi{}^t f'^{-1}(p_Y))$. Since f is non-characteristic for F one also has the isomorphism $f^{-1}F\cong\oplus_j f^{-1}F_j$ in $\mathrm{D}^b(Y;p_Y)$ and hence we get the following chain of isomorphisms:

$$\begin{aligned}
\mathrm{R}{}^t f'_! f_\pi^{-1}\,\mu\mathrm{hom}(K,F)_{p_Y} &\cong(\mathrm{R}{}^t f'_! f_\pi^{-1}\oplus_{j=1}^r\mu\mathrm{hom}(K,F_j))_{p_Y}\\
&\cong\oplus_{j=1}^r(f_\pi^{-1}\,\mu\mathrm{hom}(K,F_j))_{p_j}\\
&\cong\oplus_{j=1}^r\mu\mathrm{hom}(K,F_j)_{p_X}\\
&\cong\oplus_{j=1}^r\mu\mathrm{hom}(f^{-1}_{\mu,p_j}K,f^!_{\mu,p_j}F_j)_{p_Y}\otimes\omega_{Y|X}^{\otimes-1}\\
&\cong\oplus_{j=1}^r\mu\mathrm{hom}(f^{-1}_{\mu,p_j}K,f^{-1}_{\mu,p_j}F_j)_{p_Y}\\
&\cong\oplus_{j=1}^r\mu\mathrm{hom}(L,f^{-1}F_j)_{p_Y}\\
&\cong\mu\mathrm{hom}(L,f^{-1}F)_{p_Y}.
\end{aligned}$$

Here the first isomorphism is due to the fact that f is non-characteristic for F and that μhom is a microlocal functor, the fourth to Theorem 2.2.3 and assumptions (ii), (iv), the fifth to assumption (i) and the sixth to assumption (v).

Q.E.D.

§4. Applications to the Cauchy Problem

We said that Theorem 3.1.1 generalizes the corresponding result of [D'A-S]. As it was for [D'A-S], we are then able to recover (and even extend to general systems) the classical results of [H-L-W] (cf. also [K-S 1]) on the initial value problem for a linear partial differential operator when the data are ramified along the characteristic hypersurfaces as well as a result of [Sc] that shows how the holomorphic solution for the Cauchy problem can be expressed as a sum of functions which are holomorphic in domains whose boundary is given by the real characteristic hypersurfaces issued from the boundary of a strictly pseudoconvex domain where the data are defined.

Moreover we get the following results.

§4.1. Other Applications

a) Our aim here is to recover the results of [K-S 2] concerning hyperbolic systems (cf. [B-S] for the case of a single differential operator).

Let N and M be two real analytic manifolds, and let f be a real analytic map from N to M, which extends to a holomorphic map from Y to X. Here Y and X are complexifications of N and M respectively. Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module.

Definition 4.1.1. One says that $\mathcal{M}$ is hyperbolic with respect to f if the following conditions are satisfied.

(i) f is non-characteristic for $\mathcal{M}$,

(ii) ${}^t f'^{-1}(T_N^* Y) \cap f_\pi^{-1}(\mathrm{char}(\mathcal{M})) \subset f_\pi^{-1}(T_M^* X)$,

(iii) f_π is non-characteristic for $C(T_M^* X, \mathrm{char}(\mathcal{M}))$.

Recall that the sheaf of Sato's hyperfunctions on M is defined by $\mathcal{B}_M := \mathrm{R}\Gamma_M(\mathcal{O}_X) \otimes \omega_{M/X}^{\otimes -1}$. (Here $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X.)

We can now state the well-poseness for the hyperbolic Cauchy problem in the hyperfunction frame (cf. [K-S 2, Corollary 2.1.2]).

Proposition 4.1.2. *Let $\mathcal{M}$ be a hyperbolic system with respect to f. Then the natural morphism*:

$$f^{-1}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \longrightarrow \mathrm{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N),$$

is an isomorphism.

Proof. One has the isomorphisms:

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \cong \mathrm{R}\mathcal{H}om(\omega_{M/X}, \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))$$

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N) \cong \mathrm{R}\mathcal{H}om(\omega_{N/Y}, f^{-1}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))$$

(in the second isomorphism we used the hypothesis (i) of Definition 4.1.1 and the Cauchy-Kowalevski-Kashiwara's theorem). We then have to show that we can apply Theorem 3.1.1, for the choice $F = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$, $K = \omega_{M/X}$, $L = \omega_{N/Y}$. Hypotheses (i)–(iii) as well as (vi) are easily verified, hypotheses (iv) and (v) follow from the next Lemma 4.1.3, while hypothesis (vii) follows from Lemma 4.1.4. Q.E.D.

Lemma 4.1.3. *Let $p \in N \times_M T_M^* X \setminus T_Y^* X$, then $f_{\mu,p}^{-1}(A_M)$ is represented by A_N in $\mathrm{D}^b(Y; p_Y)$.*

Proof. We can choose a refined cutting pair (γ, ω) on X at p_X so that $f_\pi^{-1}(\gamma^{\circ a}) \cap T_Y^* X \subset \{0\}$. The map f is then non-characteristic for $\Phi_X(\gamma, \omega; A_M)$ and hence we have:

$$\begin{aligned} \mathrm{SS}(f^{-1}\Phi_X(\gamma, \omega; A_M)) &\subset {}^t f' f_\pi^{-1}(\mathrm{SS}(\Phi_X(\gamma, \omega; A_M))) \\ &\subset {}^t f' f_\pi^{-1}(T_M^* X) \subset T_N^* Y. \end{aligned}$$

Here the last inclusion follows from the fact that f is induced by a map from N

to M. Due to [K-S 3, Proposition 6.2.2] we then have the isomorphism at $p_Y: f^{-1}\Phi_X(\gamma, \omega; A_M) \cong M^{\cdot}_N$ for a complex of A-modules $M^{\cdot}$. Computing the fiber, we get the result. Q.E.D.

Lemma 4.1.4. *One has the isomorphism*: $R\Gamma_{\{y\}}(L \otimes \omega_Y) \cong R\Gamma_{\{x\}}(K \otimes \omega_X)$.

Proof. One has the isomorphisms: $R\Gamma_{\{y\}}(L \otimes \omega_Y) \cong R\Gamma_{\{y\}}\omega_N \cong A \cong R\Gamma_{\{x\}}\omega_M \cong R\Gamma_{\{x\}}(K \otimes \omega_X)$. Q.E.D.

Remark 4.1.5. It would be possible to treat micro-hyperbolic systems and recover Theorem 2.3.1 of [K-S 2] by exactly the same method. Details are left to the reader.

b) A similar result to that of [Sc] holds in the real case. Let N be a real analytic hypersurface of an open subset M of $\mathbf{R}^n$ and ω an open subset of N with smooth boundary. Let P be a linear differential operator with analytic coefficients for which N is hyperbolic. Assume P to have real characteristics with constant multiplicities transversal to $N \times_M T^*M$. Following the same line as above one can get a statement analogous the theorem of [Sc] mentioned at the beginning of §4 in the frame of hyperfunctions.

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