Publ. RIMS, Kyoto Univ.
27 (1991), 551–575

On Nagumo's H^s-Stability in Singular Perturbations

Dedicated to Professor Shigetake Matsuura on the sixtieth anniversary of his birthday

By

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§ 1. Introduction

In [5], Nagumo defined the H^s-stability in singular perturbations. Here $H^s=H^s(\mathbf{R}^{n-1}_{x'})$ is the global Sobolev space with the norm

$$\|u(x')\|_s=\left((2\pi)^{-n+1}\int|\hat{u}(\xi')|^2(1+|\xi'|^2)^s d\xi'\right)^{1/2}.$$

We shall generalize the notion of H^s-stability in some sense.

Let us consider the following linear partial differential operator with constant coefficients containing a small positive parameter ε $(0\leqq\varepsilon<1)$:

$$L_\varepsilon(D)=\varepsilon\cdot P_1(D)+P_2(D).$$

Denote by m the order of $P_1(D)$ with respect to D_1 and by m' that of $P_2(D)$. Put $m''=m-m'$ and assume that $m>m'>0$. Then the order of L_0 is less than that of L_ε for $\varepsilon\neq0$. Such an operator as L_ε is called a singularly perturbed operator.

We shall study the following so-called singulary perturbed Cauchy problem for $L_\varepsilon(D)$:

$$\text{(CP)}\qquad\begin{cases}L_\varepsilon(D)u(x)=f_\varepsilon(x), & \text{in }\ [0, T]\times\mathbf{R}^{n-1}_{x'};\\ D_1^{j-1}u(0,x')=\phi_{\varepsilon,j}(x'), & j=1,\cdots,m,\end{cases}$$

and the following so-called reduced Cauchy problem for (CP):

Communicated by S. Matsuura, March 12, 1990. Revised November 26, 1990.
1991 Mathematics Subject Classification: 35B25.
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(RCP)
$$\begin{cases} L_0(D)u(x) = f_0(x)\,, & \text{in} \quad [0,\, T]\times \mathbf{R}^{n-1}_{x'}\,; \\ D_1^{j-1}u(0,\, x') = \phi_{0,j}(x')\,, & j = 1, \cdots, m'\,. \end{cases}$$

The following assumption on P_1 and P_2 will be required.

Assumption 1. (A1): The symbols of $P_1(D)$ and $P_2(D)$ are represented as

$$P_1(\xi) = \textstyle\sum_{j=0}^{m} p_{1,j}(\xi')\xi_1^{m-j}\,,$$
$$P_2(\xi) = \textstyle\sum_{j=0}^{m'} p_{2,j}(\xi')\xi_1^{m'-j}\,,$$

where $p_{1,0}$ and $p_{2,0}$ are non-zero constants.
(A2): ($m''=2$ and $p_{2,0}/p_{1,0}$ is negative real number) or
($m''=1$ and the imaginary part of $p_{2,0}/p_{1,0}$ is non-positive).

The following assumption on the Cauchy data and on the solvability of (CP) and (RCP) will be required.

Assumption 2. There exist real numbers s and s' such that (CP) is uniquely solvable in $C([0, T]; H^s)$ and (RCP) is uniquely solvable in $C([0, T]; H^s)$ for the Cauchy data $\phi_{\varepsilon,j}(x')$ and $\phi_{0,j}(x')$ belong to $H^{s'}$ and $f_\varepsilon(x)$ and $f_0(x)$ belong to $C([0, T]; H^{s'})$.

Nagumo defined the H^s-stability of (CP) with respect to a particular solution u_0 of (RCP) in [5] as follows:

Definition 1. Let Assumption 2 be satisfied for $s'=s$.

The Cauchy problem (CP) is said to be *H^s-stable* in $0\leqq x_1\leqq T$ for $\varepsilon\downarrow 0$ with respect to a particular solution $u_0(x)$ of the reduced Cauchy problem (RCP) in $C^m([0, T]; H^s)$ if

(D1)
$$\sup_{0\leqq x_1\leqq T} \|u_\varepsilon(x_1, \cdot)-u_0(x_1, \cdot)\|_s \to 0$$

whenever $u_\varepsilon(x)$ are solutions of (CP) in $C^m([0, T]; H^s)$ satisfying the following three conditions:

(D2)
$$\sup_{0\leqq x_1\leqq T} \|f_\varepsilon(x_1, \cdot)-f_0(x_1, \cdot)\|_s \to 0;$$

(D3)
$$\|\phi_{\varepsilon,j}-\phi_{0,j}\|_s \to 0\,, \qquad j = 1, \cdots, m';$$

(D4)
$$\|\phi_{\varepsilon,j}(\cdot)-D_1^{j-1}u_0(0, \cdot)\|_s \to 0\,, \qquad j = m'+1, \cdots, m\,.$$

If $f_0(x)$ belongs to $C^{m-m'}([0, T]; H^{s'})$ then the initial values $D_1^{j-1}u_0(0, x')$, $j=m'+1, \cdots, m$ are uniquely determined and represented as a sum of derivatives of $f_0(x)$ and $\phi_{0,j}(x')$, $j=1, \cdots, m'$. When (D4) is required, then the Cauchy data $\phi_{\varepsilon,j}(x')$, $j=m'+1, \cdots, m$ are very restricted. For example, when $f_0=0$ and

$\phi_{0,j}=0$, $j=1, \cdots, m'$, (D4) implies that $\phi_{\varepsilon,j}\to 0$, $j=1, \cdots, m$. Hence another definition of the stability whose convergence on the Cauchy data $\phi_{\varepsilon,j}(x')$, $j=m'+1, \cdots, m$ are different from Nagumo's is needed.

Definition 2. Let Assumption 2 be satisfied.

The Cauchy problem (CP) is said to be (s, s')-*stable* in $0\leqq x_1\leqq T$ for $\varepsilon\downarrow 0$ with respect to a particular solution $u_0(x)$ of the reduced Cauchy problem (RCP) in $C^m([0, T]; H^{\max\{s,s'\}})$ if

(D1)
$$\sup_{0\leqq x_1\leqq T}||u_\varepsilon(x_1, \cdot)-u_0(x_1, \cdot)||_s\to 0,$$

whenever $u_\varepsilon(x)$ are solutions of (CP) in $C^m([0, T]; H^{\max\{s,s'\}})$ satisfying the following three conditions:

(D5)
$$\sup_{0\leqq x_1\leqq T}||f_\varepsilon(x_1, \cdot)-f_0(x_1, \cdot)||_{s'}\to 0;$$

(D6)
$$||\phi_{\varepsilon,j}-\phi_{0,j}||_{s'}\to 0, \qquad j=1, \cdots, m';$$

(D7): There exists a positive number M, which may depend on the choice of the initial data $\phi_{\varepsilon,j}$, $\phi_{0,j}$, and f_0 such that

$$||\phi_{\varepsilon,j}(\cdot)-D_1^{j-1}u_0(0, \cdot)||_{s'}\leqq M, \qquad j=m'+1, \cdots, m.$$

The Cauchy problem (CP) is said to be $(s, s'+0)$-*stable* in $0\leqq x_1\leqq T$ for $\varepsilon\downarrow 0$ with respect to a particular solution $u_0(x)$ of (RCP) in $C^m([0, T]; H^{\max\{s,s'\}})$ if (D1) whenever $u_\varepsilon(x)$ are solutions of (CP) in $C^m([0, T]; H^{\max\{s,s'\}})$ satisfying (D5), (D6), and

(D8): There exist positive numbers δ and M, which may depend on the choice of the initial data $\phi_{\varepsilon,j}$, $\phi_{0,j}$, and f_0 such that

$$||\phi_{\varepsilon,j}(\cdot)-D_1^{j-1}u_0(0, \cdot)||_{s'+\delta}\leqq M, \qquad j=m'+1, \cdots, m.$$

Remark. For every positive number δ, the (s, s')-stability implies the $(s, s'+0)$-stability, the $(s, s'+0)$-stability implies the $(s, s'+\delta)$-stability, and the (s, s')-stability implies the $(s-\delta, s')$-stability.

It will be shown that requiring (A2) is natural when we deal with the (s, s')-stability with respect to solutions of (RCP) for various Cauchy data. Following to the definition of the C-admissibility of (CP) with respect to (RCP) in [4], we shall define the $C([0, T]; H^s)$-admissibility of (CP) with respect to (RCP).

Definition 3. Let Assumption 2 be satisfied. The Cauchy problem (CP) is said to be $C([0, T]; H^s)$-*admissible* in $[0, T]\times R^{n-1}$ with the Cauchy data space $(H^{s'})^m$ with respect to (RCP) if for every Cauchy datum $(\psi_1, \cdots, \psi_m)\in(H^{s'})^m$, the solutions u_ε of (CP) with $\phi_{\varepsilon,j}=\psi_j$, $j=1,\cdots,m$ and $f_\varepsilon=0$ converge in $C([0, T]; H^s)$ to the solution u_0 of (RCP) with $\phi_{0,j}=\psi_j$, $j=1, \cdots, m'$ and $f_0=0$.

By looking into the proof of Theorem in [2] and § 2 and § 3 in [3], we can prove that (A2) remains a necessary condition for the $C([0, T]; H^s)$-admissibility with the Cauchy data space $(H^\infty)^m$ when P_1 and P_2 satisfy (A1). We do not give the proof in this paper.

In [5], Nagumo gave a necessary and sufficient condition for the H^s-stability for more general system in the form of inequalities which must be satisfied by the solutions of (CP) with the initial conditions:

$$D_1^{j-1}u(0, x') = \delta_{i,j}\cdot\delta(x')\,, \qquad i, j = 1, \cdots, m\,,$$

where $\delta_{i,j}$ is Kronecker's delta and $\delta(x')$ is the Dirac measure. We have succeeded in seeking a necessary and sufficient condition for the $(s, s'+0)$-stability but a necessary and sufficient condition for the (s, s')-stability is open. Our condition for the $(s, s'+0)$-stability which will be found in § 2 is Nagumo type. As a corollary, we can show that Nagumo's H^s-stability implies the $(s, s+0)$-stability. In [6], Kumano-go applied Nagumo's result to the following operator:

$$\varepsilon\cdot D_1^2+q\cdot D_1+Q(D')\,,$$

where q is a complex number and $Q(D')$ is a polynomial of D'. Kumano-go deduced conditions for the H^s-stability on the complex constant q and on the structure of the polynomial $Q(\xi')$. In § 3, we shall give another example for the H^s-stability.

Acknowledgement

The author expresses his deep gratitude to Professor Shigetake Matsuura for his encouragement and helpful comments.

§ 2. The $(s, s'+0)$-Stability

We shall use the notation and the result in Appendix. Denote the roots of $L_\varepsilon(\xi)=0$ with respect to ξ_1 by $\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$ and those of $L_0(\xi)=P_2(\xi)=0$ with respect to ξ_1 by $\sigma_j(\xi')$, $j=1, \cdots, m'$, respectively. It is well known that $\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$ are continuous in (ε, ξ') for $\varepsilon \neq 0$ and $\sigma_j(\xi')$, $j=1, \cdots, m'$ are continuous in ξ'. Put

$$b(\tau) = (\tau^{j-1}; j \downarrow 1, \cdots, m) \quad \text{and} \quad c_j = (\delta_{j,k}, k \downarrow 1, \cdots, m),$$

where $\delta_{j,k}$ is Kronecker's delta. Other notation can be found in Appendix. Denote by $Y_j(\varepsilon, x_1, \xi')$, $j=1, \cdots, m$ the fundamental solutions of the following ordinary differential equation with parameter (ε, ξ'):

$$L_\varepsilon(D_1, \xi')Y(\varepsilon, x_1, \xi') = 0$$

with initial conditions:

$$D_1^{k-1}Y(\varepsilon, 0, \xi') = \delta_{j,k}, \qquad j, k = 1, \cdots, m.$$

Then Cramer's formula implies that if $\tau_i \neq \tau_j$, $1 \leqq i < j \leqq m$ then

$$\begin{aligned} &Y_j(\varepsilon, x_1, \xi') \\ &= \textstyle\sum_{k=1}^m \exp i\tau_k x_1 \cdot \dfrac{\det(b(\tau_1), \cdots, b(\tau_{k-1}), c_j, b(\tau_{k+1}), \cdots, b(\tau_m))}{\det(b(\tau_1), \cdots, b(\tau_m))} \\ &= \dfrac{\det^t({}^t a(0), {}^t a(1), \cdots, {}^t a(j-2), {}^t e, {}^t a(j), \cdots, {}^t a(m-1))}{A(0, 1, \cdots, m-1)} \\ &= (-1)^{j-1} \cdot D(0, 1, \cdots, j-2, j, \cdots, m-1)(\tau_1, \cdots, \tau_m, x_1), \qquad j = 1, \cdots, m. \end{aligned}$$

But the last representations remain valid without any restriction on τ_j, $j=1, \cdots, m$. Denote by l the maximum of the polynomial orders of the coefficients $p_{1,j}(\xi')$, $j=0, \cdots, m$ in the symbol $P_1(\xi)$ and put

$$\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}.$$

Then we have the following theorem whose proof will be found at the end of this section.

Theorem 1. *Let Assumptions* 1 *and* 2 *be satisfied. Then the following four conditions are equivalent*:

(C1) *The Cauchy problem* (*CP*) *is* $(s, s'+0)$*-stable in* $[0, T]$ *for* $\varepsilon \downarrow 0$ *with respect to a particular solution* $u_0(x)$ *of* (*RCP*) *belonging to* $C^m([0, T]; H^{\max\{s,s'\}+l})$.

(C2) *The Cauchy problem* (*CP*) *is* $(s, s'+0)$*-stable in* $[0, T]$ *for* $\varepsilon \downarrow 0$ *with respect to every solution* $u_0(x)$ *of* (*RCP*) *belonging to* $C^m([0, T]; H^{\max\{s,s'\}+l})$.

(C3) *There exist positive numbers* ε_0 *and* C_0 *such that*

(E1) $$\sup_{0<\varepsilon\leqq\varepsilon_0,\ \xi'\in \mathbf{R}^{n-1}} \int_0^T \frac{1}{\varepsilon} \cdot |Y_m(\varepsilon, x_1, \xi')\langle\xi'\rangle^{s-s'}|\,dx_1 \leqq C_0,$$

(E2) $$\sup_{1\leqq j\leqq m',\ 0<\varepsilon\leqq\varepsilon_0,\ 0\leqq x_1\leqq T,\ \xi'\in \mathbf{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi')\langle\xi'\rangle^{s-s'}| \leqq C_0,$$

and for every positive number δ there exist positive numbers ε_δ and C_δ such that

(E3) $$\sup_{m'+1\leqq j\leqq m,\, 0<\varepsilon\leqq\varepsilon_\delta,\, 0\leqq x_1\leqq T,\, \xi'\in\mathbb{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi')\langle\xi'\rangle^{s-s'-\delta}| \leqq C_\delta .$$

(C4) *There exist positive numbers ε'_0, R_0, and C'_0 such that*

(E4) $$\sup_{0<\varepsilon\leqq\varepsilon'_0,\, R_0\leqq|\xi'|} \int_0^T \frac{1}{\varepsilon}\circ|Y_m(\varepsilon, x_1, \xi')\langle\xi'\rangle^{s-s'}|\,dx_1 \leqq C'_0,$$

(E5) $$\sup_{1\leqq j\leqq m',\, 0<\varepsilon\leqq\varepsilon'_0,\, 0\leqq x_1\leqq T,\, R_0\leqq|\xi'|} |Y_j(\varepsilon, x_1, \xi')\langle\xi'\rangle^{s-s'}| \leqq C'_0,$$

and for every positive number δ there exist positive numbers ε'_0, R_δ, and C' such that

(E6) $$\sup_{m'+1\leqq j\leqq m,\, 0<\varepsilon\leqq\varepsilon'_\delta,\, 0\leqq x_1\leqq T,\, R_\delta\leqq|\xi'|} |Y_j(\varepsilon, x_1, \xi')\langle\xi'\rangle^{s-s'-\delta}| \leqq C'_\delta .$$

Remark. Nagumo studied the H^s-stability in the following general situation:

$$L_\varepsilon = \sum_{j=0}^m L_j(\varepsilon, D')D_1^{m-j},$$

where the symbols $L_j(\varepsilon, \xi')$ are matrices of polynomials in ξ' with constant coefficients which depend continuously on the parameter $\varepsilon\geqq 0$. He proved the equivalence between the following two conditions:

(C5) The Cauchy problem (CP) is H^s-stable in $[0, T]$ for $\varepsilon\downarrow 0$ with respect to a particular solution $u_0(x)$ of (RCP) belonging to $C^m([0, T]; H^{s+l})$.

(C6) There exist positive numbers ε_0 and C_0 such that

(E7) $$\sup_{0<\varepsilon\leqq\varepsilon_0,\, \xi'\in\mathbb{R}^{n-1}} \int_0^T \frac{1}{\varepsilon}\circ|Y_m(\varepsilon, x_1, \xi')|\,dx_1 \leqq C_0;$$

(E8) $$\sup_{1\leqq j\leqq m,\, 0<\varepsilon\leqq\varepsilon_0,\, 0\leqq x_1\leqq T,\, \xi'\in\mathbb{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi')| \leqq C_0 .$$

Corollary 1. *Let Assumptions* 1 *and* 2 *be satisfied and $u_0(x)$ be a solution of (RCP) belonging to $C^m([0, T]; H^{s+l})$. If the Cauchy problem (CP) is H^s-stable in* $[0, T]$ *for $\varepsilon\downarrow 0$ with respect to a particular solution u_0, then the Cauchy problem (CP) is $(s, s+0)$-stable in* $[0, T]$ *for $\varepsilon\downarrow 0$ with respect to a particular solution u_0.*

Proof. Since Nagumo's theorem can be applied to our problem and obviously (E8) implies (E2) for $s=s'$ and (E3) for $s=s'$. Q.E.D.

To prove Theorem 1 we need several steps. For the solution u_0 of the reduced Cauchy problem (RCP), we shall consider the following singulary perturbed Cauchy problem:

$$\text{(CP1)} \qquad \begin{cases} L_\varepsilon(D)u(x) = f_\varepsilon(x)\,, & \text{in } [0, T]\times \boldsymbol{R}^{n-1}; \\ D_1^{j-1}u(0, x') = \phi_{\varepsilon,j}(x')\,, & j = 1, \cdots, m' \\ D_1^{j-1}u(0, x') = D_1^{j-1}u_0(0, x')\,, & j = m'+1, \cdots, m\,. \end{cases}$$

Here the initial values $D_1^{j-1}u(0, x')$, $j=m'+1, \cdots, m$ are fixed. The reduced Cauchy problem for (CP1) is (RCP). Denote by $u_{\varepsilon,1}(x)$ the solution of (CP1).

Lemma 1 (due to Nagumo). *Let* (*A*1) *and Assumption* 2 *be satisfied. Then the following two conditions are equivalent*:

(C7) *The Cauchy problem* (*CP*1) *is* (s, s')*-stable in* $[0, T]$ *for* $\varepsilon \downarrow 0$ *with respect to a particular solution* $u_0(x)$ *of* (*RCP*) *belonging to* $C^m([0, T]; H^{\max\{s,s'\}+l})$.

(C8) *There exist positive numbers* ε_0 *and* C_0 *such that*

$$\text{(E1)} \qquad \sup_{0<\varepsilon\leqq\varepsilon_0,\, \xi'\in\boldsymbol{R}^{n-1}} \int_0^T \frac{1}{\varepsilon}\cdot |Y_m(\varepsilon, x_1, \xi')\langle\xi'\rangle^{s-s'}|\,dx_1 \leqq C_0\,,$$

$$\text{(E2)} \qquad \sup_{1\leqq j\leqq m',\, 0<\varepsilon\leqq\varepsilon_0,\, 0\leqq x_1\leqq T,\, \xi'\in\boldsymbol{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi')\langle\xi'\rangle^{s-s'}| \leqq C_0\,.$$

Proof. First we shall show (C8) implies (C7). Put

$$v_\varepsilon(x) = u_{\varepsilon,1}(x) - u_0(x)\,,$$
$$g_\varepsilon(x) = L_0(D)u_0(x) - L_\varepsilon(D)u_0(x) + f_\varepsilon(x) - f_0(x)\,.$$

Denote by $\hat{u}(x_1, \xi')$ the Fourier transform of $u(x)$ with respect to x' and by $\mathcal{F}^{-1}_{\xi'\to x'}$ the inverse Fourier transformation. Then $v_\varepsilon(x)$ is given by

$$v_\varepsilon(x) = \mathcal{F}^{-1}_{\xi'\to x'}\big(\textstyle\sum_{j=1}^{m'} Y_j(\varepsilon, x_1, \xi')(\hat{\phi}_{\varepsilon,j}(\xi') - \hat{\phi}_{0,j}(\xi'))\big)$$
$$+\mathcal{F}^{-1}_{\xi'\to x'}\left(\int_0^{x_1} \frac{1}{p_{1,0}\cdot\varepsilon}\cdot Y_m(\varepsilon, x_1 - t, \xi')\hat{g}_\varepsilon(t, \xi')dt\right).$$

Since

$$|\hat{v}_\varepsilon(x_1, \xi')|\langle\xi'\rangle^s \leqq \textstyle\sum_{j=1}^{m'} |Y_j(\varepsilon, x_1, \xi')\langle\xi'\rangle^{s-s'}|\,|\hat{\phi}_{\varepsilon,j}(\xi') - \hat{\phi}_{0,j}(\xi')|\langle\xi'\rangle^{s'}$$
$$+\int_0^{x_1} \frac{1}{|p_{1,0}|\cdot\varepsilon}\cdot |Y_m(\varepsilon, x_1 - t, \xi')\langle\xi'\rangle^{s-s'}|\,|\hat{g}_\varepsilon(t, \xi')|\langle\xi'\rangle^{s'}dt,$$

it implies that

$$\|v_\varepsilon(x_1, \cdot)\|_s \leqq C_0\cdot\textstyle\sum_{j=1}^{m'}\|\phi_{\varepsilon,j} - \phi_{0,j}\|_{s'} + \dfrac{C_0}{|p_{1,0}|}\displaystyle\int_0^{x_1}\|g_\varepsilon(t, \cdot)\|_{s'}dt\,.$$

By (D6), we have $\sum_{j=1}^{m'}\|\phi_{\varepsilon,j} - \phi_{0,j}\|_{s'} \to 0$. Since u_0 belongs to $C^m([0, T]; H^{\max\{s,s'\}+l})$, it implies that

$$\sup_{0\leqq x_1\leqq T}||L_0(D)u_0(x_1,\ \cdot)-L_\varepsilon(D)u_0(x_1,\ \cdot)||_{s'}\to 0\ .$$

Hence (D5) implies that $\sup_{0\leqq x_1\leqq T}||g_\varepsilon(x_1,\ \cdot)||_{s'}\to 0$. Thus we have

$$\sup_{0\leqq x_1\leqq T}||v_\varepsilon(x_1,\ \cdot)||_s\to 0\ .$$

Next we shall show (C7) implies (C8). Assume that (E2) is not satisfied. Then, for a certain j with $1\leqq j\leqq m'$, there exist sequences $\{\varepsilon_n\}$ with $\varepsilon_n\downarrow 0$ and $\{t_n\}$ with $0\leqq t_n\leqq T$ and a sequence of open balls $\{S_n\}$, $S_n=\{|\xi'-\xi'_n|<r_n\}$ such that

$$|Y_j(\varepsilon_n,\ t_n,\ \xi')\langle\xi'\rangle^{s-s'}|>n \qquad \text{for} \quad \xi' \text{ in } S_n\ , \tag{2.1}$$

$$2^{-1}<(\langle\xi'\rangle/\langle\xi'_n\rangle)^{s'}<2 \qquad \text{for} \quad \xi' \text{ in } S_n\ . \tag{2.2}$$

Put

$$u_n(x)=c_n\cdot\mathcal{F}^{-1}_{\xi'\to x'}(Y_j(\varepsilon_n,\ x_1,\ \xi')\cdot\chi(\xi';\ S_n))\ ,$$

where $c_n=n^{-1}\cdot|S_n|^{-1/2}\langle\xi'_n\rangle^{-s'}$. Then $u_n(x)$ satisfies $L_{\varepsilon_n}(D)u(x)=0$. Since

$$\begin{aligned}|\hat{u}_n(t_n,\ \xi')|\langle\xi'\rangle^s&=n^{-1}\cdot|S_n|^{-1/2}\langle\xi'_n\rangle^{-s'}|Y_j(\varepsilon_n,\ t_n,\ \xi')|\chi(\xi';\ S_n)\langle\xi'\rangle^s\\&=n^{-1}\cdot|S_n|^{-1/2}(\langle\xi'\rangle/\langle\xi'_n\rangle)^{s'}|Y_j(\varepsilon_n,\ t_n,\ \xi')\langle\xi'\rangle^{s-s'}|\chi(\xi;\ S_n)\ ,\end{aligned}$$

(2.1) and (2.2) imply that

$$\sup_{0\leqq x_1\leqq T}||u_n(x_1,\ \cdot)_s\geqq||u_n(t_n,\ \cdot)||_s\geqq 1/2\ .$$

Since

$$\begin{aligned}|D_1^{j-1}\hat{u}_n(0,\ \xi')|\langle\xi'\rangle^{s'}&=c_n\cdot\chi(\xi';\ S_n)\langle\xi'\rangle^{s'}\\&=n^{-1}\cdot|S_n|^{-1/2}\cdot\chi(\xi';\ S_n)(\langle\xi'\rangle/\langle\xi'_n\rangle)^{s'}\ ,\end{aligned}$$

(2.2) implies that $||D_1^{j-1}u_n(0,\ \cdot)||_{s'}\leqq 2/n\to 0$. For $k\neq j$, we have $||D_1^{k-1}u_n(0,\ \cdot)||_{s'}=0$. Put $u_{\varepsilon_n}(x)=u_n(x)+u_0(x)$. Then we have a contradiction to (D1), (D5), (D6), and (D7).

Assume that (E1) is not satisfied. Then there exist a sequence $\{\varepsilon_n\}$ with $\varepsilon_n\downarrow 0$ and a sequence of open balls $\{S_n\}$, $S_n=\{\xi'\in\mathbb{R}^{n-1};\ |\xi'-\xi'_n|<r_n\}$ such that

$$\int_0^T\frac{1}{|p_{1,0}|\cdot\varepsilon_n}\cdot|Y_m(\varepsilon_n,\ T-x_1,\ \xi')\langle\xi'\rangle^{s-s'}|dx_1>n\ , \tag{2.3}$$

for ξ' in S_n. We choose $\phi_{\varepsilon,j}(x')=D_1^{j-1}u_0(0,\ x')$, $j=1,\cdots,m'$. Then the solutions of (CP1) for $\{\varepsilon_n\}$ are given by

$$u_n(x) = u_0(x) + \mathcal{F}^{-1}_{\xi' \to x'}\left(\int_0^{x_1} \frac{1}{p_{1,0} \circ \varepsilon_n} \cdot Y_m(\varepsilon_n, x_1 - t, \xi') \hat{g}_{\varepsilon_n}(t, \xi') dt\right).$$

Put

$$y_n(x_1, \xi') = \frac{1}{p_{1,0} \cdot \varepsilon_n} \circ Y_m(\varepsilon_n, T - x_1, \xi').$$

As we shall show later by (2.5) in the proof of Lemma 3 that $Y_m(\varepsilon, x_1, \xi')$ is continuous in (x_1, ξ') for fixed ε, it implies that $y_n(x_1, \xi')$ is continuous in (x_1, ξ') for every positive integer n. For $E = \{(x_1, \xi'); y_n(x_1, \xi') \neq 0\}$, denote by $\chi((x_1, \xi'); E)$ the characteristic function of the set E. Put

$$H_n(x_1, \xi') = \chi((x_1, \xi'); E) \cdot \overline{y_n(x_1, \xi')} / |y_n(x_1, \xi')| .$$

Then $|H_n(x_1, \xi')| \leqq 1$ and (2.3) implies

$$\left| \int_0^T y_n(x_1, \xi') \langle \xi' \rangle^{s-s'} H_n(x_1, \xi') dx_1 \right| > n ,$$

for ξ' in S_n. Approximate $H_n(x_1, \xi')$ in the sense of $L^1([0, T])$ valued in bounded functions in ξ' by the mollifier $\rho_\delta(x_1)*$ with respect to x_1. Put

$$h_{\delta,n}(x_1, \xi') = \int_R \rho_\delta(x_1 - t) H_n(t, \xi') dt .$$

Then $h_{\delta,n}(x_1, \xi')$ are continuous functions with respect to x_1 in $[0, T]$ satisfying $|h_{\delta,n}(x_1, \xi')| \leqq 1$. Since

$$\left| \left| \int_0^T y_n(x_1, \xi') \langle \xi' \rangle^{s-s'} H_n(x_1, \xi') dx \right| - \left| \int_0^T y_n(x_1, \xi') \langle \xi' \rangle^{s-s'} h_{\delta,n}(x_1, \xi') dx_1 \right| \right|$$
$$\leqq \sup_{0 \leqq x_1 \leqq T} |y_n(x_1, \xi')| \cdot \langle \xi' \rangle^{s-s'} \cdot \int_0^T |h_{\delta,n}(x_1, \xi') - H_n(x_1, \xi')| dx_1 ,$$

it implies that for ξ' in S_n there exist positive numbers $\delta_n(\xi')$ such that

$$\left| \int_0^T y_n(x_1, \xi') \langle \xi' \rangle^{s-s'} h_{\delta_n(\xi'),n}(x_1, \xi') dx_1 \right| > n ,$$

for ξ' in S_n. Put

$$h_n(x_1, \xi') = h_{\delta_n(\xi'),n}(x_1, \xi') ,$$
$$g_{\varepsilon_n}(x) = \mathcal{F}^{-1}_{\xi' \to x'}(n^{-1} |S_n|^{-1/2} h_n(x_1, \xi') \langle \xi' \rangle^{-s'} \chi(\xi'; S_n)) ,$$

where $|S_n|$ denotes the measure of S_n and $\chi(\xi'; S_n)$ is the characteristic function of the ball S_n. We set $f_{\varepsilon_n} = f_0 + g_{\varepsilon_n}$. Then

$$\|g_{\varepsilon_n}(x_1, \cdot)\|_{s'} \leqq \frac{1}{n} \to 0 .$$

Since

$$
\begin{aligned}
&(\hat{u}_n(T, \xi')-\hat{u}_0(T, \xi'))\langle\xi'\rangle^s \\
&= \int_0^T y_n(x_1, \xi')\langle\xi'\rangle^{s-s'} \cdot \hat{g}_{\varepsilon_n}(x_1, \xi')\langle\xi'\rangle^{s'} dx_1 \\
&= \int_0^T y_n(x_1, \xi')\langle\xi'\rangle^{s-s'} \circ h_n(x_1, \xi') dx_1 \cdot n^{-1} |S_n|^{-1/2} \chi(\xi'; S_n),
\end{aligned}
$$

it implies that $\|u_n(T, \cdot)-u_0(T, \cdot)\|_s \geqq 1$. This contradicts (D1), (D5), (D6), and (D7). Q.E.D.

Put

$$
\begin{aligned}
&B_R = \{|\xi'| \leqq R\},\ p = p_{2,0}/p_{1,0},\ \theta = \arg -p,\ \Theta = \exp i\theta/m'', \\
&\zeta = \exp 2\pi i/m'',\ \text{and}\ \tau'_j = \zeta^{j-m'-1},\ j=m'+1, \cdots, m.
\end{aligned}
$$

By the same argument as in Lemma 2.2 in [3], it implies the following lemma whose proof is omitted.

Lemma 2. *Let (A1) in Assumption 1 be satisfied. Then, for every positive number R, there exist a positive number ε_R with $\varepsilon_R<1$ and continuous functions $\tau_{j,1}(\varepsilon, \xi')$, $j=1, \cdots, m$ on $[0, \varepsilon_R]\times B_R$ satisfying*

$$
\lim_{\varepsilon \downarrow 0} \sup_{\xi' \in B_R} |\tau_{j,1}(\varepsilon, \xi')| = 0, \qquad \text{for} \quad j = 1, \cdots, m
$$

such that for $m'+1 \leqq i < j \leqq m$ and for $1 \leqq i \leqq m'$, $m'+1 \leqq j \leqq m$

$$
\tau_i(\varepsilon, \xi') \neq \tau_j(\varepsilon, \xi') \quad \text{on} \quad (0, \varepsilon_R]\times B_R,
$$

and

$$
\begin{aligned}
\tau_j(\varepsilon, \xi') &= \sigma_j(\xi')+\tau_{j,1}(\varepsilon, \xi'), && \text{for} \quad j = 1, \cdots, m'; \\
\varepsilon^{1/m''} \cdot \tau_j(\varepsilon, \xi') &= \Theta\tau'_j \cdot |p|^{1/m''}+\tau_{j,1}(\varepsilon, \xi'), && \text{for} \quad j = m'+1, \cdots, m.
\end{aligned}
$$

Lemma 3. *Let Assumption 1 be satisfied and ε_R be the same as in Lemma 2. For every positive number R, there exists a positive number $C_{1,R}$ such that*

$$
\sup_{0<\varepsilon\leqq\varepsilon_R,\ 0\leqq x_1\leqq T,\ |\xi'|\leqq R} \varepsilon^{-\max\{(j-m'),0\}/m''} |Y_j(\varepsilon, x_1, \xi')| \leqq C_{1,R}, \tag{2.4}
$$
$$
\text{for} \quad j = 1, \cdots, m.
$$

Proof. Fix an arbitrary positive number R and assume that $0<\varepsilon\leqq\varepsilon_R$. For arbitrary roots $\tau_j=\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$, which do not need to be distinct,

$$
Y_j(\varepsilon, x_1, \xi') = (-1)^{j-1} \cdot D(0, 1, \cdots, j-2, j, \cdots, m-1)(\tau_1, \cdots, \tau_m, x_1), \tag{2.5}
$$
$$
j = 1, \cdots, m.
$$

As we have already shown in Theorem in [2], (A2) in Assumption 1 implies that the imaginary parts of $\Theta\tau'_j$, $j=m'+1, \cdots, m$ are non-negative. Put $\eta=\varepsilon^{1/m''}$, $\eta_R=\varepsilon_R^{1/m''}$, $z_j=\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$, and $w_j=\varepsilon^{1/m''}\cdot\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$. Then Assumption 1 implies that for every positive number R, there exist positive numbers M_R, M'_R and c_R such that (A.8) in Lemma A.3 in Appendix is satisfied for $M=M_R$, $M'=M'_R$, $c=c_R$, and $\eta_0=\eta_R$. Hence Lemma A.3 can be applied to (2.5). Since $D(\rho(1), \rho(2), \cdots, \rho(m'-1))(z', x_1)$, ρ in $\mathcal{S}_2$ are entire in z' and continuous in x_1 for $0\leqq x_1\leqq T$, it implies that there exists a positive number $C_{2,R}$ such that

$$\max_{\rho\in\mathcal{S}_2}|D(\rho(1), \rho(2), \cdots, \rho(m'-1))(\tau_1, \cdots, \tau_{m'}, x_1)|\leqq C_{2,R},$$

on $[0, \varepsilon_R]\times[0, T]\times B_R$. Since $E(w)$ is holomorphic for $w_i\neq w_j$, $1\leqq i\leqq m'$ and $m'+1\leqq j\leqq m$, Lemma 2 implies that there exists a positive number $C_{3,R}$ such that for $j=1, \cdots, m'$

$$\begin{aligned}&|D(0, 1, \cdots, j-2, j, \cdots, m'-1)(\tau_1, \cdots, \tau_{m'}, x_1)\\ &\quad\times((\varepsilon^{1/m''}\cdot\tau_{m'+1})\cdot\cdots\cdot(\varepsilon^{1/m''}\cdot\tau_m)^{m'}\cdot E(\varepsilon^{1/m''}\cdot\tau_1, \cdots, \varepsilon^{1/m''}\cdot\tau_m)|\leqq C_{3,R},\end{aligned}$$

on $[0, \varepsilon_R]\times[0, T]\times B_R$. Then

$$|D(0, 1, \cdots, j-2, j, \cdots, m-1)(\tau_1, \cdots, \tau_m, x_1)|\leqq C_{3,R}+(C_1+C_2\cdot C_{2,R})\cdot\varepsilon^{1/m''},$$

for $j=1, \cdots, m'$ and

$$|D(0, 1, \cdots, j-2, j, \cdots, m-1)(\tau_1, \cdots, \tau_m, x_1)|\leqq(C_1+C_2\cdot C_{2,R})\cdot\varepsilon^{(j-m')/m''},$$

for $j=m'+1, \cdots, m$. Put $C_{1,R}=C_{3,R}+C_1+C_2\cdot C_{2,R}$, then we have (2.4).

Q.E.D.

Denote by $y_j(x_1, \xi')$, $j=1, \cdots, m'$ the fundamental solutions of the following ordinary differential equation with parameter ξ':

$$L_0(D_1, \xi')y(x_1, \xi')=0$$

with initial conditions:

$$D_1^{k-1}y(0, \xi')=\delta_{j,k}, \qquad j, k=1, \cdots, m',$$

where $\delta_{j,k}$ is Kronecker's delta. As we have already shown

$$y_j(x_1, \xi')=(-1)^{j-1}\cdot D(0, 1, \cdots, j-2, j, \cdots, m'-1)(\sigma_1, \cdots, \sigma_{m'}, x_1), \quad j=1, \cdots, m', \tag{2.6}$$

where $\sigma_j=\sigma_j(\xi')$, $j=1, \cdots, m'$ are roots appearing in Lemma 2.

Lemma 4. *Let Assumption* 1 *be satisfied and* ε_R *be the same as in Lemma* 2. *Then*

$$Y_j(\varepsilon, x_1, \xi') \to y_j(x_1, \xi'), \qquad j=1, \cdots, m'; \tag{2.7}$$

$$Y_j(\varepsilon, x_1, \xi') \to 0, \qquad j=m'+1, \cdots, m, \tag{2.8}$$

uniformly on $[0, T]\times B_R$ *when* $\varepsilon \downarrow 0$.

Moreover, $Y_j(\varepsilon, x_1, \xi'), j=1, \cdots, m$ *satisfy*

$$\sup_{1\leqq j\leqq m,\ 0<\varepsilon\leqq\varepsilon_0,\ 0\leqq x_1\leqq T,\ \xi'\in\mathbf{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi')| \leqq C_0 \tag{E8}$$

then $y_j(x_1, \xi'), j=1, \cdots, m'$ *satisfy*

$$\sup_{1\leqq j\leqq m',\ 0\leqq x_1\leqq T,\ \xi'\in\mathbf{R}^{n-1}} |y_j(x_1, \xi')| \leqq C_0. \tag{E9}$$

Proof. By Lemma 3, (2.8) is obvious and it suffices to show that for $j=1, \cdots, m'$

$$(-1)^{j-1}\circ D(0, 1, \cdots, j-2, j, \cdots, m'-1)(\tau_1, \cdots, \tau_{m'}, x_1)$$
$$\times((\varepsilon^{1/m''}\circ\tau_{m'+1})\circ\cdots\circ(\varepsilon^{1/m''}\circ\tau_m))^{m'}\circ E(\varepsilon^{1/m''}\circ\tau_1, \cdots, \varepsilon^{1/m''}\circ\tau_m) \to y_j(x_1, \xi').$$

Since $\tau_j(\varepsilon, \xi')\to\sigma_j(\xi')$, $j=1, \cdots, m'$ uniformly on B_R when $\varepsilon \downarrow 0$ by Lemma 2, it implies that for $j=1, \cdots, m'$

$$(-1)^{j-1}\circ D(0, 1, \cdots, j-2, j, \cdots, m'-1)(\tau_1, \cdots, \tau_{m'}, x_1) \to y_j(x_1, \xi').$$

On the other hand,

$$((\varepsilon^{1/m''}\circ\tau_{m'+1})\circ\cdots\circ(\varepsilon^{1/m''}\circ\tau_m))^{m'} \to ((\Theta\circ\tau'_{m'+1}\circ|p|^{1/m''})\circ\cdots\circ(\Theta\circ\tau'_m\circ|p|^{1/m''}))^{m'}$$

and

$$\begin{aligned}&E(\varepsilon^{1/m''}\circ\tau_1, \cdots, \varepsilon^{1/m''}\circ\tau_m)\\ &\to E(0, \cdots, 0, (\Theta\circ\tau'_{m'+1}\circ|p|^{1/m''}), \cdots, (\Theta\circ\tau'_m\circ|p|^{1/m''}))\\ &= 1/((\Theta\circ\tau'_{m'+1}\circ|p|^{1/m''})\circ\cdots\circ(\Theta\circ\tau'_m\circ|p|^{1/m''}))^{m'}.\end{aligned}$$

Thus we have (2.7).

Since R is arbitrary, (2.7) and (E8) imply (E9). Q.E.D.

Let us consider the following singulary perturbed Cauchy problem:

$$\begin{cases} L_\varepsilon(D)u(x)=0, & \text{in } [0, T]\times\mathbf{R}^{n-1}_{x'};\\ D_1^{j-1}u(0, x')=0, & j=1, \cdots, m'\\ D_1^{j-1}u(0, x')=\phi_{\varepsilon,j}(x'), & j=m'+1, \cdots, m,\end{cases} \tag{CP2}$$

and its reduced Cauchy problem:

$$\text{(RCP2)} \qquad \begin{cases} L_0(D)u(x)=0\,, & \text{in} \quad [0, T]\times \boldsymbol{R}_{x'}^{n-1}\,; \\ D_1^{j-1}u(0, x')=0\,, & j=1, \cdots, m'\,. \end{cases}$$

Denote by $u_{\varepsilon,2}(x)$ the solution of (CP2) and by $u_{0,2}(x)$ the solution of (RCP2). Then $u_{0,2}(x)=0$.

Lemma 5. *Let Assumption* 1 *be satisfied and* ε_R *be the same as in Lemma* 2. *Assume that every support of the datum* $\hat{\phi}_{\varepsilon,j}(\xi')$, $j=m'+1, \cdots, m$ *in* (*CP*2) *is contained in the closed ball* B_R. *Then, for arbitrary real numbers s and s′ there exists a positive number* K_R *which is independent of* ε *such that for* $0<\varepsilon\leqq\varepsilon_R$,

$$\text{(2.9)} \qquad \sup_{0\leqq x_1\leqq T} \|u_{\varepsilon,2}(x_1, \cdot)\|_s \leqq K_R\cdot\sum_{l=m'+1}^{m} \varepsilon^{(l-m')/m''}\cdot\|\phi_{\varepsilon,l}\|_{s'}\,.$$

Remark. Here we do not use any conditions on the fundamental solutions Y_j but use (A2) in Assumption 1. Lemma 4 shows that (A2) ensures the boundedness of Y_j on $[0, T]\times B_R$ when $\varepsilon\downarrow 0$.

Proof of Lemma 5. It is well known that the solution $u_{\varepsilon,2}(x)$ of (CP2) satisfies

$$\hat{u}_{\varepsilon,2}(x_1, \xi')=\sum_{j=m'+1}^{m} Y_j(\varepsilon, x_1, \xi')\cdot\hat{\phi}_{\varepsilon,j}(\xi')\,.$$

Lemma 3 implies

$$|\hat{u}_{\varepsilon,2}(x_1, \xi')|\leqq C_{1,R}\cdot\sum_{l=m'+1}^{m} \varepsilon^{(l-m')/m''}\cdot|\hat{\phi}_{\varepsilon,l}|\,,$$

on $[0, T]\times B_R$. Thus

$$(2\pi)^{-n+1}\int_{|\xi'|\leqq R} |\hat{u}_{\varepsilon,2}(x_1, \xi')\langle\xi'\rangle^s|^2 d\xi'$$
$$\leqq C_{1,R}^{\,2}\cdot m''\cdot\sum_{l=m'+1}^{m} (2\pi)^{-n+1}\int_{|\xi'|\leqq R} |\varepsilon^{(l-m')/m''}\cdot\hat{\phi}_{\varepsilon,l}(\xi')\langle\xi'\rangle^s|^2 d\xi'\,.$$

Put $K_R=C_{1,R}\cdot m''^{1/2}\cdot\sup_{|\xi'|\leqq R}\langle\xi'\rangle^{s-s'}$. Then we have (2.9). Q.E.D.

The following corollary shows us that the stability is very strong when the Cauchy problem is amdissible.

Corollary 2. *Let Assumption* 1 *be satisfied and* ε_R *be the same as in Lemma* 2. *Then, for every positive number* ε *with* $\varepsilon\leqq\varepsilon_R$, *there exist Cauchy data* $\phi_{\varepsilon,j}$, $j=m'+1, \cdots, m$ *belonging to* H^∞ *such that for arbitrary real numbers s and s′,*

$$||\phi_{\varepsilon,j}||_{s'} \to \infty\,, \qquad j = m'+1, \cdots, m;$$

$$\sup_{0 \leqq x_1 \leqq T} ||u_{\varepsilon,2}(x_1, \cdot)||_s \to 0\,,$$

where $u_{\varepsilon,2}$ are the solutions of (CP2) for these data $\phi_{\varepsilon,j}$, $j=m'+1, \cdots, m$.

Proof. Choose non-trivial $C_0^\infty(B_R)$-functions $\psi_j(\xi')$, $j=m'+1, \cdots, m$ and a positive number α with $\alpha<1/m''$. Put

$$\phi_{\varepsilon,j}(x') = \varepsilon^{-\alpha} \cdot \mathcal{F}^{-1}_{\xi' \to x'}(\psi_j(\xi'))\,, \qquad j = m'+1, \cdots, m\,,$$

which are rapidly decreasing functions. If $s'<0$, then

$$||\phi_{\varepsilon,j}||_{s'} \geqq \varepsilon^{-\alpha} \cdot \langle R \rangle^{s'} ||\mathcal{F}^{-1}(\psi_j)||_0 \uparrow \infty\,,$$

when $\varepsilon \downarrow 0$. If $s'>0$, then

$$||\phi_{\varepsilon,j}||_{s'} \geqq ||\phi_{\varepsilon,j}||_{-s'} \geqq \varepsilon^{-\alpha} \cdot \langle R \rangle^{-s'} ||\mathcal{F}^{-1}(\psi_j)||_0 \uparrow \infty\,,$$

when $\varepsilon \downarrow 0$. By (2.9),

$$\sup_{0 \leqq x_1 \leqq T} ||u_{\varepsilon,2}(x_1, \cdot)||_s \leqq \varepsilon^{1/m''-\alpha} \cdot K_R \cdot \sum_{j=m'+1}^m ||\mathcal{F}^{-1}(\psi_j)||_{s'} \downarrow 0\,,$$

when $\varepsilon \downarrow 0$. Q.E.D.

Lemma 6. *Let the same assumption as in Theorem 1 be satisfied. Consider the singulary perturbed Cauchy problem (CP2) and the reduced Cauchy problem (RCP2) for (CP2). Assume that for the Cauchy data $\phi_{\varepsilon,j}$, $j=1, \cdots, m$ there exist positive numbers δ and M such that $\sup_{1 \leqq j \leqq m} ||\phi_{\varepsilon,j}||_{s'+\delta} \leqq M$. Then the following two conditions are equivalent:*

(C9) The Cauchy problem (CP2) is $(s, s'+0)$-stable in $[0, T]$ for $\varepsilon \downarrow 0$ with respect to a particular solution $u_{0,2}=0$ of (RCP2).

(C10) For every positive number δ there exist positive numbers ε_δ and C_δ such that

$$\text{(E3)} \qquad \sup_{m'+1 \leqq j \leqq m,\, 0<\varepsilon \leqq \varepsilon_\delta,\, 0 \leqq x_1 \leqq T,\, \xi' \in \mathbb{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi') \langle \xi' \rangle^{s-s'-\delta}| \leqq C_\delta\,.$$

Proof. First we shall show (C10) implies (C9). We have only to show that if $\sup_{1 \leqq j \leqq m} ||\phi_{\varepsilon,j}||_{s'+\delta} \leqq M$ then $\sup_{0 \leqq x_1 \leqq T} ||u_{\varepsilon,2}(x_1, \cdot)||_s \to 0$. As we have already shown in the proof of Lemma 1, the solution $u_{\varepsilon,2}(x)$ of (CP2) satisfies

$$\hat{u}_{\varepsilon,2}(x_1, \xi') = \sum_{j=m'+1}^m Y_j(\varepsilon, x_1, \xi') \cdot \hat{\phi}_{\varepsilon,j}(\xi')\,.$$

Denote by $\chi(\xi'; B_R)$ the characteristic function of the ball B_R. Put

$$\hat{v}_{\varepsilon,2}(x_1, \xi') = \hat{u}_{\varepsilon,2}(x_1, \xi')\cdot\chi(\xi'; B_R),$$
$$\hat{w}_{\varepsilon,2}(x_1, \xi') = \hat{u}_{\varepsilon,2}(x_1, \xi')\cdot(1-\chi(\xi'; B_R)).$$

Then $v_{\varepsilon,2}(x)=\mathcal{F}^{-1}_{\xi'\to x'}(\hat{v}_{\varepsilon,2}(x_1, \xi'))$ is the solution of (CP2) with the initial conditions:

$$D_1^{j-1}u(0, x') = 0, \qquad j = 1, \cdots, m';$$
$$D_1^{j-1}u(0, x') = \mathcal{F}^{-1}_{\xi'\to x'}(\hat{\phi}_{\varepsilon,j}(\xi')\cdot\chi(\xi'; B_R)), \qquad j = m'+1, \cdots, m.$$

Since the supports of the Fourier transforms of these Cauchy data are contained in the ball B_R, we can apply Lemma 5 to $v_{\varepsilon,2}(x)$. Obviously

$$||\mathcal{F}^{-1}_{\xi'\to x'}(\hat{\phi}_{\varepsilon,l}(\xi')\cdot\chi(\xi'; B_R))||_{s'}\leqq||\phi_{\varepsilon,l}||_{s'},$$

(2.9) and $0<\varepsilon\leqq\varepsilon_R<1$ imply that

$$\text{(2.10)}\quad \sup_{0\leqq x_1\leqq T}||v_{\varepsilon,2}(x_1,\cdot)||_s\leqq K_R\cdot\varepsilon^{1/m''}\cdot\sum_{l=m'+1}^{m}||\mathcal{F}^{-1}_{\xi'\to x'}(\hat{\phi}_{\varepsilon,l}(\xi')\cdot\chi(\xi'; B_R))||_{s'}$$
$$\leqq K_R\cdot\varepsilon^{1/m''}\cdot\sum_{l=m'+1}^{m}||\phi_{\varepsilon,l}||_{s'}.$$

Choose a positive number δ' satisfying $\delta'<\delta$ and put $\delta''=\delta-\delta'$. Since

$$|\hat{w}_{\varepsilon,2}(x_1, \xi')\cdot\langle\xi'\rangle^s|\leqq\sum_{j=m'+1}^{m}|Y_j(\varepsilon, x_1, \xi')\cdot\langle\xi\rangle^{s-s'-\delta'}|$$
$$\cdot|\hat{\phi}_{\varepsilon,j}(\xi')\cdot\langle\xi'\rangle^{s'+\delta}|\cdot|1-\chi(\xi'; B_R)|\cdot\langle\xi'\rangle^{-\delta''},$$

the estimate (E3) for $\delta=\delta'$ implies that

$$|\hat{w}_{\varepsilon,2}(x_1, \xi')\cdot\langle\xi'\rangle^s|\leqq\sum_{j=m'+1}^{m}C_{\delta'}\cdot|\hat{\phi}_{\varepsilon,j}(\xi')\cdot\langle\xi'\rangle^{s'+\delta}|\cdot|1-\chi(\xi'; B_R)|\cdot R^{-\delta''}.$$

Hence

$$\text{(2.11)}\qquad \sup_{0\leqq x_1\leqq T}||w_{\varepsilon,2}(x_1, \cdot)||_s\leqq C_{\delta'}\cdot R^{-\delta''}\cdot\sum_{j=m'+1}^{m}||\phi_{\varepsilon,j}||_{s'+\delta}.$$

Thus

$$\text{(2.12)}\qquad \sup_{0\leqq x_1\leqq T}||u_{\varepsilon,2}(x_1, \cdot)||_s\leqq(K_R\cdot\varepsilon^{1/m''}+C_{\delta'}\cdot R^{-\delta''})\cdot M\cdot m''.$$

First take the upper limit of ε in (2.12) and next let $R\uparrow\infty$, then

$$\overline{\lim_{\varepsilon\downarrow 0}}\sup_{0\leqq x_1\leqq T}||u_{\varepsilon,2}(x_1, \cdot)||_s = 0.$$

Next we must show (C9) implies (C10). Assume that (C10) is not satisfied. Then there exists a positive number δ such that (E3) is not satisfied. Replacing s' by $s'+\delta$ in (2.2) and (2.3) in the proof of Lemma 1, we have a sequence of solutions $u_n(x)$ of (CP2) such that

$$\sup_{0\leq x_1\leq T}||u_n(x_1, \cdot)||_s \geqq 1/2 ,$$

$$||D_1^{j-1}u_n(0, \cdot)||_{s'+\delta} \to 0 , \qquad j = 1, \cdots, m .$$

This contradicts (D1), (D5), (D6), and (D7). Q.E.D.

Proof of Theorem 1. First we shall show the equivalence between (C1) and (C3). Denote by $u_{\varepsilon,1}(x)$ the solution of (CP1) and by $u_{\varepsilon,2}(x)$ the solution of (CP2) with the initial conditions:

$$D_1^{j-1}u(0, x') = 0 , \qquad j = 1, \cdots, m' ;$$

$$D_1^{j-1}u(0, x') = \phi_{\varepsilon,j}(x') - D_1^{j-1}u_0(0, x') , \qquad j = m'+1, \cdots, m .$$

Then the solution $u_\varepsilon(x)$ of (CP) is given by $u_{\varepsilon,1}(x)+u_{\varepsilon,2}(x)$. Apply Lemma 1 and Lemma 6. The condition (C3) is equivalent to the (s, s')-stability of (CP1) with respect to a particular solution u_0 of (RCP) and the $(s, s'+0)$-stability of (CP2) with respect to a particular solution $u_{0,2}=0$ of (RCP2). By the definition, the (s, s')-stability implies the $(s, s'+0)$-stability. Hence we can easily show that (C3) is equivalent to the $(s, s'+0)$-stability of (CP) with respect to a particular solution u_0 of (RCP).

Since (C3) is independent of the choice of a particular solution u_0 of (RCP), it implies that (C2) is equivalent to (C1).

Finally we shall show the equivalence between (C3) and (C4). We have only to show that (C4) implies (C3). Apply Lemma 3 for $R=R_0$. Then we have (E1) and (E2) for $\varepsilon_0=\min\{\varepsilon_0', \varepsilon_{R_0}\}$ and

$$C_0 = \max\{1, T\} \cdot \max\{C_0', C_{1,R_0}\cdot(1+R_0^2)^{\max\{(s-s'),0\}/2}\} .$$

Apply Lemma 3 for $R=R_\delta$. Then we have (E6) for $\varepsilon_\delta=\min\{\varepsilon_\delta', \varepsilon_{R_\delta}\}$ and

$$C_\delta = \max\{C_\delta', C_{1,R_\delta}\cdot(1+R_\delta^2)^{\max\{(s-s'-\delta),0\}/2}\} .$$

Q.E.D.

By the same argument as Theorem 1 we have the following theorem whose proof is omitted.

Theorem 2. *Let Assumption* 1 *and* 2 *be satisfied for* $s'=s$. *Then the following three conditions are equivalent*:

(*C5*) *The Cauchy problem* (*CP*) *is* H^s*-stable in* $[0, T]$ *for* $\varepsilon \downarrow 0$ *with respect to a particular solution* $u_0(x)$ *of* (*RCP*) *belonging to* $C^m([0, T]; H^{s+l})$.

(*C11*) *The Cauchy problem* (*CP*) *is* H^s*-stable in* $[0, T]$ *for* $\varepsilon \downarrow 0$ *with respect to every solution* $u_0(x)$ *of* (*RCP*) *belonging to* $C^m([0, T]; H^{s+l})$.

(C12) *There exist positive numbers* ε_0', R_0, *and* C_0' *such that*

$$(E10)\qquad \sup_{0<\varepsilon\leqq\varepsilon_0,\, R_0\leqq|\xi'|}\int_0^T \frac{1}{\varepsilon}\cdot|Y_m(\varepsilon, x_1, \xi')|\,dx_1 \leqq C_0',$$

$$(E11)\qquad \sup_{1\leqq j\leqq m,\, 0<\varepsilon\leqq\varepsilon_0,\, 0\leqq x_1\leqq T,\, R_0\leqq|\xi'|}|Y_j(\varepsilon, x_1, \xi')| \leqq C_0'.$$

§ 3. An Example for Nagumo's H^s-Stability

Let $P_1(\xi)$ and $P_2(\xi)$ satisfy Assumption 1 and

$$\operatorname{ord} p_{1,j}(\xi')\leqq j,\ j=0,\cdots,m;\ \operatorname{ord} p_{2,j}(\xi')\leqq j,\ j=0,\cdots,m'.$$

Then $P_1(D)$ and $P_2(D)$ are Kowalewskian operators. Put

$$L(\xi, \lambda) = P_1(\xi)+\lambda^{m''}\cdot P_2(\xi),$$

$N'=(1, 0)$ in $\mathbf{R}_{\xi_1}\times\mathbf{R}_{\xi'}^{n-1}$, and $N=(N', 0)$ in $\mathbf{R}_\xi^n\times\mathbf{R}_\lambda$. Denote by $\mathring{L}(\xi, \lambda)$ the principal symbol of $L(\xi, \lambda)$ with respect to (ξ, λ) and by $\mathring{P}_i(\xi)$, $i=1, 2$ those of $P_i(\xi)$, $i=1, 2$, respectively. Then

$$\mathring{L}(\xi, \lambda) = \mathring{P}_1(\xi)+\lambda^{m''}\cdot\mathring{P}_2(\xi).$$

It must be remarked that $\mathring{L}(N)=p_{1,0}\neq 0$ and $\mathring{P}_2(N')=p_{2,0}\neq 0$. Kevorkian and Cole's suggestive example in §4.1.2. in [7] is as follows.

Example 1 (Kevorkian and Cole).

Let $P_1(\xi_1, \xi_2)=\xi_1^2-\xi_2^2$, which is the simple wave operator, and $P_2(\xi_1, \xi_2)=\sqrt{-1}\cdot(a\cdot\xi_1+b\cdot\xi_2)$, where a and b are real numbers. Let us consider the solutions $u_\varepsilon(x_1, x_2)$ through a fixed point $P(x_1^0, x_2^0)$ of the following equation:

$$\varepsilon\cdot(P_1(D_1, D_2)+P_2(D_1, D_2))u(x_1, x_2) = 0.$$

If there exists a convergent sequence of $u_\varepsilon(x_1, x_2)$, then the limit $u_0(x_1, x_2)$ must satisfy the reduced equation

$$P_2(D_1, D_2)u(x_1, x_2) = 0.$$

Since the general solution of the reduced equation has the form: $u_0(x_1, x_2)=f(b\cdot x_1-a\cdot x_2)$ and the subcharacteristic of the reduced equation has the form: $b\cdot x_1-a\cdot x_2=$constant, if $|a/b|>1$ then the subcharacteristic to P lies outside the usual domain of dependence of P for the simple wave operator. Hence $u_0(x_1, x_2)$ can not be approximated by $u_\varepsilon(x_1, x_2)$ when $|a/b|>1$.

Thus even when $\mathring{P}_1$ and $\mathring{P}_2$ are strictly hyperbolic, we need some additional assumption on the propagation speeds. Therefore we require the following assumption.

Assumption 3.

(A3): The polynomial $\mathring{L}(\xi_1+\tau, \xi', \lambda)$ has only simple real zero for every (ξ, λ) in $\boldsymbol{R}^n\times\boldsymbol{R}-\{(0, 0)\}$. That is, $L(\xi, \lambda)$ is a strictly hyperbolic polymonial in (ξ, λ) with respect to N.

(A4): There exists a positive number T_1 such that if $\operatorname{Im}\tau<-T_1$ then $P_2(\xi_1+\tau, \xi')\neq 0$ for all ξ in $\boldsymbol{R}^n$. That is, $P_2(\xi)$ is a hyperbolic polynomial in ξ with respect to N' in the sense of Gårding.

Remark. Since

$$\begin{aligned}\mathring{L}(0+\tau, 0, \lambda) &= p_{1,0}\cdot\tau^m+\lambda^{m''}\cdot p_{2,0}\cdot\tau^{m'}\\ &= \tau^{m'}(p_{1,0}\cdot\tau^{m''}+\lambda^{m''}\cdot p_{2,0}),\end{aligned}$$

(A.3) implies that $m'\leqq 1$.

Theorem 3. *Let Assumption* 1 *and* 3 *be satisfied and s be an arbitrary real number. Then the Cauchy problem* (*CP*) *is H^s-stable* (*and therefore* $(s, s+0)$-*stable*) *in* $0\leqq x_1\leqq T$ *for* $\varepsilon\downarrow 0$ *with respect to every solution* u_0 *of* (*RCP*) *belonging to* $C^m([0, T]; H^{s+m})$.

Proof. By Theorem 2, it suffices to show that Assumption 2, which is the assumption on the unique solvability, and (C12) are satisfied. First we shall show (C12). Denote by $t_j(\xi', \lambda)$, $j=1, \cdots, m$ the roots of $L(\xi, \lambda)=0$ with respect to ξ_1. When $\varepsilon^{-1}=\lambda^{m''}$, we may write

$$t_j(\xi', \lambda)=\tau_j(\varepsilon, \xi'), \qquad j=1, \cdots, m \tag{3.1}$$

for $\varepsilon\neq 0$ by choosing the suffixes $\{j\}$ of $t_j(\xi', \lambda)$ properly. The strict hyperbolicity of $L(\xi, \lambda)$ implies that there exist positive numbers R_1, c_1, and M_1 such that

$$\inf_{j\neq k, 1\leqq j, k\leqq m, |(\xi', \lambda)|\geqq R_1} |t_j(\xi', \lambda)-t_k(\xi', \lambda)|/|(\xi', \lambda)|\geqq c_1; \tag{3.2}$$

$$\sup_{1\leqq j\leqq m, |(\xi', \lambda)|\geqq R_1} |t_j(\xi', \lambda)|/|(\xi,' \lambda)|\leqq M_1. \tag{3.3}$$

(For example, if we look carefully into the proof of Theorem 4.10 in [8], we can find this fact easily.) Hence the roots $\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$ of $L_\varepsilon(\xi)=0$ with respect to ξ_1 are distinct for $\varepsilon\neq 0$ and $R_1\leqq|\xi'|$. The hyperbolicity of $L(\xi, \lambda)$ implies that there exists a positive number C_3 such that

$$(3.4) \qquad \sup_{1\leqq j\leqq m, (\xi', \lambda)\in \boldsymbol{R}^{n-1}\times \boldsymbol{R}} |\mathrm{Im}\, t_j(\xi', \lambda)| \leqq C_3 .$$

Put $\rho=|(\xi', \lambda)|$. Then (A.4) in Appendix implies that for $\varepsilon\neq 0$, $R_1\leqq|\xi'|$, $0\leqq x_1\leqq T$, and $j=1, \cdots, m$,

$$\begin{aligned} |Y_j(\varepsilon, x_1, \xi')| &= |(-1)^{j-1}\cdot D(0, 1, \cdots, j-2, j, \cdots, m-1)(t_1, \cdots, t_m, x_1)| \\ &\leqq M(0, 1, \cdots, j-2, j, \cdots, m-1)\cdot |(t_1, \cdots, t_m)|^{m-j} \\ &\quad \times \textstyle\sum_{l=1}^m \exp(-\mathrm{Im}\, t_l x_1)/\prod_{k\neq l, 1\leqq k\leqq m} |t_l - t_k| \\ &\leqq \rho^{1-j}\cdot M(0, 1, \cdots, j-2, j, \cdots, m-1)\cdot |(t_1/\rho, \cdots, t_m/\rho)|^{m-j} \\ &\quad \times \textstyle\sum_{l=1}^m \exp(-\mathrm{Im}\, t_l x_1)/\prod_{k\neq l, 1\leqq k\leqq m} |t_l/\rho - t_k/\rho| \\ &\leqq \rho^{1-j}\cdot C_4 , \end{aligned}$$

where

$$C_4 = M(0, 1, \cdots, j-2, j, \cdots, m-1)\cdot m^{(m-j)/2}\cdot M_1{}^{m-j}\cdot m\cdot(\exp C_3 T)\cdot c_1{}^{1-m} .$$

Since $R_1\leqq|\xi'|\leqq\rho$ and $\lambda\leqq\rho$, it implies that $\rho^{1-j}\leqq R_1{}^{1-j}, j=1, \cdots, m$, and $\varepsilon^{-1}\cdot\rho^{1-m}=\lambda^{m''}\cdot\rho^{1-m}\leqq\lambda^{m''+1-m}$. Hence

$$\sup_{0\leqq\varepsilon\leqq\varepsilon_{R_1}, 0\leqq x_1\leqq T, R_1\leqq|\xi'|} |Y_j(\varepsilon, x_1, \xi')| \leqq C_4 , \qquad j = 1, \cdots, m;$$

$$\sup_{0\leqq\varepsilon\leqq\varepsilon_{R_1}, 0\leqq x_1\leqq T, R_1\leqq|\xi'|} \frac{1}{\varepsilon}\cdot |Y_j(\varepsilon, x_1, \xi')| \leqq C_4 .$$

Next we shall show that the unique solvability. Since (C12) and Lemma 3 imply (C6), Lemma 4 can be applied. It is well-known that (E8) and (E9) imply the unique solvability. Q.E.D.

Remark. If $\phi_{\varepsilon,j}, j=1, \cdots, m$ and $\phi_{0,j}, j=1, \cdots, m'$ belong to $H^\infty(\boldsymbol{R}^{n-1})$ and f_ε and f_0 belong to $H^\infty(\boldsymbol{R}^n)$ then u_ε belong to $C^m([0, T]; H^s)$ and u_0 belongs to $C^m([0, T]; H^{s+m})$.

Appendix

Let $z=(z_1, z_2, \cdots, z_n)$ be complex variables. For a non-negative integer l, denote

$$a(l)(z) = ((z_j)^l; j\rightarrow 1, \cdots, n)$$

and for non-negative integers $l_1, l_2, \cdots, l_n$ satisfying $0\leqq l_1<l_2<\cdots<l_n$, denote

$$A(l_1, l_2, \cdots, l_n)(z) = \det\,(a(l_i)(z); i\downarrow 1, \cdots, n) .$$

In particular, $A(0, 1, \cdots, n-1)(z)$ is the Vandermonde determinant and represented as the difference product $\prod_{1\leq i<j\leq n}(z_j-z_i)$. Let $i=\sqrt{-1}$ and x_1 be a real parameter. Denote

$$e(z, x_1) = (\exp iz_jx_1; j \to 1, \cdots, n)$$

and for non-negative integers $l_1, l_2, \cdots, l_{n-1}$ satisfying $0\leqq l_1<l_2<\cdots<l_{n-1}$, denote

$$B(l_1, l_2, \cdots, l_{n-1})(z, x_1) = \det {}^t({}^te(z, x_1), {}^ta(l_1)(z), \cdots, {}^ta(l_{n-1})(z)) .$$

Expand the determinant $B(l_1, l_2, \cdots, l_{n-1})(z, x_1)$ with respect to the first row. Then

(A.1) $$B(l_1, l_2, \cdots, l_{n-1})(z, x_1) = \sum_{j=1}^n (-1)^{1+j}\cdot A(l_1, l_2, \cdots, l_{n-1})(z(j))\cdot \exp iz_jx_1 ,$$

where $z(j)=(z_1, z_2, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n)$. Denote

$$C(l_1, l_2, \cdots, l_n)(z) = A(l_1, l_2, \cdots, l_n)(z)/A(0,1, \cdots, n-1)(z)$$

and

$$D(l_1, l_2, \cdots, l_{n-1})(z, x_1) = B(l_1, l_2, \cdots, l_{n-1})(z, x_1)/A(0, 1, \cdots, n-1)(z) .$$

Then $C(l_1, l_2, \cdots, l_n)(z)$ is a homogeneous symmetric polynomial in $\mathbb{Z}[z]$ of order $l_1+l_2+\cdots+l_n-(n-1)n/2$, which is called a Schur function. Since $B(l_1, l_2, \cdots, l_{n-1})(z, x_1)$ is an entire function of z and vanishes on the zeros of irreducible polynomials z_j-z_i, $1\leqq i<j\leqq n$, Nullstellensatz implies that $B(l_1, l_2, \cdots, l_{n-1})(z, x_1)$ is divided by $A(0, 1, \cdots, n-1)(z)$ in the ring of entire functions. Hence $D(l_1, l_2, \cdots, l_{n-1})(z, x_1)$ is an entire function. If $z_i\neq z_j$, $1\leqq i<j\leqq n$, then (A.1) implies that

(A.2) $$\begin{aligned} &D(l_1, l_2, \cdots, l_{n-1})(z, x_1) \\ &= \sum_{j=1}^n (-1)^{1+j}\cdot C(l_1, l_2, \cdots, l_{n-1})(z(j))\cdot \exp iz_jx_1\cdot E_j(z) , \end{aligned}$$

where $E_j(z)=1/\{(-1)^{n-j}\cdot\prod_{k\neq j,1\leqq k\leqq n}(z_j-z_k)\}$. Put

$$M(l_1, l_2, \cdots, l_n) = \max_{|z|=1} |C(l_1, l_2, \cdots, l_n)(z)| .$$

Then

(A.3) $$|C(l_1, l_2, \cdots, l_n)(z)| \leqq M(l_1, l_2, \cdots, l_n)\cdot |z|^L,$$

where $L=l_1+l_2+\cdots+l_n-(n-1)n/2$ and

(A.4) $$\begin{aligned} &|D(l_1, l_2, \cdots, l_{n-1})(z, x_1)| \\ &\leqq M(l_1, l_2, \cdots, l_{n-1})|z|^{L'}\cdot\sum_{j=1}^n \exp(-\operatorname{Im} z_jx_1)/\prod_{k\neq j,1\leqq k\leqq n}|z_j-z_k| , \end{aligned}$$

where $L'=l_1+l_2+\cdots+l_{n-1}-(n-2)(n-1)/2$.

Let m, m', and m'' be positive integers such that $m=m'+m''$. Denote $z'=(z_1, z_2, \cdots, z_{m'})$, $z''=(z_{m'+1}, z_{m'+2}, \cdots, z_m)$, and $z=(z', z'')$. Let $l_1, l_2, \cdots, l_{m-1}$ be non-negative integers satisfying $0\leqq l_1<l_2<\cdots<l_{m-1}$. Let $\mathcal{S}_1$ be the set of all bijections ρ from $\{1, 2, \cdots, m-1\}$ onto $\{l_1, l_2, \cdots, l_{m-1}\}$ satisfying

$$\rho(1)<\rho(2)<\cdots<\rho(m');$$
$$\rho(m'+1)<\rho(m'+2)<\cdots<\rho(m-1)$$

and $\mathcal{S}_2$ be the set of all bijections ρ from $\{1, 2, \cdots, m-1\}$ onto $\{l_1, l_2, \cdots, l_{m-1}\}$ satisfying

$$\rho(1)<\rho(2)\cdots<\rho(m'-1);$$
$$\rho(m')<\rho(m'+1)<\cdots<\rho(m-1)\,.$$

There are one-to-one correspondence between the bijections in $\mathcal{S}_1$ and the selections of $m-1$ objects taken m' at a time and between the bijections in $\mathcal{S}_2$ and the selections of $m-1$ objects taken $m'-1$ at a time, respectively. Define the bijection π from $\{l_1, l_2, \cdots, l_{m-1}\}$ onto $\{2, 3, \cdots, m\}$ as

$$\pi(l_j)=j+1\,, \qquad j=1, \cdots, m-1\,.$$

Denote

$$I(\rho)=\textstyle\sum_{j=1}^{m'} \pi(\rho(j))+m'(m'+1)/2$$

and

$$J(\rho)=1+\textstyle\sum_{j=1}^{m'-1} \pi(\rho(j))+m'(m'+1)/2\,.$$

For $z_i\neq z_j$, $1\leqq i\leqq m'$, $m'+1\leqq j\leqq m$, denote

$$E(z)=1/\textstyle\prod_{1\leqq i\leqq m',\ m'+1\leqq j\leqq m} (z_j-z_i)\,.$$

Lemma A.1. *For $z_i\neq z_j$, $1\leqq i\leqq m'$, $m'+1\leqq j\leqq m$,*

$$\begin{aligned}
&D(l_1, l_2, \cdots, l_{m-1})(z, x_1)\\
&=\textstyle\sum_{\rho\in\mathcal{S}_1}(-1)^{I(\rho)}\cdot C(\rho(1), \rho(2), \cdots, \rho(m'))(z')\\
&\quad\times D(\rho(m'+1), \rho(m'+2), \cdots, \rho(m-1))(z'', x_1)\cdot E(z)\\
&\quad+\textstyle\sum_{\rho\in\mathcal{S}_1}(-1)^{J(\rho)}\cdot D(\rho(1), \rho(2), \cdots, \rho(m'-1))(z', x_1)\\
&\quad\times C(\rho(m'), \rho(m'+1), \cdots, \rho(m-1))(z'')\cdot E(z)\,.
\end{aligned} \tag{A.5}$$

Proof. Apply the Laplace expansion theorem to $B(l_1, l_2, \cdots, l_{m-1})(z, x_1)$. The minors of order m' of the original matrix ${}^t({}^te(z, x_1), {}^ta(l_1)(z), \cdots, {}^ta(l_{m-1})(z))$ of order m are

$$A(\rho(1), \rho(2), \cdots, \rho(m'))(z'), \qquad \text{for } \rho \text{ in } \mathcal{S}_1,$$
$$B(\rho(1), \rho(2), \cdots, \rho(m'-1))(z'', x_1), \qquad \text{for } \rho \text{ in } \mathcal{S}_2,$$

and those cofactors of order m'' are

$$(-1)^{I(\rho)}\cdot B(\rho(m'+1), \rho(m'+2), \cdots, \rho(m-1))(z', x_1), \qquad \text{for } \rho \text{ in } \mathcal{S}_1,$$
$$(-1)^{J(\rho)}\cdot A(\rho(m'), \rho(m'+1), \cdots, \rho(m-1))(z''), \qquad \text{for } \rho \text{ in } \mathcal{S}_2,$$

respectively. Hence

$$\begin{aligned}(\text{A.6}) \qquad & B(l_1, l_2, \cdots, l_{m-1})(z, x_1) \\ &= \textstyle\sum_{\rho\in\mathcal{S}_1} (-1)^{I(\rho)}\cdot A(\rho(1), \rho(2), \cdots, \rho(m'))(z') \\ &\quad \times B(\rho(m'+1), \rho(m'+2), \cdots, \rho(m-1))(z'', x_1) \\ &\quad + \textstyle\sum_{\rho\in\mathcal{S}_2} (-1)^{J(\rho)}\cdot B(\rho(1), \rho(2), \cdots, \rho(m'-1))(z', x_1) \\ &\quad \times A(\rho(m'), \rho(m'+1), \cdots, \rho(m-1))(z'').\end{aligned}$$

Divide (A.6) by

$$\begin{aligned}(\text{A.7}) \qquad & A(0, 1, \cdots, m-1)(z) \\ &= A(0, 1, \cdots, m'-1)(z')\cdot A(0, 1, \cdots, m''-1)(z'')/E(z),\end{aligned}$$

we have (A.5). Q.E.D.

Denote

$$L'(\rho) = \begin{cases} \rho(1)+\rho(2)+\cdots+\rho(m')-(m'-1)m'/2, & \text{for } \rho \text{ in } \mathcal{S}_1 \\ \rho(1)+\rho(2)+\cdots+\rho(m'-1)-(m'-1)m'/2, & \text{for } \rho \text{ in } \mathcal{S}_2, \end{cases}$$

and

$$L''(\rho) = \begin{cases} \rho(m'+1)+\rho(m'+2)+\cdots+\rho(m-1)-(m''-1)m''/2, & \text{for } \rho \text{ in } \mathcal{S}_1 \\ \rho(m')+\rho(m'+1)+\cdots+\rho(m-1)-(m''-1)m''/2, & \text{for } \rho \text{ in } \mathcal{S}_2. \end{cases}$$

Put

$$\begin{aligned}&\tilde{M}(l_1, l_2, \cdots, l_{m-1}) \\ &= \max\{\max_{\rho\in\mathcal{S}_1} M(\rho(1), \rho(2), \cdots, \rho(m')), \\ &\quad \max_{\rho\in\mathcal{S}_1} M(\rho(m'+1), \rho(m'+2), \cdots, \rho(m-1)), \\ &\quad \max_{\rho\in\mathcal{S}_2} M(\rho(m'), \rho(m'+1), \cdots, \rho(m-1))\}.\end{aligned}$$

For a positive parameter η, put $w_j=\eta\cdot z_j$, $j=1, \cdots, m$.

Lemma A.2. *Assume that* $z_i \neq z_j$, *for* $1\leqq i\leqq m'$, $m'+1\leqq j\leqq m$ *and for* $m'+1\leqq i<j\leqq m$. *Then*

$$\begin{aligned}&|C(\rho(1), \rho(2), \cdots, \rho(m'))(z')\\&\times D(\rho(m'+1), \rho(m'+2), \cdots, \rho(m-1))(z'', x_1)\cdot E(z)|\\ \leqq &\tilde{M}(l_1, l_2, \cdots, l_{m-1})^2\cdot|z'|^{L'(\rho)}\cdot|w''|^{L''(\rho)+(m''-1)}\cdot|E(w)|\\&\times\eta^{m'm''-L''(\rho)}\cdot(\textstyle\sum_{j=m'+1}^{m}\exp(-\mathrm{Im}\, w_j x_1/\eta)/\prod_{k\neq j,\, m'+1\leqq k\leqq m}|w_j-w_k|)\,,\end{aligned}$$

for ρ in $\mathcal{S}_1$ and

$$\begin{aligned}&|D(\rho(1), \rho(2), \cdots, \rho(m'-1))(z', x_1)\\&\times C(\rho(m'), \rho(m'+1), \cdots, \rho(m-1))(z'')\cdot E(z)|\\ \leqq &|D(\rho(1), \rho(2), \cdots, \rho(m'-1))(z', x_1)|\\&\times\tilde{M}(l_1, l_2, \cdots, l_{m-1})\cdot|w''|^{L''(\rho)}\cdot|E(w)|\cdot\eta^{m'm''-L''(\rho)}\,,\end{aligned}$$

for ρ in $\mathcal{S}_2$.

Proof. Since

$$C(l_1, l_2, \cdots, l_n)(z) = \eta^{-L}\cdot C(l_1, l_2, \cdots, l_n)(\eta\cdot z)\,,$$

where $L=l_1+l_2+\cdots+l_n-(n-1)n/2$,

$$D(l_1, l_2, \cdots, l_{n-1})(z, x_1) = \eta^{-L''}\cdot D(l_1, l_2, \cdots, l_{n-1})(\eta\cdot z, x_1/\eta)\,,$$

where $L''=l_1+l_2+\cdots+l_{n-1}-(n-1)n/2$, and $E(z)=\eta^{m'm''}\cdot E(w)$, it implies that

$$\begin{aligned}&C(\rho(1), \rho(2), \cdots, \rho(m'))(z')\\&\times D(\rho(m'+1), \rho(m'+2), \cdots, \rho(m-1))(z'', x_1)\cdot E(z)\\ =&C(\rho(1), \rho(2), \cdots, \rho(m'))(z')\\&\times D(\rho(m'+1), \rho(m'+2), \cdots, \rho(m-1))(w'', x_1/\eta)\cdot E(w)\cdot\eta^{m'm''-L''(\rho)}\,,\end{aligned}$$

for ρ in $\mathcal{S}_1$ and

$$\begin{aligned}&D(\rho(1), \rho(2), \cdots, \rho(m'-1))(z', x_1)\\&\times C(\rho(m'), \rho(m'+1), \cdots, \rho(m-1))(z'')\cdot E(z)\\ =&D(\rho(1), \rho(2), \cdots, \rho(m'-1))(z', x_1)\\&\times C(\rho(m'), \rho(m'+1), \cdots, \rho(m-1))(w'')\cdot E(w)\cdot\eta^{m'm''-L''(\rho)}\,,\end{aligned}$$

for ρ in $\mathcal{S}_2$. By using (A.3) and (A.4), we come to the conclusion. Q.E.D.

Lemma A.3. *Assume that $z_i\neq z_j$, for $1\leqq i\leqq m'$, $m'+1\leqq j\leqq m$ and for $m'+1\leqq i<j\leqq m$. Let*

$$\{l_1, l_2, \cdots, l_{m-1}\} = \{0, 1, \cdots, k-1, k+1, \cdots, m-1\}\,.$$

Assume that there exist positive numbers M, M', c, and η_0 with $\eta_0\leqq 1$ such that for every η satisfying $0<\eta\leqq\eta_0$, the following estimates are satisfied:

(A.8)
$$|z'|\leqq M;\quad |w''|\leqq M;$$
$$\sum_{j=m'+1}^{m}\exp(-\mathrm{Im}\,w_j x_1/\eta)\leqq M';$$
$$\inf_{m'+1\leqq i<j\leqq m}|w_i-w_j|\geqq c;\quad \inf_{1\leqq i\leqq m',\,m'+1\leqq j\leqq m}|w_i-w_j|\geqq c\,.$$

Denote

$$\tilde{M}=\max_{0\leqq k\leqq m-1}\tilde{M}(0,1,\cdots,k-1,k+1,\cdots,m-1)\,,$$
$$C_1=\frac{(m-1)!}{m'!(m''-1)!}\circ\tilde{M}^2\circ M^{m'm''-k+m''-1}\circ c^{-m'm''-m''+1}\circ M'\,,$$

and

$$C_2=\frac{(m-1)!}{(m'-1)!m''!}\circ\tilde{M}\circ M^{m'm''}\circ c^{-m'm''}\,.$$

Then

(A.9)
$$\begin{aligned}&|D(0,1,\cdots,k-1,k+1,\cdots,m-1)(z,x_1)\\&-D(0,1,\cdots,k-1,k+1,\cdots,m'-1)(z',x_1)\\&\times(w_{m'+1}\circ w_{m'+2}\circ\cdots\circ w_m)^{m'}\circ E(w)|\\&\leqq(C_1+C_2\circ\max_{\rho\in\mathcal{S}_2}|D(\rho(1),\rho(2),\cdots,\rho(m'-1))(z',x_1)|)\circ\eta\,,\end{aligned}$$

for $k=0,\cdots,m'-1$ *and*

(A.10)
$$\begin{aligned}&|D(0,1,\cdots,k-1,k+1,\cdots,m-1)(z,x_1)|\\&\leqq(C_1+C_2\circ\max_{\rho\in\mathcal{S}_2}|D(\rho(1),\rho(2),\cdots,\rho(m'-1))(z',x_1)|)\circ\eta^{k-m'+1}\,,\end{aligned}$$

for $k=m',\cdots,m-1$. *Here* ρ *in* $\mathcal{S}_2$ *are bijections from* $\{1,2,\cdots,m-1\}$ *onto* $\{0,1,\cdots,k-1,k+1,\cdots,m-1\}$ *satisfying*

$$\rho(1)<\rho(2)<\cdots<\rho(m'-1);$$
$$\rho(m')<\rho(m'+1)<\cdots<\rho(m-1)\,.$$

Proof. First it must be remarked that

$$m'm''-L''(\rho)\geqq m'm''-m'-(m'+1)-\cdots-(m-1)+(m''-1)m''/2=0\,,$$

where the equality holds if and only if

(A.11)
$$k=0,1,\cdots,m'-1\,,$$
$$\rho\in\mathcal{S}_2\,,$$
$$\rho(j)=\begin{cases}j-1\,,& j=1,\cdots,k;\\ j\,,& j=k+1,\cdots,m-1\,.\end{cases}$$

Since

$$C(m', m'+1, \cdots, m-1)(z'') = (z_{m'+1} \cdot z_{m'+2} \cdot \cdots \cdot z_m)^{m'},$$

it implies that for ρ satisfying (A.11),

$$\begin{aligned}
&(-1)^{J(\rho)} \cdot D(\rho(1), \rho(2), \cdots, \rho(m'-1))(z', x_1) \\
&\times C(\rho(m'), \rho(m'+1), \cdots, \rho(m-1))(z'') \cdot E(z) \\
=\ &(-1)^{m'(m'+1)} \cdot D(0, 1, \cdots, k-1, k+1, \cdots, m'-1)(z', x_1) \\
&\times (w_{m'+1} \circ w_{m'+2} \circ \cdots \circ w_m)^{m'} \cdot E(w).
\end{aligned}$$

For ρ not satisfying (A.11), Lemma A.2 implies that

$$\begin{aligned}
&|C(\rho(1), \rho(2), \cdots, \rho(m'))(z') \cdot D(\rho(m'+1), \rho(m'+2), \cdots, \rho(m-1))(z'', x_1) \cdot E(z)| \\
&\leqq \tilde{M}^2 \cdot M^{L'(\rho)+L''(\rho)+m''-1} \cdot |E(w)| \cdot \eta^{m'm''-L''(\rho)} \cdot M' \cdot c^{-m''+1},
\end{aligned}$$

for ρ in $\mathcal{S}_1$ and

$$\begin{aligned}
&|D(\rho(1), \rho(2), \cdots, \rho(m'-1))(z', x_1) \cdot C(\rho(m'), \rho(m'+1), \cdots, \rho(m-1))(z'') \cdot E(z)| \\
&\leqq |D(\rho(1), \rho(2), \cdots, \rho(m'-1))(z', x_1)| \cdot \tilde{M} \cdot M^{L''(\rho)} \cdot |E(w)| \cdot \eta^{m'm''-L''(\rho)},
\end{aligned}$$

for ρ in $\mathcal{S}_2$. If (A.11) is not satisfied, then $m'm''-L(\rho) \geqq 1$. If $k=m', \cdots, m-1$, then

$$\begin{aligned}
&m'm''-L''(\rho) \geqq m'm''-(m'-1)-m'-\cdots-(m-1)+k+(m''-1)m''/2 \\
&=k-m'+1.
\end{aligned}$$

Since $|E(w)| \leqq c^{-m'm''}$ and $L'(\rho)+L''(\rho)=m'm''-k$, for ρ in $\mathcal{S}_1$, Lemma A.1 implies the conclusion. Q.E.D.

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