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Weak Equivalence and the Structures of Cocycles of an Ergodic Automorphism

By

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Abstract

Let (X, μ) be a Lebesgue space, Γ an approximately finite ergodic group of the automorphisms, α a cocycle on $X\times\Gamma$ with values in an arbitrary abelian l.c.s. group G, and ρ the Radon-Nikodym cocycle on $X\times\Gamma$. The concept of weak equivalence of the pairs (Γ, α) is introduced and studied, which generalizes the concept of trajectory equivalence of automorphism groups. It is proved that the pairs (Γ_1, α_0^1) and (Γ_2, α_0^2) $(\alpha_0^i=(\alpha, \rho))$ are (stably) weakly equivalent iff the corresponding Mackey pairs $W_1(G_0)$ and $W_2(G_0)$ of the group $G_0=G\times\boldsymbol{R}$ are isomorphic. It is proved that any ergodic action of $G\times\boldsymbol{R}$ (or G) is isomorphic to the Mackey action associated with a certain pair (Γ, α_0). The structure of cocycles of approximately finite equivalence relations is studied. The relationship between the type of the group Γ and that of the corresponding Mackey action is considered.

§0. Introduction

The present paper is a study of the countable approximately finite (a.f.) groups Γ of automorphisms of measure spaces and the cocycles α for Γ taking the values in an abelian locally compact separable (l.c.s.) group G. Such cocycles were thoroughly studied in the book by K. Schmidt [15]. He in part considered the important classes of cocycles: transient, lacunary, etc. We shall use the set of all pairs (Γ, α) to study an equivalence relation which is called the weak equivalence (see Section 1). In the case where the cocycle α is the Radon-Nikodym cocycle ρ, weak equivalence of pairs (Γ_1, ρ_1) and (Γ_2, ρ_2) is the same as the well-known orbital equivalence relation of the groups of automorphisms Γ_1 and Γ_2 [9]. The weak equivalence was introduced in [5] (see also [6]) in the study of the pairs (Γ, α), where Γ is a group of measure-preserving automorphisms and α a cocycle with the dense range in a l.c.s. amenable group

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G. It was also proved there that all such pairs are weakly equivalent for the fixed G. Later the weak equivalence of the pairs (Γ, α) was studied in [1, 2, 7] and elsewhere.

The main purpose of this paper is to describe the structures of cocycles of a.f. groups of automorphisms with values in a l.c.s. group. It appears that any cocycle is weakly equivalent to a cocycle which in a natural way composed of simpler cocycles: a transient cocycle and a cocycle with a dense range. In solving this problem, we introduced and studied measurable fields of cocycles which have a dense range in an arbitrary subgroup of the group $G_0 = G \times \mathbb{R}$.

Every pair (Γ, α_0), where $\alpha_0 = (\alpha, \rho)$, defines in a natural way an action of the group G_0 which is called the Mackey action (or refered to as associated with the pair (Γ, α_0)) [11]. As a consequence of the above result on the structures of cocycles, we obtain the solution of the problem of finding the necessary and sufficient conditions of weak equivalence of the pairs (Γ_1, α_0^1) and (Γ_2, α_0^2). These conditions consists in the isomorphism of the corresponding Mackey actions of the group G_0. These studies are based on the methods developed in [1, 5, 6]. Another approach to solution of a similar problem is proposed by A.L. Fedorov [2].

Our results are easy to be extended to the case of the pairs (Γ, α), where the cocycle α takes the values in an arbitrary l.c.s. amenable group G and the Mackey action either is free or has a closed normal subgroup of $G \times \mathbb{R}$ in the capacity of the stabilizer.

The paper is organized as follows. Section 1 presents the information on cocycles, needed for subsequent arguments and taken mainly from [15], and introduced the concept of weak equivalence of the pairs (Γ, α). In Section 2 we study the transient cocycles defined for an arbitrary countable group of automorphisms and taking values in a l.c.s. group G. In Section 3, cocycles with a dense range are constructed for an arbitrary closed subgroup $H_0 \subset G \times \mathbb{R}$ and measurable fields of cocycles with a dense range in H_0 are studied. It is found that such fields of cocycles are weakly equivalent to a constant field of cocycles. The results of this section are applied to study the cocycles, for which the Mackey action is transitive, in Section 4. Section 5 studies the lacunary cocycles on a.f. groups of measure preserving automorphisms. They correspond to the free Mackey actions of the group G, that generally speaking have a quasi-invariant measure. In the subsequent two sections, the general case is considered where the group Γ has the quasi-invariant measure and the Mackey action of the group $G_0 = G \times \mathbb{R}$ is non-free. In the last section we consider results on the relation of the types of the Mackey actions and groups Γ.

§1. Preliminaries. Weak Equivalence

1.1. In this section we shall provide the preliminary facts from the ergodic theory that we shall need for the subsequent arguments. The definitions and more detailed results can be found in [8, 9, 14, 15].

The set of all non-singular automorphisms of a Lebesgue space $(X, \mathscr{B}, \mu)$ with a continuous measure μ will be denoted by $\mathrm{Aut}(X, \mathscr{B}, \mu)$. We shall identify automorphisms differing on a measure 0 set. Let Γ be a countable subgroup of $\mathrm{Aut}(X, \mathscr{B}, \mu)$. The set $[\Gamma]=\{g\in \mathrm{Aut}(X, \mathscr{B}, \mu)\colon gx\in \Gamma x$ for μ-a.a $x\in X\}$, where $\Gamma x=\{\gamma x\colon \gamma\in\Gamma\}$ is the orbit of x, is called the full group of automorphisms generated by Γ. The set $N[\Gamma]=\{R\in \mathrm{Aut}(X, \mathscr{B}, \mu)\colon R[\Gamma]R^{-1}=[\Gamma]\}$, which is also a subgroup of $\mathrm{Aut}(X, \mathscr{B}, \mu)$ is called the normalizer of $[\Gamma]$. The group of automorphisms Γ is called approximately finite (a.f.), if there exists an automorphism $T\in \mathrm{Aut}(X, \mathscr{B}, \mu)$ such that $[\Gamma]=[T]$, where $[T]=[\{T^n\colon n\in \boldsymbol{Z}\}]$.

The two groups of automorphisms $\Gamma_1\subset \mathrm{Aut}(X_1, \mathscr{B}_1, \mu_1)$ and $\Gamma_2\subset \mathrm{Aut}(X_2, \mathscr{B}_2, \mu_2)$ are called orbital equivalent, if there exists a one-to-one measurable map $\theta\colon X_1\to X_2$, such that $\theta[\Gamma_1]\theta^{-1}=[\Gamma_2]$ and the measures μ_2 and $\theta\circ\mu_1$ are equivalent.

The ergodic group $\Gamma\subset \mathrm{Aut}(X, \mathscr{B}, \mu)$ is called a type II_1 (II_∞) group, if there exists a measure $\nu\sim\mu$, such that $\gamma\circ\nu=\nu$ for all $\gamma\in\Gamma$ and the measure $\nu(X)$ is finite (infinite). If there is no Γ-invariant measure equivalent to the measure μ, then Γ is said to be of type III. Type III may be further classified (see below).

We shall also use sometimes the terminology and facts of the measurable groupoid theory (see, e.g. [3, 12]). The result of this paper may also be fully expressed in terms of this theory. However, as a rule, we use the standard approach to the study of countable groups of automorphisms, since we proceed from the definitions and facts of [8, 9, 14, 15, etc.].

1.2. We shall cite the definition of the array as in [10].

Let Γ be an ergodic group of automorphisms of $(X, \mathscr{B}, \mu)$. The expression

$$\xi = (A, \Xi, A(\cdot), \gamma(\cdot, \cdot)) \tag{1.1}$$

will be called the Γ-array of the set $A\subset X$ ($\mu A>0$) provided that the following conditions are fulfilled:

(i) Ξ is a finite set of indices;

(ii) $\bigcup_{i\in\Xi} A(i)=A$, $A(i)\cap A(j)=\emptyset$ $(i\neq j)$, $\mu(A(i))>0$;

(iii) $\gamma(i,j)$ are non-singular maps such that $\gamma(i,j)A(j)=A(i)$, $\gamma(i,i)=\mathbb{1}$, $\gamma(i_2,j)\gamma(j,i_1)=\gamma(i_2,i_1)$, $\gamma(i,j)x\in\Gamma x$ for a.a. $x\in A(j)$.

Denote by $\mathcal{G}(\xi)$ the finite group of automorphisms of A generated by $\gamma(i,j)$, $i,j\in\Xi$ and $\mathcal{P}(\xi)$ the collection of sets of the form $\bigcup_{i\in\Delta}A(i)$, where Δ is an arbitrary subset in Ξ.

The pairs $(A(i), \gamma(j, i))$, $i, j\in\Xi$ will be called elements of ξ.

Let there be defined the two Γ-arrays: $\xi_1=(A, \Xi, A(\cdot), \gamma(\cdot, \cdot))$ and $\xi_2=(A(i_0), \Omega, B(\cdot), \delta(\cdot, \cdot))$, where $i_0\in\Xi$. Define a new Γ-array $\xi_1\times\xi_2$ which will be called a refinement of ξ_1 with respect to ξ_2, according to the equality

$$\xi_1\times\xi_2=(A, \Xi\times\Omega, C(\cdot, \cdot), \tau(\cdot, \cdot; \cdot, \cdot)),$$

where $C(i,n)=\gamma(i,i_0)B(n)$, $\tau(i_1,n_1;i,n)=\gamma(i_1,i_0)\delta(n_1,n)\gamma(i_0,i)$, $i_0, i, i_1\in\Xi$, $n, n_1\in\Omega$.

If the Γ-array (1.1) is defined, then it will be said to be defined over the partition $(A, \Xi, A(\cdot))$.

1.3. Let, as earlier, Γ be a countable ergodic group of automorphisms of $(X, \mathcal{B}, \mu)$ acting freely and let G be an arbitrary l.c.s. abelian group.

Definition 1.1. *A measurable map $\alpha: X\times\Gamma\to G$ is called a cocycle, if for any $\gamma_1, \gamma_2\in\Gamma$ and μ-a.a $x\in X$*

$$\alpha(x, \gamma_1\gamma_2) = \alpha(\gamma_2 x, \gamma_1)\,\alpha(x, \gamma_2)\,. \tag{1.2}$$

The set of all cocycles will be denoted by $Z^1(X\times\Gamma, G)$.

An example of a cocycle is the following cocycle $\rho: X\times\Gamma\to\mathbf{R}$

$$\rho(x, \gamma) = \log\frac{d\gamma^{-1}\circ\mu}{d\mu}(x)$$

which is called the Radon-Nikodym cocycle.

By $\mathcal{R}(\Gamma)$ we shall denote the measurable ergodic equivalence relation on X generated by partition into orbits of the group Γ. Then, any cocycle $\alpha\in Z^1(X\times\Gamma, G)$ defines the map $u_\alpha: \mathcal{R}(\Gamma)\to G$ which is called an orbital cocycle and is defined by the formula

$$u_\alpha(y, x) = \alpha(x, \gamma)\,, \tag{1.3}$$

where $y=\gamma x$ and $\gamma\in\Gamma$ is found for x, y uniquely, because Γ acts freely. From (1.2) and (1.3) it follows that the orbital cocycle $u: \mathcal{R}(\Gamma)\to G$ satisfies the relation $u(z, x)=u(z, y)\,u(y, x)$, where (z, x), (z, y), $(y, x)\in\mathcal{R}(\Gamma)$. The reverse statement is also true: for any measurable orbital cocycle u: $\mathcal{R}(\Gamma)\to G$ there exists a cocycle $\alpha\in Z^1(X\times\Gamma, G)$ such that $\alpha(x, \gamma)=u(\gamma x, x)$ [15].

For the freely acting group Γ, any cocycle α may be enlarged in a natural way to the full group $[\Gamma]$. Therefore, wherever convenient, we shall believe that $\alpha\in Z^1(X\times[\Gamma], G)$.

The two cocycles α and β from $Z^1(X\times\Gamma, G)$ are called Γ-cohomologous, if there exists a measurable function f: $X\to G$ such that

$$\alpha(x, \gamma) = f(\gamma x)\,\beta(x, \gamma)\,f(x)^{-1}. \tag{1.4}$$

A cocycle α is called a coboundary, if it is Γ-cohomologous to the unit cocycle, i.e. $\alpha(x, \gamma)=f(\gamma x)f(x)^{-1}$ for a measurable function f: $X\to G$.

1.4. Let the countable ergodic groups of automorphisms $\Gamma_i\subset \mathrm{Aut}(X_i, \mathcal{B}_i, \mu_i)$, $i=1, 2$ be orbital equivalent, i.e. let there exists a one-to-one map θ: $X_1\to X_2$ such that $\theta[\Gamma_1]\theta^{-1}=[\Gamma_2]$ and $\theta\circ\mu_1\sim\mu_2$. Let there be defined a cocycle $\beta\in Z^1(X_2\times[\Gamma_2], G)$ then, by the map θ, the cocycle β can be "transfered" to the group $[\Gamma_1]$:

$$\theta^{-1}\circ\beta(x_1, \gamma_1) = \beta(\theta x_1, \theta\gamma_1\theta^{-1}),\ (x_1, \gamma_1)\in X_1\times[\Gamma_1]\,. \tag{1.5}$$

Relation (1.5) defines the one-to-one correspondence between the cocycles from $Z^1(X_1\times[\Gamma_1], G)$ and those from $Z^1(X_2\times[\Gamma_2], G)$. In this case, the Γ_1-cohomologous cocycles correspond to Γ_2-cohomologous cocycles and conversely. This is presented in more detail in [16].

We shall consider all the pairs (Γ, α), where Γ is a countable ergodic group of automorphisms of $(X, \mathcal{B}, \mu)$, $\alpha\in Z^1(X\times\Gamma, G)$ and define, on such the set of pairs, an equivalence relation generalizing the orbital equivalence of the groups of automorphisms. Then, we shall develop a complete system of invariants of such the equivalence relation.

Definition 1.2. *Let there be the two pairs (Γ_i, α_i), $i=1, 2$, where Γ_i is a freely acting group of automorphisms of $(X_i, \mathcal{B}_i, \mu_i)$ and $\alpha_i\in Z^1(X_i\times\Gamma_i, G)$. We shall call the pairs (Γ_1, α_1) and (Γ_2, α_2) weakly equivalent, if there exists a map θ: $X_1\to X_2$ which implies the orbital equivalence of Γ_1 and Γ_2 and is such that the cocycle $\theta^{-1}\circ\alpha_2$ is Γ_1-cohomologous to the cocycle α_1.*

If the cocycles $\theta^{-1}\circ\alpha_2$ and α_1 are Γ_1-cohomologous, then the cocycles $\theta\circ\alpha_1$ and α_2 are Γ_2-cohomologous. Thus, Definition 1.2 indeed suggests the equivalence relation on the set of pairs (Γ, α).

If (Γ, α_1) and (Γ, α_2) are weakly equivalent, then the cocycles α_1 and α_2 will also be called weakly equivalent.

In the case $\alpha_i(x_i, \gamma_i)=\rho_i(x_i, \gamma_i)$ Definition 1.2 coincides with the definition of the orbital equivalence of groups of automorphisms.

1.5. Consider on the group $\boldsymbol{Z}$ the Haar measure $\chi_{\boldsymbol{Z}}$, i.e. $\chi_{\boldsymbol{Z}}(i)=1$, $i\in\boldsymbol{Z}$. Denote by τ the shift on $\boldsymbol{Z}$: $\tau(i)=i+1$. Let $\Gamma\subset\mathrm{Aut}\,(X, \mathcal{B}, \mu)$ be a countable ergodic group of automorphisms and consider the direct product $\tilde{\Gamma}=\Gamma\times\{\tau^n: n\in\boldsymbol{Z}\}\subset\mathrm{Aut}\,(X\times\boldsymbol{Z}, \mu\times\chi_{\boldsymbol{Z}})$. If $\alpha\in Z^1(X\times\Gamma, G)$, then define the cocycle $\tilde{\alpha}$ for $\tilde{\Gamma}$: $\tilde{\alpha}(x, n, \gamma, \tau^k)=\alpha(x, \gamma)$, where $(x, n)\in X\times\boldsymbol{Z}$, $k\in\boldsymbol{Z}$, $\gamma\in\Gamma$. Thus, $\tilde{\alpha}\in \boldsymbol{Z}^1(X\times\boldsymbol{Z}\times\tilde{\Gamma}, G)$. The pair $(\tilde{\Gamma}, \tilde{\alpha})$ will be called the countable expansion of (Γ, α).

Definition 1.3. *Call the two pairs (Γ_1, α_1) and (Γ_2, α_2) stably weakly equivalent, if their countable expansions $(\tilde{\Gamma}_1, \tilde{\alpha}_1)$ and $(\tilde{\Gamma}_2, \tilde{\alpha}_2)$ are weakly equivalent.*

Let $B\subset X(\mu B>0)$ and $\Gamma\subset\mathrm{Aut}\,(X, \mathcal{B}, \mu)$. Then, there exists in $[\Gamma]$ a countable group Γ_B, such that $[\Gamma_B]=[\Gamma]_B$, where $[\Gamma]_B=\{\gamma\in[\Gamma]: \gamma x=x, x\in X-B\}$ [9]. If Γ is of type III, then Γ_B and Γ are orbital equivalent and there exists a one-to-one measurable map θ: $X\to B$ such that $\theta x\in\Gamma x$ for μ-a.a. $x\in X$. If $\alpha\in Z^1(X\times\Gamma, G)$, then a cocycle $\alpha_B\in Z^1(X\times[\Gamma]_B, G)$ can be defined as $\alpha_B(x, \gamma_B)=\alpha(x, \gamma)$, $(x, \gamma_B)\in X\times\Gamma_B$, where $\gamma\in\Gamma$ can be found from the condition $\gamma_B x=\gamma x$.

Proposition 1.4. *Let there be a pair (Γ, α), where $\Gamma\subset\mathrm{Aut}\,(X, \mathcal{B}, \mu)$ and let $B\subset X$, $\mu(B)>0$. Then,* (1) *if Γ is of type III or Γ is of type II_∞ and $\mu(B)=\infty$, then (Γ, α) and (Γ_B, α_B) are weakly equivalent*; (2) *if Γ is of type II_1 or II_∞ and $\mu(B)<\infty$, then (Γ, α) and (Γ_B, α_B) are stably weakly equivalent.*

Proof. The proof is simple. □

Corollary 1.5. *If Γ_i, $i=1, 2$ are groups of automorphisms of type III or II_∞, then the pairs (Γ_1, α_1) and (Γ_2, α_2) are weakly equivalent if and only if they are stably weakly equivalent.*

If the group $\Gamma\subset\mathrm{Aut}\,(X, \mathcal{B}, \mu)$ is of type III, i.e. has the nontrivial Radon-Nikodym cocycle $\rho(x, \gamma)$, then it is natural to consider, along with the cocycle $\alpha\in Z^1(X\times\Gamma, G)$ the cocycle $\alpha_0\in Z^1(X\times\Gamma, G\times\boldsymbol{R})$ defined by the formula $\alpha_0(x, \gamma)=(\alpha(x, \gamma), \rho(x, \gamma))$.

Proposition 1.6. *The pairs (Γ_1, α_1) and (Γ_2, α_2), where $\Gamma_i\subset\mathrm{Aut}\,(X_i, \mathcal{B}_i, \mu_i)$,*

$i=1$, 2 *are weakly equivalent if and only if the pairs* $(\Gamma_1, \alpha_{1,0})$ *and* $(\Gamma_2, \alpha_{2,0})$ *are* weakly equivalent.

Proof. The proof is simple. □

We notice that there is an example of the pairs, which are stably weakly equivalent, but are not weakly equivalent [2].

1.6. Consider in more detail the properties of the cocycles with values in an abelian l.c.s. group G. By $\bar{G}=G\cup\{\infty\}$, we shall denote the one-point compactification of G.

Definition 1.7. *Let* Γ *be a countable ergodic group of automorphisms of* $(X, \mathcal{B}, \mu)$ *and* $\alpha\in Z^1(X\times\Gamma, G)$. *An element* $f\in\bar{G}$ *is called the essential value of the cocycle* α, *if for any neighborhood* V_f *of* f *in* $\bar{G}$ *and any set* $B\in\mathcal{B}$, $\mu(B)>0$ *we have*

$$\mu(\bigcup_{\gamma\in\Gamma}(B\cap\gamma^{-1}B\cap\{x\in X\colon \alpha(x, \gamma)\in V_f\}))>0\,.$$

The set of all essential values of the cocycle α will be denoted by $\bar{r}(\Gamma, \alpha)$. Put $r(\Gamma, \alpha)=\bar{r}(\Gamma, \alpha)\cap G$.

If $r(\Gamma, \alpha)=G$, then we shall say that the cocycle α has a dense range in the group G.

Lemma 1.8. [15] *The following statements are true*: (i) *if the pairs* (Γ_1, α_1) *and* (Γ_2, α_2) *are weakly equivalent, then* $\bar{r}(\Gamma_1, \alpha_1)=\bar{r}(\Gamma_2, \alpha_2)$; (ii) $r(\Gamma, \alpha)$ *is a closed subgroup of* G; (iii) *a cocycle* α *is a coboundary if and only if* $\bar{r}(\Gamma, \alpha)=\{0\}$.

Applying Lemma 1.8 to the pair (Γ, ρ), where Γ is a group of type III, we conclude that $r(\Gamma, \rho)$ can be only one of the following groups: $\{0\}$, $\{n\log\lambda\colon n\in\boldsymbol{Z}\}$ $(0<\lambda<1)$ and $\boldsymbol{R}$. Accordingly, Γ is called a type III_0, III_λ and III_1 group. The group Γ is said to be of type I, if the partition into its orbit is measurable.

Lemma 1.9. *For any pair* (Γ, α) *we have* $\bar{r}(\Gamma, \alpha)=\bar{r}(\tilde{\Gamma}, \tilde{\alpha})$.

Proof. Straightforward. □

Since $r(\Gamma, \alpha)$ is a closed subgroup of G, then the quotient group $\hat{G}=G/r(\Gamma, \alpha)$ and the cocycle $\hat{\alpha}\in Z^1(X\times\Gamma, \hat{G})$ can be considered, setting

$$\hat{\alpha}(x, \gamma) = \alpha(x, \gamma)+r(\Gamma, \alpha)\,. \tag{1.6}$$

Lemma 1.10. [15] *For any pair* $(\Gamma, \hat{\alpha})$ *always* $r(\Gamma, \hat{\alpha})=\{\hat{0}\}$, *where* $\hat{0}$ *is the identity in* $\hat{G}$.

Lemma 1.10 is the basis of the following definition.

Definition 1.11. *A cocycle* $\alpha \in Z^1(X \times \Gamma, G)$ *is called regular, if* $\bar{r}(\Gamma, \hat{\alpha}) = \{\hat{0}\}$ *and nonregular, if* $\bar{r}(\Gamma, \hat{\alpha}) = \{\hat{0}, \infty\}$.

Lemma 1.12. [15] *For any pair* (Γ, α) *the following conditions are equivalent*: (i) α *is a regular cocycle*; (ii) (Γ, α) *is weakly equivalent to* (Γ, α_1), *where a cocycle* $\alpha_1(x, \gamma)$ *takes values in* $r(\Gamma, \alpha)$ *for any* $\gamma \in \Gamma$ *and a.a.* $x \in X$.

Definition 1.13. *Let* Γ *be a freely acting conservative* (*i.e. not of type I*) *group of automorphisms of* $(X, \mathcal{B}, \mu)$. *A cocycle* $\alpha \in Z^1(X \times \Gamma, G)$ *is called recurrent, if for any set* $B \in \mathcal{B}$, $\mu(B) > 0$ *and any neighborhood* V *of the identity in* G

$$\mu\Big(\bigcup_{\gamma \in \Gamma} (B \cap \gamma^{-1} B \cap \{x \in X: \alpha(x, \gamma) \in V\})\Big) > 0 .$$

If a cocycle α *is not recurrent, then it is called transient.*

Lemma 1.14. *If* α *is a recurrent cocycle, then the cocycle* $\hat{\alpha}$ *defined according to* (1.6) *will also be recurrent.*

Proof. Straightforward. □

The following statement may be regarded as another definition of the transient cocycles.

Proposition 1.15. [15] *A cocycle* $\alpha \in Z^1(X \times \Gamma, G)$ *is transient if and only if there exists a measurable set* $B_0 \subset X$, $\mu(B_0) > 0$ *and there is a neighborhood* V_0 *of the identity in* G *both such that*

$$\mu\Big(\bigcup_{\gamma \in \Gamma} (B_0 \cap \gamma^{-1} B_0 \cap \{x \in X: \alpha(x, \gamma) \in V_0\})\Big) = 0 \tag{1.7}$$

It follows from (1.7) that for the group $\Gamma_{B_0} \subset [\Gamma]$ the cocycle α_{B_0} does not take the values in V_0.

For a transient cocycle $\alpha \in Z^1(X \times \Gamma, G)$ always $\bar{r}(\Gamma, \alpha) = \{0, \infty\}$.

Definition 1.16. *A cocycle* $\alpha \in Z^1(X \times \Gamma, G)$ *is called lacunary, if there exists a neighborhood* V_0 *of the identity in* G *such that*

$$\mu\Big(\bigcup_{\gamma \in \Gamma} \{x \in X: \alpha(x, \gamma) \in V_0 - \{0\}\}\Big) = 0 .$$

Lemma 1.17. [15] *A cocycle* $\alpha \in Z^1(X \times \Gamma, G)$ *is cohomologous to a lacunary cocycle, if and only if there exists a neighborhood* V_1 *of the identity in* G *such that* $r(\Gamma, \alpha) \cap V_1 = \{0\}$.

§2. Associated Actions and Transient Cocycles

2.1. Everywhere in this section we shall assume that Γ is a freely acting ergodic group of automorphisms of a Lebesgue space $(X, \mathscr{B}, \mu)$, G an arbitrary l.c.s. group, and a cocycle $\alpha\in Z^1(X\times\Gamma, G)$. By χ_G the Haar measure on the group G will be denoted.

Let $\Gamma(\alpha)\subset\text{Aut}\,(X\times G, \mu\times\chi_G)$ be the skew product constructed by the group Γ and the cocycle α: for $\gamma(\alpha)\in\Gamma(\alpha)$, $(x, g)\in X\times G$

$$\gamma(\alpha)\,(x, g) = (\gamma x, \alpha(x, \gamma)g)\,, \quad \gamma\in\Gamma\,. \tag{2.1}$$

Lemma 2.1. [15] *$\Gamma(\alpha)$ is conservative if and only if α is a recurrent cocycle, and $\Gamma(\alpha)$ is of type I if and only if α is a transient cocycle.*

It follows from Lemma 2.1 that the properties of recurrence and transientness of cocycles are invariants of weak equivalence.

Define an action V of the group G on $(X\times G, \mu\times\chi_G)$:

$$V(g)\,(x, h) = (x, hg^{-1})\,, \quad g\in G\,. \tag{2.2}$$

It follows from (2.1) and (2.2) that the groups of automorphisms $\Gamma(\alpha)$ and $V(G)$ commutate elementwise. Let ξ be the measurable hull of partition into the orbits of $\Gamma(\alpha)$. Then, the group $V(G)$ generates on the quotient space $(\Omega, \nu)=((X\times G)/\xi, (\mu\times\chi_G)/\xi)$ a new action of G which will be denoted by $W_{(\Gamma,\alpha)}(G)$ or just $W(G)$.

Definition 2.2. *The action $W_{(\Gamma,\alpha)}(G)$ of the group G is called the action associated with the pair (Γ, α) or the Mackey action.*

Proposition 2.3. *If the pairs (Γ_1, α_1) and (Γ_2, α_2) are weakly equivalent, then the associated actions $W_{(\Gamma_1, \alpha_1)}(G)$ and $W_{(\Gamma_2, \alpha_2)}(G)$ are isomorphic.*

Proof. It follows from the above condition that there exists a one-to-one measurable map $\varphi: X_1\to X_2$ such that $\varphi[\Gamma_1]\varphi^{-1}=[\Gamma_2]$, $\varphi^{-1}\circ\mu_2\sim\mu_1$ and for μ_1-a.a. $x\in X_1$

$$\alpha_2(\varphi x, \varphi\gamma_1\varphi^{-1}) = f(\gamma_1 x)\,\alpha_1(x, \gamma_1)f(x)^{-1}, \quad \gamma_1\in[\Gamma_1]\,, \tag{2.3}$$

where $\Gamma_i\subset\text{Aut}\,(X_i, \mathscr{B}_i, \mu_i)$, $i=1, 2$ and $f: X_1\to G$ is a measurable map. Define

$$\Phi(x, h) = (\varphi x, f(x)h)\,, \quad (x, h)\in X_1\times G\,. \tag{2.4}$$

Therefore, the quotient map $\tilde{\Phi}$: $(\Omega_1, \nu_1)\to(\Omega_2, \nu_2)$ satisfies the equality $\tilde{\Phi}W_{(\Gamma_1,\alpha_1)}(g)\,\tilde{\Phi}^{-1}=W_{(\Gamma_2,\alpha_2)}(g)$, $g\in G$. □

Proposition 2.4. (1) *Let* $(\tilde{\Gamma}, \tilde{\alpha})$ *be a countable expansion of a pair* (Γ, α). *Then,* $W_{(\tilde{\Gamma},\tilde{\alpha})}(G)$ *is isomorphic to* $W_{(\Gamma,\alpha)}(G)$. (2) *If the pairs* (Γ_1, α_1), *and* (Γ_2, α_2) *are stably weakly equivalent, then the associated actions* $W_{(\Gamma_1,\alpha_1)}(G)$ *and* $W_{(\Gamma_2,\alpha_2)}(G)$ *are isomorphic.*

Proof. Straightforward. □

2.2. Below we shall consider the situation, where a cocycle α from a pair (Γ, α) is transient and takes the values in the l.c.s. group G.

It follows from Lemma 2.1 that the partition into orbits of the group $\Gamma(\alpha) \subset \mathrm{Aut}\,(X\times G, \mu\times\chi_G)$ is measurable. Then, the quotient space Ω can be regarded as a measurable subset of positive measure in $X\times G$, which intersect with each orbit of $\Gamma(\alpha)$ exactly at one point. Then, $X\times G=\bigcup_{\gamma\in\Gamma}\gamma(\alpha)\Omega$ and the measure $\nu=(\mu\times\chi_G)\,|_{\Omega}$. The action $W(G)=W_{(\Gamma,\alpha)}(G)$ associated with (Γ, α) will be written as follows. Let $(x, h)\in\Omega$, $g\in G$, then

$$W(g)\,(x, h) = \gamma(\alpha)\,(x, hg^{-1})\,, \tag{2.5}$$

where $\gamma(\alpha)$ is an element of $\Gamma(\alpha)$ such that $(x, hg^{-1})\in\gamma(\alpha)^{-1}\Omega$.

The further arguments are convenient to be made in terms of the measurable groupoid theory (see [3] and [12]). Denote by $\mathcal{H}$ a measurable groupoid with discrete orbits which is defined by the action of Γ on X. Any cocycle $\alpha\in Z^1(X\times\Gamma, G)$ will define a homomorphism of the groupoid $\mathcal{H}$ into the group G. Denote by $\mathcal{G}$ a measurable groupoid with continuous orbits generated by the group $W(G)$ of automorphisms of Ω.

According to Proposition 1.15, for a transient cocycle α there exists in X a measurable subset B $(\mu(B)>0)$ for which $\alpha(x, \gamma_B)\notin V_0$, where $x\in B$, $\gamma_B\in[\Gamma]_B$ and V_0 is a neighborhood of the identity in G. Consider the pair (Γ_B, α_B) and let $\mathcal{G}_B$ denote the measurable groupoid generated by the action $W_B(G)=W_{(\Gamma_B,\alpha_B)}(G)$ associated with (Γ_B, α_B), which was defined on the quotient space (Ω_B, ν_B), $\Omega_B\subset B\times G$.

Recall the definition of a return cocycle. Let $U(H)$ be a group of Borel automorphisms of $(X, \mathcal{B}, \mu)$ (not necessarily countable), which is a free Borel action of a l.c.s. group H. A set $E\subset X$ is called a complete lacunary section for $U(H)$ if $\mu(X-U(H)E)=0$ and there is a neighborhood V of the identity of H such that $U(V)x\cap E=\{x\}$ for all $x\in E$ [3]. There arises on E a countable measurable equivalence relation $\mathcal{R}$ and hence so does a group Γ of automorphisms of the set E generating the equivalence relation $\mathcal{R}$. Put $u(y, x)=h$ for $(y, x)\in\mathcal{R}$, if $U(h)x=y$. Since $U(H)$ acts freely, the orbital cocycle u is defined uniquely.

Definition 2.5. *The cocycle u: $\mathcal{R}\rightarrow H$ constructed in the above way is called a return cocycle (or homomorphism) for the action $U(H)$ with respect to the set E (or in short, a return cocycle on E).*

If $\mathcal{G}$ is a measurable groupoid and E a subset of its set of units, then $\mathcal{G}|_E$ will denote the reduction of the groupoid $\mathcal{G}$ on E.

The following lemma is a slightly modified version of Lemma 7.4 of [3].

Lemma 2.6. *Let the groupoids $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{G}_B$ be as above, α: $\mathcal{H}\rightarrow G$ be a transient homomorphism, and $\alpha_B \notin V_0$, where V_0 is a neighborhood of the identity in G. There exists a canonical isomorphism of the groupoid $\mathcal{H}|_B = B\times\Gamma_B$ onto the groupoid $\mathcal{G}_B|_E$, where E is a complete lacunary section of the action $W_{(\Gamma_B, \alpha_B)}(G) = W_B(G)$ on Ω_B, such that for this isomorphism the homomorphism α_B: $\mathcal{H}|_B \rightarrow G$ transforms into the return homomorphism of the action $W_B(G)$ on E,*

Proof. Let us introduce a measurable map θ: $B\rightarrow\Omega_B$ putting $\theta x=(y, h)$, if there exists an automorphism $\gamma_B(\alpha_B)\in\Gamma_B(\alpha_B)$ depending on the point $x\in B$, for which $\gamma_B(\alpha_B)\,(x, e)=(y, h)$. Prove that $\theta(B)$ is a complete lacunary section for $W_B(G)$ on (Ω_B, ν_B). Indeed, since $B\times\{e\}$ intersects every orbit of the group $V(G)$, then $\theta(B)$ will also intersects every orbit of the group $W_B(G)$, which means completeness of the section $\theta(B)$. Describe now the equivalence relation which generates on $\theta(B)$ the associated action $W_B(G)$. If $y=\gamma x$, where $x, y\in B$, $\gamma\in[\Gamma]_B$, then we shall show that

$$\theta(y) = W_B(\alpha_B(x, \gamma))\,\theta(x)\,. \tag{2.6}$$

Let $\theta(x)=(\gamma_1 x, \alpha_B(x, \gamma_1))$, $\theta(y)=(\gamma_2 y, \alpha(y, \gamma_2))$, $\gamma_1, \gamma_2\in[\Gamma]_B$. Then,

$$W_B(\alpha_B(x, \gamma))\,\theta(x) = \gamma_0(\alpha_B)\,(\gamma_1 x, \alpha_B(x, \gamma_1)\,\alpha_B(x, \gamma)^{-1}), \tag{2.7}$$

where the automorphism $\gamma_0(\alpha_B)$ is chosen from the condition $\gamma_0(\alpha_B)\,(\gamma_1 x, \alpha_B(x, \gamma_1)\,\alpha_B(x, \gamma^{-1}))\in\Omega_B$. It is easy to see that the role γ_0 can be played by $\gamma_2\gamma\gamma_1^{-1}$. Indeed,

$$\gamma_2\gamma\gamma_1^{-1}(\alpha_B)\,(\gamma_1 x, \alpha_B(x,\gamma_1)\,\alpha_B(x, \gamma)^{-1}) = (\gamma_2 y, \alpha_B(y_1, \gamma_2)) = \theta(y)\,, \tag{2.8}$$

By comparing (2.8) and (2.7), we conclude that (2.6) is true.

Vice versa, if $W_B(h)\theta(x)=\theta(y)$, then similar arguments can show that there exists an automorphism $\gamma\in[\Gamma]_B$ for which $\gamma x=y$ and

$$h = \alpha_B(x, \gamma)\,. \tag{2.9}$$

Thus, it is proved that the orbits of the equivalence relation on $\theta(B)$ are countable, because the cocycle $\alpha_B \notin V_0$. Therefore, the section $\theta(B)$ of the action $W_B(G)$ is lacunary. Besides, it follows from (2.6) and (2.9) that it is the cocycle α_B that corresponds to the return cocycle of the action $W_B(G)$ on $\theta(B)$. Consider the measure $\theta\circ\mu$ on $\theta(B)$, and let E be a Borel subset of $\theta(B)$ which is full with respect to $\theta\circ\mu$. Because the later is zero on sets from $\theta(B)$ if and only if the measure μ is zero on the respective sets from B, then $\theta\circ\mu$ is equivalent to the projection of the measure $(\mu\times\chi_G)|_{\Omega_B}$ on the section E along the orbits of $W_B(G)$. Therefore, the map $(y, x)\to(\theta(y), \theta(x))$ is the desired isomorphism of the groupoids $\mathcal{H}|_B$ and $\mathcal{G}_B|_E$. $\square$

Remark 2.7. Proposition 1.4, 2.3 and 2.4 imply that for any group $\Gamma\subset \mathrm{Aut}(X, \mathcal{B}, \mu)$ and any $B\subset X$ $(\mu(B)>0)$ the associated actions $W_{(\Gamma,\alpha)}(G)$ and $W_{(\Gamma_B,\alpha_B)}(G)$ are isomorphic. Therefore, everywhere in this section we shall denote the associated action by $W(G)$.

Lemma 2.8. *Let $W(G)$ be a free non-singular action of a l.c.s. group G on (Ω, ν). Let also E_1 and E_2 be complete sections of the action $W(G)$ on Ω and $\mathcal{G}_1=\mathcal{G}|_{E_1}$, $\mathcal{G}_2=\mathcal{G}|_{E_2}$ be the reductions of the groupoid $\mathcal{G}=\Omega\times W(G)$ on E_1 and E_2. Assume that there exists an isomorphism $\psi\colon \mathcal{G}_1\to\mathcal{G}_2$ and let ψ_0 be the restriction of the map ψ onto E_1, i.e. $\psi_0\colon E_1\to E_2$. If ψ_0 is an inner automorphism of the groupoid $\mathcal{G}$, i.e. the points x and $\psi_0(x)$ lie in one and the same orbit of $W(G)$, then the return homomorphisms $\pi_1\colon \mathcal{G}_1\to G$ and $\psi^{-1}\circ\pi_2\colon \mathcal{G}_1\to G$ are equivalent (or in other words, the cocycles π_1 and $\psi^{-1}\circ\pi_2$ are cohomologous).*

Proof. Define the homomorphism $\pi\colon \mathcal{G}\to G$ with putting for $(\omega, W(h))\in \mathcal{G}$ $(\omega\in\Omega)$

$$\pi(\omega, W(h)) = h\,. \tag{2.10}$$

Then the return homomorphisms π_1 and π_2 are related to π as follows

$$\pi_1 = \pi/\mathcal{G}_1, \quad \pi_2 = \pi/\mathcal{G}_2\,. \tag{2.11}$$

By the condition of the lemma, the points x and $\psi_0(x)$, $x\in E_1$ are related as $\psi_0(x)=W(h)x$, where $h=h(x)\in G$. The measurability of the function $h(x)$ follows from the measurability of the maps $x\to\psi_0(x)$, $(x, h)\to W(h)x$. By the definition of the cocycle $\psi^{-1}\circ\pi_2$, we have the following for $(x, W(g))\in\mathcal{G}_1$:

$$\psi^{-1}\circ\pi_2(x, W(g)) = h(W(g)x)\, gh(x)^{-1}, \tag{2.12}$$

because $\psi_0 W(g)\,\psi_0^{-1}(\psi_0(x))\in E_2$. Formula (2.12) can be written, according to

(2.10) and (2.11), as $\psi_0^{-1}\circ\pi_2(x, W(g))=h(W(g)x)\pi_1(x, W(g))h(x)^{-1}$, whence follows the statement of the lemma. □

2.3. If Γ is a countable group of automorphisms of $(X, \mathcal{B}, \mu)$ and a cocycle $\alpha\in Z^1(X\times\Gamma, G)$, then the pair (Γ, α) corresponds to the pair $(\mathcal{H}, \alpha)$, where $\mathcal{H}$ is a measurable groupoid generated by the action Γ on X and α a homomorphism from $\mathcal{H}$ into G. The two pairs $(\mathcal{H}_1, \alpha_1)$ and $(\mathcal{H}_2, \alpha_2)$ will be weakly equivalent, if there exists an isomorphism φ of the groupoids $\mathcal{H}_1$ and $\mathcal{H}_2$ such that the homomorphisms α_1 and $\varphi^{-1}\circ\alpha_2$ are equivalent. (The definition of equivalent homomorphisms see e.g. in [3]).

Theorem 2.9. *Let there be defined the two pairs $(\mathcal{H}_1, \alpha_1)$ and $(\mathcal{H}_2, \alpha_2)$, where $\mathcal{H}_i$ is a discrete measurable groupoid generated by an action of the group $\Gamma_i\subset$ $\mathrm{Aut}(X_i, \mathcal{B}_i, \mu_i)$ and α_i is transient homomorphism from $\mathcal{H}_i$ into G, $i=1,2$. Then, if the associated actions $W_1(G)=W_{(\mathcal{H}_1,\alpha_1)}(G)$ and $W_2(G)=W_{(\mathcal{H}_2,\alpha_2)}(G)$ are isomorphic, then the pairs $(\mathcal{H}_1, \alpha_1)$ and $(\mathcal{H}_2, \alpha_2)$ are stably weakly equivalent.*

Proof. Since α_i $(i=1, 2)$ is a transient homomorphism, then there exists a set B_i and a neighborhood V_0 of the identity in G, such that $\alpha_i(x_i, \gamma_i)\notin V_0$ for $(x_i, \gamma_i)\in\mathcal{H}_i|_{B_i}$, $i=1,2$. We shall consider the pairs $(\mathcal{H}_1|_{B_1}, \alpha_{1B_1})$ and $(\mathcal{H}_2|_{B_2}, \alpha_{2B_2})$, Let $W_{iB_i}(G)$, $\mathcal{H}_i|_{B_i}$, $\mathcal{G}_i|_{B_i}$ and $(\Omega_{B_i}, \nu_{B_i})$, $i=1,2$ be as in Lemma 2.6. It follows from the condition of the theorem and Proposition 2.4 that there exists a one-to-one measurable map $\psi\colon \Omega_{B_1}\to\Omega_{B_2}$ such that $\psi\circ\nu_{B_1}\sim\nu_{B_2}$ and

$$\psi W_{1B_1}(g) = W_{2B_2}(g)\psi\,, \quad g\in G\,, \tag{2.13}$$

where $W_{iB_i}(G)$ is the action of G associated with $(\mathcal{H}_i|_{B_i}, \alpha_{iB_i})$, $i=1, 2$. In view of (2.13) the isomorphism Ψ of the groupoids $\mathcal{G}_1|_{B_1}=\mathcal{G}_{B_1}$ and $\mathcal{G}_2|_{B_2}=\mathcal{G}_{B_2}$ is defined by the map

$$\Psi\colon ((x, h), g) = (\psi(x, h), g)\,, \quad (x, h)\in\Omega_{B_1}\,. \tag{2.14}$$

By applying Lemma 2.6, we find that there exist canonical isomorphisms j_1: $\mathcal{H}_1|_{B_1}\to\mathcal{G}_{B_1}|_{E_1}$ and j_2: $\mathcal{H}_2|_{B_2}\to\mathcal{G}_{B_2}|_{E_2}$, where E_i is a complete lacunary section for $W_{iB_i}(G)$ such that $j_i\circ\alpha_{iB_i}$ is a return cocycle for restriction of $W_{iB_i}(G)$ on E_i, $i=1, 2$. From relation (2.14) it follows that the map Ψ also defines the isomorphism of the groupoids $\mathcal{G}_{B_1}|_{E_1}$ and $\mathcal{G}_{B_2}|_{\psi(E_1)}$.

Put $\tilde{\mathcal{H}}_i|_{B_i}=\mathcal{H}_i|_{B_i}\times Z$, where $Z=\boldsymbol{Z}\times\boldsymbol{Z}$ is the transitive discrete groupoid generated by the shift on $\boldsymbol{Z}$. Consider two groupoids with the discrete orbits $\mathcal{G}_{B_2}|_{E_2}\times Z$ and $\mathcal{G}_{B_2}|_{\psi(E_1)}\times Z$. In view of Theorem 4.12 of [3], there exists an isomorphism τ can be inner for the groupoid $\mathcal{G}_{B_2}$. Let $J=[0, 1]\times[0, 1]$ be transitive groupoid with continuous orbits. It follows from [3, Corollary 4.4] that the groupoid $\mathcal{G}_{B_2}$ is isomorphic to the groupoid $\mathcal{G}_{B_2}|_{E_2}\times Z\times J$ and to keep the notation simple, we shall assume the groupoids to coincide. Putting $\tilde{\tau}=\tau\times\mathbb{1}\times\mathbb{1}$ enlarge τ to the automorphism $\tilde{\tau}$ of $\mathcal{G}_2$. By Theorem 1 of [7], there exists an automorphism q inner for $\mathcal{G}_{B_2}$ and such that $q\circ\tilde{\tau}=a\times\mathbb{1}$, where a is an automorphism of the groupoid $\mathcal{G}_{B_2}|_{E_2}\times Z$. With account of the equalities

$$q\circ\tilde{\tau}\,|_{\mathcal{G}_{B_2}|_{E_2}\times Z}=q\circ\tau\,|_{\mathcal{G}_{B_2}|_{E_2}\times Z}=a$$

we find that q maps $\mathcal{G}_{B_2}|_{\psi(E_1)}\times Z$ into $\mathcal{G}_{B_2}|_{E_2}\times Z$. Thus, it is proved that for the groupoid $\mathcal{G}_{B_2}$ there are two lacunary sections such that the groupoids $\mathcal{G}_{B_2}|_{\psi(E_1)}\times Z$ and $\mathcal{G}_{B_2}|_{E_2}\times Z$ defined on them are isomorphic, this isomorphism being inner with respect to $\mathcal{G}_{B_2}$. It follows from Lemma 2.6, that the homomorphisms $j_2\circ\tilde{\alpha}_{2B_2}$ and $\Psi\circ j_1\circ\tilde{\alpha}_{1B_1}$ are return ones for the groupoids $\mathcal{G}_{B_2}|_{E_2}\times Z$ and $\mathcal{G}_{B_2}|_{\psi(E_1)}\times Z$ (here $\tilde{\alpha}_{iB_i}$, $i=1, 2$ is the countable expansion of the cocycle α_{iB_i}). According to Lemma 2.8, the homomorphisms $j_2\circ\tilde{\alpha}_{2B_2}$ and $\psi\circ j_1\circ\tilde{\alpha}_{1B_1}$ are equivalent. It now remains to note that $\varphi=j_2^{-1}\circ q\circ\Psi\circ j_1$ is the isomorphism of the groupoids $\tilde{\mathcal{H}}_1|_{B_1}$ and $\mathcal{H}_2|_{B_2}$, and therefore the pairs $(\mathcal{H}_1, \alpha_1)$ and $(\mathcal{H}_2, \alpha_2)$ are stably weakly equivalent. □

Remark. The isomorphism τ (the subject of the proof of Theorem 2.9) may be directly chosen to be inner, as is obvious from simple considerations.

Our proof of Theorem 2.9 is similar in the idea to that in [7]; another proof was given in [2].

2.4. Here we shall consider the property of transientness for the cocycle $\alpha_0=(\alpha, \rho)$.

Proposition 2.10. *Let Γ be a countable group of automorphisms of $(X, \mathcal{B}, \mu)$. A cocycle $\alpha_0\in Z^1(X\times\Gamma, G\times\boldsymbol{R})$ is transient if and only if the cocycle $\alpha\in Z^1(X\times\Gamma, G)$ is transient.*

Proof. Let α be a transient cocycle. Consider the group of automorphisms $\Gamma_d\subset\mathrm{Aut}\,(X\times\boldsymbol{R}, \mu\times\chi_{\boldsymbol{R}})$ dual to Γ, whose elements act as $\gamma_d(x, u)=(\gamma x, u+\rho(x, \gamma))$, $\gamma\in\Gamma$ i.e. $\Gamma_d=\Gamma(\rho)$. The cocycle α is enlarged to the cocycle α' of the group Γ_d by the formula $\alpha'(x, u, \gamma_d)=\alpha(x, \gamma)$. The transientness of α means transientness of α'. Therefore, by Lemma 2.1, the partition into orbits of the group $\Gamma_d(\alpha')\subset\mathrm{Aut}\,(X\times\boldsymbol{R}\times G, \mu\times\chi_{\boldsymbol{R}}\times\chi_G)$ is measurable. As is easy to see, $\Gamma_d(\alpha')$ and $\Gamma(\alpha_0)$ coincide. By applying again Lemma 2 1, we find that α_0 is transient. Conversely, let α_0 be a transient cocycle. This means that $\Gamma(\alpha_0)$ is of type I group of automorphisms. It follows from the equality $\Gamma(\alpha_0)=\Gamma(\alpha)_d$ that the group $\Gamma(\alpha)_d$ is also of type I; but then it is evident that in this case the group $\Gamma(\alpha)\subset\text{Aut}(X\times G, \mu\times\chi_G)$ can only be of type I as well. Therefore, the cocycle α is transient. □

Corollary 2.11. *Let the pairs* $(\mathcal{H}_1, \alpha_1)$ *and* $(\mathcal{H}_2, \alpha_2)$ *be as in Theorem* 2.9. *Then, for* $G_0=G\times\boldsymbol{R}$: (i) *if the associated actions* $W_{(\mathcal{H}_1,(\alpha_1)_0)}(G_0)$ *and* $W_{(\mathcal{H}_2,(\alpha_2)_0)}(G_0)$ *are isomorphic. then* $(\mathcal{H}_1, \alpha_1)$ *and* $(\mathcal{H}_2, \alpha_2)$ *are stably weakly equivalent*; (ii) *if* $W_{(\mathcal{H}_1,\alpha_1)}(G)$ *and* $W_{(\mathcal{H}_2,\alpha_2)}(G)$ *are isomorphic, then so are the actions* $W_{(\mathcal{H}_1,(\alpha_1)_0)}(G_0)$ *and* $W_{(\mathcal{H}_2,(\alpha_2)_0)}(G_0)$.

§3. Measurable Fields of Cocycles

3.1. In this and next sections we shall only consider the cocycles α which take values in an abelian l.c.s. group G. However the findings are valid in more general situation (see the end of this section).

Now we shall construct, for any closed subgroup H_0 of $G_0=G\times\boldsymbol{R}$, an approximately finite (a.f.) ergodic group of automorphisms $\Delta\subset\text{Aut}(Y_0, p)$ and a cocycle β, both such that $\beta_0=(\beta, \rho)$ takes the values in H_0 and $r(\Delta, \beta_0)=H_0$.

Let $\{h_0(n)\}_{n=1}^{\infty}$ be a dense sequence of group elements from H_0, such that every member of this sequence occurs in it an infinite number of times. Every element $h_0(n)$ can be represented as $(h_G(n), h_{\boldsymbol{R}}(n))$, $n\in\boldsymbol{N}$. Choose $\{h_0(n)\}_{n=1}^{\infty}$ to belong in turn to a countable subgroup H_0' (dense in H_0). The projections of the groups H_0 and H_0' onto G and $\boldsymbol{R}$ will be denoted by H_G, $H_{\boldsymbol{R}}$ and H_G', $H_{\boldsymbol{R}}'$ respectively. The group H_0 and its projections H_G and $H_{\boldsymbol{R}}$ can be either discrete or continuous. The closed H_0, generally speaking, does not mean that H_G and $H_{\boldsymbol{R}}$ are closed.

Put $Y_0=\{0, 1\}^{\boldsymbol{N}}$ and $A_n(0)=\{y\in Y_0: y_n=0\}$, $A_n(1)=\{y\in Y_0: y_n=1\}$, $n\in\boldsymbol{N}$, where $y\in Y_0$ is $\{y_n\}_{n=1}^{\infty}$. Then, $A_n(0)\cap A_n(1)=\varnothing$ and $A_n(0)\cup A_n(1)=Y_0$, $n\in\boldsymbol{N}$. Consider on Y_0 the probability product-measure p, for which

$$p(A_n(1)) = \exp(h_{\boldsymbol{R}}(n))\, p(A_n(0))\,, \tag{3.1}$$

where the sequence $\{h_0(n)=(h_G(n), h_{\boldsymbol{R}}(n))\}_{n=1}^{\infty}$ has been chosen above. Introduce the automorphisms $\delta_n\in\text{Aut}(Y_0, p)$, $n\in\boldsymbol{N}$ such that $\delta_n\{y_k\}=\{y_k'\}$, where $y_k'=y_k$ with $k\neq n$ and $y_n'=y_n+1 \pmod 2$. It follows from (3.1) that

$$\rho(y, \delta_n) = h_{\boldsymbol{R}}(n), \ \ y\in A_n(0), \ \ n\in\boldsymbol{N}. \tag{3.2}$$

Denote by Δ the group of automorphisms of (Y_0, p) generated by δ_n, $n\in\boldsymbol{N}$.

As is known, Δ is ergodic and a.f. [9]. If $H'_{\mathbf{R}}$ is a dense subgroup of $\mathbf{R}$, then Δ is of type III_1; if $H'_{\mathbf{R}}=\{n\log\lambda: h\in\mathbf{Z}\}$, then Δ is of type III_λ $(0<\lambda<1)$; and if $H'_{\mathbf{R}}=\{0\}$ (i.e. $H_0\subset G$), then Δ is of type II_1. Consider the case of the type III group Δ; type II group Δ is considered similarly.

Define the cocycle $\beta\in Z^1(Y_0\times\Delta, H_G)$ on the generators of the group Δ:

$$\beta(y,\delta_n)=\begin{cases}h_G(n), & y\in A_n(0)\\ -h_G(n), & y\in A_n(1),\quad n\in\mathbf{N}.\end{cases} \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$\beta_0(y,\delta_n)=h_0(n),\quad y\in A_n(0),\quad n\in\mathbf{N}. \tag{3.4}$$

Lemma 3.1. *Let Δ_0 be the subgroup of $[\Delta]$ specified as $\Delta_0=\{\gamma\in[\Delta]: \beta_0(y,\gamma)=0\}$. Then, Δ_0 is the ergodic group of automorphisms of (Y_0, p).*

Proof. By the choice of the sequence $\{h_0(n)\}_{n=1}^{\infty}$, there are infinitely many numbers n and n_1, such that $h_0(n)=h_0(n_1)$. Put for such n and n_1

$$(\gamma(n,n_1)\,y)_k=\begin{cases}y_{n_1}, & k=n\\ y_n, & k=n_1\\ y_k, & k\neq n, n_1.\end{cases} \tag{3.5}$$

From (3.2), (3.3), (3.4) and (3.5), we find that $\beta_0(y,\gamma(n,n_1))=0$ for all $y\in Y_0$, i.e. $\gamma(n,n_1)\in\Delta_0$. The group Δ_0 is ergodic; it is proved in same way, as in [9, Example 1]. □

Theorem 3.2. *The cocycle $\beta_0\in Z^1(Y_0\times\Delta, H_0)$ speciefied by* (3.4) *has a dense range in H_0,* i.e. $r(\Delta,\beta_0)=H_0$.

The proof follows in a transparent way form Lemma 3.1.

Corollary 3.3. *The action of the group $G_0=G\times\mathbf{R}$ associated with the pair (Δ,β_0) is isomorphic to the transitive action of G_0 on the quotient space G_0/H_0.*

The proof follows from Theorem 3.2 (see also Theorem 4.1 below).

3.2. We shall use the following notation: $(X_0,\mathcal{B}_0,\mu_0)$ and $(Y,\mathcal{F},\nu)$ are the Lebesgue spaces with probability measures and $(X,\mathcal{B},\mu)=(X_0\times Y,\mathcal{B}_0\times\mathcal{F},\mu_0\times\nu)$; S is an ergodic automorphism of $(Y,\mathcal{F},\nu)$ and $S_0=\mathbf{1}\times S\in\mathrm{Aut}(X,\mathcal{B},\mu)$. Let, as previously, H_0 be a closed subgroup of $G_0=G\times\mathbf{R}$.

Assume that for μ_0-a.a. $x_0\in X_0$ the cocycle $\alpha(x_0)\in Z^1(Y\times[S], G)$ is defined. We shall say that in this case a field of cocycles $x_0\to\alpha(x_0)$ $(x_0\in X_0)$ with values in G is defined. To each field of cocycles $x_0 \to \alpha(x_0)$ corresponds a map $\alpha: X \times [S_0] \to G$

$$\alpha(x_0, y, S_0^n) = \alpha(x_0)(y, S^n), \tag{3.6}$$

which satisfies the identity for the cocycles.

Definition 3.4. *A field of cocycles $x_0 \to \alpha(x_0)$, $x_0 \in X_0$ is called measurable, if the cocycle α defined by* (3.6) *is measurable.*

Definition 3.5. *A measurable field of cocycles $x_0 \to \alpha(x_0)$ has a dense range in a group $H \subset G$ (the notation $r(\{\alpha(\cdot)\}) = H$), if for a.a. $x_0 \in X_0$ the cocycle $\alpha(x_0)$ has a dense range in H.*

Each measurable field of cocycles $x_0 \to \alpha(x_0)$ generates also a measurable field $x_0 \to \alpha_0(x_0) = (\alpha(x_0), \rho)$ of cocycles taking values in the group H_0.

Below we shall consider only measurable fields of cocycles and assume S to be of type II_1 or III_λ $(0 < \lambda \leqslant 1)$.

Lemma 3.6. *The following statements are equivalent:*

(*i*) *a measurable field of cocycles $x_0 \to \alpha_0(x_0)$ has a dense range in H_0;*

(*ii*) *for any $h_0 \in H_0$, any set $A \subset X_0 \times Y$ of positive measure and any neighborhood V of the identity in G_0, there exist a measurable field of automorphisms $s_0 = (x_0 \to s(x_0)) \in [S_0]$ and a set $B \subset A$, such that $s_0 B \subset A$ and $\alpha_0(x_0)(y, s(x_0)) \in h_0 + V$ for a.a. $(x_0, y) \in B$;*

(*iii*) *for any $h_0 \in H_0$, any two sets A and B in $X_0 \times Y$ such that $\nu(A(x_0)) > 0 \Leftrightarrow \nu(B(x_0)) > 0$ (where $A(x_0)$ and $B(x_0)$ are x_0-sections of A and B) and any neighborhood U of the identity in G_0, there exist a set $A' \subset A$ of positive measure and an element $s_0' = (x_0 \to s'(x_0)) \in [S_0]$ such that $s_0' A' \subset B$ and $\alpha_0(x_0)(y, s'(x_0)) \in h_0 + U$ for a.a. $(x_0, y) \in A'$.*

The proof of this lemma is similar to that of Lemma 2.1 from [5].

Choose in the group G an invariant metric d compatible with the topology of the group G.

Next lemma is formulated and proved for the case of S of type III; the case of type II automorphism S is to be considered similarly.

Lemma 3.7. *Let a measurable field of cocycles $x_0 \to \alpha_0(x_0)$ be such that $r(\{\alpha_0(\cdot)\}) = H_0$ and $h_0 = (h_G, h_R) \in H_0$. Let A and B be subsets of $X = X_0 \times Y$, such that $\nu(A(x_0)) = \nu(B(x_0)) e^{h_R}$, where $A(x_0)$ and $B(x_0)$ are x_0-sections of A and B. Then, for any $\varepsilon > 0$, there exists an automorphism $s_0 \in [S_0]$ for which $s_0 A = B$*

and

$$\alpha_0(x_0)\,(y, s(x_0)) \in (h_G, h_R) + (V(\varepsilon) \times (-\varepsilon, \varepsilon))\,, \tag{3.7}$$

where $s_0 = (x_0 \to s(x_0))$ *and* $V(\varepsilon) = \{g \in G\colon d(0, g) < \varepsilon\}$.

Proof. Let us use statement (iii) of Lemma 3.6 and construct a map $t_1 \in [S_0]$ such that for a subset $A_1 \subset A$ of positive measure $t_1 A_1 = B_1 \subset B$ and

$$\alpha_0(x_0)\,(y, t_1(x_0)) \in h_0 + V_0(\varepsilon),\ (x_0, y) \in A_1\,, \tag{3.8}$$

where $t_1 = (x_0 \to t_1(x_0))$ and $V_0(\varepsilon) = V(\varepsilon) \times (-\varepsilon, \varepsilon)$. Choose here A_1 so that

$$\mu(t_1 A_1) \leqslant e^{h_R} \mu(A_1)\,. \tag{3.9}$$

Consider now the set of pairs (A_1, t_1) satisfying (3.8) and (3.9) and define on this set a partial order relation, assuming $(A_1', t_1') < (A_1'', t_1'')$ if $A_1' \subset A_1''$ and $t_1' = t_1''$ on A_1'. By Zorn lemma, there exists a maximal pair (A_0, t_0) with respect to such the order relation. Therefore, we readily conclude that $A_0 = A$ by mod 0 and the automorphism $t_0 = (x_0 \to t(x_0))$ satisfies the relation

$$\alpha_0(x_0)\,(y, t(x_0)) \in h_0 + V_0(\varepsilon) \tag{3.10}$$

for a.a. $(x_0, y) \in A$. Besides, $\mu(t_0 A) \leqslant \mu(B)$.

For the symmetry reasons, there exists an automorphism $w_0 = (x_0 \to w(x_0)) \in [S_0]$ such that $w_0 B \subset A$ and

$$\alpha_0(x_0)\,(y, w(x_0)) \in -h_G + V_0(\varepsilon) \tag{3.11}$$

for a.a. $(x_0, y) \in B$. From the maps t_0 and w_0, as in [9] we shall construct the Bernstein map $s_0 = (x_0 \to s(x_0)) \in [S_0]$ which is the one-to-one map from A onto B:

$$s_0(x_0, y) = \begin{cases} t_0(x_0, y),\ (x_0, y) \in \bigcup_{i=0}^{\infty} [(w_0 t_0)^i A - w_0 (t_0 w_0)^i B] \cup \bigcap_{i=0}^{\infty} (w_0 t_0)^i A \\ w_0^{-1}(x_0, y),\ (x_0, y) \in \bigcup_{i=0}^{\infty} (w_0 (t_0 w_0)^i B - (w_0, t_0))^{i+1} A)\,. \end{cases}$$

From (3.10) and (3.11), follows (3.7). □

Further it will be convenient to believe the cocycle α_0 to take values in the countable subgroup H_0' dense in H_0 (see Subsection 3.1). This assumption is not restrictive [6].

Lemma 3.8. *Let A be a measurable subset of positive measure in $X_0 \times Y$ and $\{h_0(i)\}_{i=1}^{N}$ a set of elements of the group H_0', where $h_0(i) = (h_G(i), h_R(i))$ and let the* *range of a measurable field of cocycles* $x_0 \to \alpha_0(x_0)$ *be the group* H_0. *Assume that* $\xi=(A, [0, N-1], B(i), r(\cdot, \cdot))$ *is an* S_0*-array of* A *such that* $\mu(B(i))=\exp(h_R(i))$ $\mu(B(0))$ *and for certain* $\varepsilon>0$ *and a.a.* $(x_0, y)\in B(0)$

$$\alpha_0(x_0)(y, r(i, 0))\in h_0(i)+V_0(\varepsilon), \quad i\in[1, N-1], \tag{3.12}$$

where $V_0(\varepsilon)=V(\varepsilon)\times(-\varepsilon, \varepsilon)$. *Then, there exists a measurable function* $f_0: A\to H_0'$ *such that* $f_0(x_0, y)=0$ *for a.a.* $(x_0, y)\in B(0)$ *and*

$$f_0(x_0, y)\in V_0(\varepsilon), \quad (x_0, y)\in A \tag{3.13}$$

and for the measurable field of cocycles

$$\beta_0(x_0)(y, s) = f_0(x_0, sy)+\alpha_0(x_0)(y, s)-f_0(x_0, y), \quad s\in[S] \tag{3.14}$$

the following relations are true for a.a. $(x_0, y)\in B(0)$:

$$\beta_0(x_0)(y, r(i, 0)) = h_0(i), \quad i = 1, 2, \cdots, N-1. \tag{3.15}$$

Proof. Put for $i=1, 2, \cdots, N-1$ the following

$$f_0(x_0, y) = \begin{cases} 0 \\ h_0(i)-\alpha_0(r(i, 0)^{-1}(x_0, y), r(i, 0)), (x_0, y)\in B(i), \end{cases} \tag{3.16}$$

where α_0 satisfies (3.6). Simple check shows that (3.13) and (3.15) follow from (3.12), (3.14) and (3.16). □

In other words, Lemma 3.8 states that the cocycle α may be replaced by the S_0-cohomologous cocycle β and the measure ν by the measure ν' equivalent to it, so that on the elements of the S_0-array ξ the cocycle β_0 should have constant values, the Radon-Nikodym cocycle in particular also becoming constant on such elements. In this case, the function f_0 performing the cohomologous replacement takes values in the prescribed neighborhood of the identity in H_0.

3.3. Before starting to prove the uniqueness theorem for fields of cocycles, let us consider the uniqueness theorem for individual cocycles with a dense range in the prescribed group $H_0\subset G_0$. The proof method of this theorem will then be extended, in a transparent way, to the case of measurable fields of cocycles.

Let Γ be an ergodic a.f. group of automorphisms of $(X, \mathcal{B}, \mu)$, $\mu(X)=1$ and an automorphism $T\in \mathrm{Aut}(X, \mathcal{B}, \mu)$ be such that $[\Gamma]=[T]$; let H_0 be a closed subgroup of G_0. Assume the cocycle $\alpha_0=(\alpha, \rho)$ to take values in H_0 and $H_0=r(\Gamma, \alpha)$. As has been said, according to whether the group H_R (where H_0

$=(H_G, H_R))$ is trivial, or discrete of the form $\mathbb{Z}\log\lambda$, or dense in $\mathbb{R}$, the group Γ is of types II_1, III_λ $(0<\lambda<1)$, or III_1. The case of Γ of type II was considered in [6].

Here we shall prove the following uniqueness theorem for the group Γ of automorphisms of type III.

Theorem 3.9. *Let the groups of automorphisms $\Gamma_i\subset\mathrm{Aut}\,(X_i, \mathcal{B}_i, \mu_i)$ and the cocycles α_0^i, $i=1, 2$ satisfy the conditions: $\alpha_0^i\in Z^1(X_i\times\Gamma_i, H_0)$, $r(\Gamma_i, \alpha_0^i)=H_0$, i.e. let the associated actions $W_{(\Gamma_i, \alpha_0^i)}(G_0)$ coincide with the transitive action of the group G_0 on G_0/H_0. Then, the pairs (Γ_1, α_0^1) and (Γ_2, α_0^2) are weakly equivalent.*

This theorem will be proved in several steps. Note now that Lemmas 3.6, 3.7, and 3.8 remain valid, if the field of cocycles in their formulations is replaced by one fixed cocycle.

Lemma 3.10. *Let $\xi=(X,[1,N], A(\cdot), \gamma(\cdot, \cdot))$ be a Γ-array on the set X, such that for $\varepsilon>0$ and a set $D\subset X$ of positive measure, the following is true:*

$$\mu(\{x\in X\colon Tx\in\mathcal{G}(\xi)x\})>1-\varepsilon\,,$$
$$\mu(D\Delta D_1)<\varepsilon\,, \tag{3.17}$$

where $D_1\in\mathcal{P}(\xi)$ and Γ, T and α_0 satisfy the conditions enumerated above. Then, there exists a Γ-array ξ_1 such that

$$\mu(\{x\in X\colon Tx\in\mathcal{G}(\xi_1)x\})>1-2\varepsilon\,,$$
$$\mu(D\Delta D_1')<2\varepsilon\,, \tag{3.18}$$

where $D_1'\in\mathcal{P}(\xi_1)$. Moreover, the cocycle α_0 is cohomologous to a cocycle α_0' which takes constant values on all elements of ξ_1, a function f_0 specifying the cohomologous replacement takes values in the neighborhood $V_0(2\varepsilon)=V(2\varepsilon)\times(-2\varepsilon, 2\varepsilon)$ of the identity in G_0.

Proof. Since α_0 takes values in the countable subgroup $H_0'\subset H_0$, then the functions $f^i(x)=\alpha_0(x, \gamma(i, 1))$, $i=1, 2, \cdots, N$, where $x\in A(1)$, are piecewise constant. Let $\{E_\omega\colon \omega\in\Omega\}$ be the partition of $A(1)$ into the sets on which $f^i(x)$, $i=1, 2, \cdots, N$ are constant. The set Ω is finite or countable (we shall naturally assume it countable). Let Ω' be a finite subset of Ω, such that

$$\mu\Big(\bigcup_{\omega\in\Omega'}\bigcup_{i=1}^{N}\gamma(i, 1)\,E_\omega\Big)>1-\frac{\varepsilon}{2}. \tag{3.19}$$

Let us believe for definiteness that $f^i(x)=(h_\omega^i, a_\omega^i)$ for $x\in E_\omega$, $\omega\in\Omega$, $i\in[1, N]$. Then, for the same values of ω, i

$$\mu(\gamma(i, 1)\, E_\omega) = e^{a^i_\omega}\mu(E_\omega)\,. \tag{3.20}$$

Put $q_\omega=\mu(E_\omega)$, $\omega\in\Omega$. The condition (3.19) then is

$$\sum_{\omega\in\Omega'}\sum_{i=1}^{N} e^{a^i_\omega} q_\omega > 1-\frac{\varepsilon}{2}\,.$$

From the given numbers $\varepsilon>0$ and $\{q_\omega\}_{\omega\in\Omega'}$, we can find the numbers $q>0$ and $n_\omega\in \boldsymbol{N}$ such that

$$\begin{aligned} &n_\omega q\leqslant q_\omega<(n_\omega+1)\, q\,,\\ &\sum_{i=1}^{N}\sum_{\omega\in\Omega'} e^{a^i_\omega}(q_\omega-n_\omega q)<\frac{\varepsilon}{2}\,, \end{aligned} \tag{3.21}$$

Let $F_\omega\subset E_\omega$, $\mu(F_\omega)=q$, $\omega\in\Omega'$. Inequalities (3.21) show that there exists a collection of sets $\{F_\omega(k)\}_{k=1}^{n_\omega}$ from E_ω such that $F_\omega(1)=F_\omega$, $F_\omega(k_1)\cap F_\omega(k_2)=\varnothing$ $(k_1\neq k_2)$, $\mu(F_\omega(k))=q$ for $k=1, 2, \cdots, n_\omega$, $\omega\in\Omega'$. Put

$$A'(1) = \bigcup_{\omega\in\Omega'}\bigcup_{k=1}^{n_\omega} F_\omega(k)\,,$$

$$A''(1) = (\bigcup_{\omega\in\Omega\setminus\Omega'} E_\omega)\cup(\bigcup_{\omega\in\Omega'}(E_\omega-\bigcup_{k=1}^{n_\omega} F_\omega(k)))\,. \tag{3.22}$$

Then, $A(1)=A'(1)\cup A''(1)$ and $A'(1)\cap A''(1)=\varnothing$. From (3.19), (3.20) and (3.22), it follows that

$$\mu(\bigcup_{j=1}^{N}\gamma(j, 1)\, A''(1))<\varepsilon\,. \tag{3.23}$$

Let us number the sets $F_\omega(k)$ successively: $F(1), F(2), \cdots, F(M)$, where $M=\sum_{\omega\in\Omega'} n_\omega$. Also, denote the functions $f^i(x)$ on $F(j)$ in a different way, putting $f^i(x)=(g^i_j, b^i_j)$ for $x\in F(j)$. The collection $\{(g^i_j, b^i_j)\colon i=1, 2, \cdots, N; j=1, 2, \cdots, M\}$ obviously coincides with the collection $\{(h^i_\omega, a^i_\omega)\colon i=1, 2, \cdots, N;\ \omega\in\Omega'\}$. Therefore, (3.20) leads to $\mu(\gamma(i, 1)\, F(j))=q\exp(b^i_j)$.

By applying Lemma 3.7, consider over the partition $(A'(1), [1, M], F(\cdot))$ a Γ-array η'_1, whose automorphisms $\delta(\cdot, \cdot)$ have the property

$$\alpha_0(x, \delta(j, 1))\in V_0(\varepsilon)\,,\quad j = 1, 2, \cdots, M \tag{3.24}$$

for a.a. $x\in F(1)$. Denote by η' a refinement of the array ξ by the array η'_1 defined on the set $A=\bigcup_{i=1}^{N}\gamma(i, 1)\, A'(1)$. The array η' consists of the sets $\gamma(i, 1)\, F(j)$ with the measure $e^{b^i_j}q$, $i=1, 2, \cdots, N$; $j=1, 2, \cdots, M$ and the group $\mathcal{G}(\eta')$ consists of the automorphisms $t'(\cdot, \cdot)$ defined by relations of the form $\gamma(i, 1)\, \delta(\cdot, \cdot)\, \gamma(i, 1)^{-1}$, $i, j\in 1, 2, \cdots, M$. For convenience, let us number again the sets of η' successively: $\eta'=(A, [1, MN], C'(\cdot), t'(\cdot, \cdot))$. Let us find out what the values are that the cocycle α_0 takes on elements of ξ'. Let for definiteness,

$C'(1)=F(1)$. If $C'(n)\subset A'(1)$, then, in view of (3.24) $\alpha_0(x, t'(n, 1))\in V_0(\varepsilon)$ for $x\in C'(1)$; if $C'(n)\subset \gamma(i, 1)\, A'(1)$, then for $x\in C'(1)$

$$\alpha_0(x, t'(n, 1))=\alpha_0(x, \gamma(i, 1)\,\delta(k, 1))\in(g_k^i, b_k^i)+V_0(\varepsilon)\,,$$

where k is specified by the equality $F(k)=\gamma(i, 1)^{-1}C'(n)$. Therefore, put $(g(n), b(n))=(0, 0)$ for $C'(n)\subset A'(n)$ and $(g(n), b(n))=(g_k^i, b_k^i)$ for $C'(n)=\gamma(i, 1)\, F(k)$. Thus, for a.a. $x\in C(1)$

$$\alpha_0(x, t'(n, 1))\in(g(n), b(n))+V_0(\varepsilon), \quad n = 1, 2, \cdots, NM\,. \tag{3.25}$$

Then, $\mu(C'(n))=e^{b(n)}q=r_n$.

Using (3.23), calculate the measure of the set $B=\bigcup_{i=1}^{N} \gamma(i, 1)A''(1)$:

$$\mu(B) = 1-\sum_{n=1}^{NM} e^{b(n)}q = 1-\sum_{n=1}^{NM} r_n<\varepsilon\,.$$

Subdivide B into the nonintersecting subsets $C''(n)$, $n\in[1, NM]$ such that

$$\mu(C''(n)) = \mu(C'(n))\, \mu(B)\, \mu(A)^{-1}. \tag{3.26}$$

Since $b(n)=\log r_n r_1^{-1}$, then for $r_n'=\mu(C''(n))$ it follows from (3.26) that $b(n)=\log r_n'(r_n')^{-1}$, i.e. $r_n r_1^{-1}=r_n'(r_1')^{-1}$. Since $(g(n), b(n))\in H_0'$, then, by Lemma 3.7, over the partition $(B, [1, NM], C''(\cdot))$ a Γ-array η'' can be defined having the automorphisms $t''(\cdot)$ such that for a.a. $x\in C''(1)$

$$\alpha_0(x, t''(n, 1))\in(g(n), b(n))+V_0(\varepsilon)\,, \quad n\in[1, NM]\,. \tag{3.27}$$

Construct the Γ-array $\xi_1=(X, [1, NM], C(\cdot), t(\cdot, \cdot))$ from the arrays η' and η'', putting

$$C(n) = C'(n)\cup C''(n)\,,$$
$$t(n, 1)\, x = \begin{cases} t'(n, 1)\, x\,, & x\in C'(1) \\ t''(n, 1)\, x\,, & x\in C''(1)\,, \quad n = 1, 2, \cdots, NM\,. \end{cases}$$

Then, it follows from (3.25) and (3.27) that

$$\alpha_0(x, t(n, 1))\in(g(n), b(n))+V_0(\varepsilon)\,, \quad n = 1, 2, \cdots, NM\,. \tag{3.28}$$

Apply Lemma 3.8 to the array ξ_1. Then, there exists a measurable function $f_0(x)$ taking values in $V_0(\varepsilon)$ and such that the cocycle

$$\alpha_0'(x, t) = f_0(tx)+\alpha_0(x, t)-f_0(x)\,, \quad t\in[T] \tag{3.29}$$

has the property

$$\alpha_0'(x, t(n, 1)) = (g(n), b(n)), \quad n = 1, 2, \cdots, NM\,, \tag{3.30}$$

for a.a. $x\in C(1)$. It follows from (3.17) and (3.23) that the array ξ_1 satisfies (3.18). Equalities (3.30) show that the cocycle α'_0 cohomologic to α_0 takes constant values on elements of ξ_1.

To conclude the proof of the lemma, let us estimate the change of the measure of X as a result of replacement of μ by μ' by formula (3.29) by means of the function $f_{\boldsymbol{R}}(x)$, where $f_0(x)=(f_G(x), f_{\boldsymbol{R}}(x))$. It follows from (3.29) that

$$e^{-\varepsilon}<\mu'(X)=\int_X \exp(f_{\boldsymbol{R}}(x))\,d\mu(x))<e^{\varepsilon}. \tag{3.31}$$

Normalize the measure μ', putting $\mu'_1(E)=\mu'(E)\,\mu'(X)^{-1}$ for $E\in\mathcal{B}$. This means that $f_{\boldsymbol{R}}(x)$ is replaced by the function $f'_{\boldsymbol{R}}(x)$ such that $\exp(f'_{\boldsymbol{R}}(x))=\mu'(X)^{-1}\exp(f_{\boldsymbol{R}}(x))$. It follows from (3.31) that $f'_{\boldsymbol{R}}(x)\in(-2\varepsilon, 2\varepsilon)$. □

Lemma 3.11. *Let $(X, \mathcal{B}, \mu)$, Γ, T, α_0, H_0 and H'_0 be as above. Then, there exists a sequence $\{\xi_n\}_{n=1}^{\infty}$ of Γ-arrays and a cocycle α'_0 cohomologous to the cocycle α_0, both such that*

(i) ξ_{n+1} *is the refinement of* ξ_n, $n\in\boldsymbol{N}$;

(ii) $\{T_X^m\colon m\in\boldsymbol{Z}\}=\bigcup_{n=1}^{\infty}\mathcal{G}(\xi_n)x$ *for a.a.* $x\in X$;

(iii) $\mathcal{B}=\sigma(\bigcup_{n=1}^{\infty}\mathcal{P}(\xi_n))$;

(iv) *on any element of* ξ_n, $n\in\boldsymbol{N}$ *the cocycle* α'_0 *takes a constant value.*

Proof. Let a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ monotonically converges to zero and $\sum_{n=1}^{\infty}\varepsilon_n<\infty$. Let $\{D_n\}_{n=1}^{\infty}$ be a dense sequence of sets in $\mathcal{B}$, whose every element occurs in it an infinite number of times. Apply Lemma 3.10 and construct a Γ-array $\xi_1=(X, [1, N_1], A_1(\cdot), \gamma_1(\cdot, \cdot))$ and a cocycle $\alpha_0^{(1)}$ cohomologous to α_0, such that the group $\mathcal{G}(\xi_1)$ approximates the orbits of T accurate to ε_1 and $\mathcal{P}(\xi_1)$ approximates the set D_1 also accurate to ε_1 (i.e. inequalities similar to (3.17) are true). The cocycle $\alpha_0^{(1)}$ takes constant values on elements of ξ_1 and is obtained from the cocycle α_0 by the cohomologous transition defined by a transfer function $f_0^{(1)}(x)$. This function has the properties: $f_0^{(1)}(x)\in V_0(\varepsilon)$ and $f_0^{(1)}(x)=0$ for $x\in A_1(1)$. Then, construct on $A_1(1)$ a Γ-array ξ'_1 such that the array $\xi_2=\xi_1\times\xi'_1$ approximates the orbits of T and the set D_1 accurate to ε_2. Here, if $\xi'_1=(A_1(1), [1, N'_1], A'_1(\cdot), \gamma'_1(\cdot, \cdot))$, then on the set $A'_1(1)$ one can define the function $\bar{f}_0^{(1)}(x)$ such that $\bar{f}_0^{(1)}(x)=0$ on $A'_1(1)$ and $\bar{f}_0^{(1)}(x)\in V_0(\varepsilon_2)$. Denote by $f_0^{(2)}(x)$ the $\mathcal{G}(\xi_1)$-invariant function obtained from $\bar{f}_0^{(1)}(x)$ by shifts by $\gamma_1(i, 1)$, $i=1, 2, \cdots, N_1$. According to Lemma 3.10 $\bar{f}_0^{(1)}(x)$ and ξ'_1 can be chosen to be such that the cocycle $\alpha_0^{(2)}(x, t)=f_0^{(2)}(tx)+\alpha_0^{(1)}(x, t)-f_0^{(2)}(x)$ takes constant values on elements of the array ξ_2, $t\in[T]$.

Continuing the procedure let us construct the sequence of the arrays $\{\xi_n\}_{n=1}^{\infty}$

satisfying conditions (i)-(iii) of the present lemma and also the sequence of the functions $\{f_0^{(n)}(x)\}_{n=1}^{\infty}$ such that the series $\sum_{n=1}^{\infty} f_0^{(n)}(x)$ converges to a function $f_0(x)$. Then, according to the choice of $f_0^{(n)}(x)$, the cocycle $\alpha_0'(x, t)=f_0(tx)+\alpha_0(x, t)-f_0(x)$, $t\in[T]=[\Gamma]$ takes the constant values on all elements of the array ξ_n, $n\in N$. □

Remark 3.12. It follows from the construction of the function f_0 that, generally speaking, $f_0 \notin H_0'$. However, the cocycle α_0' takes, as α_0 does, its values in the group H_0' because the functions $f_0^{(n)}(x)$ are invariant with respect to automorphisms from the group $\mathcal{G}(\xi_i)$, $i=1, 2, \cdots, n$; $n\in N$.

Proof of Theorem 3.9. Recall that the cocycles α_0^i, $i=1, 2$ take the values in the group $H_0'\subset H_0$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ and $\{D_n\}_{n=1}^{\infty}$ be sequences the same as in proof of Lemma 3.11. Define for ε_1, D_1 and α_0^1 a Γ_1-array ξ_1 and a Γ_1-cohomologous cocycle $\bar{\alpha}_0^1(1)$ satisfying Lemma 3.10. Then, $\bar{\alpha}_0^1(1)$ takes constant values on the elements of ξ_1, and ξ_1 approximates T_1 ($[T_1]=[\Gamma_1]$) and D_1 accurate to ε_1. Using Lemmas 3.6, 3.7 and 3.8, construct for α_0^2 a Γ_2-array η_1 with the same number of sets as in the array ξ_1 and define a cocycle $\bar{\alpha}_0^2(1)$, Γ_2-cohomologous to α_0^2, both such that on sets with the same indices the values of $\bar{\alpha}_0^1(1)$ and $\bar{\alpha}_0^2(1)$ coincide. That this construction is possible follows from the facts that $r(\Gamma_1, \alpha_0^1)=r(\Gamma_2, \alpha_0^2)$ and the groups Γ_1 and Γ_2 are weakly equivalent. Then, define a refinement η_2 of the array η_1 so that the approximations of T_2 ($[T_2]=[\Gamma_2]$) and of D_1 should have the accuracy to ε_2. Next, construct a cocycle $\bar{\alpha}_0^2(2)$ cohomologous to $\bar{\alpha}_0^2(1)$, so that the values of $\bar{\alpha}_0^2(2)$ should be constant on elements of η_2. From the proof of Lemma 3.11, it follows that $\bar{\alpha}_0^2(2)$ takes constant values on elements of η_1 as well. Returning to the Γ_1-cocycle $\bar{\alpha}_0^1(1)$ and the Γ_1-array ξ_1, define a refinement ξ_2 of ξ_1 and a cocycle $\bar{\alpha}_0^1(2)$, so that the array ξ_2 should have as many sets as η_2 has and that on sets with the same indices the values of $\bar{\alpha}_0^1(2)$ (Γ_1-cohomologous to $\bar{\alpha}_0^1(1)$) should coincide with the values of $\bar{\alpha}_0^2(2)$. In transition to cohomologous cocycles, as in Lemma 3.11, the functions $f_0^1(1), f_0^1(2)$ and $f_0^2(1), f_0^2(2)$ are constructed that define cohomologous equivalence of cocycles and are such that $f_0^i(j)\in V_0(\varepsilon_j)$, $i, j=1, 2$.

By repeating the above procedure a countable number of times, we obtain two sequences of the arrays $\{\xi_n\}_{n=1}^{\infty}$, $\{\eta_n\}_{n=1}^{\infty}$ two sequences of the cocycles $\{\bar{\alpha}_0^1(i)\}_{i=1}^{\infty}$, $\{\bar{\alpha}_0^2(i)\}_{i=1}^{\infty}$ and two sequences of the functions $\{f_0^1(k)\}_{k=1}^{\infty}$, $\{f_0^2(k)\}_{k=1}^{\infty}$ corresponding to the groups Γ_1 and Γ_2, respectively. The arrays ξ_n and η_n satisfy conditions (i)-(iv) of Lemma 3.11, and the functions $f_0^i(k)$ and the cocycles $\alpha_0^j(k)$, $j=1, 2$ are related as

$$\bar{\alpha}_0^j(k)(x, t) = \sum_{i=1}^{k} f_0^j(i)(tx) + \alpha_0^j(x, t) - \sum_{i=1}^{k} f_0^j(i)(x),\ t \in [\Gamma_j] \tag{3.32}$$

and

$$f_0^j(i)(x) \in V_0(\varepsilon_i)\,. \tag{3.33}$$

Besides, the cocycles $\bar{\alpha}_0^1(k)$ and $\bar{\alpha}_0^2(k)$ have equal values on elements of ξ_k and η_k that have the same indices. Since $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, then in view of (3.33), we can assume $f_0^j(x) = \sum_{i=1}^{\infty} f_0^j(i)(x)$, $j=1, 2$. Then, it follows from (3.32) that for $j=1, 2$ the cocycles $\bar{\alpha}_0^j(x, t) = f_0^j(tx) + \alpha_0^j(x, t) - f_0^j(x)$, $t \in [\Gamma_j]$ are defined and their values are constant on all elements of the arrays ξ_n and η_n $n \in \boldsymbol{N}$. Besides, if take elements of ξ_n and η_n having the same indices, then the cocycles $\bar{\alpha}_0^1$ and $\bar{\alpha}_0^2$ take on them equal values (see Remark 3.12).

The above enumerated properties enable conclusion (as in [1]) that there exists an automorphism θ for which $\theta[\Gamma_2]\theta^{-1} = [\Gamma_1]$ and $\bar{\alpha}_0^1(\theta x, \theta t \theta^{-1}) = \bar{\alpha}_0^2(x, t)$, $t \in [\Gamma_2]$. Putting it otherwise, the pairs (Γ_1, α_0^1) and (Γ_2, α_0^2) are weakly equivalent. □

3.4. Let us come back to considering the fields of cocycles which we began in Subsection 3.2. To prove Theorem 3.9 and Lemmas 3.10 and 3.11, we used the results of the said subsection (Lemmas 3.6, 3.7 and 3.8) that are true for the fields of cocycles. Therefore, the proofs of the results of Subsection 3.3 formulated for individual cocycles can be extended without changes to the case of the fields of cocycles. Then, we obtain validity of

Theorem 3.13. *Let $(X_0, \mathscr{B}_0, \mu_0)$ and $(Y, \mathscr{F}, \nu)$ be Lebesgue spaces, S an ergodic automorphism of $(Y, \mathscr{F}, \nu)$, $x_0 \to \alpha_0^i(x_0)$, $i=1, 2$ measurables fields of cocycles with values in a group H_0 such that $r(\{\alpha_0^1(\cdot)\}) = r(\{\alpha_0^2(\cdot)\}) = H_0$. Then, there exists a measurable field of automorphism $P_0 = (x_0 \to P(x_0))$ such that $P(x_0) \in N[S]$ and the cocycle $P_0 \circ \alpha_0^1$ is S_0-cohomologous to the cocycle α_0^2, where $S_0 = \boldsymbol{1} \times S$. In other words, (S_0, α_0^1) and (S_0, α_0^2) are weakly equivalent.*

In Subsection 3.1, the cocycle β_0 with values in H_0 and a dense range in H_0 was constructed. By β_0 we shall also denote the constant field of cocycles, each equal to β_0.

Corollary 3.14. *Let the conditions of Theorem 3.13 be fulfilled and $r(\{\alpha_0(\cdot)\}) = H_0$. Then, there exists a measurable field of automorphisms $P_0 = (x \to P(x_0)) \in N[S_0]$ such that the cocycle $P_0 \circ \alpha_0$ is S_0-cohomologous to the constant* *field of cocycles* β_0.

Corollary 3.15. *Let a measurable field of cocycles* $x_0 \to \alpha_0(x_0)$ $(x_0 \in X_0)$ *has a dense range in* H_0 *and takes values in* H_0. *Then, it can be replaced by a cohomologous field of cocycles* $x_0 \to \alpha_0'(x_0)$ *(thereby, the measure on* Y *will also be replaced by an equivalent one), so that the group* $\{s \in [S]: \alpha_0'(x_0)(y, s) = 0, y \in Y\}$ *will be ergodic for a.a.* $x_0 \in X_0$.

Remark 3.16. Analysis of Theorems 3.9 and 3.13 shows that they remain true also when the cocycle $\alpha_0 = (\alpha, \rho)$ takes values in a closed normal amenable subgroup H_0 of $G \times \mathbf{R}$ so that the group H_0 is dense range of α_0 and G is an arbitrary amenable l.c.s. group.

§4. Transitive Associated Actions

4.1. Let Γ be an arbitrary countable ergodic group of automorphisms of $(X, \mathcal{B}, \mu)$; G a nonabelian l.c.s. group and $\alpha \in Z^1(X \times \Gamma, G)$.

Theorem 4.1. *Let the associated action* $W_{(\Gamma,\alpha)}(G)$ *for a pair* (Γ, α) *be isomorphic to the transitive action of the group G on a quotient space* G/H, *where* H *is a proper closed subgroup of* G. *Then, the cocycle* α *is* Γ*-cohomologous to a cocycle* β *taking all its values in* H. *If* G *is abelian, then* $r(\Gamma, \alpha) = r(\Gamma, \beta) = H$.

Proof. As in Section 2, we shall consider the skew product $\Gamma(\alpha) \subset \mathrm{Aut}(X \times G, \mu \times \chi_G)$ and the action V of G which are defined by (2.1) and (2.2). Let ξ be a measurable hull of partition into orbits of the group $\Gamma(\alpha)$. By the condition of the theorem, the quotient space $(\Omega, \nu) = ((X \times G)/\xi, (\mu \times \chi_G)/\xi)$ is isomorphic to the quotient space G/H, on which the measure is the projection of the Haar measure χ_G. Therefore, we shall believe that $\Omega = G/H$. Denote by q the quotient map from $X \times G$ into G/H. From the definition of the associated action $W_{(\Gamma,\alpha)}(G) = W(G)$ it follows that

$$q(V(g)(x, h)) = W(g)\, q(x, h), \quad g \in G. \tag{4.1}$$

Besides,

$$q(\gamma(\alpha)(x, h)) = q(x, h), \quad \gamma(\alpha) \in \Gamma(\alpha) \tag{4.2}$$

for a.a. $(x, h) \in X \times G$. Thus, it follows from (4.2) that there is $g_0 \in G$ such that for a.a. $x \in X$

$$q(\gamma x, \alpha(x, \gamma)\, g_0) = q(x, g_0), \quad \gamma \in \Gamma. \tag{4.3}$$

Relations (2.1), (2.2) and (4.1)–(4.3) show that for a.a. $x\in X$

$$q(\gamma x, \alpha(x, \gamma)\, g_0) = W(g_0)^{-1}\, W(\alpha(x, \gamma)^{-1})q(\gamma x, e)\,,$$

$$q(x, g_0) = W(g_0^{-1})q(x, e)\,,$$

where e is the identity in G. Therefore,

$$W(\alpha(x, \gamma))\, q(x, e) = q(\gamma x, e)\,, \quad \gamma\in\Gamma\,. \tag{4.4}$$

Let $\omega_0\in G/H$ be the point in the quotient space into which the group H is projected. Then,

$$H = \{g\in G\colon W(g)\,\omega_0=\omega_0\}\,. \tag{4.5}$$

By the theorem on the measurable choice, there exists a measurable map $\theta\colon G/H\to G$ such that $W(\theta(\omega))\,\omega_0=\omega$ since $W(g)\,(g\in G)$ is the shift into G/H. For $x\in X$ we put $f(x)=\theta(q(x, e))$ and define the cocycle β, which is Γ-cohomologous to α, by the formula: $\beta(x, \gamma)=f(\gamma x)^{-1}\,\alpha(x, \gamma)\,f(x)$. Let us check that all values of β lie in H:

$$\begin{aligned} W(\beta(x, \gamma))\,\omega_0 &= W(\theta(q(\gamma x, e))^{-1})\; W(\alpha(x, \gamma))\, q(x, e) \\ &= W(\theta(q(\gamma x, e)))^{-1}\, q(\gamma x, e) = \omega_0\,. \end{aligned} \tag{4.6}$$

Here we have used equalities (4.4) and (4.5). Relation (4.6) means that $\beta(x, \gamma)\in H$ for a.a. $x\in X$ and all $\gamma\in\Gamma$.

Thus, $r(\Gamma, \alpha)=H$ (assuming now G to be an abelian group). From results of Section 5 (see Theorem 5.9) it immediately follows that indeed $r(\Gamma, \alpha)=H$. □

4.2. Below the group G is assumed to be abelian.

Theorem 4.2. *Let $\Gamma\subset\mathrm{Aut}(X, \mathcal{B}, \mu)$ be an arbitrary ergodic group and $\alpha\in Z^1(X\times\Gamma, G)$. The action $W_{(\Gamma,\alpha)}(G)$ associated with the pair (Γ, α) is transitive if and only if the cocycle α is regular.*

Proof. It follows from Theorem 4.1 and Lemma 1.12. □

Proposition 4.3. *Let there be defined a pair (Γ, α), where $\Gamma\subset\mathrm{Aut}(X, \mathcal{B}, \mu)$, $\alpha\in Z^1(X\times\Gamma, G)$ and $r(\Gamma, \alpha_0)=H_0\subset G_0$. Then, regularity of α_0 means regularity of α. The reverse is wrong.*

Proof. Since α_0 is regular, all values of the cocycle α_0 may be thought to lie in H_0. Use, as in Section 3, the notation $H_0=(H_G, H_R)$. Show that $r(\Gamma, \alpha)=\bar{H}_G$, where $\bar{H}_G$ is the closure of the group H_G in G. As all values of α

are in the group H_G, then evidently $r(\Gamma, \alpha)\subset\bar{H}_G$. Let U be an arbitrary neighborhood of the identity in G and $h_0\in\bar{H}_G$. Then, there is an element $h_1\in H_G$ such that $h_1\in h_0+U$. For a certain $u_1\in H_R$ the element $(h_1, u_1)\in H_0$. Therefore, for any $A\subset X$ and $U_0=U\times(-\varepsilon, \varepsilon)$ there is a subset $B\subset A$ of positive measure and an automorphism $\gamma\in[\Gamma]$ such that $\gamma B\subset A$ and $\alpha_0(x, \gamma)\in(h_1, u_1)+U_0$, $x\in B$. Thus, $\alpha(x, \gamma)\in h_1+U\subset h_0+2U$ for a.a. $x\in B$, i.e. $h_0\in r(\Gamma, \alpha)$. Therefore, we obtain that the cocycle α is regular.

An example showing that regular α does not necessarily mean regularity of α_0 will be provided in Section 7. □

Corollary 4.4. *Let the pairs (Γ_1, α_0^1) and (Γ_2, α_0^2) be such that the cocycles α_0^i, $i=1, 2$ are regular and the associated actions $W_{(\Gamma_1,\alpha_0^1)}(G_0)$ and $W_{(\Gamma_2,\alpha_0^2)}(G_0)$ be isomorphic. Then, so are the actions $W_{(\Gamma_1,\alpha^1)}(G)$ and $W_{(\Gamma_2,\alpha^2)}(G)$.*

The proof follows from Theorem 4.2 and Proposition 4.3.

The statement reverse to Corollary 4.4 is wrong.

Using the results of Section 3 (see Theorem 3.9), one can prove the uniqueness theorem for regular cocycles defined on a.f. groups of automorphisms.

Theorem 4.5. *Let there be the pairs (Γ_1, α_0^1) and (Γ_2, α_0^2) such that the cocycles α_0^i, $i=1, 2$ are regular and let $r(\Gamma_1, \alpha_0^1)=r(\Gamma_2, \alpha_0^2)=H_0$ (i.e. the associated actions $W_{(\Gamma_1,\alpha_0^1)}(G_0)$ and $W_{(\Gamma_2,\alpha_0^2)}(G_0)$ be isomorphic). Then, the pairs (Γ_1, α_0^1) and (Γ_2, α_0^2) are weakly equivalent.*

Proof. The statement of the theorem follows from Theorem 3.9, because α_0^i, $i=1, 2$ may be thought to take values in H_0. □

Remark 4.6. In [4], for an arbitrary l.c.s. group G an analogy of the set $r(\Gamma, \alpha)$ was introduced, viz. the set $\sigma(\Gamma, \alpha)$, where the cocycle $\alpha\in Z^1(X\times\Gamma, G)$ and Γ is an arbitrary countable group of automorphisms. The set $\sigma(\Gamma, \alpha)$ is a closed normal subgroup of G and has the same properties as $r(\Gamma, \alpha)$ (see Section 1). Theorems 4.1, 4.2 and 4.5 are true also in the assumption that H is an amenable normal subgroup of G (or H_0 of G_0). In this case, $r(\Gamma, \alpha)$ should be replaced by $\sigma(\Gamma, \alpha)$. Note also that in all the theorems of this section, the group Γ can be of any type.

§5. Free Associated Actions. Type II

5.1. Recall our standard notation: Γ is an ergodic a.f. countable group of automorphisms of $(X, \mathcal{B}, \mu)$; G a l.c.s. abelian group, and a cocycle $\alpha\in$

$Z^1(X\times\Gamma, G)$. It will be assumed that the cocycle α is recurrent and nonregular (see Definitions 1.11 and 1.13), because the cases of the transient and regular cocycles were considered in Sections 2 and 4. It will also be assumed that Γ is a type II group of automorphisms. Without loss of generality it may be thought that Γ is of type II_1 and $\Gamma\circ\mu=\mu$, $\mu(X)=1$.

Let $r(\Gamma, \alpha)=H\subset G$. The case, where the group $H=\{0\}$, will be considered simultaneously with the general case. Section 3 treats the situation of $H=G$ and therefore H will be assumed to be a closed proper subgroup of G. As earlier in Section 1, we shall define the cocycle $\hat{\alpha}(x, \gamma)=\alpha(x, \gamma)+H$, $\gamma\in[\Gamma]$, which takes values in the group $\hat{G}=G/H$. By Lemma 1.14, $\hat{\alpha}$ is recurrent, as α. By the Definition 1.11 $\bar{r}(\Gamma, \hat{\alpha})=\{\hat{0}, \infty\}$, where $\hat{0}$ is the unit in $\hat{G}$.

Lemma 5.1. *The pair (Γ, α) is weakly equivalent to (Γ, α_1) for which the cocycle $\hat{\alpha}_1\in Z^1(X\times\Gamma, \hat{G})$ is lacunary.*

Proof. It follows from Lemma 1.17 that the cocycle $\hat{\alpha}$ is Γ-cohomologous to a lacunary cocycle $\hat{\alpha}_1\in Z^1(X\times\Gamma, \hat{G})$: $\hat{\alpha}_1(x, \gamma)=\hat{f}(\gamma x)+\hat{\alpha}(x, \gamma)-\hat{f}(x)$, where $\hat{f}: X\to\hat{G}$ is a measurable map. According to the theorem on measurable choice, there exists a measurable map $\psi: X\to G$ such that $\hat{f}=\psi+H$. Put $\alpha_1(x, \gamma)=\psi(\gamma x)+\alpha(x, \gamma)-\psi(x)$. □

On the basis of this lemma, we shall always believe the cocycle $\hat{\alpha}$ to be lacunary. In other words, there exists a neighborhood V_0 of the identity in $\hat{G}$, for which $\hat{\alpha}(x, \gamma)\notin V_0-\{\hat{0}\}$, $x\in X$, $\gamma\in[\Gamma]$.

Consider the orbital cocycle $\hat{u}_{\hat{\alpha}}=\hat{u}$ corresponding to $\hat{\alpha}$. Let $\mathcal{R}(\Gamma)$ be the measurable equivalence relation on X defined by partition of X into orbits of Γ. Put $\mathcal{P}=\{(x_1, x_2)\in\mathcal{R}(\Gamma): \hat{u}(x_1, x_2)=0\}$. Obviously, $\mathcal{P}$ is also a measurable equivalence relation and $\mathcal{P}\subset\mathcal{R}(\Gamma)$. It follows from the results of [4] that in $[\Gamma]$ there exists an freely acting automorphism S_0 such that $\mathcal{R}(S_0)=\mathcal{P}$. Denote the σ-algebra of measurable S_0-invariant subsets in X by $\mathcal{B}_0$. Then, $\mathcal{B}_0$ corresponds to the partition ξ of X into ergodic components of S_0. Put $X_0=X/\xi$, $\mu_0=\mu_\xi$ and let $\pi: X\to X_0$ be the natural projection onto X_0.

Lemma 5.2 [15]. *Let $\mu=\int_{X_0}\nu_{x_0}\,d\mu_0(x_0)$ be the expansion of the measure μ into the canonical system of measures with respect to the partition ξ. Then, the recurrence of $\hat{\alpha}$ means that the measure μ_0 and the measures $\{\nu_{x_0}\}_{x_0\in X_0}$ are probability and nonatomic for μ_0-a.a. $x_0\in X_0$. Besides, the automorphism S_0 is conservative (i.e. not of type I), and ν_{x_0} is S_0-ergodic and S_0-invariant for μ_0-a,.a. $x_0\in X_0$.*

We consider the Lebesgue space $(\pi^{-1}(x_0), \nu_{x_0})$ for every fixed $x_0 \in X_0$. It follows from Lemma 5.2 that for μ_0-a.a. $x_0 \in X_0$ the space $(\pi^{-1}(x_0), \nu_{x_0})$ is isomorphic to certain standard Lebesgue space (Y, ν), $\nu(Y)=1$. Let $S(x_0)$ be the restriction of the automorphism S_0 onto $\pi^{-1}(x_0)$. According to Lemma 5.2, for μ_0-a.a. $x_0 \in X_0$, the automorphism $S(x_0)$ is ergodic, of type II_1. From Theorem 2.5 of [10] it follows existence of a measurable field of isomorphisms $x_0 \to P(x_0)$ $(x_0 \in X_0)$ such that $P(x_0)$: $Y \to \pi^{-1}(x_0)$, $P(x_0)\circ\nu = \nu_{x_0}$ and $P(x_0)^{-1}[S(x_0)]\, P(x_0)=[S]$ for μ_0-a.a. $x_0 \in X_0$, where S is an ergodic type II_1 automorphism of (Y, ν). E.g., the space (Y_0, p) and the automorphism group Δ of Subsection 3.1 may be taken as (Y, ν) and S assuming P to be Δ-invariant.

Each point $x \in X$ can be represented as $x=(\pi(x), y)$, where $y \in \pi^{-1}(\pi(x))$. Then, the transformation P: $x \to (\pi(x), P(\pi(x))^{-1} y)$ maps X into $X_0 \times Y$ and the measure μ into $\mu_0 \times \nu$. We have the automorphism group $P\Gamma P^{-1}$ and the cocycle $P\circ\alpha$, both defined on the space $(X_0 \times Y, \mu_0 \times \nu)$. Thus, the pair (Γ, α) transforms into $(P\Gamma P^{-1}, P\circ\alpha)$. For simplicity, we shall believe that on $(X_0 \times Y, \mu_0 \times \nu)$ the automorphism group Γ acts with the cocycle $\alpha \in Z^1(X_0 \times Y \times \Gamma, G)$, and these have the following properties: S_0: $(x_0, y) \to (x_0, Sy)$ belongs to $[\Gamma]$, and $\hat{\alpha}(x_0, y, S_0)=\hat{0}$.

Lemma 5.3 [15]. *On the Lebesgue space $(X_0, \mathcal{B}_0, \mu_0)$, there exists an a.f. countable ergodic group Γ_0 of automorphisms such that*

$$\pi(\Gamma_X) = \Gamma_0\, \pi(x),\ x \in X = X_0 \times Y\,, \tag{5.1}$$

and there exists an orbital transient cocycle u_0: $\mathcal{R}(\Gamma_0) \to \hat{G}$ such that for any $\gamma \in [\Gamma]$

$$\hat{\alpha}(x, \gamma) = u_0(\pi(\gamma x), \pi(x))\,, \quad x \in X\,. \tag{5.2}$$

Relations (5.1) and (5.2) imply the following. The orbits of Γ_0 is formed by the projection π of the orbits of Γ onto X_0. The cocycle u_0 has the same set of values as the cocycle $\hat{\alpha}$. Therefore, there exists a neighborhood V_0 of the identity in $\hat{G}$ for which $u_0(\gamma_0\, x_0, x_0) \notin V_0$, $\gamma_0 \in [\Gamma_0]$ $(\gamma_0 \neq 1)$, $x_0 \in X_0$. From Lemma 5.3 it follows that there exists an ergodic automorphism $Q \in \mathrm{Aut}(X_0, \mathcal{B}_0, \mu_0)$ for which $[\Gamma_0]=[Q]$. Put $\varphi(x_0)=u_0(Q\, x_0, x_0)$, $x_0 \in X_0$.

Lemma 5.4. *In the full group $[\Gamma]$ there exists an automorphism R, such that*

$$\hat{\alpha}(x, R) = \varphi(\pi(x))\,, \quad x \in X\,. \tag{5.3}$$

Proof. Since all elements of Γ_0 result from projection of Γ onto X_0, then for the automorphism $Q\in[\Gamma_0]$ there exists an automorphism $R\in[\Gamma]$ such that

$$R\colon \{x_0\}\times Y \to \{Q\, x_0\}\times Y\,, \quad x_0\in X_0\,. \tag{5.4}$$

By using equality (5.2), we obtain it that $\hat{\alpha}(x, R)=u_0(\pi(R\,x), \pi(x))=u_0(Q\,\pi(x), \pi(x))=\varphi(\pi(x))$. □

Consider the automorphism

$$R(Q^{-1}\times\mathbf{1})\colon (x_0, y) \to (x_0, y')\,, \tag{5.5}$$

which for a.a. $x_0\in X_0$ defines a measurable one-to-one map $U(x_0)\colon y—y'$. Relation (5.5) shows that $x_0\to U(x_0)$ $(x_0\in X_0)$ is the measurable field of automorphisms of (Y, ν).

Lemma 5.5. *The following statements are true: $R\in N[S_0]$ and $U(x_0)\in N[S]$ for μ_0-a.a. $x_0\in X_0$.*

Proof. By using (5.3), (5.4) and (5.5), we obtain that $\hat{\alpha}(x, RS_0R^{-1})=\hat{0}$. Therefore, $RS_0R^{-1}\in[S_0]$. Then, $R(Q^{-1}\times\mathbf{1})\in N[S_0]$, which, together with (5.5), leads to $U(x_0)\in N[S]$, $x_0\in X_0$. □

Introduce the notation: $Q_0(x_0, y)=(Qx_0, U(x_0)\,y)$. Unite now the above proved results in the following theorem on the structure of the cocycles $\hat{\alpha}$.

Theorem 5.6. *Let a pair (Γ, α) be as above. The full group $[\Gamma]$ is generated by the action of two automorphisms Q_0 and S_0 on $(X_0\times Y, \mu_0\times\nu)$ as follows $Q_0(x_0, y)=(Qx_0, U(x_0)\,y)$, $S_0(x_0, y)=(x_0, Sy)$ with $Q_0\in N[S_0]$. The cocycle $\hat{\alpha}\in Z^1(X\times\Gamma, G)$ has the properties: $\hat{\alpha}(x_0, y, Q_0)=\varphi(x)$, $\hat{\alpha}(x_0, y, S_0)=\hat{0}$, where $\varphi(x_0)\notin V_0$ (V_0 is a neighborhood of the identity in $\hat{G}$). The set of values of $\hat{\alpha}$ coincides with that of the orbital cocycle u_0.*

Proof. We have to prove only the equality

$$\{\gamma x\colon \gamma\in\Gamma\} = \{Q_0^m\, S_0^k(x_0, y)\colon m, k\in\mathbf{Z}\}\,,\; x=(x_0, y)\,, \tag{5.6}$$

since all the other statements of the theorem follow from Lemmas 5.2–5.5. Assume (5.6) to be wrong; i.e. the set

$$A = \bigcup_{\gamma\in\Gamma} \{x\in X\colon \gamma x\notin \{Q_0^m\, S_0^k x\colon m, k\in\mathbf{Z}\}\}$$

has a positive measure. As has been said, the cocycle α (and thus $\hat{\alpha}$) may be thought to take on only a countable number of values. The condition of the theorem means that for any fixed $\gamma\in\Gamma$ there exists a measurable function

$m=m(\pi(x))$ for which

$$\hat{\alpha}(x, r) = u_0(Q^{m(\pi(x))}\,\pi(x), \pi(x) = \hat{\alpha}(x, Q_0^{m(\pi(x))})\,. \tag{5.7}$$

Then, there is an automorphism $r_1=r_1(x)\in[\Gamma]$ such that $rx=r_1\,Q_0^m x$ and for $x\in A$

$$\hat{\alpha}(x, r) = \hat{\alpha}(Q_0^m\, x, r_1)+\hat{\alpha}(x, Q_0^m)\,. \tag{5.8}$$

Comparing (5.7) and (5.8), we conclude that $\hat{\alpha}(Q_0^m\, x, r_1)=\hat{0}$, i.e. $r_1\in[S_0]$; this is contradiction to the above assumption and thus proves (5.6). □

5.2. Here it will be assumed that the cocycle $\alpha\in Z^1(X\times\Gamma, G)$ is lacunary (it is also recurrent and nonregular). This means that $\bar{r}(\Gamma, \alpha)=\{0,\infty\}$. Therefore, all the results formulated and proved for $\hat{\alpha}$ in Subsection 5.1 are also true for α because $\hat{\alpha}=\alpha$ for the group $H=\{0\}$.

Let us first consider the case where the action $W_{(\Gamma,\alpha)}(G)$ associated with the pair (Γ, α) preserves measure.

Lemma 5.7. *Let Γ be an ergodic a.f. type II_1 group of automorphisms of $(X, \mathcal{B}, \mu)$, let $\alpha\in Z^1(X\times\Gamma, G)$ be a lacunary cocycle, and the action $W_{(\Gamma,\alpha)}(G)$ preserves measure. Then, in the condition of Theorem* 5.6 *the automorphism Q_0 may be chosen to be $Q\times\mathbb{1}$.*

Proof. We shall calculate the associated action $W_{(\Gamma,\alpha)}(G)$. From the ergodicity of S on (Y, ν) and the triviality of α on S_0, it follows that the action $W_{(\Gamma,\alpha)}(G)$ is defined on the quotient space of $(X_0\times G, \mu_0\times\chi_G)$ by the partition into orbits of the automorphism $Q(u_0)\colon (x_0, g)\to(Qx_0, g+u_0(Qx_0, x_0))$. This partition is measurable, because the cocycle u_0 on Q_0 is transient and $\alpha(x, Q_0)=u_0(Qx_0, x_0)=\varphi(x_0)\notin V_0$, where V_0 is a certain neighborhood of the identity in G. Hence, $W_{(\Gamma,\alpha)}(G)=W_{(Q,u_0)}(G)$. The action $W_{(Q,u_0)}(G)$ preserves measure if and only if Q preserves measure μ_0. Now, since the probability measure ν on Y is S-invariant and S-ergodic, then the condition $U(x_0)\in N[S]$ means that $U(x_0)\circ\nu=\nu$ for μ_0-a.a. $x_0\in X_0$. Applying the cohomology theorem [10] (see also [1]), we obtain existence of a measurable field of automorphisms $x_0\to P(x_0)\in N[S]$ such that $P(Q\,x_0)^{-1}\,U(x_0)\,P(x_0)\in[S]$, $x_0\in X_0$. This means that the transformation $P\colon (x_0, y)\to(x_0, P(x_0)\,y)$ maps $[\Gamma]$ into the group generated by $Q\times\mathbb{1}=Q_0$ and $\mathbb{1}\times S=S_0$. This does not change the cocycle α. □

The proofs of Lemma 5.7 and Theorem 2.9 lead to

Proposition 5.8. *Let the pairs (Γ_i, α_i), $i=1, 2$ satisfy the following condi-*

tions: Γ_i is an a.f. countable ergodic type II group of automorphisms of $(X_i, \mathcal{B}_i, \mu_i)$, $\alpha_i \in Z^1(X_i \times \Gamma_i, G)$ is a lacunary cocycle, and the action $W_i(G) = W_{(\Gamma_i, \alpha_i)}(G)$ preserves measure. The actions $W_1(G)$ and $W_2(G)$ are isomorphic if and only if the pairs (Q^1, u_0^1) and (Q^2, u_0^2) are stably weakly equivalent, where (Q^i, u_0^i), $i=1, 2$ are defined by (Γ_i, α_i), as in Theorem 5.6.

Theorem 5.9. (theorem of uniqueness). *Let the pairs (Γ_i, α_i), $i=1, 2$ be the same as in Proposition 5.8 and the associated action $W_i(G) = W_{(\Gamma_i, \alpha_i)}(G)$ preserves measure. The pairs (Γ_1, α_1) and (Γ_2, α_2) are stably weakly equivalent if and only if $W_1(G)$ and $W_2(G)$ are isomorphic.*

Proof. Obviously, we may take $X_1 = X_2 = X$. Then, it follows from Theorem 5.6 that $Q^1 = Q^2 = Q$, $X_0^1 = X_0^2 = X_0$, $\mu_0^1 = \mu_0^2 = \mu_0$, $Y_1 = Y_2 = Y$, $\nu_1 = \nu_2 = \nu$, $S^1 = S^2 = S$. The pairs (Γ_1, α_1) and (Γ_2, α_2) differ only in the values of α_1 and α_2 on $Q_0 = Q \times \mathbf{1}$. The pairs (Γ_1, α_1) and (Γ_2, α_2) are stably equivalent if and only if so are (Q, u_0^1) and (Q, u_0^2). Indeed, it follows from the structure of α_1 and α_2 that if ψ_0: $X_0 \times \boldsymbol{Z} \to X_0 \times \boldsymbol{Z}$ is a map responsible for weak equivalence of the cocycles $\tilde{u}_0^1$ and $\tilde{u}_0^2$, then the map ψ: $(x_0, n, y) \to (\psi_0(x, n), y)$ will define weak equivalence of the pairs $(\tilde{\Gamma}_1, \tilde{\alpha}_1)$ and $(\tilde{\Gamma}_2, \tilde{\alpha}_2)$ (recall that $\tilde{u}_0^i$ and $\tilde{\alpha}_i$ are the countable expansions of u_0^i and α_i, $i=1, 2$). Therefore, by Proposition 5.8, the isomorphism of $W_1(G)$ and $W_2(G)$ means stable weak equivalence of (Γ_1, α_1) and (Γ_2, α_2). The reverse statement was proved in Proposition 2.3. □

Let us show now that any free measure-preserving action $W(G)$ of G may be regarded as associated with a pair (Γ, α), where Γ is a type II group and α a lacunary cocycle. Namely, we shall prove

Theorem 5.10 (theorem of existence). *Let $W(G)$ be an ergodic free action of G on a Lebesgue space (Ω, p) preserving the probability measure p. Then, there exists an ergodic countable a.f. group $\Gamma \subset \mathrm{Aut}(X, \mathcal{B}, \mu)$, $\mu(X)=1$, preserving the measure μ, and there exists a lacunary cocycle $\alpha \in Z^1(X \times \Gamma, G)$, both such that the action $W(G)$ is isomorphic to the action $W_{(\Gamma, \alpha)}(G)$ associated with the pair (Γ, α).*

Proof. Choose a complete lacunary Borel section $X_0 \subset \Omega$ of the action $W(G)$. The measure μ_0 on X_0 will be defined as the image of the measure p. There exists on X_0 a countable ergodic equivalence relation $\mathcal{R}$, and let Q be such an automorphism of X_0 that $\mathcal{R}(Q) = \mathcal{R}$ [4]. Clearly, Q preserves the measure μ_0, and $\mu_0(X_0)=1$. Define the return cocycle $u_0 \in Z^1(X_0 \times [Q], G)$ for the action Q on X_0, assuming $u_0(Q\, x_0, x_0) = g$, where $g \in G$ satisfies the equality

$W(g)\,x_0 = Q\,x_0$. Since the action $W(G)$ is free, then $g = g(x_0)$ is defined unambiguously.

Let (Y, ν), $\nu(Y)=1$ be a Lebesgue space, and $S \in \mathrm{Aut}(Y, \nu)$ be an ergodic automorphism preserving ν. Define on $(X, \mu) = (X_0 \times Y, \mu_0 \times \nu)$ an group Γ of automorphisms of type II_1 generated by Q_0 and S_0:

$$Q_0(x_0, y) = (Q\,x_0, y)\,, \quad S_0(x_0, y) = (x_0, Sy)\,. \tag{5.9}$$

The group Γ is ergodic and a.f. Put

$$\alpha(x_0, y, Q_0) = u_0(Q\,x_0, x_0)\,, \quad \alpha(x_0, y, S_0) = 0\,. \tag{5.10}$$

Since the section X_0 is lacunary, then the cocycle u_0 is transient and the cocycle α is lacunary. It is now transparent that the associated action $W_{(\Gamma,\alpha)}(G)$ is isomorphic to $W(G)$ (see e.g. [3] and also Section 2). □

There exists an example of a type II group Γ and a lacunary cocycle $\alpha \in Z^1/(X \times \Gamma, G)$ such that the associated action $W_{(\Gamma,\alpha)}(G)$ is non-singular (i.e. of type III). Moreover, such a pair (Γ, α) may be constructed by any non-singular action $W(G)$ of G.

Theorem 5.11. *Let $W(G)$ be an ergodic free action of G on an Lebesgue space (Ω, p), $p(\Omega)=1$ with a non-singular measure p. Then, there exists a piar (Γ, α), where Γ is an ergodic a.f. group of automorphisms of $(X, \mathcal{B}, \mu)$ preserving a σ-finite measure μ and $\alpha \in Z^1(X \times \Gamma, G)$ is a lacunary cocycle, and this pair is such that $W_{(\Gamma,\alpha)}(G)$ and $W(G)$ are isomorphic.*

Proof. As in the proof of Theorem 5.10, let us define the following objects: (X_0, μ_0), Q, $\mathcal{R}$ and u_0. The automorphism Q has, generally speaking, the non-trivial Radon-Nikodym cocycle $\rho(x_0, Q) = \log \dfrac{d\,Q_0^{-1} \circ \mu_0}{d\,\mu_0}(x_0)$. Consider a Lebesgue space (Y, ν) with σ-finite measure ν and an ergodic automorphism S of (Y, ν) preserving ν. Let $x_0 \to U(x_0)$ $(x_0 \in X_0)$ be a measurable field of automorphisms of (Y, ν) such that $U(x_0) \in N[S]$ and

$$\Phi(U(x_0)) \equiv \log \frac{d\,U(x_0)^{-1} \circ \nu}{d\,\nu} = -\rho(x_0, Q)\,. \tag{5.11}$$

Put $(X, \mu) = (X_0 \times Y, \mu_0 \times \nu)$ and $Q_0(x_0, y) = (Q\,x_0, U(x_0)\,y)$, $S_0(x_0, y) = (x_0, Sy)$. Then, $Q_0 \in N[S_0]$. Denote by Γ the a.f. type II_∞ group of automorphisms generated by Q_0 and S_0. Define the cocycle α for Γ, according to formulae (5.10). The pair (Γ, α) will satisfy the conditions of the theorem. □

Let us prove now the uniqueness theorem for the pairs (Γ, α), whose associated actions are non-singular.

Theorem 5.12. *Let Γ_i be an ergodic a.f. type II group of automorphisms of $(X_i, \mathcal{B}_i, \mu_i)$, and $\alpha_i \in Z^1(X_i \times \Gamma_i, G)$ be a lacunary cocycle, $i=1, 2$. Assume $W_{(\Gamma_i, \alpha_i)}(G)$ associated with (Γ_i, α_i) to be non-singular, $i=1, 2$. The pairs (Γ_1, α_1) and (Γ_2, α_2) are stably weakly equivalent if and only if $W_{(\Gamma_1, \alpha_1)}(G)$ and $W_{(\Gamma_2, \alpha_2)}(G)$ are isomorphic.*

Proof. Since we are interested in the stable weak equivalence relation, then we may consider the pair $(\tilde{\Gamma}_i, \tilde{\alpha}_i)$ instead of (Γ_i, α_i), $i=1, 2$, i.e. believe that Γ_i is of type II_∞ and μ_i is Γ_i-invariant and infinite. Theorem 5.6 naturally remains valid in this case as well, and the measure ν on the space Y is also infinite (see Subsection 5.1). The automorphism $Q_0^i(x_0, y)=(Q^i x_0, U_i(x_0) y)$ preserves the measure $\mu_i=\mu_0^i \times \nu$, therefore, $\Phi(U_i(x_0))=-\rho(x_0, Q^i)$ (here $(x_0, y) \in X$). Based on the isomorphism of $W_{(\Gamma_1, \alpha_1)}(G)$ and $W_{(\Gamma_2, \alpha_2)}(G)$, it may be assumed, as in Theorem 5.9, that $\Gamma_1=\Gamma_2=\Gamma$, $Q^1=Q^2=Q$, $\alpha_1=\alpha_2=\alpha$, $(X_1, \mu_1)=(X_2, \mu_2)=(X, \mu)$. Thus, $\Phi(U_1(x_0))=\Phi(U_2(x_0))$. By the cohomology theorem [10] there exists a measurable field of automorphisms $x_0 \to P(x_0)$ $(x_0 \in X_0)$ such that $P(x_0) \in N[S]$ and for a.a. $x_0 \in X_0$

$$P(Q\, x_0)^{-1} U_1(x_0) P(x_0) = U_2(x_0)\, s(x_0)\,, \tag{5.12}$$

where $x_0 \to s(x_0) \in [S]$. Equality (5.12) shows that the transformation P: $(x_0, y) \to (x_0, P(x_0)\, y)$ belongs to $N[\Gamma]$ and $P \circ \alpha = \alpha$, because the cocycle α is completely defined by the action of Q on X_0. Thus, P maps the generators (Q_0^1, S_0) of $[\Gamma]$ into the generators (Q_0^2, S) without α being replaced. This proof is concluded, as that of Theorem 5.9, by consideration of the corresponding transient cocycles. □

Corollary 5.13. *Let $\alpha \in Z^1(X \times \Gamma, G)$ be a lacunary cocycle, where Γ is of type II. There exists a cocycle α_1, which is stably weakly equivalent to α and such that α_1 is trivial on S_0 and α_1 is a transient cocycle on Q_0, where Q_0 and S_0 are the generators of $[\Gamma]$.*

§6. Non-free Associated Actions. Type II

6.1. Let Γ be an ergodic a.f. group of automorphisms, $G_0=G \times \boldsymbol{R}$, and a cocycle $\alpha_0 \in Z^1(X \times \Gamma, G_0)$. Assume α_0 to be nonregular and $r(\Gamma, \alpha_0)=H_0$, and Γ to be of type III. If Γ is a type II group, then the proofs below, will be simplified.

Recall that, according to Theorem 5.6, Γ is generated by Q_0 and S_0 which act on the Lebesgue space $(X_0\times Y, \mu_0\times\nu)$ as follows

$$Q_0(x_0, y) = (Qx_0, U(x_0)y), \quad S_0(x_0, y) = (x_0, Sy). \tag{6.1}$$

The cocycle $\hat{\alpha}\in Z^1(X_0\times Y\times\Gamma, \hat{G}_0)$, $\hat{G}_0=G_0/H_0$ takes on Q_0 and S_0 the following values

$$\hat{\alpha}_0(x_0, y, Q_0) = \varphi(x_0), \quad \hat{\alpha}_0(x_0, y, S_0) = \hat{0}, \tag{6.2}$$

where $\varphi(x_0)\notin V_0$ and V_0 is a neighborhood of identity in $\hat{G}_0$.

It follows from (6.2) that the cocycle α_0 takes the values on S_0 from the group H_0. Thus, α_0 on $[S_0]$ defines a measurable field of cocycles $x_0\to\alpha_0(x_0)(y, s)$, $s\in[S_0]$, $x_0\in X_0$ with values in H_0.

Lemma 6.1. *The range of the field of cocycles* $x_0\to\alpha_0(x_0)(y, s)$, $s\in[S]$ *is the group* H_0.

Proof. It follows from (6.2) that there exists such a neighborhood V of the identity in G_0 that the values of α_0 on Γ do not belong to the set $(H_0+V)-H_0$. Let h_0 be an arbitrary element of H_0 and W an arbitrary neighborhood of the identity in G_0. The condition $r(\Gamma, \alpha_0)=H_0$ means that for any set $A\subset X_0\times Y$, $(\mu_0\times\nu)(A)>0$ there exist a subset $B\subset A$, $(\mu_0\times\nu)(B)>0$ and an automorphism $\gamma\in[\Gamma]$, such that $\gamma B\subset A$ and $\alpha_0(x_0, y, \gamma)\in h_0+W$ for a.a. $(x_0, y)\in B$. Since $\gamma(x_0, y)=S_0^m Q_0^n(x_0, y)$, where $m=m(x_0, y)$, $n=n(x_0, y)$, then we obtain for $(x_0, y)\in B$

$$\alpha_0(Q_0^n(x_0, y), S_0^m)+\alpha_0(x_0, y, Q_0^n)\in h_0+W.$$

Since α_0 on Q_0 takes values in H_0, then $\alpha_0(x_0, y, Q_0^n)\in H_0+W$. Let $W\subset V$; we obtain that $\alpha_0(x_0, y, Q_0^n)\in H_0$. It follows from (6.2) that in this case $n=0$ and then $\gamma(x_0, y)=S_0^m(x_0, y)$. Hence, the field of cocycles $x_0\to\alpha_0(x_0)$ $(x_0\in X_0)$ has the property $r(\{\alpha_0(\cdot)\})=H_0$. □

The above properties of Γ and α_0 show that the ergodic automorphism $S\in\mathrm{Aut}(Y, \mathcal{F}, \nu)$ may be only either of type II or of type III$_\lambda$ $(0<\lambda\leq 1)$. This depends on the group H_R, where $H_0=(H_G, H_R)$ (see Subsection 3.1). In Section 3 we introduced the standard cocycle β_0 defined on the a.f. group of automorphisms, its range coinciding with H_0. Application of Theorem 3.13 yields the following result: there exists a measurable field of automorphisms $x_0\to R(x_0)\in N[S]$ such that the cocycle $R\circ\alpha_0$ is S_0-cohomologous to a constant field of cocyles β_0, where $R(x_0, y)=(x_0, R(x_0)\,y)$. In other words, there exists a measurable function f: $X_0\times Y\to H_0$ such that for $s_0\in[S_0]$, $(x_0, y)\in X_0\times Y$

$$f(s_0(x_0, y))+R\circ\alpha_0(x_0, y, s_0)-f(x_0, y) = \beta_0(y, s(x_0))\,, \tag{6.3}$$

where $s_0(x_0, y)=(x_0, s(x_0)\,y)$. Under the action of R the group Γ will transform into the group $\Gamma'=R\Gamma R^{-1}$ generated by S_0 and Q_0', where $Q_0'(x_0, y)=(Q\,x_0, U'(x_0)\,y)$, $U'(x_0)=R(Q\,x_0)\,U(x_0)\,R(x_0)^{-1}$. Since R preserves the measure ν (see Section 3), then for a.a. $x_0\in X_0$, $\Phi(U(x_0))=\Phi(U'(x_0))$. The cocycle α_0 will be replaced by α_0' defined on Γ':

$$\alpha_0'(x_0, y, r') = f(r'(x_0, y))+R\circ\alpha_0(x_0, y, r')-f(x_0, y)\,, \tag{6.4}$$

where f is the same as in (6.3), i.e. α_0' coincides with β_0 on $[S_0]$. Thus, we have proved the following

Lemma 6.2. *The pair (Γ, α_0), where $r(\Gamma, \alpha_0)=H_0$, is weakly equivalent to the pair (Γ', α_0') for which $\alpha_0'(x_0, y, S_0)=\beta_0(y, S_0)$.*

Consider first the value of α_0' on Q_0'. Let σ: $G_0/H_0\to G_0$ be a measurable section of G_0 over G_0/H_0. This means that $\sigma(\hat{g})\in G_0$ and $\pi(\sigma(\hat{g}))=\hat{g}$, where $\hat{g}\in\hat{G}_0$ and π: $G_0\to\hat{G}_0$ is a natural projection. Thus, any element $g_0\in G_0$ can be represented as

$$g_0 = \sigma(\hat{g}_0)+h_0(g_0)\,, \tag{6.5}$$

where $h_0(g_0)\in H_0$. We have, in view of (6.2) and (6.5) $\hat{\alpha}_0(x_0, y, Q_0)=\sigma(\varphi(x_0))+h_0(x_0, y)$, where $h_0(x_0, y)$ is a measurable function from $X_0\times Y$ into H_0. The value of $R\circ\alpha_0$ on Q_0' is easy to calculate: $R\circ\alpha_0(x_0, y, Q_0')=\sigma(\varphi(x_0))+h_0(x_0, R(x_0)^{-1}y)$. Then, according to (6.4),

$$\begin{aligned}\alpha_0'(x_0, y, Q_0') &= f(Q\,x_0, U'(x_0)\,y)+\sigma(\varphi(x_0))\\ &\quad +h_0(x_0, R(x_0)^{-1}\,y)-f(x_0, y)\,.\end{aligned} \tag{6.6}$$

Lemma 6.3. *Let $l(x_0, y)=-f(Q\,x_0, U'(x_0)\,y)-h_0(x_0, R(x_0)^{-1}\,y)+f(x_0, y)$. Then, in $[S_0]$ there exists a measurable field of automorphisms $s_0=(x_0\to s(x_0))$ such that for a.a. $(x_0, y)\in X_0\times Y$*

$$\alpha_0'(Q_0'(x_0, y), s_0) = l(x_0, y)\,. \tag{6.7}$$

The proof of the lemma is transparent enough, so we shall only provide a sketch of it. As earlier, taking into consideration the results of [6], the cocycle α_0 may be assumed to take values in a countable group H_0' which is dense in

H_0. In this case the function f can be chosen so that α_0' should be aslo take values in H_0' (see (6.4) and Remark 3.12). Therefore, $f(r'(x_0, y))-f(x_0, y)$ belongs to H_0'. These observations enable conclusion that the function l is piecewise constant and has values in H_0'. Since the group $[S_\beta]=\{s\in[S], \beta_0(y, s)=0, y\in Y\}$ is ergodic on (Y, ν), then equality (6.7) is easy to obtain for any fixed $x_0\in X_0$. Because $l(x_0, y)$ is the measurable function, the corresponding field of automorphisms $s_0=(x_0\rightarrow s(x_0))$ may be chosen to be measurable. □

Put $\bar{Q}_0'(x_0, y)=(Q\,x_0, s(Q\,x_0)\,U'(x_0)\,y)=s_0\,Q_0'(x_0, y)$, where s_0 is the same as in Lemma 6.3. Then, it follows from (6.6) and (6.7) that $\alpha_0'(x_0, y, \bar{Q}_0')=\sigma(\varphi(x_0))\equiv\bar{\varphi}(x_0)$. Thereby the following result is proved.

Theorem 6.4. *For a pair (Γ, α_0) having the properties described in the beginning of Subsection* 6.1, *there exists a pair (Γ', α_0') weakly equivalent to (Γ, α_0) and such that Γ' is generated by $\bar{Q}_0'$ and S_0, so that*

$$\alpha_0'(x_0, y, \bar{Q}_0') = \bar{\varphi}(x_0)\,, \quad \alpha_0'(x_0, y, S_0) = \beta_0(y, S)\,. \tag{6.8}$$

6.2. We shall calculate the associated action $W_{(\Gamma,\alpha_0)}(G_0)$ of G_0, where (Γ, α_0) is the same as in Subsection 6.1.

Proposition 6.5. *For a pair (Γ, α_0), $r(\Gamma, \alpha_0)=H_0$, the associated action $W_{(\Gamma,\alpha_0)}(G_0)$ has the group H_0 as a stabilizer, i.e. $W_{(\Gamma,\alpha_0)}(G_0/H_0)$ is isomorphic to the free action of $\hat{G}_0=G_0/H_0$ associated with (Q, u_0), where $u_0(Q\,x_0, x_0)=\varphi(x_0)$.*

Proof. The proof follows from Theorem 6.4, Lemma 6.1 and Proposition 5.8. □

After the preparations made, let us consider the existence and uniqueness theorems (analogous to the theorems of Section 5) for the case, where Γ is an ergodic a.f. type II group of automorphisms and α a nonregular cocycle from $Z^1(X\times\Gamma, G)$ such that $r(\Gamma, \alpha)=H$.

Let there be defined two pairs (Γ_i, α_i), $i=1, 2$, and let $r(\Gamma_1, \alpha_1)=r(\Gamma_2, \alpha_2)=H$. According to Theorem 6.4, we can transfer to weakly equivalent pairs which has the following properties $\Gamma_1=\Gamma_2=\Gamma$, $\alpha_1(x_0, y, S_0)=\alpha_2(x_0, y, S_0)=\beta(y, S)$ (in the case where Γ is of type II, β_0 coincides with β).

Theorem 6.6. *Let Γ be an ergodic a.f. type II_1 group of automorphisms of $(X, \mathcal{B}, \mu)$ and $\alpha_i\in Z^1(X\times\Gamma, G)$, $i=1, 2$. Let $W_{(\Gamma,\alpha_1)}(G)$ and $W_{(\Gamma,\alpha_2)}(G)$ be isomorphic and preserve measure; then $r(\Gamma, \alpha_1)=r(\Gamma, \alpha_2)=H$ and the pairs (Γ, α_1) and (Γ, α_2) are stably weakly equivalent.*

Proof. As has been mentioned, the group Γ can be believed to be generated by $\bar{Q}_0'$ and S_0 and the cocycles α_i, $i=1,2$ to satisfy relations (6.8). The isomorphism of $W_1(G)=W_{(\Gamma,\alpha_1)}(G)$ and $W_2(G)=W_{(\Gamma,\alpha_2)}(G)$, and Proposition 6.5, and (6.8) lead to the equalities: $Q_1=Q_2=Q$, $\varphi_1(x_0)=\varphi_2(x_0)=\varphi(x_0)$, $\alpha_1(x_0, y, \bar{Q}_0^1)=\alpha_2(x_0, y, \bar{Q}_0^2)$, where $\bar{Q}_0^i(x_0, y)=(Q\,x_0, s_i(x_0)\,U_i(x_0)\,y)$, $i=1, 2$. Note that $\bar{Q}_0^i$, $i=1, 2$ and Q preserve measure, since also so do Γ and $W_i(G)$ (see Proposition 6.5). Therefore, we obtain that $\Phi(U_1(x_0))=\Phi(U_2(x_0))=1$ for a.a. $x_0\in X_0$. Thus, we can consider the weakly equivalent pairs for which $U_1(x_0)=U_2(x_0)=\mathbf{1}$. Then, it is sufficient (as in Section 5) to consider the automorphism $Q_0=Q\times\mathbf{1}$ instead of $\bar{Q}_0^1$, defining $\alpha_1(x_0, y, Q_0)=\bar{\varphi}(x_0)=\sigma(\varphi(x_0))$, where $\sigma\colon G/H\to G$ was defined in Subsection 6.1. If the statement of the theorem is proved on this assumption, then it will obviously be valid in the general case as well. Thus, the generators of $[\Gamma]$ are chosen in two ways: (Q_0, S_0) and $(\bar{Q}_0, S_0)$, where $\bar{Q}_0=s_0Q_0$, and α_1 and α_2 coincide on S_0 and are related as follows:

$$\alpha_2(x_0, y, Q_0) = \alpha_1(x_0, y, Q_0)+h(x_0, y) \tag{6.9}$$

on Q_0, where $h(x_0, y)$ is a measurable function with values in H.

Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be a sequence of positive numbers monotonically converging to 0 and $\{\mathcal{D}_k\}_{k=1}^{\infty}$ a sequence of sets, which is dense in $\mathcal{B}$, each term occuring an infinite number of times in it. Construct a Γ-array ζ_1 such that the cocycle α_1 takes constant values on each of its elements. Besides, the group of automorphisms $\mathcal{G}(\zeta_1)$, approximates the orbits of Γ with error ε_1, and in $\mathcal{P}(\zeta_1)$ there is a set D_1' which approximates D_1 with the error ε_1. Such the array exists because the function $\bar{\varphi}(x_0)$ may be thought to be piecewise constant. Since α_1 on $\{Q_0^n\colon n\in\boldsymbol{Z}\}$ is transient, then the array ζ_1 is globally nontransitive and consists of a finite number of transitive components. According to (6.9) and the condition of the theorem, there exists a Γ-array η_1 such that: (1) η_1 has as many transitive components as ζ_1; (2) every transitive compnent of η_1 contains as many sets as the corresponding component of ζ_1; (3) the sets of η_1 can be numbered so that for the sets $E(i)$ and $F(i)$ of ζ_1 and η_1, respectively, having identical numbers, $\nu(E(i)\,(x_0))=\nu(F(i)\,(x_0))$ for a.a. $x_0\in X_0$, where $E(i)\,(x_0)$ and $F(i)\,(x_0)$ are x_0-sections of $E(i)$ and $F(i)$; (4) the values of α_1 and α_2 on elements of ζ_1, and η_1 with identical numbers coincide. E.g., η_1 can be constructed over the partition which defines ζ_1 on X. Then, refine the Γ-array η_1 and construct a Γ-array η_2 such that the cocycle α_2 takes constant values on its elements and $\mathcal{G}(\eta_2)$ approximates the orbits of Γ with the error ε_1 and $\mathcal{P}(\eta_2)$ also approximates the set D_1 with the error ε_1. To do so, let us consider a set A consisting of the union of fundamental sets of transitive components of η_1 and construct the Γ-array η_1' on A in such a way that the refinement of η_1 by η_1', which we denote by η_2, should have the above properties. Then, construct a refinement ζ_1 of ζ_2 so that the above conditions (1)—(4) should be fulfilled for the arrays ζ_2, η_2. By repeating the said procedure a countable number of times, construct two sequence of Γ-arrays $\{\zeta_k\}_{k=1}^{\infty}$ and $\{\eta_k\}_{k=1}^{\infty}$, which approximate the σ-algebra $\mathscr{B}$ and the orbits of Γ. On elements of these arrays having the same numbers, the cocycles α_1 and α_2 have the same values. Therefore, as in the proof of Theorem 2.3 of [1] and Theorem 3.9, we conclude that there exists a measure-preserving automorphism $\theta \in N[\Gamma]$ such that $\theta \circ \alpha_2 = \alpha_1$. Note that the automorphism θ represents a measurable filed of automorphisms $x_0 \to \theta(x_0) \in N[S]$. This follows from the fact that for any $k \in \boldsymbol{N}$ there exists an automorphism $s_k \in [S_0]$ such that ζ_k is mapped by s_k into η_k (see property (3) above). Since θ is the limit in the metric d on $N[\Gamma]$ of the automorphism sequence $\{s_k\}_{k=1}^{\infty}$, then θ has the above form (the metric d was defined in [9]). Thus, it is proved that the pairs (Γ_1, α_1) and (Γ_2, α_2) (under the above assumptions) are stably weakly equivalent. □

Consider the problem of existence of a pair (Γ, α) for which the associated action is isomorphic to a given action of G. We shall consider in particular non-free actions.

Theorem 6.7. *Let an ergodic action $W(G)$ of the group G on a Lebesgue space (Ω, p), $p(\Omega)=1$ be defined which preserves p and has a stabilizer $H \subset G$. Then there exists a pair (Γ, α), where Γ is a type II_1 ergodic a.f. group of automorphisms of (X, μ) and a cocycle $\alpha \in Z^1(X \times \Gamma, G)$, such that $r(\Gamma, \alpha)=H$ and the associated action $W_{(\Gamma,\alpha)}(G)$ is isomorphic to $W(G)$.*

Proof. The plan of the proof is the same as that of Theorem 5.10. Define for the free action $W(\hat{G})$ of $\hat{G}=G/H$ on (Ω, p) the following objects as in Thoerem 5.10: (X_0, μ_0), Q, u_0, (Y, ν) and $S \in \mathrm{Aut}(Y, \nu)$. Let $\beta \in Z^1(Y \times [S], H)$ be the standard cocycle defined in Subsection 3.1 such that $H=r(S, \beta)$. Let Γ be a group of automorphisms of $(X, \mu)=(X_0 \times Y, \mu_0 \times \nu)$ generated by $Q_0=Q \times \mathbb{1}$ and $S_0=\mathbb{1} \times S$. Define a cocycle α for Γ: $\alpha(x_0, y, Q_0)=\sigma(u_0(Q\, x_0, x_0))$, $\alpha(x_0, y, S_0)=\beta(y, S)$, where σ: $\hat{G} \to G$ is a measurable section of G over $\hat{G}$. As in Theorem 5.10, we see that the pair (Γ, α) is sought-for one. □

The following theorem shows that the action associated with (Γ, α) does not necessarily preserve measure, though Γ is a type II group of automorphisms.

Theorem 6.8. *Let $W(G)$ be an ergodic non-free non-singular action of G on a space (Ω, p), $p(\Omega)=1$ and let H be the stabilizer of the action of G. Then, there exists an ergodic a.f. group Γ acting on a Lebesgue space (X, μ), $\mu(X)=\infty$ and preserving the measure μ and there exists a cocycle $\alpha \in Z^1(X\times\Gamma, G)$, and both are such that $r(\Gamma, \alpha)=H$ and $W_{(\Gamma,\alpha)}(G)$ is isomorphic to $W(G)$.*

Remark. If $E\subset X$, $\mu(E)<\infty$, then for the pair (Γ_E, α_E) the associated action $W_{(\Gamma_E,\alpha_E)}(G)$ is isomorphic to $W_{(\Gamma,\alpha)}(G)$. Therefore, it is unessnetial that Γ in Theorem 6.8 is of type II_∞.

Proof. Define the objects (X_0, μ_0), Q, u_0, σ, $W(\hat{G})$ as in Theorems 5.10 and 6.7. The automorphism Q has a nontrivial Radon-Nikodym cocycle $\rho(x_0, Q^n) =\log \dfrac{d\,Q^{-n}\circ\mu_0}{d\,\mu_0}(x_0)$. Let S_1 be an ergodic automorphism of a Lebesgue space (Y_1, ν_1), $\nu_1(Y_1)=\infty$ preserving the measure ν_1. Construct a measurable filed $x_0 \to U(x_0)\in N[S_1]$ $(x_0\in X_0)$ of automorphisms of (Y_1, ν_1) such that

$$\Phi(U(x_0)) = -\log\frac{d\,Q^{-1}\circ\mu_0}{d\,\mu_0}(x_0)\,. \tag{6.10}$$

Let S, (Y, ν) and β be the same as in Thoerem 6.7. Define the automorphism group F generated by commutating automorphisms $\mathbf{1}\times S$ and $S_1\times\mathbf{1}$ on the space $(Y_1\times Y, \nu_1\times\nu)$ and the cocycle $\beta_1\in Z^1(Y_1\times Y\times F, H)$, assuming that $\beta(y, S^m)= \beta_1(y_1, y, S_1^n, S^m)$. Now it is obvious that $r(F, \beta_1)=H$.

Let us consider the following automorphisms on the space $(X, \mu)= (X_0\times Y_1\times Y, \mu_0\times\nu_1\times\nu)$:

$$\begin{aligned} Q_0(x_0, y_1, y) &= (Q\,x_0, U(x_0)\,y_1, y)\,,\\ S_{1,0}(x_0, y_1, y) &= (x_0, S_1\,y_1, y)\,,\\ S_0(x_0, y_1, y) &= (x_0, y_1, Sy)\,. \end{aligned}$$

These automorphisms generate the ergodic a.f. group Γ, which, by (6.10), is of type II_∞. Define the cocycle α on Γ:

$$\begin{aligned} \alpha(x_0, y_1, y, Q_0) &= \sigma(u_0(Q\,x_0, x_0))\,,\\ \alpha(x_0, y_1, y, S_{1,0}) &= 0\,,\\ \alpha(x_0, y_1, y, S_0) &= \beta(y, S)\,. \end{aligned} \tag{6.11}$$

It is easy checked that formulae (6.11) define the cocycle α on Γ correctly. The ergodicity of S_1 on Y and of $S(\beta)$ on $Y\times H$ means that the associated action $W_{(\Gamma,\alpha)}(G)$ of G has the stabilizer H and is defined on the quotient space $X_0\times G/H$ by the measurable partition into orbits of $(x_0, \hat{g})\to(Q\,x_0, \hat{g}+u_0(Q\,x_0, x_0))$. Since

u_0 is a return cocycle for $W(G/H)$, then $W_{(\Gamma,\alpha)}(G)$ is isomorphic to $W(G)$. □

Prove now the uniqueness theorem for pairs whose associated actions are non-free and non-singular.

It follows from Theorem 6.4 that, if a weakly equivalent pairs are considered, any pair (Γ, α) can be assumed to have the following properties: the group Γ is of type II and is generated by the automorphisms $\bar{Q}_0: (x_0, y) \to (Q x_0, \bar{U}(x_0) y)$, $S_0: (x_0, y) \to (x_0, Sy)$ acting on $(X_0 \times Y, \mu_0 \times \nu)$, and the cocycle α is given by the formulae:

$$\begin{aligned} \alpha(x_0, y, \bar{Q}_0) &= \bar{\varphi}(x_0) = \sigma(\varphi(x_0)), \\ \alpha(x_0, y, S_0) &= \beta(y, S), \end{aligned} \tag{6.12}$$

where β is the standard cocycle from Section 3, the function $\varphi: X_0 \to \hat{G}$ is outside a neighborhood of identity in G and σ is a section of G over $\hat{G}$.

Theorem 6.9. *Let there be defined the pairs* (Γ_1, α_1) *and* (Γ_2, α_2) *satisfying the above conditions and relations* (6.12). *Then, if the associated actions* $W_{(\Gamma_1,\alpha_1)}(G)$ *and* $W_{(\Gamma_2,\alpha_2)}(G)$ *are isomorphic, then* (Γ_1, α_1) *and* (Γ_2, α_2) *are stably weakly equivalent, and* $r(\Gamma_1, \alpha_1) = r(\Gamma_2, \alpha_2)$.

Proof. Obviously, it may be believed that Γ_1 and Γ_2 act on the same space $(X_0 \times Y, \mu_0 \times \nu)$ and moreover $[\Gamma_1] = [\Gamma_2] = [\Gamma]$. In view of the results of Section 5 and Proposition 6.8, by changing to stably weakly equivalent pairs, we can provide that $Q_1 = Q_2 = Q$, $\varphi_1(x_0) = \varphi_2(x_0) = \varphi(x_0)$. Thus, we are now to construct an automorphism $P = (x_0 \to P(x_0)) \in N[\Gamma]$, where $P(x_0) \in N[S]$ and $P^{-1} \circ \alpha_1 = \alpha_2$, $P^{-1} \bar{Q}_0^1 P = \bar{Q}_0^2 s_0$. It follows from the conditions of the theorem and the fact that the automorphisms $\bar{Q}_0^i$, $i = 1, 2$, preserve measure, that for a.a. $x_0 \in X_0$,

$$\Phi(\bar{U}_1(x_0)) = \Phi(\bar{U}_2(x_0)). \tag{6.13}$$

The values of α_1 and α_2 on $\bar{Q}_0^1$ and $\bar{Q}_0^2$ differ by a function taking values in the group $H = r(S, \beta)$. We may multiply the element $\bar{Q}_0^2$ by an automorphism $s_0' \in [S_0]$, so that the values of α_1 and α_2 on $\bar{Q}_0^1$ and $\bar{Q}_0^2 s_0'$ should become equal.

Now we shall do as in the proof of Theorem 6.6. Using the above properties of α_1 and α_2 and the property (6.13), construct two sets of Γ-arrays, $\{\xi_n\}_{n=1}^{\infty}$ and $\{\eta_n\}_{n=1}^{\infty}$ satisfying the conditions: (1) $\bigcup_{n=1}^{\infty} \mathcal{Q}(\xi_n) x = \bigcup_{n=1}^{\infty} \mathcal{Q}(\eta_n) x = \Gamma x$; (2) $\sigma(\bigcup_{n=1}^{\infty} \mathcal{P}(\xi_n)) = \sigma(\bigcup_{n=1}^{\infty} \mathcal{P}(\eta_n)) = \mathcal{B}$; (3) the arrays ξ_n and η_n, $n \in N$ have an equal number of sets and $\nu(E(i)(x_0)) = \nu(F(i)(x_0))$ for a.a. $x_0 \in X_0$, where

$E(i)$ and $F(i)$ are sets from ξ_n and η_n; (4) on elements of ξ_n and η_n having the same serial numbers the cocycles α_1 and α_2 assume equal values. Therefore, there exists an automorphism $P=(x_0 \rightarrow P(x_0)) \in N[\Gamma]$, which transforms the generators $(\bar{Q}_0^1, S_0)$ of Γ into $(\bar{Q}_0^2, S_0)$ and is such that $P^{-1} \circ \alpha_1 = \alpha_2$. □

§7. Associated Actions for Type III Groups Γ

7.1. In this section we shall consider the existence and uniqueness theorems for the pairs (Γ, α_0), where Γ is a type III a.f. ergodic group of automorphisms. In Subsection 7.2 we shall construct an example of (Γ, α_0) for which $r(\Gamma, \alpha)=G$, $r(\Gamma, \rho)=\boldsymbol{R}$ (i.e. Γ is of type III_1), but $r(\Gamma, \alpha_0)=\{0\}$. In Subsection 7.4 we shall study the interrelation between the type of Γ and that of an associated action.

We shall use essentailly the results of the two preceding sections. First, we shall consider free associated actions of $G_0=G \times \boldsymbol{R}$, where G is an arbitrary l.c.s. abelian group. Recall that we deal here with nonregular recurrent cocycle α_0.

It follows from Theorem 5.6 that, if (Γ, α_0) is such that $W_{(\Gamma, \alpha_0)}(G_0)$ is free, i.e. $r(\Gamma, \alpha_0)=\{0\}$, then Γ acts on the space $(X_0 \times Y, \mu_0 \times \nu)$, and (Γ, α_0) has the following properties: (1) Γ is generated by the automorphisms Q_0 and S_0 (see (6.1)); (2) α_0 is defined on Q_0 and S_0 by the formulae: $\alpha_0(x_0, y, Q_0)=\varphi(x_0)$, $\alpha_0(x_0, y, S_0)=0$, where the function $\varphi(x_0)$ is separated from the identity in G_0 by a neighborhood V_0.

Property (2) implies that α_0, when considered on $\{Q_0^n: n \in \boldsymbol{Z}\}$, is there transient, and the ergodic automorphism S is of type II, i.e. $S \circ \nu=\nu$. Assume for convenience the measure ν to be infinite. Besides, since $\varphi(x_0)=(\varphi_G(x_0), \varphi_{\boldsymbol{R}}(x_0))$, then

$$\varphi_{\boldsymbol{R}}(x_0) = \log \frac{d\, Q^{-1} \circ \mu_0}{d\, \mu_0}(x_0) + \Phi(U(x_0)) .$$

The above properties of (Γ, α_0) suggest validity of the following lemma (which is proved in the same way as Lemma 5.8).

Lemma 7.1. *The free action $W_{(\Gamma, \alpha_0)}(G_0)$ of G associated with (Γ, α_0) is isomorphic to the associated action of G_0 constructed by the pair $(Q, \varphi(x_0))$.*

Theorem 7.2. *Let there be two pairs (Γ_1, α_0^1) and (Γ_2, α_0^2), which satisfy the above conditions and are such that $r(\Gamma_1, \alpha_0^1)=r(\Gamma_2, \alpha_0^2)=\{0\}$. Then, the isomorphism of the associated actions $W_{(\Gamma_1, \alpha_0^2)}(G_0)$ and $W_{(\Gamma_2, \alpha_0^2)}(G_0)$ means that (Γ_1, α_0^1)*

and (Γ_2, α_0^2) *are weakly equivalent.*

The proof is essentially the same as that of Theorem 5.9.

Theorem 7.3. *Let* $W(G_0)$ *be a free ergodic action of* G_0 *on a Lebesgue space* (Ω, p). *Then, there exists a pair* (Γ, α_0), *where* Γ *is an ergodic a.f. group of automorphisms of* $(X, \mathcal{B}, \mu)$ *and* $\alpha_0 \in Z^1(X \times \Gamma, G_0)$ *both such that the associated action* $W_{(\Gamma, \alpha_0)}(G_0)$ *is isomorphic to* $W(G_0)$.

Proof. We introduce, as in Theorem 5.11, the following objects: (X_0, μ_0), $Q, u_0, (Y, \nu), S$. Denote

$$u_0(Q\,x_0, x_0) = \varphi(x_0) = (\varphi_G(x_0), \varphi_R(x_0))\,. \tag{7.1}$$

Choose a measurable field of automorphisms $x_0 \to U(x_0) \in N[S]$ such that

$$\Phi(U(x_0)) = \varphi_R(x_0) - \log \frac{d\,Q^{-1} \circ \mu_0}{d\,\mu_0}(x_0)\,.$$

Let Γ be the group of automorphisms with the generators Q_0 and S_0, where Q_0 and S_0 are defined by (6.1). The cocycle α is defined by the formulae: $\alpha(x_0, y, Q_0) = \varphi_G(x_0)$, $\alpha(x_0, y, S_0) = 0$. The action $W_{(\Gamma, \alpha_0)}(G_0)$ is isomorphic to $W(G)$. □

In Subsection 7.4 we shall consider the problem of the type of the group Γ whose existence was proved in Theorem 7.3.

7.2. Let us construct an example of a type III_1 automorphism group Γ and a cocycle $\alpha \in Z^1(X \times \Gamma, G)$, which are such that the ranges of the components α and ρ of α_0 are G and $\mathbf{R}$, respectively, while $\alpha_0 = (\alpha, \rho)$ is lacunary in G_0.

Example 7.4. Let there be defined a Lebesgue space $(X, \mu) = (X_0 \times Y, \mu_0 \times \nu)$, $\mu_0(X_0) = \nu(Y) = \infty$, an ergodic automorphism $Q \in \mathrm{Aut}(X_0, \mu_0)$, $Q \circ \mu_0 = \mu_0$, an ergodic automorphism $S \in \mathrm{Aut}(Y, \nu)$, $S \circ \nu = \nu$, and an automorphism $U \in \mathrm{Aut}(Y, \nu)$, the latter such that $U \in N[S]$ and $U \circ \nu = \lambda\, \nu$, where $\lambda \in (0, 1)$. Put as usual $Q_0(x_0, y) = (Q\,x_0, Uy)$, $S_0(x_0, y) = (x_0, Sy)$. Since Q is of type II_∞, then there exists an action $l(G)$ of G on (X_0, μ_0), which preserves the measure μ_0 and is such that $l(g) \in N[Q]$, $g \in G$ and $l(g) \notin [Q]$, $g \neq e$ (see e.g. [6]). Put $l_0(g)(x_0, y) = (l(g)\,x_0, U(g)\,y)$, $g \in G$, where $U(g) \in N[S]$ and $\Phi(U(g)) = \log \lambda(g)$, the numbers $\log \lambda$, $\log \lambda(g)$, $g \in G$ being rationally independent, and assume that there exists an element $g_1 \in G$ such that $\lambda(g_1) \neq 1$. Consider equivalence relation $\mathcal{E}$ with continuous orbits, which is generated on (X, μ) by the action of Q_0, S_0 and $l_0(g) = l(g) \times U(g)$, $g \in G$. The above properties of $\lambda(g)$ show that the equivalence relation $\mathcal{E}$ is of type III_1. Consider the cocycle α deffned on the generators of $\mathcal{E}$ as follows: $\alpha(x_0, y, Q_0)=\alpha(x_0, y, S)=0$, $\alpha(x_0, y, l_0(g))=g$, $g\in G$. Obviously, $r(\mathcal{E}, \alpha)=G$. Calculate now the range of $\alpha_0=(\alpha, \rho)$. Show that the associated action $W_{(\mathcal{E},\alpha_0)}(G_0)$ is free, whence it will follow that $r(\mathcal{E}, \alpha_0)=\{0\}$. The orbits of the equivalence relation $\mathcal{E}(\alpha_0)$ are generated by the automorphisms:

$$\begin{aligned} Q_0(\alpha_0)\,(x_0, y, h, u) &= (Q_0(x_0, y), h, u+\log\lambda)\,, \\ S_0(\alpha_0)\,(x_0, y, h, u) &= (x_0, Sy, h, u)\,, \\ l_0(g)\,(\alpha_0)\,(x_0, y, h, u) &= (l(g)\,x_0, U(g)\,y, h+g, u+\log\lambda(g)) \end{aligned} \tag{7.2}$$

(recall that $\mathcal{E}(\alpha_0)$ defines the partition into equivalence classes on the space $X_0\times Y\times G\times \boldsymbol{R}$). Let us find now the quotient space by the measurable hall of partition into orbits of the equivalence relation $\mathcal{E}(\alpha_0)$ on $X_0\times Y\times G\times\boldsymbol{R}$. Since S acts ergodically on Y and in view of (7.2), this quotient space should be sought for in the set $X_0\times G\times\boldsymbol{R}$, i.e. in "the plane perpendicular to Y". Put $E=X_0\times\{0\}\times[0, -\log\lambda)$. It follows from (7.2) that the set E intersects the orbits of $Q_0(\alpha_0)$ and $l_0(g)\,(\alpha_0)$ exactly at one point. Therefore, E can be identified with the desired quotient space, and $W_{(\mathcal{E},\alpha_0)}(G_0)$ can be thought to be defined on E. Let $(g_0, u_0)\in G_0$, $(x_0, 0, u)\in E$; then

$$W_{(\mathcal{E},\alpha_0)}(g_0, u_0)\,(x_0, 0, u) = Q_0^n(\alpha_0)\,(l(-g_0)\,x_0, 0, u+u_0-\log\lambda(g_0))\,,$$

where the number n is chosen by the condition $u+u_0-\log\lambda(g_0)+n\log\lambda\in[0, -\log\lambda]$. Therefore, the automorphism $W_{(\mathcal{E},\alpha_0)}(g_0, u_0)$ acts identically if and only if $Q^n\,l(-g_0)\,x_0=x_0$. However, since $l(G)$ was chosen to be strictly outer to $[Q]$, the latter equality is not true.

Let us construct the countable group Γ of type III_1 in the following way. Let $\mathcal{G}_1$ be the measurable groupoid generated by $[Q]$ and $l(g)$, $g\in G$. Then $\mathcal{G}_1$ is isomorphic to the groupoid $X'\times\boldsymbol{T}\times\Gamma\times\boldsymbol{T}$, where a countable automorphism group Γ acts on the Lebesgue space X' [3, 17]. The cocycle α defined on $\mathcal{G}_1$ can be replaced by the cohomologous cocycle α' so that it should become trivial for the action of the circle $\boldsymbol{T}$ on itself, i.e. α' should be concentrated on $X'\times\Gamma$ (simple arguments omitted). The pair (Γ, α') will have the properties: $r(\Gamma, \alpha')=G$, $r(\Gamma, \alpha_0')=\{0\}$. □

The latter example suggests that regularity of α does not imply regularity of α_0.

7.3. Consider the uniqueness and existence theorems for (Γ, α_0) for the case of type III group Γ and a non-free associated action $W_{(\Gamma,\alpha_0)}(G_0)$.

Recall the results of Theorem 6.4. By changing to the weakly equivalent pair, one can provide the following properties of any (Γ, α_0) such that $r(\Gamma, \alpha_0)= H_0$: (1) the group Γ is generated by Q_0 and S_0 acting on $(X_0\times Y, \mu_0\times\nu)$ as $Q_0(x_0, y)=(Q x_0, U(x_0) y)$, $S_0(x_0, y)=(x_0, Sy)$, $U(x_0)\in N[S]$, (2) α_0 is defined on Γ as

$$\begin{aligned}\alpha_0(x_0, y, Q_0) &= \varphi(x_0) = (\varphi_G(x_0), \varphi_{\mathbf{R}}(x_0))\,,\\ \alpha_0(x_0, y, S_0) &= \beta_0(y, S)\,,\end{aligned} \tag{7.3}$$

where β_0 is the standard cocycle from Section 3, for which $r(S, \beta_0)=H_0$. For S there are two alternatives: it is either of type II or of type III$_\lambda$ $(0<\lambda\leq 1)$, according to the form of $H_0\subset G\times\mathbf{R}$.

Theorem 7.5. *Let the pairs* (Γ_i, α_0^i), $i=1, 2$ *have the above properties and* $r(\Gamma_1, \alpha_0^1)=r(\Gamma_2, \alpha_0^2)=H_0$. *Then, if the associated actions* $W_{(\Gamma_1,\alpha_0^1)}(G_0)$ *and* $W_{(\Gamma_2,\alpha_0^2)}(G_0)$ *are isomorphic, then* (Γ_1, α_0^1) *and* (Γ_2, α_0^2) *are weakly equivalent.*

Proof. It follows from the conditions of the theorem and relations (7.3) that for both the pairs (Γ_1, α_0^1) and (Γ_2, α_0^2) it can be believed that the Lebesgue spaces coincide in which the groups Γ_1 and Γ_2 act, and also $S_0^1=S_0^2$. Since S is of type II or III$_\lambda$ $(0<\lambda\leq 1)$, then there exists in $[S]$ an ergodic subgroup on which the cocycle β_0 is trivial (see Corollary 3.15). Thus, as in Lemma 6.5, the associated action $W_{(\Gamma_i,\alpha_0^i)}(G_0)$ is isomorphic to the action of G_0 on the quotient space of $X_0\times\hat{G}_0$ by the measurable partition into orbits of $(x_0, \hat{g}_0)\to(Q^i x_0, \hat{g}+\hat{\varphi}^i(x_0))$, $i=1, 2$, where $\hat{\varphi}^i(x_0)$ is separated from zero. By changing mentally to weakly equivalent pairs, we find that $Q^1=Q^2=Q$ and $\hat{\varphi}^1=\hat{\varphi}^2=\varphi$. The latter equality, combined with (7.3), means that φ^1 and φ^2 differ by an element of H_0, and thus,

$$\varphi_G^1(x_0) = \varphi_G^2(x_0)=h_G(x_0)\,, \quad \varphi_{\mathbf{R}}^1(x_0) = \varphi_{\mathbf{R}}^2(x_0) = h_{\mathbf{R}}(x_0)\,. \tag{7.4}$$

Since $\varphi_{\mathbf{R}}^i(x_0) = \rho(x_0, Q)+\Phi(U^i(x_0))$, $i=1, 2$, then (7.4) leads to

$$\Phi(U^1(x_0)) = \Phi(U^2(x_0))+h_{\mathbf{R}}(x_0)\,. \tag{7.5}$$

Relation (7.5) means that by multiplying Q_0^2 by an element $s_0\in[S_0]$ (as in the proof of Theorem 6.4), we shall obtain that α_0^1 on Q_0^1 and α_0^2 on $\bar{Q}_0^2=s_0 Q_0^2$ coincide. Now, in view of Theorems 6.6 and 6.9, there exists a transformation $P=(x_0\to P(x_0))$ with the following properties: $P\in N[S_0]$, $\Phi(P(x_0))=1$, $P Q_0^1 P^{-1} =s_0' Q_0^2$, $P^{-1}\circ\alpha_0^1=\alpha_0^2$. By repeating the arguments of Theorem 6.9, we find that

(Γ_1, α_0^1) and (Γ_2, α_0^2) are weakly equivalent. □

The proof of Theorem 7.5 shows that its conditions provide for orbital equivalence of Γ_1 and Γ_2.

Theorem 7.6. *Let $W(G_0)$ be an ergodic non-free action of G_0 on a Lebesgue space (Ω, p), for which p is non-singular, and H_0 the stabilizer of $W(G_0)$. Then, there exist an ergodic a.f. group Γ acting on (X, μ) and a cocycle $\alpha_0 \in Z^1(X\times\Gamma, G_0)$, and both are such that $r(\Gamma, \alpha_0)=H_0$ and the associated action $W_{(\Gamma,\alpha_0)}(G_0)$ is isomorphic to $W(G_0)$.*

Proof. This one is partly similar to those of Theorems 6.8 and 7.3. □

7.4. In conclusion, let us consider several results on the relationship between the types of Γ and $W_{(\Gamma,\alpha_0)}(G_0)$. Let the group Γ be generated, as before, by Q_0 and S_0 and the cocycle α_0 satisfy relations (7.3).

Proposition 7.7. *Let Γ be an arbitrary countable ergodic group of automorphisms of $(X, \mathcal{B}, \mu)$ and α a transient cocycle from $Z^1(X\times\Gamma, G)$, where G is an arbitrary l.c.s. group. The type of Γ coincides with that of the associated action $W_{(\Gamma,\alpha)}(G)$ (and also with that of $W_{(\Gamma,\alpha_0)}(G_0)$).*

The proof follows from the results of Section 2. □

Corollary 7.8. *The type of the associated action $W_{(\Gamma,\alpha_0)}(G_0)$ for the group Γ generated by Q_0 and S_0 and for the cocycle α_0 having the properties mentioned in the beginning of this section coincides with the type of the automorphism Q.*

Proof. The statement follows from Proposition 6.5. □

Consider the following problem: let an action $W(G_0)$ be defined on a Lebesgue space (Ω, p); then, what type may Γ be of, if $W_{(\Gamma,\alpha_0)}(G_0)$ is isomorphic to $W(G_0)$? Let us first dwell on some particular cases.

Lemma 7.9. *For any pair (Γ, α_0), we have $r(\Gamma, \alpha_0)\subset r(\Gamma, \alpha)\times r(\Gamma, \rho)$.*

Proposition 7.10. *Let the pair (Γ, α_0) be such that Γ is of type II or III_0 and the associated action $W_{(\Gamma,\alpha_0)}(G_0)$ is non-free, i.e. $r(\Gamma, \alpha_0)=H_0\neq\{0\}$. Then, the group H_0 is $H_G\times\{0\}$ where H_G is a closed subgroup of G.*

The proof immediately follows from Lemma 7.9. □

Theorem 7.11. *Let Γ be an arbitrary ergodic a.f. group of automorphisms of $(X, \mathcal{B}, \mu)$ and G an arbitrary l.c.s. abelian group. There exists a cocycle*

$\alpha_0 \in Z^1(X \times \Gamma, G)$ such that the associated action $W_{(\Gamma,\alpha_0)}(G_0)$ is of an arbitrary type, i.e. the type of $W_{(\Gamma,\alpha_0)}(G_0)$ does not depend on the type of Γ.

Proof. (1) Let Γ be of type II. It can be thought of as generated by Q_0 and S_0 preserving measure on $(X_0 \times Y, \mu_0 \times \nu)$ where $Q_0(x_0, y) = (Q\, x_0, U(x_0)\, y)$, $S_0(x_0, y) = (x_0, Sy)$, $U(x_0) \in N[S]$, $\Phi(U(x_0)) = -\rho(x_0, Q)$. Let the cocycle α_0 be a transient on Q_0 and be zero on S_0. Then, the statement of the theorem follows from Corollary 7.8 and Theorem 5.11, because Q can be of an arbitrary type.

(2) Let Γ be of type III_λ $(0<\lambda<1)$ or III_1. Then Γ is orbitally equivalent to $\Gamma \times \Gamma_1$, where Γ_1 is of type II group. Let a cocycle $\alpha \in Z^1(X \times \Gamma \times \Gamma_1, G)$ be zero on Γ and be concentrated on Γ_1. Then the associated action $W_{(\Gamma \times \Gamma_1, \alpha_0)}(G_0)$ is isomorphic to $W_{(\Gamma,\rho)}(\boldsymbol{R}) \times W_{(\Gamma_1,\alpha)}(G)$. For the cases under consideration, $W_{(\Gamma,\rho)}(\boldsymbol{R})$ is either a transitive, or a trivial flow. Thus, the type of $W_{(\Gamma \times \Gamma_1, \alpha_0)}(G_0)$ is determined by that of $W_{(\Gamma,\alpha)}(G)$ which, by (1), can be arbitrary.

(3) Let Γ be of type III_0. It can be shown, by the same method as in (2), that if $W_{(\Gamma,\rho)}(\boldsymbol{R})$ preserves measure, then $W_{(\Gamma,\alpha_0)}(G_0)$ may be of an arbitrary type. Consider the case where $W_{(\Gamma,\rho)}(\boldsymbol{R})$ is non-singular. Use the Krieger representation of a type III_0 group Γ as a group $\mathcal{G}(Q, \varphi)$ generated on $(X_0 \times Y, \mu_0 \times \nu)$, $\nu(Y) = \infty$ by the automorphisms $Q_0(x_0, y) = (Q\, x_0, U(x_0)\, y)$, $S_0(x_0, y) = (x_0, Sy)$ such that $\rho(x_0, y, Q_0) = \rho(x_0, Q) + \Phi(U(x_0)) = \varphi(x_0) > C > 0$ and $\rho(x_0, y, S_0) = 0$ [10]. The flow $W_{(\Gamma,\rho)}(\boldsymbol{R})$ is a special flow constructed from the basis automorphism Q and the ceiling function $\varphi(x_0)$. Replace $\mathcal{G}(Q, \varphi)$ by an orbitally equivalent group. Consider on $(X_0 \times Y \times Y_1, \mu_0 \times \nu \times \nu_1)$ an automorphism group Γ' whose generators act as follows

$$Q_0'(x_0, y, y_1) = (Q\, x_0, U_2(x_0)\, y, U_1(x_0)\, y_1)\,,$$
$$S_0'(x_0, y, y_1) = (x_0, Sy, y_1)\,,$$
$$S_0''(x_0, y, y_0) = (x_0, y, S_1\, y_1)\,,$$

where $S_1 \circ \nu_1 = \nu_1$, $U_1(x_0) \in N[S_1]$, $U_2(x_0) \in N[S]$, $\Phi(U_2(x_0)) = \varphi(x_0)$, $\Phi(U_1(x_0)) = -\rho(x_0, Q)$. It is easy to calculate that $W_{(\Gamma',\rho)}(\boldsymbol{R}) = W_{(\Gamma,\rho)}(\boldsymbol{R})$. Define a cocycle $\alpha \in Z^1(X_0 \times Y \times Y_1 \times \Gamma', G)$ assuming α to be zero on Q_0' and S_0' to be equal to g on S_0'' where $g \in G$ (G is assumed non-compact). Calculate $W_{(\Gamma',\alpha_0)}(G_0)$. The cocycle α_0 is zero on S_0'. Let $(X_0 \times Y, \mu_0 \times \nu_1) = (X_0', \mu_0')$ and the full automorphism group $[Q']$ of X_0' be generated by $(x_0, y_1) \to (Q\, x_0, U_1(x_0)\, y)$ and $(x_0, y_1) \to (x_0, S_1\, y_1)$. The automorphism Q' obviously preserves the measure μ_0'. Since $U_1(x_0) \in N[S_1]$ and $\Phi(U_1(x_0)) \neq 1$ for a.a. $x_0 \in X_0$, the cocycle $\alpha_0 = (\alpha, \rho)$ is transient on $\bar{Q}_0': (x_0', y) \to (Q' x_0', V(x_0) y)$, where $V(x_0) \in N[S]$ and is constructed from $U_2(x_0)$. Because Q' preserves measure, then $W_{(\Gamma', \alpha_0)}(G_0)$ will also be of type II (see (7.8)).

(4) By combining the methods of (2) and (3), we can provide that the action $W_{(\Gamma, \alpha_0)}(G_0)$ (for Γ of type III_0) should have type III_λ $(0 \leq \lambda \leq 1)$. □

References

[1] Bezuglyi, S.I. and Golodets, V.Ya., Outer conjugacy for actions of countable amenable groups on measure space, (in Russian), *Izv. AN SSSR, Ser. Mat.*, **50** (1986), 604–621.

[2] Fedorov, A.L., The Krieger theorem for cocycles, (in Russian) *Dep. at VINITI* 25.02.85, No. 1406–85 Dep.

[3] Feldman, J., Hahn P. and Moore, C.C., Orbit structure and countable sections for actions of continuous group, *Adv. Math.*, **28** (1978), 186–230.

[4] Feldman, J. and Moore, C.C., Ergodic equivalence relations, cohomology, and von Neumann algebras, I, *Trans. Amer. Math. Soc.*, **234** (1977), 289–324.

[5] Golodets, V.Ya. and Sinelshchikov, S.D., *Existence and uniqueness of cocycles of an ergodic automorphism with dense images in amenable groups*, (in Russian), Kharkov, 1983 (Preprint of FTINT AN USSR, 19–83).

[6] ———, Outer conjugacy for actions of continuous amenable groups, *Publ. RIMS, Kyoto Univ.*, **23** (1987), 737–769.

[7] ———, Structure of automorphisms of measurable groupoids and comparison of transient cocycles, (in Russian), *Dokl. AN USSR, ser. A*, (1987), No. 5, 3–5.

[8] Kornfeld, I.P., Sinay, Ya. G. and Fomin, S.V., *Ergodic theory*, (in Russian), Nauka, Moscow, 1980.

[9] Hamachi, T. and Osikawa, M., Ergodic groups of automorphisms and Krieger's theorems, *Sem. Math. Sci. Keio Univ.*, (1981), No. 3, 1–113,

[10] Krieger, W., On ergodic flows and isomorphisms of factors, *Math. Ann.*, **223** (1976), 19–70.

[11] Mackey, G.W., Virtual groups and group actions, *Math. Ann.* **166** (1966), 187–207.

[12] Ramsay, A., Virtual groups and group actions, *Adv. Math.*, **6** (1971), 253–322.

[13] ———, *Topologies on measured groupoids*, Univ. of Colorado, 1983.

[14] Rokhlin, V.A., Selected problems of the metrical theory of dynamical systems, (in Russian), *Uspekhi Mat. Nauk*, **4** (1949), 57–128.

[15] Schmidt, K., *Lecture on cocycles of ergodic transformation groups*, Univ. of Warwick, 1976.

[16] Stepin, A.M., Cohomologies of groups of automorphisms of the Lebesgue space, *Funkts. Anal. i Prilozh.*, (in Russian) **5** (1971), 91–92.