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Finiteness Obstructions of Equivariant Fibrations

By

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Abstract

Let G be a compact Lie group and $E \to B$ a G-fibration. We define a homomorphism $Wa^G(B) \oplus U^G(B)$ into $Wa^G(E) \oplus U^G(E)$ sending the pair of the finiteness obstruction of B and the equivariant Euler characteristic of B to that of E. Here Wa^G is the functor from the G-homotopy category of finitely dominated G-CW complexes into the category of abelian groups given by W. Lück. By making use of this, we show that if H and K are closed subgroups with H or K normal such that $W(HK)$ is not finite, $G \times_H X$ is K-homotopy equivalent to a finite K-CW complex.

Introduction

Let G be a compact Lie group. Assume that B is a finitely G-dominated G-CW complex. Lück [3] has given a functor Wa^G from the G-homotopy category of finitely dominated G-CW complexes into the category of abelian groups and has introduced the equivariant finiteness obstruction $w^G(B) \in Wa^G(B)$ with a geometrical approach to Wall's finiteness obstructions. In the case when G is the trivial group, $Wa^G(B)$ is isomorphic to $\tilde{K}_0(Z[\pi_1(B)])$ and this isomorphism sends $w^G(B)$ to the Wall's finiteness obstruction $O(B)$ [6].

Let G_b denote the isotropy subgroup $\{g \in G \mid g \cdot b = b\}$ at $b \in B$. A G-map $p\colon E \to B$ is said to be a G-fibration [7], if it satisfies the G-homotopy lifting property for any G-CW complexes. We say that p is a G-fibration with fibre F if there is some action of G_b on F satisfying that $p^{-1}(b)$ is G_b-homotopy equivalent to F for each $b \in B$. In this paper, for a G-fibration $p\colon E \to B$, it is assumed that the base space B is a finitely dominated G-CW complex and that the fibre of p is weakly finitely dominated. The notion of weakly finitely domination is introduced in the first section.

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One purpose of this paper is to describe the finiteness obstruction of E by that of B and F for a G-fibration $F \to E \to B$, as the diagonal product formula [3, Theorem 6.3].

This paper is organized as follows. In Section 1, we prepare for a construction of a homomorphism from $Wa^G(B)$ into $Wa^G(E)$. We introduce the equivariant Euler characteristic, given by Lück [3, 4], which is a functorial additive invariant. In Section 2, we define a transfer $p(Wa^G)$ from $Wa^G(B)$ into $Wa^G(E)$ by making use of the properties of the equivariant Euler characteristics. When G is the trivial group, this homomorphism coincides with the transfer map defined by Ehrlich [1]. In Section 3, we obtain that $G \times_H X$ has the K-homotopy type of a finite K-CW complex for some closed subgroups K and H of G.

§1. Preliminaries

Let X and Z be G-spaces. We say that Z dominates X, if there exist G-maps $s: X \to Z$ and $r: Z \to X$ such that $r \circ s$ is G-homotopic to the identity map over X. In this case r is called a domination with section s. We call X finitely dominated, if there exists a finite G-CW complex Z which dominates X. We say that the fibre F of p is *weakly finitely dominated* (resp. *weakly finite*) if for each $b \in B$, F is G_b-homotopy equivalent to a finitely G_b-dominated (resp. finite) G_b-CW complex with respect to the given G_b-action. The fibre of p is weakly finitely dominated if it is finitely H-dominated for each maximal orbit type (H) of $G \cdot V$ for any connected component V in the G-CW complex B, since B has finitely many orbit types. In particular if B has a fixed point of each element of G (that is $B^G \neq \emptyset$) and B/G is connected, the condition "finitely dominated" implies the condition "weakly finitely dominated".

Proposition 1.1. *A G-space dominated by a G-CW complex has the G-homotopy type of a G-CW complex.*

Proof. Let X be a G-space dominated by a G-CW complex Z. Then there exist G-maps $s: X \to Z$ and $r: Z \to X$ such that $r \circ s \underset{G}{\simeq} 1_X$. By G-approximation theorem, there exist a G-CW complex Y and a weak G-homotopy equivalence $f: Y \to X$. Take a G-map $\phi: Z \to Y$ with $f \circ \phi \underset{G}{\simeq} r$. Clearly $f \circ \phi \circ s \underset{G}{\simeq} r \circ s \underset{G}{\simeq} id_X$ and $f \circ \phi \circ s \circ f \underset{G}{\simeq} f$. Since $f_*: [Y, Y]_G \to [Y, X]_G$ is a bijection, we have $\phi \circ s \circ f \underset{G}{\simeq} id_X$. Then $\phi \circ s$ is a G-homotopy inverse of f and so f is a G-homotopy equivalence.

Let G-$\mathcal{FDCW}$ be the G-homotopy category of finitely dominated G-CW

complexes.

Lemma 1.2. *Let B be a finitely G-dominated G-space. If a G-fibration $p: E \to B$ has a fibre which is weakly finitely dominated, then E is finitely G-dominated.*

Proof. Since a G-map obtained from the pullback of p with respect to the domination of B is a domination of E, we can assume that B is a finite G-CW complex. Let $G/H \times I^n$ be an open n-cell of B and let $i: G/H \times I^n \to B$ be an inclusion map. Then E_i is G-homeomorphic to $G \times_H F \times I^n$. Since the pushout construction is closed for the category G-$\mathcal{FDCW}$, we can prove the lemma by using induction on the number of cells in B.

We introduce the equivariant Euler characteristics. Let X and Y be G-CW complexes. G-maps $f: X \to B$ and $g: Y \to B$ are said to be *equivalent*, $f \sim g$, if there is a G-homotopy equivalence $h: X \to Y$ with $f \underset{G}{\simeq} g \circ h$. We define $\pi_0(G, B)$ as the set of equivalence classes as follows.

$$\pi_0(G, B) := \{G/H \to B \text{ a } G\text{-map} \mid H \leq G\} / \sim$$

Let $U^G(B)$ be the free abelian group generated by the set $\pi_0(G, B)$. We identify $U^G(B)$ with the group consisting of maps from $\pi_0(G, B)$ into the set of integers. A G-map $f: B \to B'$ induces a homomorphism $f_*: U^G(B) \to U^G(B')$ by composition: for any a in $U^G(B)$, $f_*(a)([x: G/H \to B']) = \sum a([y])$, where the sum is taken over all $[y] \in \pi_0(G, B)$ with $f \circ y \sim x$.

Definition. (cf. Definition 5.3 [4]) Let D be a G-subcomplex of B. We denote the connected component of B^H containing $x(H)$ by V_x^H. Let $V_x^{(H)} = G \cdot V_x^H$ and $V_x^{>(H)} = G \cdot V^{>H}$. We define $\chi^G(B, D) \in U^G(B)$ by

$$\chi^G(B, D)(x: G/H \to B) := \chi(V_x^{(H)}/G, (V_x^{>(H)} \cup (D \cap V_x^{(H)}))/G).$$

We call $\chi^G(B, D) \in U^G(B)$ the equivariant Euler characteristic.

Let $i: D \to B$ be an inclusion map. Then $i_* \chi^G(D)$ is the element represented by the assignment $y \mapsto \chi((V_y^{(H)} \cap D)/G, (V_y^{>(H)} \cap D)/G)$.

Lemma 1.3. (cf. [4, Theorem 5.4]) *Let the following diagram be a pushout diagram for pairs of finitely dominated G-CW complexes with k a G-cofibration.*

$$\begin{array}{ccc} (B_0, D_0) & \overset{k}{\hookrightarrow} & (B_2, D_2) \\ \downarrow & \searrow^{j_0} & \downarrow j_2 \\ (B_1, D_1) & \overset{j_1}{\to} & (B, D) \end{array}$$

Then we have

(1) $\chi^G(B, D)=\chi^G(B)-i_*\chi^G(D)$, *where* $i\colon D\to B$ *is an inclusion.*

(2) $\chi^G(B_2, D_0)=\chi^G(B_2, B_0)+k_*\chi^G(B_0, D_0)$.

(3) $\chi^G(B, D)=j_{1*}\chi^G(B_1, D_1)+j_{2*}\chi^G(B_2, D_2)-j_{0*}\chi^G(B_0, D_0)$.

Definition. (cf. [3, Definition 2.1]) Let $\mathcal{C}$ be a small full subcategory of the category of G-spaces containing $\emptyset$ and $\{pt\}$. Let L be a functor from $\mathcal{C}$ to the category of abelian groups and let l be an assignment associating to an object X in $\mathcal{C}$ an element $l(X)$ in $L(X)$. If the pair (L, l) satisfies the following condition (a), (b), and (c), we call (L, l) a *functorial additive invariant* for $\mathcal{C}$.

(a) Homotopy invariance.

(i) If $f\colon X\to Y$ is a G-homotopy equivalence in $\mathcal{C}$, then $L(f)\,(l(X))=l(Y)$.

(ii) If f and g is G-homotopic, then $L(f)=L(g)$.

(b) Additivity. Given a G-pushout in $\mathcal{C}$ with k a G-cofibration,

$$\begin{array}{ccc} X_0 & \overset{k}{\hookrightarrow} & X_1 \\ \downarrow & \overset{j_0}{\searrow} & \downarrow j_1 \\ X_2 & \overset{j_2}{\to} & X \end{array}$$

then $l(X)=L(j_1)\,(l(X_1))+L(j_2)\,(l(X_2))-L(j_0)\,(l(X_0))$.

(c) $l(\emptyset)=0$.

For example, by Lemma 1.3, the pair (U^G, χ^G) is a functorial additive invariant for G-$\mathcal{FDCW}$.

§2. Transfer of a G-fibration

For any G-map $f\colon X\to B$, we define $\bar{f}\colon E_f\to E$ as a map obtained from the pullback of p with respect to f.

$$\begin{array}{ccc} E_f & \overset{\bar{f}}{\to} & E \\ \downarrow & P.B. & \downarrow p \\ X & \overset{f}{\to} & B \end{array}$$

Lemma 2.1. *Let* $p\colon E\to B$ *be a* G*-fibration with a weakly finitely dominated fibre* F. *Let* B *be obtained from* D *by attaching a finite number of cells and* $i\colon D\to B$ *an inclusion map. Then for any functorial additive invariant* (L, l) *for* G-$\mathcal{FDCW}$, *we have*

$$l(E)=L(\bar{i})\,(l(E_i))+\sum_{x\in\pi_0(G,B)}\chi^G(B,D)\,(x)\;L(\bar{x})\,(l(E_x))\,.$$

For any functorial additive invariant (L, l), we often abbreviate $L(\bar{f})$ to $\bar{f}_*$.

Proof. We prove it by induction on the number of cells in $B—D$. In the case of $B=D$, it is trivial. In the case of $B=G/H\times S^n$ we obtain that $l(E)$ equals $(1+(-1)^n)\,\bar{j}_*\,l(G\times_H F)$, since E is G-homeomorphic to $G\times_H F\times S^n$. Let B be obtained from $M(\supset D)$ by attaching one cell $G/H\times I^n$.

$$\begin{array}{ccccc} G/H & \hookrightarrow & G/H\times S^{n-1} & \overset{k}{\hookrightarrow} & G/H\times I^n \\ & & \downarrow & & \downarrow j \\ & & M & \overset{m}{\rightarrow} & B \end{array}$$

By making use of the proof of Proposition 1 in [5], the following diagram is a G-pushout.

$$\begin{array}{ccc} E_{j\circ k} & \overset{\bar{k}}{\hookrightarrow} & E_j \\ \downarrow & & \downarrow \tilde{j} \\ E_m & \overset{\bar{m}}{\rightarrow} & E \end{array}$$

Then we have

$$\begin{aligned} l(E) &= \bar{m}_*\,l(E_m)+\bar{j}_*\,l(E_j)-\overline{j\circ k}_*\,l(E_{j\circ k}) \\ &= \bar{m}_*\,l(E_m)+\bar{\varphi}_*\,l(E_\varphi)-(1+(-1)^{n-1})\,\bar{\varphi}_*\,l(E_\varphi) \\ &= \bar{m}_*\,l(E_m)+(-1)^n\,\bar{\varphi}_*\,l(E_\varphi)\,, \end{aligned}$$

where φ is the G-map from G/H into B.

On the other hand, it is easy to see that $\chi^G(B, M)=(-1)^n\,\varphi_*\,\chi^G(G/H)$, that is,

$$\chi^G(B,M)\,(x\colon G/L\to B)=\begin{cases}(-1)^n & \text{if } (L)=(H) \text{ and } x=\varphi\,,\\ 0 & \text{otherwise.}\end{cases}$$

As the assumption of the induction, we suppose that

$$l(E_m)=\bar{d}_*\,l(E_i)+\sum_{y\in\pi_0(G,M)}\chi^G(M,D)\,(y)\;\bar{y}_*\,l(E_{m\circ y})\,,$$

where $d\colon D\hookrightarrow M$ is an inclusion map. Then

$$\begin{aligned} l(E) &= \bar{m}_*\,l(E_m)+\chi^G(B,M)\,(\varphi)\,\bar{\varphi}_*\,l(E_\varphi) \\ &= \bar{i}_*\,l(E_i)+\sum_{y\in\pi_0(G,M)}\chi^G(M,D)\,(y)\;\overline{m\circ y}_*\,l(E_{m\circ y})+\chi^G(B,M)\,(\varphi)\,\bar{\varphi}_*\,l(E_\varphi)\,. \end{aligned}$$

Let $y: G/H \to M$ and $z: G/K \to M$ be any G-maps. If $[y]=[z]$ in $\pi_0(G, M)$, then there is a G-homotopy equivalence $\bar{\sigma}: E_y \to E_z$ such that $\bar{z}\circ\bar{\sigma}$ and $\bar{y}$ are G-homotopic. Since (L, l) has the homotopy invariance, we get $\bar{y}_* l(E_y) = \bar{z}_* \circ \bar{\sigma}_* l(E_y) = \bar{z}_*\, l(E_z)$ in $L(E)$. If $b \in \pi_0(G, B)$ is not in the image of m_*, we easily obtain $m_* \chi^G(M, D)(b)=0$. By Lemma 1.3 (1), we have

$$\begin{aligned} l(E) &= \bar{i}_*\, l(E_i) + \sum_{b\in\pi_0(G,B)} m_*\chi^G(M, D)(b)\, \bar{b}_*\, l(E_b) + \chi^G(B, M)(\varphi)\, \bar{\varphi}_*\, l(E_\varphi) \\ &= \bar{i}_*\, l(E_i) + \sum_{b\in\pi_0(G,B)} (m_*\chi^G(M, D) + \chi^G(B, M))(b)\, \bar{b}_*\, l(E_b) \\ &= \bar{i}_*\, l(E_i) + \sum_{b\in\pi_0(G,B)} \chi^G(B, D)(b)\, \bar{b}_*\, l(E_b)\,. \end{aligned}$$

This completes the proof.

Lück has defined $Wa^G(B)$ by the set of equivalence classes of the set of G-maps $f: X \to B$ with X finitely dominated and $w^G(B)$ by the equivalence class containing the identity 1_B of B. Here two G-maps $f_0: X_0 \to B$ and $f_4: X_4 \to B$ are equivalent, if there exists a commutative diagram

$$\begin{array}{ccccccccc} X_0 & \subset & X_1 & \underset{G}{\xrightarrow{\simeq}} & X_2 & \underset{G}{\xleftarrow{\simeq}} & X_3 & \supset & X_4 \\ & {\scriptstyle f_0}\searrow & {\scriptstyle f_1}\searrow & & \downarrow{\scriptstyle f_2} & & \swarrow{\scriptstyle f_3} & \swarrow{\scriptstyle f_4} & \\ & & & & B & & & & \end{array}$$

such that (X_1, X_0) and (X_3, X_4) are finite relative G-CW complexes, and $X_1 \to X_2$ and $X_3 \to X_2$ are G-homotopy equivalences. For a G-map $f: Y \to X$ with Y finitely dominated, we denote by $[f: Y \to X]$ its represented element of $Wa^G(X)$. The additive structure on $Wa^G(X)$ is given by a disjoint sum:

$$[f: Y \to X] + [g: Z \to X] = [f \amalg g: Y \amalg Z \to X]\,.$$

The pair (Wa^G, w^G) is a functorial additive invariant for G-$\mathcal{FDCW}$ ([3, Theorem 1.1]). The element $w^G(X)$ is zero if and only if X has the G-homotopy type of a finite G-CW complex.

Theorem 2.2. *Let (L, l) be a functorial additive invariant for G-$\mathcal{FDCW}$. For a G-fibration $p: E \to B$, a map $p(L): Wa^G(B) \to L(E)$ which sends $[f: X \to B]$ to $L(\bar{f})(l(E_f)) - \sum_{b\in\pi_0(G,B)} f_*\chi^G(X)(b)\, L(\bar{b})(l(E_b))$, is a homomorphism.*

Proof. We show that $p(L)$ is well-defined. Let $f: X \to B$ and $g: Y \to B$ be G-maps. If there exists a G-homotopy equivalence $h: X \to Y$ such that f and $g\circ h$ are G-homotopic, then we have obviously $p(L)(f)=p(L)(g)$. Suppose that Y is obtained from X by attaching finitely many cells and g is an extension of f. Let $i\colon X\to Y$ be an inclusion. For $x, x'\in\pi_0(G, X)$ with $f_*(x)=f_*(x')\in \pi_0(G, B)$, we get $\overline{f\circ x}_*\, l(E_{f\circ x})=\overline{f\circ x'}_*\, l(E_{f\circ x'})$. Then

$$\sum_{x\in\pi_0(G,X)} \chi^G(X)\,(x)\,\overline{f\circ x}_*\, l(E_{f\circ x}) = \sum_{b\in\pi_0(G,B)} f_*\chi^G(X)\,(b)\,\bar{b}_*\, l(E_b)\,.$$

By applying Lemma 2.1 to the G-fibration $E_g\to Y$, we have

$$\begin{aligned}\bar{f}_*\, l(E_f) &= \bar{g}_*\circ\bar{i}_*\, l(E_f) = \bar{g}_*(l(E_g)-\sum_{y\in\pi_0(G,Y)} \chi^G(Y, X)\,(y)\,\bar{y}_*\, l(E_{g\circ y}))\\ &= \bar{g}_*\, l(E_g)-\sum_{y\in\pi_0(G,Y)} (\chi^G(Y)-i_*\chi^G(X))\,(y)\,\overline{g\circ y}_*\, l(E_{g\circ y})\\ &= \bar{g}_*\, l(E_g)-\sum_{b\in\pi_0(G,B)} (g_*\chi^G(Y)-f_*\chi^G(X))\,(b)\,\bar{b}_*\, l(E_b)\,,\end{aligned}$$

and so $p(L)(f)=p(L)\,(g)$.

By the definition of $p(L)$, we easily obtain the following.

Proposition 2.3.

(1) $l(E)=p(L)\,(w^G(B))+\sum_{x\in\pi_0(G,B)} \chi^G(B)\,(x)\; L(\bar{x})\,(l(E_x))$.

(2) *Let p_i be G-fibrations and $j_i\colon E_i\to E_1\cup_{E_0} E_2$ the natural inclusions.*

(a) *Suppose these are with the same fibre. Let $p=p_1\cup_{p_0} p_2$ be the pushout G-fibration of the following commutative diagram. Then we have*

$$p(L)\,(w^G(B)) = j_{1*}\, p_1(L)\,(w^G(B_1))+j_{2*}\, p_2(L)\,(w^G(B_2))-j_{0*}\, p_0(L)\,(w^G(B_0))\,.$$

(b) *Suppose these are with the same base space. Let $p=p_1\cup_{p_0} p_2$. Then we have*

$$p(L) = j_{1*}\, p_1(L)+j_{2*}\, p_2(L)-j_{0*}\, p_0(L)\,.$$

$$\begin{array}{ccccc} F_1 & \leftarrow & F_0 & \rightarrow & F_2\\ \downarrow & & \downarrow & & \downarrow\\ E_1 & \leftarrow & E_0 & \rightarrow & E_2\\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_2}\\ B_1 & \leftarrow & B_0 & \rightarrow & B_2\end{array}$$

Theorem 2.4. *There exists a homomorphism from $Wa^G(B)\oplus U^G(B)$ to $L(E)$ sending $(w^G(B), \chi^G(B))$ to $l(E)$. In particular, there is a homomorphism from $Wa^G(B)\oplus U^G(B)$ to $Wa^G(E)\oplus U^G(E)$ sending $(w^G(B), \chi^G(B))$ to $(w^G(E), \chi^G(E))$.*

Proof. For any $a\in U^G(B)$, there exists a G-map $h\colon Z\to B$ such that $w^G(Z)=0$ and $a=h_*(\chi^G(Z))$. Then any element of $Wa^G(B)\oplus U^G(B)$ can be written as $([f\colon Y\to B], f_*\chi^G(Y))=f_*(w^G(Y), \chi^G(Y))$. By the well-definedness of $p(L)$, a homomorphism $Wa^G(B)\oplus U^G(B)\to L(E)$ which sends $([f\colon Y\to B], f_*\chi^G(Y))$ to $\bar{f}_* l(E_f)$ is the required homomorphism.

Corollary 2.5. *Let B be a finite G-CW complex. If $\chi^G(B)=0$ or F is weakly finite, then E has the G-homotopy type of a finite G-CW complex.*

Proof. If F is a finite H-CW complex, then $G\times_H F$ is a finite G-CW complex. Then if F is weakly finite, we have $w^G(E_x)=0$ for any $x\in\pi_0(G, B)$. By Proposition 2.3 (1), we have the result.

The following result is an equivariant version of Lal's theorem [2].

Corollary 2.6. *Let B be a connected finite G-CW complex with a trivial G-action. Then we have*

$$w^G(E) = \chi(B)\cdot i_* w^G(F)\,,$$

where $i\colon F\to E$ is an inclusion.

§3. Applications to Some Equivariant Fibrations

We use the following lemma to give some equivariant fibrations.

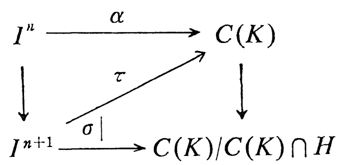
Lemma 3.1. ([8]) *Let K and H be closed subgroups of G with $K\leq H$. The component of $(G/H)^K$ which includes H is precisely $(C(K)/C(K)\cap H)_0$, where $C(K)$ is the centralizer of K in G.*

Proposition 3.2. *Let X be a H-CW complex. Then $p\colon G\times_H X\to G/H$ is a G-fibration with fibre X.*

Proof. It is sufficient to show the homotopy lifting property for G-maps $\sigma\colon G/K\times I^{n+1}\to G/H$ and $\rho\colon G/K\times I^n\to G\times_H X$ with $p\circ\rho=\sigma$ over $G/K\times I^n$.

$$\begin{array}{ccc} G/K\times I^n & \xrightarrow{\rho} & G\times_H X \\ \downarrow & & \downarrow p \\ G/K\times I^{n+1} & \xrightarrow{\sigma} & G/H \end{array}$$

We may suppose that $\sigma(K, 0, 0)=H$. Then we have $K\leq H$. By Lemma 3.1, there exist continuous maps $\alpha\colon I^n\to C(K)$ and $\beta\colon I^n\to X$ such that $\rho(K, t)=[\alpha(t), \beta(t)]$. Since $C(K)\to C(K)/C(K)\cap H$ is a fibration, there exists a map $\tau\colon I^{n+1}\to C(K)$ such that the following diagram commutes.



We define $\tilde{\rho}\colon G/K\times I^{n+1}\to G\times_H X$ by $\tilde{\rho}(gK, t, s)=[g\cdot\tau(t, s), \beta(t)]$. Since the isotropy group of $\beta(t)$ in H contains K for any $t\in I^n$, it is well-defined and is the required G-map.

We also have examples of equivariant fibrations.

Proposition 3.3. *Let H and K be closed subgroups of G with $K\leq H$ and X a H-CW complex. Then $G\times_K X\to G\times_H X$ is a G-fibration with fibre H/K.*

Proposition 3.4. *Let H and K be closed subgroups of G with $K\leq H$ and X a H-CW complex. Then $G\times_K X\to G/H$ is a G-fibration with fibre $H\times_K X$.*

For $H\leq G$ and a finitely dominated H-CW complex X, define a homomorphism $\mathrm{Ind}_H^G(X)\colon Wa^H(X)\to Wa^G(G\times_H X)$ by

$$\mathrm{Ind}_H^G(X)([f\colon Y\to X]) = [id\times_H f\colon G\times_H Y\to G\times_H X]\,.$$

For $g\in G$, we denote by gX a gHg^{-1}-space $gH\times_H X\subset G\times_H X$ and define a map $F(g)\colon Wa^H(X)\to Wa^{gHg^{-1}}(gX)$ by

$$F(g)([f\colon Y\to X]) = [gf\colon gY\to gX]\,.$$

Let Y be a G-CW complex. To consider G-maps as H-maps induces a homomorphism $\mathrm{Res}_H^G(Y)\colon Wa^G(Y)\to Wa^H(Y)$.

For $x\in X$, we denote by $V_x\in\pi_0(X)$ an element which represents the connected component of X which includes x.

Theorem 3.5. *Let H and K be closed subgroups of G with H or K normal and let X be a H-CW complex. Then $\mathrm{Res}_K^G\circ\mathrm{Ind}_H^G$ has the following decomposition.*

$$\begin{aligned}&\mathrm{Res}_K^G(G\times_H X)\,\mathrm{Ind}_H^G(X)\\ &\quad=\chi((G/KH)_0)\sum_{V_{KgH}\in\pi_0(K\backslash G/H)} i_{g*}\,\mathrm{Ind}_{K\cap H^g}^K(gX)\,\mathrm{Res}_{K\cap H^g}^{H^g}(gX)\,F(g)\end{aligned}$$

Here $H^g=gHg^{-1}$ and $i_g\colon K\times_{K\cap H^g} gX\to G\times_H X$ are canonical inclusions. Further if G/KH is connected we have

$$\mathrm{Res}_K^G(G\times_H X)\,\mathrm{Ind}_H^G(X) = \chi(G/KH)\,i_*\,\mathrm{Ind}_{K\cap H}^K(X)\,\mathrm{Res}_{K\cap H}^H(X)\,.$$

Proof. Let $[f\colon Y\to X]$ be an element of $Wa^H(X)$. If K is normal, we have $G/KH=K\backslash G/H$. If H is normal, then $K/K\cap H$ acts freely on G/H and so $G/H\to K\backslash G/H$ is a K-fibration. Then in either cases $G\times_H Y\to K\backslash G/H$ is a K-fibration. By Proposition 2.3 (1), we obtain

$$w^K(G\times_H Y)=\chi((G/KH)_0)\sum_{V_{KgH}\in\pi_0(K\backslash G/H)} j_{g*}\,w^K(KgH\times_H Y)\,.$$

$$\begin{array}{ccccc}
 & & K\times_{K\cap H^g} gY & \xrightarrow{id\times_{K\cap H^g} gf} & K\times_{K\cap H^g} gX \\
 & & \downarrow & & \downarrow i_g \\
KgH\times_H Y & \xrightarrow{j_g} & G\times_H Y & \xrightarrow{id\times_H f} & G\times_H X \\
\downarrow & & \downarrow & & \downarrow \\
V_{KgH} & \subset & K\backslash G/H & = & K\backslash G/H
\end{array}$$

It is not hard to show that $KgH\times_H Y\to K\times_{K\cap H^g} gY$ sending $[kgh, y]$ to $[k, g(hy)]$ is a K-homeomorphism. Hence we have

$$\begin{aligned}
&\mathrm{Res}^G_K(G\times_H X)\,\mathrm{Ind}^G_H(X)\,([f])\\
&\quad=(id\times_H f)_*\,w^K(G\times_H Y)\\
&\quad=\chi((G/KH)_0)\sum_{V_{KgH}} i_{g*}(id\times_{K\cap H^g} gf)_*\,w^K(K\times_{K\cap H^g} gY)\\
&\quad=\chi((G/KH)_0)\sum_{V_{KgH}} i_{g*}\,\mathrm{Ind}^K_{K\cap H^g}(gX)\,\mathrm{Res}^{H^g}_{K\cap H^g}(gX)\,([gf])\,.
\end{aligned}$$

This completes the proof.

We set $\phi(G)=\{(H)\mid |WH|<\infty\}$. Suppose (H) is not in $\phi(G)$. Since $(G/H)^K$ carries a free WH-action, and so has a free S^1-action, we have $\chi((G/H)^K)=0$ for any $K\leq G$. From this and Theorem 3.5 we have the following result.

Theorem 3.6. *Let X be a finitely dominated H-CW complex. Let H and K be closed subgroups with H or K normal such that (HK) is not in $\phi(G)$. Then $G\times_H X$ is K-homotopy equivalent to a finite K-CW complex.*

Corollary 3.7. *Let X be a finitely dominated H-CW complex. If (H) is not in $\phi(G)$, then $G\times_H X$ has the homotopy type of a finite CW complex.*

Let Y be a finitely dominated H-CW complex. The assignment $f\colon X\to Y\times K$ to $f/K\colon X/K\to Y$ induces an isomorphism from $Wa^{H\times K}(Y\times K)$ to $Wa^H(Y)$. This proof is similar to [3, Theorem 5.4]. For example let $G=H\times K$ and let X be a finitely dominated H-space which is not H-homotopy equivalent to a finite

H-CW complex. It is easy to see that $G\times_H X$ is G-homeomorphic to $X\times K$, where $H\times K$ acts on the first (resp. second) factor via the projection $H\times K\rightarrow H$ (resp. $H\times K\rightarrow K$). Then $G\times_H X$ is not G-homotopy equivalent to a finite G-CW complex.

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