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New R Matrices Associated with Cyclic Representations of $U_q(A_2^{(2)})$

By

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Abstract

New R matrices are constructed as intertwiners of N-dimensional representations of $U_q(A_2^{(2)})$ at $q^N=1$. Analogous construction for $U_q(A_1^{(1)})$ reproduces the chiral Potts model.

§1. Introduction

Let V be a finite dimensional vector space and $R(\xi, \eta)$ a linear operator acting on $V\otimes V$ with parameters $\xi, \eta \in S$ (to be called spectral parameters). We call $R(\xi, \eta)$ an R matrix if the Yang-Baxter equation

$$\begin{aligned}&(R(\eta, \lambda)\otimes 1)(1\otimes R(\xi, \lambda))(R(\xi, \eta)\otimes 1)\\&\quad = (1\otimes R(\xi, \eta))(R(\xi, \lambda))\otimes 1)(1\otimes R(\eta, \lambda))\end{aligned} \tag{1.1}$$

holds.

A scheme of constructing R matrices is as follows [11]. Consider a Hopf algebra U. Suppose that a family of representations $(V, \pi_\xi)_{\xi\in S}$ of U is given in such a way that

(i) $(V\otimes V\otimes V, \pi_\xi\otimes\pi_\eta\otimes\pi_\lambda)$ is indecomposable for generic ξ, η, λ, i.e., if $F\in \mathrm{End}(V\otimes V\otimes V)$ satisfies $[F, (\pi_\xi\otimes\pi_\eta\otimes\pi_\lambda)(g)]=0$ for any $g\in U$ then F is a scalar,

(ii) there exists an intertwiner R: $V\otimes V \xrightarrow{\sim} V\otimes V$ such that

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$$R(\xi,\eta)(\pi_\xi\otimes\pi_\eta)(g)=(\pi_\eta\otimes\pi_\xi)(g)R(\xi,\eta)\qquad (g\in U)$$

for any $\xi, \eta\in S$,

(iii) $R(\xi,\xi)=1$.

Under these conditions $R(\xi,\eta)$ satisfies (1.1).

For example, take $U=U_q(A_1^{(1)})$, $V=\mathbb{C}^2$, $S=\mathbb{C}^\times=\mathbb{C}\setminus\{0\}$. Using $\xi=x$ as a coordinate of S, we set

$$\pi_\xi(e_0)=\begin{pmatrix}0&0\\x&0\end{pmatrix},\quad \pi_\xi(f_0)=\begin{pmatrix}0&x^{-1}\\0&0\end{pmatrix},\quad \pi_\xi(t_0)=\begin{pmatrix}q^{-1}&0\\0&q\end{pmatrix},$$
$$\pi_\xi(e_1)=\begin{pmatrix}0&x\\0&0\end{pmatrix},\quad \pi_\xi(f_1)=\begin{pmatrix}0&0\\x^{-1}&0\end{pmatrix},\quad \pi_\xi(t_1)=\begin{pmatrix}q&0\\0&q^{-1}\end{pmatrix}.$$

Here e_i, f_i, t_i $(i=0, 1)$ are the Chevalley generators. By this choice the conditions (i), (ii), (iii) are satisfied and $R(\xi,\eta)$ gives the Boltzmann weights of the 6 vertex model.

Bazhanov and Stroganov [5] found that the Boltzmann weights of the chiral Potts model [2], [4] are obtained by a certain algebraic procedure starting from the R matrix of the 6 vertex model. In this paper we reformulate their discovery in the above scheme by taking U to be a certain central extension of $U_q(A_1^{(1)})$ with q a root of 1. This Hopf algebra is known as the quantum double of a 'Borel' subalgebra of $U_q(A_1^{(1)})$ [6]. Let us denote it by $\tilde{U}_q(A_1^{(1)})$. In [8] we have shown that the Fateev-Zamolodchikov model, the trigonometric limit of the chiral Potts model, is obtained by the scheme (i), (ii), (iii) for $U=U_q(A_1^{(1)})$. The central extension enables us to reproduce the whole of the chiral Potts model.

The story goes as follows. If q is a primitive N-th root of 1, $U_q(A_1)=U_q(\mathfrak{sl}(2,\mathbb{C}))$ admits a 3 parameter family of N-dimensional irreducible representations [12], [13]. (See [7], [9] for general results. See also [1], [3], [8].) It is extended to a 6 parameter family of N-dimensional irreducible representations of $\tilde{U}_q(A_1^{(1)})$. The requirement (ii) restricts the parameters ξ, η to be on an algebraic surface $\mathcal{S}$. In fact, $\mathcal{S}$ factorizes essentially into two identical curves: $\mathcal{S}=\mathcal{C}\times\mathcal{C}$. Accordingly, $R(\xi,\eta)$ factorizes into 4 pieces. They are the Boltzmann weights of the chiral Potts model.

The main achievement of this paper is the construction of new R matrices corresponding to the case $A_2^{(2)}$, as opposed to $A_1^{(1)}$ for the chiral Potts model. (For generic q the intertwiner for $U_q(A_2^{(2)})$ gives the Izergin-Korepin model [10], [11] in the simplest case.) We again start from the N-dimensional irreducible representations of $U_q(A_1)$. In this case only those representations which send the Casimir element of $U_q(A_1)$ to zero can be extended to the representations of $\tilde{U}_q(A_2^{(2)})$. This restriction effects that the set $\mathcal{S}$ is only a curve, given explicitly by

$$c^N + c^{-N} = \Gamma_1(a^N + a^{-N})$$

where Γ_1 is a modulus. This is essentially the same curve as C for the chiral Potts model. The R matrix is given in Theorem 3.4. Unlike the case $A_1^{(1)}$, however, we have not found a basis of $V = \mathbf{C}^N$ for which the matrix elements of $R(\xi, \eta)$ factorize.

The plan of this paper is as follows. In Section 2 we construct a family of representations of $\tilde{U}_q(A_2^{(2)})$. In Section 3 we solve the equation for the intertwiner and give new R matrices. We show also the indecomposability of $V \otimes V \otimes V$ to prove the validity of the Yang-Baxter equation. The intertwiner for $\tilde{U}_q(A_1^{(1)})$ and the chiral Potts model are discussed in Section 4.

§2. Algebra $\tilde{U}_q(A_2^{(2)})$ and its Cyclic Representations

Let us first recall the definition of the quantized enveloping algebra of type $A_2^{(2)}$. We consider only the case when the deformation parameter q is a root of 1.

Throughout this paper we fix a positive odd integer $N \geq 3$, and a primitive N-th root of unity q. We set $\omega = q^2$.

Let $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ be the generalized Cartan matrix of type $A_2^{(2)}$, and set $q_0 = q^4$, $q_1 = q$. The quantized enveloping algebra U_q of type $A_2^{(2)}$ is a $\mathbf{C}$-algebra generated by e_i, f_i, t_i, t_i^{-1} $(i = 0, 1)$, subject to the following defining relations.

$$t_i t_i^{-1} = t_i^{-1} t_i = 1\,, \quad t_i t_j = t_j t_i\,, \tag{2.1a}$$

$$t_i e_j t_i^{-1} = q_i^{a_{ij}} e_j\,, \quad t_i f_j t_i^{-1} = q_i^{-a_{ij}} f_j\,, \quad [e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}\,, \tag{2.1b}$$

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_i e_i^l e_j e_i^{1-a_{ij}-l} = 0\,, \qquad i \neq j\,, \tag{2.1c}$$

$$\sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1-a_{ij} \\ l \end{bmatrix}_i f_i^l f_j f_i^{1-a_{ij}-l} = 0\,, \qquad i \neq j\,. \tag{2.1d}$$

Here we use the following notations.

$$\begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i \cdots [m-n+1]_i}{[n]_i \cdots [1]_i}\,, \quad [m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}\,.$$

For our purposes it is necessary to enlarge this algebra by adding 2 central elements z_i ($i=0, 1$). Let us denote the enlarged algebra by $\tilde{U}_q=\tilde{U}_q(A_2^{(2)})$.

This algebra can be endowed with a structure of a Hopf algebra. We shall need only the comultiplication $\Delta: \tilde{U}_q \to \tilde{U}_q \otimes \tilde{U}_q$ defined by

$$\Delta(e_i) = e_i \otimes 1 + z_i t_i \otimes e_i \tag{2.2a}$$

$$\Delta(f_i) = f_i \otimes t_i^{-1} + z_i^{-1} \otimes f_i , \tag{2.2b}$$

$$\Delta(t_i) = t_i \otimes t_i , \tag{2.2c}$$

$$\Delta(z_i) = z_i \otimes z_i . \tag{2.2d}$$

Note that this comultiplication coincides with that of U_q if we set $z_i=1$. As in the case of U_q we form tensor products of representations via this comultiplication.

Let us denote by $U_q(A_1)$ the subalgebra of $\tilde{U}_q$ generated by e_1, f_1, t_1, t_1^{-1}. As $q^N=1$, the powers $e_1^N, f_1^N, t_1^{\pm N}$ belong to the center of $U_q(A_1)$. We consider finite dimensional irreducible representations such that e_1^N and f_1^N are non zero constants. We call such representations *cyclic representations.*

The cyclic representation of $U_q(A_1)$ is N-dimensional and depend on 3 continuous parameters [12], [13]. This is described as follows. Let V be an N-dimensional vector space over $\mathbb{C}$. Choose two linear operators X, Z on V satisfying

$$ZX = \omega XZ , \quad X^N = Z^N = 1 .$$

Proposition 2.1. *An N-dimensional cyclic representation*

$$\pi_{a_0 a_1 x_1}: U_q(A_1) \to \mathrm{End}(V) \qquad (a_0, a_1, x_1 \in \mathbb{C}^\times)$$

is given as follows.

$$\pi_{a_0 a_1 x_1}(e_1) = x_1 X ,$$

$$\pi_{a_0 a_1 x_1}(f_1) = -\frac{x_1^{-1}}{(q-q^{-1})^2}\left(\frac{\omega a_1}{a_0} Z + \frac{a_0}{\omega a_1} Z^{-1} - a_0 a_1 - \frac{1}{a_0 a_1}\right) X^{-1} ,$$

$$\pi_{a_0 a_1 x_1}(t_1) = \frac{q a_1}{a_0} Z .$$

The Casimir element

$$q t_1 + q^{-1} t_1^{-1} + (q-q^{-1})^2 f_1 e_1$$

takes the value $a_0 a_1 + (a_0 a_1)^{-1}$ *for this representation.*

We will extend this representation to that of $\tilde{U}_q$.

Proposition 2.2. *Let $\pi_{a_0a_1x_1}$ be as in Proposition 2.1. There exists a representation π: $\tilde{U}_q(A_2^{(2)})\to \mathrm{End}(V)$ such that $\pi|_{U_q(A_1)}=\pi_{a_0a_1x_1}$ if and only if $(a_0a_1)^2=-1$. In this case π is given by*

$$\pi(e_0) = \mp\frac{x_0}{(q^4-q^{-4})^2}((qaZ+q^{-1}a^{-1}Z^{-1})X^{-1})^2\,, \quad \pi(e_1) = x_1X\,, \tag{2.3a}$$

$$\pi(f_0) = x_0^{-1}X^2\,, \quad \pi(f_1) = -\frac{x_1^{-1}}{(q-q^{-1})^2}(qaZ+q^{-1}a^{-1}Z^{-1})X^{-1}\,, \tag{2.3b}$$

$$\pi(t_0) = \pm a^{-2}Z^{-2}\,, \quad \pi(t_1) = aZ\,, \tag{2.3c}$$

$$\pi(z_0) = c_0\,, \quad \pi(z_1) = c_1\,, \tag{2.3d}$$

with some c_0, c_1, $x_0\in \mathbf{C}^\times$, and $a=qa_1/a_0$.

Proof. From the requirements $t_0e_1t_0^{-1}=q^{-4}e_1$, $[t_0, t_1]=0$, we have

$$\pi(t_0) = bZ^{-2}\,,$$

where $b\in\mathbf{C}^\times$ is to be determined. The conditions $t_1f_0t_1^{-1}=q^4f_0$, $t_0f_0t_0^{-1}=q^{-8}f_0$, $[e_1, f_0]=0$ fix the form of $\pi(f_0)$ to be

$$\pi(f_0) = x_0^{-1}X^2\,,$$

where $x_0\in\mathbf{C}^\times$ is some constant. From $t_1e_0t_1^{-1}=q^{-4}e_0$, $t_0e_0t_0^{-1}=q^8e_0$ we know that $\pi(e_0)$ has the following form

$$\pi(e_0) = \varphi(Z)X^{-2}$$

with some function φ. The condition $[e_0, f_0]=(t_0-t_0^{-1})/(q^4-q^{-4})$ implies that φ must have the form

$$\varphi(Z) = -\frac{x_0}{(q^4-q^{-4})^2}(q^{-4}bZ^{-2}+q^4b^{-1}Z^2+c)\,,$$

where $c\in\mathbf{C}$ is to be determined. Finally the requirement $[e_0, f_1]=0$ fixes the constants b and c. Namely we have

$$a_0a_1+a_0^{-1}a_1^{-1} = 0\,,$$
$$b = \pm\left(\frac{a_0}{qa_1}\right)^2, \quad c = \pm(q^2+q^{-2})\,.$$

The first relation means the Casimir element vanishes. By setting $a=qa_1/a_0$, we have (2.3). The Serre relations (2.1c), (2.1d) can be checked by using the vanishing of the Casimir element and the fact that $\pi(e_0)$ and $\pi(f_0)$ are proportional to $\pi(f_1)^2$ and $\pi(e_1)^2$, respectively. □

Hereafter we consider only representations (2.3) such that $\pi(t_0 t_1^2)=1$. We denote this representation by π_ξ where $\xi=(a, c_0, c_1, x_0, x_1)\in(\mathbb{C}^\times)^5$.

The following will be used in Section 3. Let us denote the representation with $a=c_0=c_1=1$ and $x_0=x_1=x$ by π_x.

Lemma 2.3. *Assume $N\neq 3$. Let (V', π') be a representation of $U_q(A_2^{(2)})$, and consider the equations for $F(x)\in\mathrm{End}(V\otimes V')$*

$$[(\pi_x\otimes\pi')(f_i), F(x)] = 0 \qquad (i=0, 1),$$
$$(\pi_x\otimes\pi')(t_1)F(x) = \omega^m F(x)(\pi_x\otimes\pi')(t_1). \tag{2.4}$$

Then for generic x any solution has the form $F(x)=Z^m\otimes F'(x)$, where $F'(x)\in$ $\mathrm{End}(V')$ satisfies

$$[\pi'(f_i), F'(x)] = 0 \qquad (i=0, 1),$$
$$\pi'(t_1)F'(x) = \omega^m F'(x)\pi'(t_1). \tag{2.5}$$

Proof. Clearly $Z^m\otimes F'(x)$ with $F'(x)$ satisfying (2.5) is a solution of (2.4). The coefficients of the linear equations (2.4) are polynomials in x. Therefore it is sufficient to prove the assertion for $F(x)$ which are polynomials in x. In terms of Z and X, the equations (2.4) are

$$[T_i, F(x)] = 0, \qquad (i=1, 2), \tag{2.6a}$$
$$(Z\otimes\pi'(t_1))F(x) = \omega^m F(x)\,(Z\otimes\pi'(t_1)), \tag{2.6b}$$

where

$$T_1 = X^2\otimes\pi'(t_1)^2 + x1\otimes\pi'(f_0),$$
$$T_2 = Y\otimes\pi'(t_1)^{-1} + x1\otimes\pi'(f_1),$$

and

$$Y = x\pi_x(f_1) = -(qZ+q^{-1}Z^{-1})X^{-1}/(q-q^{-1})^2.$$

It follows from (2.6b) that $F(x)$ commutes with $1\otimes\pi'(t_1)^N$. Then using

$$\begin{aligned} T_1' &= (1\otimes\pi'(t_1)^{-N})T_1^{(N+1)/2} \\ &= X\otimes\pi'(t_1) + x\frac{\omega^2}{\omega^2+1}X^{-1}\otimes\pi'(t_1^{-1}f_0) + O(x^2), \end{aligned}$$

we find that $F(x)$ commutes with

$$\begin{aligned} T_3 &= -q^{-1}\frac{q-q^{-1}}{q+q^{-1}}(\omega T_1' T_2 - T_2 T_1') \\ &= Z^{-1}\otimes 1 + x\left(\omega^3 Z + \frac{\omega^2}{\omega^2+1}Z^{-1}\right)X^{-2}\otimes\pi'(t_1^{-2}f_0) + O(x^2) \end{aligned}$$

and

$$T_4 = (\omega T_2 T_3 - T_3 T_2)/x(\omega - 1)$$
$$= Z^{-1} \otimes \pi'(f_1) + \omega^6 \frac{q+q^{-1}}{(q-q^{-1})^2} Z^2 X^{-3} \otimes \pi'(t_1^{-3} f_0) + O(x) .$$

Putting $x=0$, $F(0)$ commutes with $Z \otimes 1$, $X \otimes \pi'(t_1)$ and

$$Z^{-1} \otimes \pi'(f_1) + \omega^6 \frac{q+q^{-1}}{(q-q^{-1})^2} Z^2 X^{-3} \otimes \pi'(t_1^{-3} f_0) .$$

The commutativity with the first two operators and (2.6b) show that $F(0)$ is of the form

$$F(0) = Z^m \otimes F' ,$$

where $F' \in \mathrm{End}(V')$ satisfies

$$\pi'(t_1) F' = \omega^m F' \pi'(t_1) .$$

In the case $N \neq 3$, Z^{-1} and $Z^2 X^{-3}$ are linearly independent. From the commutativity with the last operator, it then follows that F' commutes with $\pi'(f_i)$ $(i=0, 1)$. Therefore $(F(x)-F(0))/x$ satisfies (2.4). By repeating this we have the conclusion. □

Proposition 2.4. *Assume* $N \neq 3$. *For generic* x_i, *if* $F \in \mathrm{End}(V^{\otimes n})$ *satisfies*

$$[(\pi_{x_1} \otimes \cdots \otimes \pi_{x_n})(f_i), F] = 0 \qquad (i = 0, 1) ,$$
$$(\pi_{x_1} \otimes \cdots \otimes \pi_{x_n})(t_1) F = \omega^m F (\pi_{x_1} \otimes \cdots \otimes \pi_{x_n})(t_1) ,$$

then F is a scalar. Moreover, if $m \not\equiv 0 \bmod N$ *then* $F=0$.

Proof. Thanks to the Lemma 2.3, the problem reduces to the case $n=1$. In this case, the equations are

$$[X^2, F] = [Y, F] = 0 , \quad ZF = \omega^m FZ .$$

From this, the assertion follows. □

We have also proved the following directly by using computer.

Lemma 2.5. *Suppose that* $N=3$ *and consider the representation* π_ξ *such that* $x_0 = x_1 = 1$, $c_0 = c^{-2}$ *and* $c_1 = c$. *For generic a, c the tensor product* $(V \otimes V \otimes V, \pi_\xi \otimes \pi_\xi \otimes \pi_\xi)$ *is indecomposable.*

§3. The Intertwiner for $\tilde{U}_q(A_2^{(2)})$

Now we shall solve

$$R(\xi, \tilde{\xi})(\pi_\xi \otimes \pi_{\tilde{\xi}})(g) = (\pi_{\tilde{\xi}} \otimes \pi_\xi)(g) R(\xi, \tilde{\xi}) \qquad (g \in \tilde{U}_q(A_2^{(2)})) . \tag{3.1}$$

As a result we obtain a new R matrix whose spectral parameters live on a curve. Firstly we derive necessary conditions for the existence of a solution. The following will be used frequently.

Lemma 3.1. *Let ε be a primitive N-th root of unity. If A, B are elememts of a $\mathbb{C}$-algebra satisfying $AB=\varepsilon BA$, then we have $(A+B)^N=A^N+B^N$.*

For convenience we shall call an expression *invariant* if it remains the same under the exchange of ξ and $\tilde{\xi}$.

Proposition 3.2. *Let $a^{2N} \neq -1$ and set $c=c_1$, $d=c_0 c_1^2$. For the existence of an intertwiner* (3.1), *it is necessary that the following are invariants*:

$$\Gamma_1 = \frac{c^N + c^{-N}}{a^N + a^{-N}}, \quad \Gamma_2 = x_0^N (a^{2N} - c^{2N} d^{-N}), \quad \Gamma_3 = \frac{1 - a^N c^N}{x_1^N}, \tag{3.2a}$$

$$\Gamma_4 = \frac{c^{2N} d^{-N} + c^{-2N} d^N + 2}{(a^N + a^{-N})^2}, \quad \Gamma_5 = d + d^{-1}. \tag{3.2b}$$

For generic values of the parameters, the Jacobian of the map G: $(a, c, x_0, x_1, d) \mapsto (\Gamma_i)_{1 \le i \le 5}$ is nonzero.

Proof. If $R(\pi_\xi \otimes \pi_{\tilde{\xi}})(g) = (\pi_{\tilde{\xi}} \otimes \pi_\xi)(g) R$ $(g \in \tilde{U}_q(A_2^{(2)}))$ and R is invertible, then $\operatorname{tr}(\pi_\xi \otimes \pi_{\tilde{\xi}})(g) = \operatorname{tr}(\pi_{\tilde{\xi}} \otimes \pi_\xi)(g)$. Apply this to $g = e_i^N, f_i^N, e_0 e_1^2$ and $f_0 f_1^2$. Using (2.2) and Lemma 3.1, we obtain the following invariants γ_i.

$$\gamma_1 = \frac{1 - a^{-2N} c_0^N}{x_0^N (a^N + a^{-N})^2}, \quad \gamma_2 = x_0^N (a^{2N} - c_0^{-N}),$$

$$\gamma_3 = \frac{1 - a^N c_1^N}{x_1^N}, \quad \gamma_4 = \frac{x_1^N (a^{-N} - c_1^{-N})}{a^N + a^{-N}},$$

$$\gamma_5 = \frac{1 - c_0 c_1^2}{x_0 x_1^2}, \quad \gamma_6 = \frac{x_0 x_1^2 (1 - c_0 c_1^2)}{c_0 c_1^2}.$$

The Γ_i are obtained by setting $\Gamma_1 = 1 - \gamma_3 \gamma_4$, $\Gamma_2 = \gamma_2$, $\Gamma_3 = \gamma_3$, $\Gamma_4 = 1 - \gamma_1 \gamma_2$ and $\Gamma_5 = \gamma_5 \gamma_6 + 2$. The Jacobian of the map G is found to be

$$4N^4 \frac{1 - d^2}{a c x_0 x_1 d^2} \frac{a^N - a^{-N}}{(a^N + a^{-N})^4} \frac{(d^N - 1)(c^{2N} + d^N)}{c^N d^N} \Gamma_2 \Gamma_3 .$$

This completes the proof. □

In view of the above proposition, we must impose some condition on ξ and $\tilde{\xi}$ in order to obtain an R matrix depending continuously on them. Hereafter we shall assume

$$c_0 c_1^2 = 1$$

and denote the parameters $(a, c, x_0, x_1) \in (\mathbb{C}^\times)^4$ by the same letter ξ. This choice makes Γ_5 trivial and $\Gamma_4 = (\Gamma_1)^2$. Now (3.2a) reduces to the following invariants:

$$\Gamma_1 = \frac{c^N + c^{-N}}{a^N + a^{-N}}, \quad \Gamma_2 = x_0^N (a^{2N} - c^{2N}), \quad \Gamma_3 = \frac{1 - a^N c^N}{x_1^N}. \tag{3.3}$$

This defines a family of algebraic curves $C_\Gamma \subset \{\xi = (a, c, x_0, x_1) \in (\mathbb{C}^\times)^4\}$ parametrized by $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$.

We shall show that (3.1) has a solution under (3.3). First we prepare a lemma. Let ε be a primitive N-th root of unity, and let $\mathcal{W}_\varepsilon$ be the $\mathbb{C}$-algebra generated by z and x satisfying $zx = \varepsilon xz$.

Lemma 3.3. *Let σ be a representation of $\mathcal{W}_\varepsilon$ on a vector space V such that $\sigma(z)^N = a$, $\sigma(x)^N = b$, $a, b \in \mathbb{C}$. Set $Z = \sigma(z)$ and $X = \sigma(x)$. Let $\alpha, \beta, \gamma, \delta$ be complex numbers satisfying $\alpha^N a + \beta^N = \gamma^N a + \delta^N$. Then if we define $P(Z)$ by*

$$P(Z) = \sum_{k=0}^{N-1} p_k Z^k, \quad p_k = \prod_{l=1}^{k} (\varepsilon\gamma - \varepsilon^l \alpha) \prod_{l=k+1}^{N-1} (\varepsilon^l \beta - \delta), \tag{3.4}$$

it satisfies

$$P(Z)(\alpha Z + \beta) X = (\gamma Z + \delta) X P(Z). \tag{3.5}$$

Let $\tilde{P}(Z) = \sum_{k=0}^{N-1} \tilde{p}_k Z^k$ be $P(Z)$ with α and γ, and β and δ being interchanged. Then

$$P(Z)\tilde{P}(Z) = \rho,$$

with

$$\rho = aN \frac{\gamma - \alpha}{\beta - \delta} \frac{\alpha^N \delta^N - \beta^N \gamma^N}{\alpha\delta - \beta\gamma}.$$

Proof. Without loss of generality we may assume that $ab \neq 0$ and that σ is an N-dimensional irreducible representation. Then $Z^k X^l$ $(0 \leq k, l \leq N-1)$ are linearly independent. Therefore (3.5) is equivalent to

$$\begin{aligned} p_{k+1}(\varepsilon^{k+1}\beta - \delta) &= p_k(\varepsilon\gamma - \varepsilon^{k+1}\alpha) \qquad (0 \leq k \leq N-2), \\ p_0(\beta - \delta) &= p_{N-1} a(\varepsilon\gamma - \alpha). \end{aligned} \tag{3.6}$$

This recursion relation is satisfied by (3.4). Since $P(Z)\tilde{P}(Z)$ satisfies (3.5) with

$\alpha=\gamma$ and $\beta=\delta$, (3.6) implies that it is proportional to the identity. Therefore, using the formula

$$\sum_{k=0}^{N-1}\frac{y}{\varepsilon^k x-y}=\frac{Ny^N}{x^N-y^N}$$

we find $P(Z)\tilde{P}(Z)=\rho$ with

$$\begin{aligned}\rho &= p_0\tilde{p}_0+a\sum_{k=1}^{N-1}p_k\tilde{p}_{N-k}\\ &= -\frac{a}{(\beta-\delta)^2}\sum_{k=0}^{N-1}\prod_{l=1}^{k}(\varepsilon\gamma-\varepsilon^l\alpha)\prod_{l=k+1}^{N}(\varepsilon^l\beta-\delta)\prod_{l=1}^{N-k}(\varepsilon\alpha-\varepsilon^l\gamma)\prod_{l=N-k+1}^{N}(\varepsilon^l\delta-\beta)\\ &= a\frac{(\gamma-\alpha)(\alpha^N-\gamma^N)(\beta^N-\delta^N)}{\beta-\delta}\sum_{k=0}^{N-1}\frac{\varepsilon^k}{(\varepsilon^k\alpha-\gamma)(\varepsilon^k\beta-\delta)}\\ &= aN\frac{\gamma-\alpha}{\beta-\delta}\,\frac{\alpha^N\delta^N-\beta^N\gamma^N}{\alpha\delta-\beta\gamma}. \quad \square\end{aligned}$$

Theorem 3.4. *For generic values of the parameters ξ and $\tilde{\xi}$ satisfying* (3.3), *the equation* (3.1) *has a unique invertible solution up to a scalar multiple. It is explicitly given by*

$$R(\xi,\tilde{\xi})=P^1(C_1)P^2(C_2)P^3(C_3)=P^3(\tilde{C}_3)P^1(C_1)P^2(C_2) \tag{3.7}$$

where

$$\begin{aligned}&C_1=X^2\otimes Z^2X^{-2}\,,\quad C_2=ZX^{-1}\otimes X\,,\\ &C_3=(\pi_\xi\otimes\pi_{\tilde{\xi}})(e_1)^2(\pi_\xi\otimes\pi_{\tilde{\xi}})(f_0)^{-1}/x_0\tilde{x}_0\,,\\ &\tilde{C}_3=(\pi_{\tilde{\xi}}\otimes\pi_\xi)(e_1)^2(\pi_{\tilde{\xi}}\otimes\pi_\xi)(f_0)^{-1}/x_0\tilde{x}_0\,,\\ &P^i(C)=\sum_{k=0}^{N-1}p_k^iC^k \qquad (i=1,2,3)\,,\\ &p_k^1=\prod_{l=1}^{k}(\omega^4a^2x_0-\omega^{4l}\tilde{a}^2\tilde{x}_0)\prod_{l=k+1}^{N-1}(\omega^{4l}c^2x_0-\tilde{c}^2\tilde{x}_0)\,,\\ &p_k^2=\prod_{l=1}^{k}(\omega\tilde{a}\tilde{c}x_1-\omega^l ac\tilde{x}_1)\prod_{l=k+1}^{N-1}(\omega^l x_1-\tilde{x}_1)\,,\\ &p_k^3=\prod_{l=1}^{k}(\omega^2acx_0-\omega^{2l}\tilde{a}\tilde{c}\tilde{x}_0)\prod_{l=k+1}^{N-1}\omega(\omega^{2l}\tilde{a}\tilde{c}x_1^2-ac\tilde{x}_1^2)\,,\end{aligned}$$

Proof. Set

$$\Omega_1=1\otimes X^2,\quad \Omega_2=X\otimes 1,\quad \Omega_3=C_1^{(N+1)/2}=q^{-1}X\otimes ZX^{-1}.$$

Note that

$$C_3=((\tilde{a}^2\tilde{x}_0C_1+c^2x_0)\Omega_1)^{-1}((ac\tilde{x}_1C_2+x_1)\Omega_2)^2\,,$$

and that

$$C_3^N = \frac{\tilde{a}^N \tilde{c}^N x_1^{2N} - a^N c^N \tilde{x}_1^{2N}}{a^N c^N x_0^N - \tilde{a}^N \tilde{c}^N \tilde{x}_0^N} . \tag{3.8}$$

We have the following commutation relations.

$$C_1 \Omega_1 = \omega^4 \Omega_1 C_1 , \quad C_2 \Omega_2 = \omega \Omega_2 C_2 , \quad C_3 \Omega_3 = \omega^2 \Omega_3 C_3 , \tag{3.9a}$$

$$[C_i, C_j] = [\Omega_j, \Omega_j] = [C_i, \Omega_j] = 0 \qquad (1 \leq i \neq j \leq 2) , \tag{3.9b}$$

$$[C_3, C_1^{(N+1)/2} C_2] = 0 , \quad [\Omega_3, C_1] = [\Omega_3, C_2] = 0 . \tag{3.9c}$$

Thanks to Lemma 3.4 $P^i(C_i)$ ($i=1, 2, 3$) satisfies

$$P^1(C_1)(\tilde{a}^2 \tilde{x}_0 C_1 + c^2 x_0) \Omega_1 = (a^2 x_0 C_1 + \tilde{c}^2 \tilde{x}_0) \Omega_1 P^1(C_1) , \tag{3.10a}$$

$$P^2(C_2)(ac\tilde{x}_1 C_2 + x_1) \Omega_2 = (\tilde{a}\tilde{c} x_1 C_2 + \tilde{x}_1) \Omega_2 P^2(C_2) , \tag{3.10b}$$

$$P^3(C_3) \tilde{a}\tilde{c} (\tilde{x}_0 C_3 + \omega x_1^2) \Omega_3 = ac(x_0 C_3 + \omega \tilde{x}_1^2) \Omega_3 P^3(C_3) . \tag{3.10c}$$

Now we shall show that $R = R(\xi, \tilde{\xi})$ given by (3.7) satisfies (3.1) for $g = t_i$ ($i=0,1$), f_0 and e_1. In terms of C_i and Ω_i ($i=1, 2$), the equations become

$$RC_1^{(N+1)/2} C_2 = C_1^{(N+1)/2} C_2 R ,$$
$$R(\tilde{a}^2 \tilde{x}_0 C_1 + c^2 x_0) \Omega_1 = (a^2 x_0 C_1 + \tilde{c}^2 \tilde{x}_0) \Omega_1 R ,$$
$$R(ac\tilde{x}_1 C_2 + x_1) \Omega_2 = (\tilde{a}\tilde{c} x_1 C_2 + \tilde{x}_1) \Omega_2 R .$$

They follow immediately from (3.9), (3.10a), (3.10b).

Next we shall turn to (3.1) for $g = f_1$. It is sufficient to check (3.1) for $g = f_1 e_1$. After some calculations we obtain

$$\begin{aligned} &-(q-q^{-1})^2 (\pi_\xi \otimes \pi_{\tilde{\xi}})(f_1 e_1) \\ &= q(\pi_\xi \otimes \pi_{\tilde{\xi}})(t_1) + q^{-1}(\pi_\xi \otimes \pi_{\tilde{\xi}})(t_1^{-1}) \\ &\quad + \frac{1}{x_1 \tilde{x}_1} \left(\frac{\tilde{a}}{c} (\tilde{x}_0 C_3 + \omega x_1^2) \Omega_3 + \frac{c}{\tilde{a}} (x_0 C_3 + \omega^{-1} \tilde{x}_1^2) \Omega_3^{-1} \right) . \end{aligned} \tag{3.11}$$

Note that

$$P^1(C_1) P^2(C_2) C_3 = \tilde{C}_3 P^1(C_1) P^2(C_2) . \tag{3.12}$$

Using (3.9c), (3.10c), (3.11) and (3.12) we obtain

$$R(\pi_\xi \otimes \pi_{\tilde{\xi}})(f_1 e_1) = (\pi_{\tilde{\xi}} \otimes \pi_\xi)(f_1 e_1) R .$$

Finally we shall consider (3.1) for $g = e_0$. This equation can be checked directly. In the case $N \neq 3$, it can be shown also by the following argument. Let

$$F = R^{-1}((\pi_{\tilde{\xi}} \otimes \pi_\xi)(e_0) R - R(\pi_\xi \otimes \pi_{\tilde{\xi}})(e_0)) .$$

We can easily show that F satisfies (2.4) with $m=-2$. From Proposition 2.4, it vanishes. Therefore R satisfies (3.1) for $g=e_0$. Clearly R satisfies (3.1) for $g=z_i$ $(i=0, 1)$. This completes the proof. □

Remark. If we set

$$\kappa = \tilde{a}^{2N}\tilde{x}_0^N + c^{2N}x_0^N = a^{2N}x_0^N + \tilde{c}^{2N}\tilde{x}_0^N ,$$

then $\kappa R(\xi, \tilde{\xi})$ is holomorphic on the curve C_Γ.

Remark. When $\Gamma_1=1$ and $\Gamma_2=\Gamma_3=0$, C_Γ degenerates to a rational curve. Letting $a, c\to 1$ we find that x_0/x_1 is an invariant. The R matrix becomes a polynomial in the single variable $x=x_0\tilde{x}_1/x_1\tilde{x}_0$. We call this the *trigonometric* case.

Proposition 3.5. *The obtained R matrices satisfy the following inversion relation*

$$R(\xi, \tilde{\xi})R(\tilde{\xi}, \xi) = \rho(\xi, \tilde{\xi})$$

where

$$\rho(\xi, \tilde{\xi}) = \omega^{-2}N^3(ac\tilde{a}\tilde{c})^{N-1}\frac{a^2x_0-\tilde{a}^2\tilde{x}_0}{c^2x_0-\tilde{c}^2\tilde{x}_0}\,\frac{\tilde{a}\tilde{c}x_1-ac\tilde{x}_1}{x_1-\tilde{x}_1}\,\frac{a^Nc^Nx_0^N+\tilde{a}^N\tilde{c}^N\tilde{x}_0^N}{acx_0+\tilde{a}\tilde{c}\tilde{x}_0}$$
$$\times\left(\frac{\tilde{a}^N\tilde{c}^Nx_1^{2N}-a^Nc^N\tilde{x}_1^{2N}}{\tilde{a}\tilde{c}x_1^2-ac\tilde{x}_1^2}\right)^2\frac{x_0^Nx_1^{2N}-\tilde{x}_0^N\tilde{x}_1^{2N}}{x_0x_1^2-\tilde{x}_0\tilde{x}_1^2}\,.$$

Proof. Let $\tilde{P}^i$ be P^i with ξ and $\tilde{\xi}$ being interchanged. Thanks to Lemma 3.3, $P^i(C_i)\tilde{P}^i(C_i)$ $(i=1, 2, 3)$ are proportional to the identity. Therefore, noting $[C_1, C_2]=0$, we find

$$\begin{aligned}R(\xi, \tilde{\xi})R(\tilde{\xi}, \xi) &= P^1(C_1)P^2(C_2)P^3(C_3)\tilde{P}^3(C_3)\tilde{P}^1(C_1)\tilde{P}^2(C_2)\\ &= P^1(C_1)\tilde{P}^1(C_1)P^2(C_2)\tilde{P}^2(C_2)P^3(C_3)\tilde{P}^3(C_3)\,.\end{aligned}$$

Using (3.8) and Lemma 3.3 we obtain the expression for $\rho(\xi, \tilde{\xi})$. □

In order to show that the R satisfies the Yang-Baxter equation, it remains to prove the indecomposability of the tensor products of three cyclic representations of $\tilde{U}_q(A_2^{(2)})$. Let $\mathcal{CV}=\cup_\Gamma C_\Gamma\times C_\Gamma\times C_\Gamma\subset\mathbb{C}^{15}$

Proposition 3.6. *For generic $(\Gamma, \xi, \eta, \lambda)\in\mathcal{CV}$, if $F\in\mathrm{End}(V\otimes V\otimes V)$ commutes with $(\pi_\xi\otimes\pi_\eta\otimes\pi_\lambda)(g)$ for any $g\in\tilde{U}_q$, then F is a scalar operator.*

Proof. Since C_Γ is irreducible for generic Γ, the variety $\mathcal{CV}$ is irreducible. Therefore we can show the assertion by specialization argument. This is already done in Proposition 2.4 and Lemma 2.5. □

Remark. From the R matrix of Theorem 3.4, one can get a local Hamiltonian $\sum_j H_{jj+1}$ by a standard procedure. More precisely, set

$$H = \frac{d}{d\varepsilon} \log R(\xi, \tilde{\xi})|_{\varepsilon=0},$$

where $\tilde{\xi}=\tilde{\xi}(\varepsilon)\to\xi$ as $\varepsilon\to 0$. As usual let X_j, Z_j be the operators acting as X, Z on the j-th component in the tensor product of $V^{\otimes L}$, and similarly for H_{jj+1}. Up to a scalar multiple and a term proportional to the identity,we have

$$H_{jj+1} = H^1_{jj+1}+H^2_{jj+1}+H^3_{jj+1},$$

$$H^1_{jj+1} = 2c^{2N}\sum_{k=1}^{N-1}\frac{1}{1-\omega^{4k}}\left(-\frac{\omega^4 a^2}{c^2}\right)^k C_1^k,$$

$$H^2_{jj+1} = (1+a^N c^N)\sum_{k=1}^{N-1}\frac{1}{1-\omega^k}(-\omega ac)^k C_2^k,$$

$$H^3_{jj+1} = (a^{2N}+c^{2N})\sum_{k=1}^{N-1}\frac{1}{1-\omega^{2k}}(-\omega)^k C_3^k,$$

where

$$C_1 = X_j^2 Z_{j+1}^2 X_{j+1}^{-2}, \quad C_2 = Z_j X_j^{-1} X_{j+1},$$

$$C_3 = (X_j+acZ_jX_{j+1})^2(a^2X_j^2Z_{j+1}^2+c^2X_{j+1}^2)^{-1}.$$

§4. Intertwiners for $\tilde{U}_q(A_1^{(1)})$ and the Chiral Potts Model

In this section $\tilde{U}_q$ means the algebra $\tilde{U}_q(A_1^{(1)})$. It is defined by the same formulas (2.1), (2.2), wherein $(a_{ij})=\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $q_0=q_1=q$. As in Section 2 we shall consider N-dimensional representations obtained as the extensions of Proposition 2.1 for $U_q(A_1)$, the subalgebra generated by e_1, f_1 and $t_1^{\pm 1}$. We find it more convenient to make the change of variables

$$X = r^{-1}(a_1^2 Z-1)X', \quad x_1 = \frac{r}{q-q^{-1}}x_1'.$$

Here $r^N=a_1^{2N}-1$, so that $X'^N=1$ and $ZX'=\omega X'Z$. Dropping primes we thus have $\pi\colon \tilde{U}_q\to \mathrm{End}(V)$ $(V=\mathbf{C}^N)$, where

$$\pi(e_0) = x_0 X^{-1}\frac{a_0^2 Z^{-1}-1}{q-q^{-1}}, \quad \pi(e_1) = x_1\frac{a_1^2 Z-1}{q-q^{-1}}X,$$

$$\pi(f_0) = (a_0a_1x_0)^{-1}\frac{a_1^2 Z-1}{q-q^{-1}}X, \quad \pi(f_1) = (a_0a_1x_1)^{-1}X^{-1}\frac{a_0^2 Z^{-1}-1}{q-q^{-1}},$$

$$\pi(t_0)=\frac{a_0}{qa_1}Z^{-1},\quad \pi(t_1)=\frac{qa_1}{a_0}Z,\quad \pi(z_0)=c_0,\quad \pi(z_1)=c_1.$$

The six parameters $\xi=(a_0, a_1, c_0, c_1, x_0, x_1)\in(\mathbf{C}^\times)^6$ entering π will be exhibited as π_ξ.

Given ξ, $\tilde{\xi}\in(\mathbf{C}^\times)^6$ we now look for an intertwiner R: $V\otimes V\xrightarrow{\sim}V\otimes V$ such that

$$R(\xi,\tilde{\xi})(\pi_\xi\otimes\pi_{\tilde{\xi}})(g)=(\pi_{\tilde{\xi}}\otimes\pi_\xi)(g)R(\xi,\tilde{\xi})\qquad (g\in\tilde{U}_q). \tag{4.1}$$

By a similar reasoning as given in Section 3, we have

Proposition 4.1. *Assume* $a_i^{2N}\neq 1$, $(a_0a_1)^2\neq -1$, $\tilde{a}_i^{2N}\neq 1$, $(\tilde{a}_0\tilde{a}_1)^2\neq -1$. *For an intertwiner to exist it is necessary that the following quantities are invariants:*

$$\gamma_1=\frac{1-c_0^Na_0^Na_1^{-N}}{x_0^N(a_0^{2N}-1)},\quad \gamma_2=\frac{1-c_1^Na_0^{-N}a_1^N}{x_1^N(a_1^{2N}-1)}, \tag{4.2a}$$

$$\gamma_3=\frac{(a_0a_1x_0)^N(a_0^{-N}a_1^N-c_0^{-N})}{a_1^{2N}-1},\quad \gamma_4=\frac{(a_0a_1x_1)^N(a_0^Na_1^{-N}-c_1^{-N})}{a_0^{2N}-1}, \tag{4.2b}$$

$$\gamma_5=\frac{1-c_0c_1}{x_0x_1((a_0a_1)^2+1)},\quad \gamma_6=\frac{x_0x_1(1-c_0^{-1}c_1^{-1})}{1+(a_0a_1)^{-2}}. \tag{4.2c}$$

For generic ξ the Jacobian of the map $\xi\mapsto(\gamma_i)$ is nonzero.

Hereafter we shall assume

$$c_0c_1=1.$$

This makes (4.2c) trivial, and $\gamma_1\gamma_3=\gamma_2\gamma_4$. Eqs. (4.2a), (4.2b) then define a surface $\mathcal{S}=\mathcal{S}_{\gamma_1\gamma_2\gamma_3}$ written in the coordinates $(a_0, a_1, c_0, x_0, x_1)$. As it turns out, $\mathcal{S}$ is essentially a product of two curves. To see this, consider the curve $\mathcal{C}_k$ in the coordinates $(x, y, \mu)\in(\mathbf{C}^\times)^3$ [4]

$$\mathcal{C}_k\colon x^N+y^N=k(1+x^Ny^N),\quad \mu^N=\frac{k'}{1-kx^N}=\frac{1-ky^N}{k'}. \tag{4.3}$$

Here the parameter k is a modulus and $k^2+k'^2=1$. Set

$$k^2=-\gamma_1\gamma_3,\quad \kappa_0^N=-k/\gamma_1,\quad \kappa_1^N=-k/\gamma_2.$$

Then the following gives an algebraic correspondence:

$$\mathcal{S}_{\gamma_1\gamma_2\gamma_3}\longleftrightarrow\mathcal{C}_k\times\mathcal{C}_k$$

$$(a_0, a_1, c_0, x_0, x_1)\mapsto(x, y, \mu, x', y', \mu')$$

$$x_0 = \kappa_0 x', \quad x_1 = \kappa_1 y',$$
$$a_0^2 = \frac{y}{x'\mu\mu'}, \quad a_1^2 = \frac{x\mu\mu'}{y'}, \quad c_0 a_0 a_1 = \frac{qx}{y'}.$$

Hereafter we shall use the letter $r=(x_r, y_r, \mu_r)$ to denote a point on C_k.

To describe the intertwiners, define matrices

$$S_{rs} = \sum_{a=0}^{N-1} \hat{W}_{rs}(a)(X^{-1} \otimes X)^a,$$
$$T_{rs} = \sum_{a=0}^{N-1} \overline{W}_{rs}(a) Z^a.$$

Here $r, s \in C_k$, and the coefficients $\hat{W}_{rs}(a)$ and $\overline{W}_{rs}(a)$ are defined via the recursion relations

$$\frac{\hat{W}_{rs}(a)}{\hat{W}_{rs}(a-1)} = \frac{\mu_s y_r - \mu_r y_s \omega^{a-1}}{\mu_s x_s - \mu_r x_r \omega^a},$$
$$\frac{\overline{W}_{rs}(a)}{\overline{W}_{rs}(a-1)} = \mu_r \mu_s \frac{x_r \omega - x_s \omega^a}{y_s - y_r \omega^a}.$$

Note also that $W_{rs}(l) = \sum_{a=0}^{N-1} \hat{W}_{rs}(a) \omega^{-al}$ are given by

$$\frac{W_{rs}(a)}{W_{rs}(a-1)} = \frac{\mu_r}{\mu_s} \frac{y_s - x_r \omega^a}{y_r - x_s \omega^a}.$$

These are the Boltzmann weights of the chiral Potts model [4].

Theorem 4.2. *Consider* $\xi = (r, r')$, $\tilde{\xi}' = (\tilde{r}, \tilde{r}') \in C_k \times C_k$ *where* C_k *is given by* (4.3). *Then up to a scalar multiple the intertwiner* (4.1) *is given by*

$$R(\xi, \tilde{\xi}) = S_{r'\tilde{r}}(T_{r'\tilde{r}'} \otimes T_{r\tilde{r}}) S_{r\tilde{r}'}.$$

Proof. Set $K = \mathbf{Q}(k, k')$. Let $\Phi_N(q)$ be the N-th cyclotomic polynomial in q where q is an indeterminate variable. Set $\tilde{K} = K[q]/K\Phi_N(q)$. Let $A = \tilde{K}[x, x^{-1}, y, y^{-1}, \mu, \mu^{-1}]$ be the coordinate ring of C_k over $\tilde{K}$, and $B = A \otimes_{\tilde{K}} A$ the coordinate ring of $C_k \times C_k$. We consider a B-algebra $\mathcal{W}$ generated by Z and X with the defining relations $ZX = \omega XZ$ $(\omega = q^2)$, $Z^N = X^N = 1$. We may regard

$$\frac{q-q^{-1}}{\kappa_1}(\pi_\xi \otimes \pi_{\tilde{\xi}})(e_1)$$
$$= x_r \mu_r \mu_{r'} ZX \otimes 1 - y_{r'} X \otimes 1 + x_{\tilde{r}} \mu_r \mu_{r'} \mu_{\tilde{r}} \mu_{\tilde{r}'} Z \otimes ZX - y_{\tilde{r}'} \mu_r \mu_{r'} Z \otimes X$$

as an element of $\mathcal{W} \otimes_{\tilde{K}} \mathcal{W}$, and (4.1) for $g = e_1$ as an equation in $\mathcal{W} \otimes_{\tilde{K}} \mathcal{W}$.

This equation is shown by using the following and similar identities in $\mathcal{W}$ (see Lemma 3.3).

$$S_{rs}(y_s(X^{-1}\otimes X)-\omega x_r)(Z\otimes 1)=\frac{\mu_s}{\mu_r}(y_r(X^{-1}\otimes X)-\omega x_s)(Z\otimes 1)S_{rs},$$
$$T_{rs}(\mu_r\mu_s x_s Z-y_r)X=(\mu_r\mu_s x_r Z-y_s)XT_{rs}.$$

In order to prove the case $g=e_0$ we can use the following K-linear anti-involution $*$ of $\mathcal{W}$:

$$x_r^*=y_r,\quad \mu_r^*=\mu_r^{-1},\quad x_s^*=y_s,\quad \mu_s^*=\mu_s^{-1},$$
$$Z^*=Z^{-1},\quad X^*=X^{-1},\quad q^*=q^{-1}.$$

We have

$$\left(\frac{q-q^{-1}}{\kappa_1}(\pi_\xi\otimes\pi_{\tilde{\xi}})(e_1)\right)^*=\frac{q-q^{-1}}{\kappa_0}(\pi_\xi\otimes\pi_{\tilde{\xi}})(e_0).$$

We also have

$$S_{rs}((y_s(X^{-1}\otimes X)-\omega x_r)(Z\otimes 1))^*=\left(\frac{\mu_s}{\mu_r}(y_r(X^{-1}\otimes X)-\omega x_s)(Z\otimes 1)\right)^*S_{rs},\tag{4.4a}$$
$$T_{rs}((\mu_r\mu_s x_s Z-y_r)X)^*=((\mu_r\mu_s x_r Z-y_s)X)^*T_{rs}.\tag{4.4b}$$

Therefore we obtain (4.1) for $g=e_0$.

For $g=f_0, f_1$, we use another anti-involution $\wedge$ of $\mathcal{W}$:

$$\hat{x}_r=\frac{1}{qx_r},\quad \hat{y}_r=\frac{q}{y_r},\quad \hat{\mu}_r=-\frac{qx_r\mu_r}{y_r},$$
$$\hat{x}_s=\frac{1}{qx_s},\quad \hat{y}_s=\frac{q}{y_s},\quad \hat{\mu}_s=-\frac{qx_s\mu_s}{y_s},$$
$$\hat{Z}=Z,\quad \hat{X}=X^{-1},\quad \hat{q}=q.$$

Then we have

$$(\kappa_0c_0\tilde{c}_0(\pi_\xi\otimes\pi_{\tilde{\xi}})(f_0))^\wedge=-\frac{q^2}{\kappa_0c_0\tilde{c}_0}(\pi_\xi\otimes\pi_{\tilde{\xi}})(t_0^{-1}e_0),$$
$$\left(\frac{\kappa_1}{c_0\tilde{c}_0}(\pi_\xi\otimes\pi_{\tilde{\xi}})(f_1)\right)^\wedge=-\frac{c_0\tilde{c}_0}{\kappa_1}(\pi_\xi\otimes\pi_{\tilde{\xi}})(t_1^{-1}e_1).$$

The identities (4.4) with $*$ replaced by $\wedge$ are also valid. Therefore we obtain (4.1) for $g=f_0, f_1$. □

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