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On Ground State Degeneracy of Z_2 Symmetric Quantum Spin Models

By

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Abstract

We consider a class of $\boldsymbol{Z}_2$ symmetric quantum spin Hamiltonians. Anisotropic spin 1/2 Heisenberg models are typical examples.
Proof of groound state degeneracy ($\boldsymbol{Z}_2$ symmetry breaking), construction of pure gournd states are given in a systematic way.

§ 1. Introduction

In [9], we presented a method for study of ground state structures of certain quantum spin systems. The quantum spin system of [9] has the unique ground state on the finite volume, but in the infinite volume limit, phase transitions can occur by the same mechanism of classical spin systems. However, the Hamiltonians considered in [9] do not have symmetry. Hence, it is interesting to consider a class of quantum spin systems with symmetry which contains several examples of physical interest by the methods developed in [9]. The aim of this paper is to investigate the ground state structure of certain perturbation of classical Ising Hamiltonian. The typical example we have in our mind is the spin 1/2 anisotropic Heisenberg model on the regular lattice $\boldsymbol{Z}^d$.

$$H = -\sum_{|j-j'|=1} \{\sigma_x^{(j)}\sigma_x^{(j')} + \delta\sigma_y^{(j)}\sigma_y^{(j')} + \varepsilon\sigma_z^{(j)}\sigma_z^{(j')}\} \tag{1.1}$$

where δ, ε are real (small) parameters and the sum is taken over all nearest neighbor pairs and $\sigma_\alpha^{(j)}$ $(\alpha = x, y, z)$ is the Pauli spin matrix on the site j in $\boldsymbol{Z}^d$. The Hamiltonian has the $\boldsymbol{Z}_2$ symmetry. We will give sufficient conditions for existence of long range order, mass gap and uniqueness of $\boldsymbol{Z}_2$ symmetric trans-

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lationally invariant ground state. We also construct pure ground states. Our conditions can be verified for ε and δ sufficiently small using expansions of J. Kirkwood and L. Thomas of [5].

As is the case in [9], the irreducible ground state representation (=GNS representation of a pure ground state) can be realized on L^2 space of a Gibbs measure for dimensional classical spin system, and the regularized Hamiltonian is the selfadjoint extensions of the generator of certain Markov semigroup.

We again use C^* algebra approach and we will assume that readers are familiar with basic results explained in [4] and [6].

The algebra of observables is the *UHF* C^* algebra $\mathbb{A}$

$$A = \bigotimes_{\mathbb{Z}^d} M_2(\mathbb{C}) \tag{1.2}$$

where by $M_2(\mathbb{C})$ we denote the set of all complex 2 by 2 matrices.

The Pauli spin matrix $\sigma_\alpha^{(j)}$ (j in $\mathbb{Z}^d$, $\alpha = x, y, z$) is an element of (1.2) satisfying usual relations

$$(\sigma_\alpha^{(j)})^2 = 1 \quad (\sigma_\alpha^{(j)})^* = \sigma_\alpha^{(j)} \tag{1.3a}$$

$$\sigma_x^{(j)}\sigma_y^{(j)} = i\sigma_z^{(j)} \tag{1.3b}$$

$$[\sigma_\alpha^{(j)}, \sigma_\beta^{(k)}] = \sigma_\alpha^{(j)}\sigma_\beta^{(k)} - \sigma_\beta^{(k)}\sigma_\alpha^{(j)} = 0 \qquad \text{if} \quad j \neq k\,. \tag{1.3c}$$

Let $\mathbb{A}_{\text{loc}}$ be the set of strictly local elements in $\mathbb{A}$, in another word, the polynomials of $\sigma_\alpha^{(j)}$ ($\alpha = x, y, z$). We will also consider the abelian subalgebra $\mathbb{B}$ generated by $\sigma_z^{(j)}$ (j in $\mathbb{Z}^d$), and we set

$$\mathbb{B}_{\text{loc}} = \mathbb{A}_{\text{loc}} \cap \mathbb{B}\,. \tag{1.4}$$

For a finite subset C of $\mathbb{Z}^d$, we define

$$\sigma_\alpha(C) = \prod_{C \ni j} \sigma_\alpha^{(j)} \qquad \text{for} \quad \alpha = x, y, z\,. \tag{1.5}$$

The Hamiltonian we consider is

$$H = -\sum_{\substack{\mathbb{Z}^d \supset C \\ C \neq \emptyset}} V_C(\sigma_z)\sigma_x(C) - \sum_{\mathbb{Z}^d \ni j} W_j(\sigma_z) \tag{1.6}$$

where $V_C(\sigma_z)$ and $W_j(\sigma_z)$ are in $\mathbb{B}_{\text{loc}}$.

Assumption 1.1. (i) *H of* (1.6) *is translationally invariant and of finite range.*
(ii) *$V_C(\sigma_z)$ and $W_j(\sigma_z)$ are selfadjoint and*

$$[V_C(\sigma_z), \sigma(C_x)] = 0 \tag{1.7}$$

for any finite C in $\boldsymbol{Z}^d$.

(iii)
$$V_c(\sigma_z) = 0 \qquad \text{if } |C| \text{ is odd} \tag{1.8a}$$
$$V_c(\sigma_z) \geq 0 \qquad \text{if } |C| \text{ is even} \tag{1.8b}$$
$$V_c(\sigma_z) > 0 \qquad \text{if } C = \{i, j\} \text{ with } |i-j| = 1 \tag{1.8c}$$

where $|C|$ *is the number of points in C.*

Under the above situation, the Hamiltonian H gives rise to a 1 parameter group of automorphisms of $\boldsymbol{A}$.

$$\tau_t(Q) = e^{itH} Q e^{-itH} \qquad \text{for } Q \text{ in } \boldsymbol{A}. \tag{1.9}$$

See [4].
In the Heisenberg Hamiltonian (1.1), (1.8) follows from the identity (1.10) provided $|\delta| < 1$

$$\sigma_x^{(j)} \sigma_x^{(j')} + \delta \sigma_y^{(j)} \sigma_y^{(j')} = (1 - \delta \sigma_z^{(j)} \sigma_z^{(j')}) \sigma_x^{(j)} \sigma_x^{(j')} \tag{1.10}$$

$\tau_t(\cdot)$ of (1.9) is the time evolution of observables. The ground state of τ_t is a state φ of the $\boldsymbol{C}^*$ algebra $\boldsymbol{A}$ satisfying

$$\varphi(Q^*[H, Q]) \geq 0 \qquad \text{for } Q \text{ in } \boldsymbol{A}_{\text{loc}}. \tag{1.11}$$

Let φ be a ground state and $\{\pi_\varphi(\cdot), \Omega_\varphi, H_\varphi\}$ be the associated G.N.S. triple where $\pi_\varphi(\cdot)$ is the representation of A in the Hilbert space H_φ and Ω_φ is the cyclic vector implementing the state φ. Then there exists a selfadjoint operator H such that

$$H_\varphi \geq 0 \tag{1.12a}$$
$$H_\varphi \Omega_\varphi = 0 \tag{1.12b}$$
$$e^{itH_\varphi} \pi_\varphi(Q) e^{-itH_\varphi} = \pi_\varphi(\tau_t(Q)) \qquad \text{for any } Q \text{ in } \boldsymbol{A}. \tag{1.12c}$$

H_φ plays the role of the regularized Hamiltonian.

We now consider the $\boldsymbol{Z}_2$ symmetry of the Hamiltonian (1.6). Let Θ be the automorphism of $\boldsymbol{A}$ determined by

$$\Theta(\sigma_x^{(j)}) = -\sigma_x^{(j)} \tag{1.13a}$$
$$\Theta(\sigma_z^{(j)}) = -\sigma_z^{(j)} \qquad \text{for any } j \text{ in } \boldsymbol{Z}^d. \tag{1.13b}$$

Obviously we have

$$\Theta^2 = \text{identity} \tag{1.14}$$
$$\Theta \circ \tau_t = \tau_t \circ \Theta. \tag{1.15}$$

If φ is a ground state, so is $\varphi\circ\Theta$. In other words, the quantum spin systems we consider have $\mathbb{Z}_2$ symmetry defined by Θ.

Our results of this paper are as follows. Under some conditions described in § 4, any translationally invariant, Θ invariant, ground state φ restricted to B is a Gibbs measure. φ has the long range order (as a state of $\mathcal{A}$) even if the corresponding Gibbs measure is extremal. The decomposition of the state φ into pure ground states can be given explicitly. The construction of H_φ will also be given. By our construction, we see the regularized Hamiltonian H_φ gives rise to the generator of a Markov semigroup on B. We will also give a sufficient condition of existence of mass gap of H_φ.

The basic ideas are already given in [9], but we have some complications due to $\mathbb{Z}_2$ symmetry. The positivity assumption (1.8) is essential in our approach. It is a subtle question whether we can obtain similar results without the assumption (1.8) or not. C. Albanese got some interesting results for certain Hamiltonians without our positivity. See [2].

Some results related to ours can be found in [1] and [5], however they didn't describe pure ground states and in crucial parts they assume certain high temperature condition for Gibbs measures. Spectral properties of Heisenberg models have been investigated in [5] but we believe our approach (use of Markov semigroup) is interesting in itself.

The rest of this paper is as follows. In § 2 we consider the finite volume ground state as a preliminary. § 3 is de- voted to representations of A on L^2 space of a Gibbs measure. § 4 establishes the correspondence of the Gibbs measure and translationally invariant ground states of quantum systems. We discuss Markov semigroup and existence of mass gap in § 5. § 6 is devoted to Heisenberg models as an example within the reach of our results.

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§ 2. Ground States on Finite Volume

In this section, we consider ground states on finite volumes. Let Λ be a cube in $\boldsymbol{Z}^d$. Let H_Λ be the Hamiltonian on Λ with the periodic boundary condition.

$$H_\Lambda = -\sum_{C\cap\Lambda\neq\emptyset} V_c(\sigma_z)\sigma_x(C) - \sum_{\Lambda\ni j} W_j(\sigma_z)\,. \tag{2.1}$$

We will always use the periodic boundary condition for finite systems in this paper. We regard H_Λ as a matrix acting on the finite dimensional Hilbert space $\mathcal{H}_\Lambda$,

$$\mathcal{H}_\Lambda = \bigotimes_\Lambda \boldsymbol{C}^2\,. \tag{2.2}$$

The ground state of H_Λ is the eigenvector state for the smallest eigenvalue. We fix a basis of the Hilbert space $\mathcal{H}_\Lambda$ which diagonalizes the z component of Pauli spin matrices in the following manner. Let X_Λ be the classical spin configuration space (with spin 1/2) on Λ,

$$X_\Lambda = \{1, -1\}^\Lambda\,. \tag{2.3}$$

To each point σ in X_Λ, we assign the vector $|\sigma>$ in $\mathcal{H}_\Lambda$ by the formula,

$$|\sigma> = \bigotimes_{\Lambda\ni j} e_\sigma^{(j)}\,. \tag{2.4}$$

where $\sigma^{(j)}$ is the jth coordinate of σ and

$$e_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad e_{-1} = \begin{pmatrix}0\\1\end{pmatrix}. \tag{2.5}$$

Thus the vector ξ of $\mathcal{H}_\Lambda$ is identified with a function $\xi(\sigma)$ on X_Λ via the identity,

$$\xi = \sum_{X_\Lambda\ni\sigma} \xi(\sigma)|\sigma>. \tag{2.6}$$

We consider two subspaces $\mathcal{H}_\Lambda^{(\pm)}$ of $\mathcal{H}_\Lambda$ defined by

$$\mathcal{H}_\Lambda^{(+)} = \{\eta \text{ in } \mathcal{H}_\Lambda;\ \sigma_z(\Lambda)\eta = +\eta\} \tag{2.7a}$$

$$\mathcal{H}_\Lambda^{(-)} = \{\eta \text{ in } \mathcal{H}_\Lambda;\ \sigma_z(\Lambda)\eta = -\eta\}\,. \tag{2.7b}$$

Due to the condition (1.8a), the Hamiltonian H_Λ commutes with $\sigma_z(\Lambda)$, and splits into two sectors.

Consider

$$H_\Lambda^{\text{Ising}} = -\sum_{\substack{|j-j'|=1\\ \Lambda\ni j\\ \Lambda\ni j'}} \sigma_x^{(j)}\sigma_x^{(j')} \tag{2.8}$$

where we again impose the periodic boundary condition.

Observation 1. *Let* $|\sigma>$, $|\sigma'>$ *be in* $\mathcal{H}_\Lambda^{(+)}$ *with* σ, σ' *in* X_Λ. *For positive* β, *we have*

$$\langle\sigma|e^{\beta H_\Lambda^{\text{Ising}}}|\sigma'\rangle>0\,. \tag{2.9}$$

This is because we obtain $|\sigma>$ by successive applications of $\sigma_x^{(j)}\sigma_x^{(j')}$ to $|\sigma'>$ and we have the formula

$$e^{-\beta H_\Lambda^{\text{Ising}}}=\prod_{|j-j'|=1}\{\cosh\beta+\sinh\beta\sigma_x^{(j)}\sigma_x^{(j')}\} \tag{2.10}$$

(2.9) is true for $\mathcal{H}_\Lambda^{(-)}$.

Observation 2. (2.9) *is valid for* H_Λ *in place of* H_Λ^{Ising}.

It suffices to remark that

$$\langle\sigma|e^{-\beta H_\Lambda}|\sigma'\rangle\geq\langle\sigma|e^{-x\beta H_\Lambda^{\text{Isign}}}|\sigma'\rangle+\text{non negative number} \tag{2.11}$$

where
$$x=\inf\text{Spectrum } V_C(\sigma_z)>0 \tag{2.12}$$

with $c=\{i,j\}$ $|i,j|=1$.

(2.11) is a consequence of our Assumption 1.1 (iii). By the above observation, we can apply the Perron Frobenius theorem to $-H_\Lambda$ restricted to $\mathcal{H}_\Lambda^{(\pm)}$. Thus we can conclude the following.

Lemma 2.1. (i) H_Λ *restricted to* $\mathcal{H}_\Lambda^{(\pm)}$ *has the unique positive (normalized) ground state vectorwhere the positivity refers to the choice of the basis* (2.4) *and* (2.5).
(ii) *The ground state eigenvalue (=the smallest eigenvalue) of* H_Λ *is of multiplicity at most* 2.

If $|\Lambda|$ *is odd and*

$$[\sigma_x(\Lambda),\ V_C(\sigma_z)]=0 \tag{2.13}$$

for any C, then the multiplicity of the ground state eigenvalue is 2.

The last statement follows from the commutativity,

$$[\sigma(\Lambda),\ H_\Lambda]=0 \tag{2.14}$$

and the property that $\sigma_x(\Lambda)$ is an isomorphism of $\mathcal{H}_\Lambda^{(+)}$ and $\mathcal{H}_\Lambda^{(-)}$.

Using the notation (2.6), we fix ground state vectors $\Omega^{(\pm)}$

$$\Omega^{(+)}=\Sigma^{(+)}\Omega^{(+)}(\sigma)|\sigma> \tag{2.15a}$$

$$\Omega^{(-)} = \Sigma^{(-)}\Omega^{(-)}(\sigma)|\sigma\rangle \tag{2.15b}$$

where the sum $\Sigma^{(+)}$ (resp. $\Sigma^{(-)}$) is taken over spin configurations for which the even number (resp. odd) of spin is up. Then Lemma 2.1 means

$$\Omega^{(\pm)}(\sigma) > 0\,. \tag{2.16}$$

We now define

$$h_\Lambda^{(\pm)}(\sigma) = -2\log\Omega^{(\pm)}(\sigma) \tag{2.17}$$

and

$$h_\Lambda(\sigma) = \begin{cases} h_\Lambda^{(+)}(\sigma) & \text{if } |\sigma\rangle \text{ is in } \mathcal{H}_\Lambda^{(+)} \quad (2.18a) \\ h_\Lambda^{(-)}(\sigma) & \text{if } |\sigma\rangle \text{ is in } \mathcal{H}_\Lambda^{(-)} \quad (2.18b) \end{cases}$$

$\Omega^{(\pm)}(\sigma)$(or $h_\Lambda^{(\pm)}(\sigma)$) must satisfy eigenvalue equation

$$H_\Lambda\Omega^{(+)} = E_\Lambda^{(+)}\Omega^{(+)} \tag{2.19a}$$

$$H_\Lambda\Omega^{(-)} = E_\Lambda^{(-)}\Omega^{(-)} \tag{2.19b}$$

where $E_\Lambda^{(\pm)}$ are eigenvalues.

For each point σ in X_Λ and a subset D of Λ, we define σ_D as a point of X_Λ via the formula,

$$(\sigma_D)^{(j)} = \begin{cases} -\sigma^{(j)} & \text{if } j \text{ is in } D \quad (2.20a) \\ \sigma^{(j)} & \text{if } j \text{ is not in } D \quad (2.20b) \end{cases}$$

where $(\sigma_D)^{(j)}$ is the coordinate (σ_D) at the site j in Λ. Then (2.19) leads to the following equation.

$$-\sum_{\sigma} V_C(\sigma)\exp\left[-\frac{1}{2}\{h_\Lambda^{(\pm)}(\sigma)-h_\Lambda^{(\pm)}(\sigma)\}\right]-\sum_{\Lambda\ni j} W_j(\sigma) = E_\Lambda^{(\pm)} \tag{2.21}$$

where $V_C(\sigma) = \langle\sigma|V_C(\sigma_Z)|\sigma\rangle$ (2.22a)

$$W_j(\sigma) = \langle\sigma|W_j(\sigma_Z)|\sigma\rangle\,. \tag{2.22b}$$

We can go in the reverse direction. Suppose we find a function $f(\sigma)$ satisfying

$$-\sum_{\sigma} V_C(\sigma)\exp\left[-\frac{1}{2}\{f(\sigma_c)-f(\sigma)\}\right]-\sum_{\Lambda\ni j} W_j(\sigma) = \tilde{E}$$

for some real $\tilde{E}$ and any $|\sigma\rangle$ in $\mathcal{H}^{(+)}$.

Then the vector $\tilde{\Omega}$ defined by

$$\tilde{\Omega} = \sum{}^{(+)} e^{-(1/2)f(\sigma)}|\sigma\rangle \tag{2.23}$$

is the ground state vector. This is due to the uniqueness of Perron Frobenius positive vector. If we consider the expectation for the vector Ω,

$$\Omega = \Omega^{(+)} + \Omega^{(-)} \tag{2.24}$$

then

$$(\Omega, \sigma_Z(B)\Omega) = \sum_{\sigma} e^{-h_\Lambda(\sigma)}\sigma(B)\,. \tag{2.25}$$

So we can interpret (2.25) as the integration by a Gibbs measure. In particular, if $|\Lambda|$ is odd and (2.13) is valid, $E_\Lambda^{(+)} = E_\Lambda^{(-)}$ and Ω is a ground state vector. Moreover we have another ground state vector $\tilde{\Omega}$ defined by

$$\tilde{\Omega} = \Omega^{(+)} - \Omega^{(-)}\,. \tag{2.26}$$

We will consider the same situation in the infinite volume case.
As is shown in [1] and [5], in certain examples we have (without assuming (2.13))

$$\lim_{\Lambda \to \infty} |E_\Lambda^{(+)} - E_\Lambda^{(-)}| = 0 \tag{2.27}$$

hence even though Ω or $\tilde{\Omega}$ is not a ground state vector we can carry out the analysis of [8] in the infinite volume.

§ 3. Representations of Observables and Gibbs Measures

We first recall some known facts about Gibbs measures on X,

$$X = \{1, -1\}^{Z^d}\,. \tag{3.1}$$

By the product topology, X is a compact metrizable space. The set of continuous functions $C(X)$ on X can be identified with B via the formula

$$\sigma^{(j)} = \sigma_z^{(j)} \tag{3.2}$$

where $\sigma^{(j)}$ is the coordinate function of the site j in $\boldsymbol{Z}^d$.

Let $f(\sigma)$ be a polynomial given by

$$f(\sigma) = \sum_{Z^d \supset A} f_A \sigma(A) \tag{3.3}$$

$$\sigma(A) = \prod_{A \ni j} \sigma^{(j)} \tag{3.4}$$

where A is a finite subset of $\boldsymbol{Z}^d$ and f_A is zero except a finite number of A.

We introduce a norm for $f(\sigma)$ of (3.3) as follows.

$$||f||_{\delta_1\delta_2} = \sum_{Z^d \supset A} e^{\delta_1 \mathrm{dia}(A)} e^{\delta_2 |A|} |f_A| \tag{3.5}$$

where δ_1, δ_2 are positive numbers and dia(A) is the diameter of the set A, $|A|$ is the number of points in A.

Let $\{h_A\}$ be a potential on X. We will often write

$$h(\sigma) = \sum_{Z^d \supset A} h_A \sigma(A). \tag{3.6}$$

We assume the following decay condition of $h(\sigma)$

$$\sup_{Z^d \ni j} \{\sum_{A \ni j} e^{\delta_1 \mathrm{dia}(A)} e^{\delta_2 |A|} |f_A|\} < \infty \tag{3.7}$$

where δ_1 and δ_2 are positive. We also assume translational invariance of $h(\sigma)$ or simplicity.

For a finite Λ, we define a linear operator $E_\Lambda(\ )$ on $C(X)$

$$E_\Lambda(f)(\sigma) = \frac{\int_{\Lambda \ni j} \prod d\sigma^{(j)} e^{-h_\Lambda(\sigma)} f(\sigma)}{Z(\sigma_{\Lambda^c})} \tag{3.8}$$

where

$$h_\Lambda(\sigma) = \sum_{\Lambda \cap A \neq \emptyset} h_A \sigma(A) \tag{3.9}$$

and $\prod_{\Lambda \ni j} d\sigma^{(j)}$ is the product measure with uniform distribution, $Z(\sigma_{\Lambda^c})$ is a normalization constant determined by

$$E_\Lambda(1)(\sigma) = 1\,. \tag{3.10}$$

By definition, $E_\Lambda(f)(\sigma)$ depends on the variables outside Λ. A measure $d\mu(\sigma)$ on X is a Gibbs measure if

$$\int d\mu(\sigma) f(\sigma) = \int d\mu(\sigma) E_\Lambda(f)(\sigma). \tag{3.11}$$

If $d\mu(\sigma)$ is a Gibbs measure, $d\mu(\sigma)$ and $d\mu(\sigma_c)$ are equivalent as measures (σ_c is defined in (2.20)) and

$$\frac{d\mu(\sigma_c)}{d\mu(\sigma)} = \exp\{2 \sum_{|C \cap A|:\,\mathrm{odd}} h_A(A)\}\,. \tag{3.12}$$

The converse is also true, namely, if $d\mu(\sigma)$ is a measure on X and (3.12) is valid for any $C=\{j\}$ with j in $\boldsymbol{Z}^d$, then $d\mu(\sigma)$ is a Gibbs measure. see [10].

We will use the following variant of this characterization of Gibbs measures.

Lemma 3.1. *Let $d\mu(\sigma)$ be a measure on X such that $d\mu(\sigma_c)$ and $d\mu(\sigma)$ are equivalent for any edge $C=\{i,j\}$, $|i-j|=1$ and (3.12) is valid for the same case. Assume further the following cluster property. For any $G(\sigma)$ in $C(X)$ there exists a constant C_G and a sequence of points $\{j_k k=0, 1, 2\cdots\}$ in $\mathbb{Z}^d$ such that*

$$\lim_{k\to\infty}\int F(\sigma)\tau_{j_k}(G)(\sigma)d\mu(\sigma)=\{\int F(\sigma)d\mu(\sigma)\}\times C_G \tag{3.13}$$

where $\tau_j(\circ)$ is the (lattice) translation determined by.

$$\tau_j(\sigma^{(k)})=\sigma^{(k+j)}. \tag{3.14}$$

Then $d\mu(\sigma)$ is a Gibbs measure.

Proof. By the remark given above; we have only to show

$$\int F(\sigma_j)d\mu(\sigma)=\int\exp\{2\sum_{A\ni j}h_A\sigma(A)\}F(\sigma_j)d\mu(\sigma). \tag{3.15}$$

We set

$$d_j(\sigma)=\exp\{2\sum_{A\ni j}h_A\sigma(A)\} \tag{3.16a}$$

$$\begin{aligned}I_{ij}(\sigma)&=\exp\{2(\sum_{\substack{A\ni i\\ \not\ni j}}h_A\sigma(A)+\sum_{\substack{A\ni j\\ \not\ni i}}h_A\sigma(A))\}\\ &=d_i(\sigma)d_j(\sigma)\exp[-2\sum_{\substack{A\ni i\\ \ni j}}h_A\sigma(A)].\end{aligned} \tag{3.16b}$$

Our assumption implies (3.12) for any $C=\{i,j\}$, so

$$\int F(\sigma_{\{i,j\}})d\mu(\sigma)=\int I_{ij}(\sigma)F(\sigma)\,d\mu(\sigma) \tag{3.17}$$

for any $F(\sigma)$ in $C(X)$.

We next consider the limit $j\to\infty$ in (3.17).
By (3.7), we have

$$\lim_{j\to\infty}||I_{ij}(\sigma)-d_i(\sigma)d_j(\sigma)||=||d_i(\sigma)||\lim_{j\to\infty}||d_j(\sigma)||\{\exp(\sum_{\substack{A\ni i\\ \ni j}}|h_A|)-1\}=0. \tag{3.18}$$

Hence we consider the following

$$\lim_{j\to\infty}\int I_{ij}(\sigma)F(\sigma)d\mu(\sigma)=\lim_{j\to\infty}\int d_i(\sigma)d_j(\sigma)F(\sigma)\,d\mu(\sigma). \tag{3.19}$$

By taking subsequence j_k in (3.13) with $G(\sigma)=d_0(\sigma)$,

$$(3.19) = \{\int d_i(\sigma)F(\sigma)d\mu(\sigma)\} C_G \,. \tag{3.20}$$

(Here we use the translational invariance of the potential $h(\sigma)$.)
If we set $F(\sigma)=1$ in (3.18) and (3.17)

$$\int d_i(\sigma)d\mu(\sigma) = \frac{1}{C_G}$$

for any i.

So we can set $i=j_k$ and take $k=\infty$

$$C_G^2 = 1\,, \quad C_G = 1\,. \tag{3.21}$$

We also note that if $F(\sigma)$ is continuous

$$\lim_{j\to\infty} F(\sigma_{\{i,j\}}) = F(\sigma_i)\,. \tag{3.22}$$

By (3.17) (3.19) (3.21) (3.20), we obtain (3.15). q.e.d.

Next we construct representations of observables on L^2 space of a Gibbs measure.

Let $d\mu(\sigma)$ be a Gibbs measure and $L^2(d\mu)$ be the Hilbert space of square integrable functions of $d\mu(\sigma)$.

Define

$$\pi_+(\sigma_x^{(j)})F(\sigma) = [d_j(\sigma)]^{1/2}F(\sigma_j) \tag{3.23a}$$

$$\pi_+(\sigma_z^{(j)})F(\sigma) = \sigma^{(j)}F(\sigma) \qquad \text{for } F(\sigma) \text{ in } L^2(d\mu)\,. \tag{3.23b}$$

In the same manner,

$$\pi_-(\sigma_x^{(j)})F(\sigma) = -[d_j(\sigma)]^{1/2}F(\sigma_j) \tag{3.24a}$$

$$\pi_-(\sigma_z^{(j)})F(\sigma) = \sigma^{(j)}F(\sigma)\,. \tag{3.24b}$$

Let Ω_μ be the constant function 1 as a vector of $L^2(d\mu)$.

Proposition 3.2. (i) (3.23) *gives rise to the representation* $\pi_+(\ )$ *of the* $\boldsymbol{C}^*$ *algebra* $\boldsymbol{A}$.
(ii) (3.24) *gives rise to the representation* $\pi_-(\ )$ *of* A *and* $\pi_-(Q)=\pi_+(\Theta(Q))$ *for* Q *in* $\boldsymbol{A}$ *where* Θ *is defined in* (1.13).
(iii) *The representation* $\pi_\pm(\cdot)$ *are irreducible if and only if* $d\mu$ *is an extremal Gibbs measure.*

Proof. (i) and (ii) are straight forward. (iii) is equivalent to Corollary 3.8 of [6]. Note that

$$L^\infty(d\mu) = \pi_+(B)'' = \pi-(B)\,. \tag{3.25}$$

So the center of representations $\pi_\pm(A)'' \cap \pi_\pm(A)'$ is in $L^\infty(d\mu)$. q.e.d.

Let $\varphi_\mu^{(+)}$ and $\varphi_\mu^{(-)}$ be vector states of Ω_μ for $\pi_\pm$;

$$\varphi_\mu^{(+)}(Q) = (\Omega_\mu, \pi_+(Q)\Omega_\mu) \tag{3.26a}$$

$$\varphi_\mu^{(-)}(Q) = (\Omega_\mu, \pi_-(Q)\Omega_\mu)\,. \tag{3.26b}$$

Obviously

$$\varphi_\mu^{(+)} = \varphi_\mu^{(-)}\circ\Theta\,. \tag{3.27}$$

Proposition 3.3. *Let φ_μ be a state defined by*

$$\varphi_\mu = \frac{1}{2}(\varphi_\mu^{(+)}+\varphi_\mu^{(-)})\,. \tag{3.28}$$

The GNS representation of φ_μ is given as follows.

The Hilbert space is two copy of $L^2(d\mu)$

$$L^2(d\mu)\oplus L^2(d\mu)\,.$$

The representation is the direct sum $\pi_+\oplus\pi_-$ and the GNS cyclic vector is $\Omega_\mu\oplus\Omega_\mu$. φ_μ has the long range order as a state of $\mathfrak{A}$.

The above statement looks like trivial, however it explain the reason why quantum Ising models discussed in [5] have ground states with long range order.

The existence of the potential $h(\sigma)$ will be discussed in § 4 and § 6.

§ 4. Regularized Hamiltonian

In this section, we relate the states constructed in § 3 to ground state representations. We assume the following conditions.

Assumption 4.1. Let $h_\Lambda(\sigma)$ be defined by (2.18).

For an edge $b=\{i, j\}$; $|i-j|=1$, we define

$$h_{\Lambda,b}(\sigma) = \frac{1}{2}\{h_\Lambda(\sigma)-h_\Lambda(\sigma_b)\}\,. \tag{4.1}$$

We assume there is an increasing sequence of cubes Λ_k in $\mathbf{Z}^d$ satisfying $\Lambda_k\subset \Lambda_{k+1}\subset\Lambda_{k+2}, \cdots, \bigcup_k \Lambda_k=\mathbf{Z}^d$, and

(i) $$\lim_{k\to\infty} |E_{A_k}^{(+)} - E_{A_k}^{(-)}| = 0 \tag{4.2}$$

(ii) the limit exists in the norm $\| \ \|_{\delta_1,\delta_2}$

$$\lim_{k\to\infty} h_{A_k,b}(\sigma) = h_b(\sigma) \tag{4.3}$$

(iii) $h\|_{A_k,b}(\sigma)\|_{\delta_1,\delta_2}$ is bounded uniformly in k and b.

Remark 4.2. (i) (4.2) *is automatic if* $|A_k|$ *is odd and* (2.13) *is valid.*
(ii) *For quantum Ising model with Hamiltonian,*

$$H = -\sum_{|j-j'|=1} \sigma_x^{(j)}\sigma_x^{(j')} - \varepsilon \sum_{Z^d \ni k} \sigma_z^{(k)} \tag{4.4}$$

the above assumption can be verified if ε *is small.*
See § 3 *of* [5].
The idea of [5] *works for more general Hamiltonians if Ising terms are sufficiently large. See also the examples of* § 6 *of this paper.*

Lemma 4.3. *Under the assumption* 4.1 *there exists a potential* $h(\sigma)$

$$h(\sigma) = \sum h_A \sigma(A) \tag{4.5}$$

such that $h(\sigma)$ *is translationally invariant and for any edge* b

$$h_b(\sigma) = \sum_{|b\cap A|=1} h_A \sigma(A) \tag{4.6}$$

where the left hand side is defined by (4.3), *and*

$$\sum_{A\ni j} |h_A| e^{\delta_{1'}\mathrm{diam}(A)} e^{\delta_{2'}|A|} < \infty \tag{4.7}$$

for $0 \le \delta_{j'} < \delta_j$ $(j=1, 2)$.

Proof. Suppose $h_{A_k}(\sigma)$ of Assumption 4.1 is given by

$$h_{A_k}(\sigma) = \sum_A h_{k,A}\sigma(A) \tag{4.8}$$

(4.3) implies the existence of the limit

$$\lim_{k\to\infty} h_{k,A} = h_A \tag{4.9}$$

The translational invariance of $h(\sigma)$ is due to the use of the periodic boundary condition for finite systems. (4.6) is obvious.
We must check (4.7). We take a sequence $\{j_e\}$ of point in Z^d satisfying

$$j_0 = j, \quad |j_0 - j_k| = k, \quad |j_k - j_{k+1}| = 1 \qquad j_k \neq j_1 \text{ if } k \neq 1 .$$

Then we rewrite the sum (4.7) as

$$\sum_{A\ni j_0} = \sum_{\substack{A\ni j_0\\ \not\ni j_1}} + \sum_{\substack{A\ni j_0\\ \ni j_1\\ \not\ni j_2}} + \cdots + \sum_{\substack{A\ni j_0\\ \ni j_1\\ \ni j_k\\ \not\ni j_{k+1}}} + \tag{4.10}$$

In the k th part

$$\sum_{\substack{A\ni j_0, j_1 \cdots j_k\\ \not\ni j_{k+1}}} |h_A| e^{\delta_{1'} \mathrm{diam}(A)} e^{\delta_{2'}|A|}$$

$$\leq \sum_{\substack{A\ni j_0\\ \ni j_k\\ \not\ni j_{k+1}}} |h_A| e^{\delta_{1'} \mathrm{diam}(A)} e^{\delta_{2'}|A|}$$

$$\leq e^{-(\delta_1-\delta_{1'})|j_0-j_{k+1}|} \|h_b(\sigma)\|_{0.\delta_2} < \infty \tag{4.11}$$

where $b_k = \{j_k \cdot j_{k+1}\}$. Thus by translational invariance

$$(4.7) \leq (\sum_k e^{-(\delta_1-\delta_{1'})k}) \|h_b(\sigma)\|_{0.\delta_2} < \infty. \tag{4.12}$$

q.e.d.

Following [9] we define the regularized Hamiltonian as follows. Let A be a cube. Then,

$$H_{A,\mathrm{reg}} = \frac{1}{2} \sum_{\sigma \cap A \neq \emptyset} C_C^* V_C(\sigma_Z) C_C \tag{4.13}$$

where

$$C_C = \{1-\sigma_x(C)\} \exp\left[\frac{1}{2} \sum_{|\sigma \cap A| \text{ odd}} h_A \sigma_Z(A)\right]. \tag{4.14}$$

It is easy to see

$$[C_C, V_C(\sigma_Z)] = 0 \tag{4.15}$$

$$\frac{1}{2} C_C^* V_C(\sigma_Z) C_C = -V_C(\sigma_Z)\sigma_x(C) + V_C(\sigma_Z) \exp[\sum_{|\sigma \cap A| \text{ odd}} h_A \sigma_Z(A)]. \tag{4.16}$$

We can prove the following in the same manner of [9].

Lemma 4.4. *For Q in $\mathcal{A}_{\mathrm{loc}}$,*

$$[H, Q] = \lim_{A\to\infty} [H_{A,\mathrm{reg}}, Q]. \tag{4.17}$$

Proposition 4.5. *Let $d\mu(\sigma)$ be a Gibbs measure for* (4.5).
(i) *The states $\varphi_\mu^{(+)}$ and $\varphi_\mu^{(-)}$ defined in* (3.26) *are ground states for* (1.6). *Furthermore, we have*

$$\varphi_\mu^{(+)}(H_{A,\mathrm{reg}}) = 0 \tag{4.18}$$

for any finite A.

(ii) *On $L^2(d\mu)$, the following limits exist in the sense of strong resolvent convergence, give rise to a positive selfadjoint operator H_μ.*

$$\lim_{\Lambda\to\infty} \pi_\pm(H_{\Lambda,\mathrm{reg}}) = H_\mu\,. \tag{4.19}$$

We also have the property

$$e^{itH_\mu}\pi_\pm(Q)e^{-itH_\mu} = \pi_\pm(\tau_t(Q)) \tag{4.20}$$

for Q in $\mathcal{A}$

$$H_\mu \Omega_\mu = 0\,. \tag{4.21}$$

H_μ of (4.19) is the selfadjoint extension of the generator of Markov semigroup studied in [6]. We can apply results (or ideas) of Chapter 1 and 4 of [6]. See also § 5 of this paper. We dont't give the proof of Proposition 4.5.

We next define $\boldsymbol{J}$ as the set of states satisfying (4.18).

$$\boldsymbol{J} = \left\{\begin{matrix}\varphi(\cdot)\text{ state of }\mathcal{A}\text{ such that }\varphi(H_{\Lambda,\mathrm{reg}})=0\\ \text{for any finite }\Lambda\text{ in }\boldsymbol{Z}^d\end{matrix}\right\}. \tag{4.22}$$

We also set

$$\boldsymbol{G} = \text{the closed convex hull of }\left\{\begin{matrix}\varphi_\mu^{(\pm)}\ \ \mu\text{: Gibbs measure}\\ \text{for } h(\sigma)\end{matrix}\right\}. \tag{4.23}$$

Theorem 4.6. (i) $\boldsymbol{I}=\boldsymbol{G}$.
(ii) *For any translationally invariant ground state φ of τ_t, there exists a Gibbs measure μ and a real number λ such that* $0\leq\lambda\leq 1$

$$\varphi = \lambda\varphi_\mu^{(+)}+(1-\lambda)\varphi_\mu^{(-)}\,. \tag{4.24}$$

To prove Theorem 4.6 we use the following lemma.

Lemma 4.7. *$\boldsymbol{J}$ is convex, closed, compact in the weak* topology. An extremal point of $\boldsymbol{J}$ is a pure state of $\mathcal{A}$.*

Proof. The first claim is obvious. To prove the second claim, we consider an extremal point of J, say, φ. If φ is not pure, we have states φ_1 φ_2 and λ with $0\leq\lambda\leq 1$ satisfying

$$\varphi = \lambda\varphi_1+(1-\lambda)\varphi_2\,. \tag{4.25}$$

As $H_{\Lambda,\mathrm{reg}}$ is a positive operator, we have

$$\varphi_1(H_{\Lambda,\mathrm{reg}}) = \varphi_2(H_{\Lambda,\mathrm{reg}}) = 0 \tag{4.26}$$

hence φ_1 and φ_2 are in $\boldsymbol{J}$. This is a contradiction. q.e.d.

Proof of Theorem 4.6 (*i*). We must show $J \subset G$. Let φ be a pure state in J. The restriction of φ to B is a Gibbs measure $d\mu(\sigma)$ due to Lemma 3.1. As φ is in J, for any edge $b=\{i, j\}$

$$\varphi(C_b^* C_b) = 0\,. \tag{4.27}$$

This implies

$$\varphi(\sigma_x^{(i)}\sigma_x^{(j)}\sigma_Z(A)) = \int \exp\,[\sum_{|B\cap b|=1} h_B\,\sigma(B)]\times\sigma(A)d\mu(\sigma)\,. \tag{4.28}$$

As any $\sigma_x(A)$ with $|A|$ even is a product of $\sigma_x(b)$ (b edge of $\mathbf{Z}^d$), we can show

$$\varphi(\sigma_x(A)\,\sigma_Z(B)) = \int \exp\{\sum_{|A\cap C|\mathrm{odd}} h_c\,\sigma(c)\}\,\sigma_Z(B)\,d\mu(\sigma) \tag{4.29}$$

if $|A|$ is even.
Then if $|A|$ is odd, $\frac{1}{2}(\varphi+\Theta\varphi)(\sigma_x(A)\sigma_Z(B))=0$, so

$$\frac{1}{2}\{\varphi+\varphi\circ\Theta\} = \frac{1}{2}(\varphi_\mu^{(+)}+\varphi_\mu^{(-)}) \tag{4.30}$$

where $d\mu$ is given by the restriction of φ to B.
We claim that $d\mu(\sigma)$ is an extremal Gibbs measure if φ is pure. This is because purity implies

$$\lim_{k\to\infty}\{\varphi(Q_1\tau_k(Q_2))-\varphi(Q_1)\varphi(\tau_k(Q_2))\} \tag{4.31}$$

for $Q_1\,Q_2$ in A and τ_k lattice translation.
So if $Q_1\,Q_2$ are in B, the observables at infinity wich respect to $d\mu(\sigma)$ are trivial. (See [4]).

Thus states φ, $\varphi\circ\Theta$, $\varphi_\mu^{(+)}$, $\varphi_\mu^{(-)}$ are pure. φ and $\varphi\circ\Theta$ are not equivalent because if they are equivalent, the left-hand side of (4.30) is a factor state, however

$$\lim_{j\to\infty}\frac{1}{2}(\varphi+\varphi\circ\Theta)(\sigma_x^{(0)}\,\sigma_x^{(j)}) = \lim_{j\to\infty}\varphi_\mu^{(+)}(\sigma_x^{(0)})\varphi_\mu^{(+)}(\sigma_x^{(j)}) > \varepsilon^2 \tag{4.32}$$

where we used (4.30), (3.17), (3.18) and

$$\begin{aligned}\varphi_\mu^{(+)}(\sigma_x^{(j)}) &= \varphi_\mu^{(-)}(\sigma_x^{(j)})\\ &= \int \exp\{\sum_{B\ni j} h_B\sigma(B)\}\,d\mu(\sigma) > \exp\{\sum_{B\ni j}|h_B|\} = \varepsilon\,.\end{aligned} \tag{4.33}$$

(Note that ε is dependent of j due to translational invariance of $h(\sigma)$.)

The left-hand side of (4.32) is zero if $\frac{1}{2}\{\varphi+\varphi\circ\Theta\}$ is a factor state, so we have a contradiction.

The pure state φ is disjoint from $\varphi\circ\Theta$. In view of (4.30), we can conclude $\varphi=\varphi_{\mu}^{(+)}$ or $\varphi_{\mu}^{(-)}$.

As a consequence all the extremal points of $\boldsymbol{J}$ are in $\boldsymbol{G}$. This completes the proof of (i). q.e.d.

Proof of Theorem 4.6 (*ii*). Let φ be a translationally invariant ground state. We have only to show that φ is in $\boldsymbol{J}$. Note the following identities.

$$\frac{1}{|\Lambda|}\varphi(H_{\Lambda,\mathrm{reg}})=\sum_{C\ni 0}\varphi\left(\frac{C_C^* V_C(\sigma_Z)C_C}{|C|}\right) \tag{4.34}$$

$$\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\|(H_\Lambda-E_\Lambda^{(+)})-H_{\Lambda,\mathrm{reg}}\|=0 \tag{4.35}$$

(4.35) can be proved in the same way of Lemma 4.3 of [9].
By a result of [2], we have (due to (4.2))

$$\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\varphi(H_\Lambda-E_\Lambda^{(+)})=0\,. \tag{4.36}$$

If we combine (4.36) (4.35) (4.34), we can prove

$$\varphi(C_C^* V_C(\sigma_Z)C_C)=0\,. \tag{4.37}$$

This implies φ is in $\boldsymbol{J}$. q.e.d.

§ 5. Markov Semigroups

In this section, we discuss the relation of quantum Spin Hamiltonian considered in § 4 and Markov processes of X. For simplicity of exposition, we consider the Hamiltonian

$$H=-\sum_{b\,:\,\mathrm{edge}}V_b(\sigma_Z)\sigma_x(b)-\sum_{\boldsymbol{Z}^d\ni j}W_j(\sigma_Z) \tag{5.1}$$

where the sum for b is taken in all the edges $b=\{i,j\}$ $|i-j|=1$ (nearest neighbor and we assume (1.7), translational invariance and finite range property in (5.1).

Recall that the abelian C^* algebra $\boldsymbol{B}$ generated by $\sigma_Z^{(j)}$ (j in $\boldsymbol{Z}^d$) is the set of continuous functions $C(X)$ on $X=\{1,-1\}^{\boldsymbol{Z}^d}$.

By our Assumption 4.1, we have the potential $h(\sigma)$ on X

$$h(\sigma) = \sum_{A} h_A \sigma(A) . \tag{5.2}$$

For any edge b, we define $\Delta_f(b)$ for $f(\sigma)$ in $C(X)$ by

$$\Delta_f(b) = \frac{1}{2} \sup_{x \ni \sigma} | f(\sigma) - f(\sigma_b) | . \tag{5.3}$$

Then we introduce $D(X)$

$$D(X) = \{f(\sigma);\ \sum_b \Delta_f(b) < \infty\} \tag{5.4a}$$

$$|||F||| = \sum_b \Delta_f(b) \qquad \text{for } F \text{ in } D(X) . \tag{5.4b}$$

The Markov (pre) generator L we will consider is

$$LF(\sigma) = \sum_b d_b(\sigma)\{f(\sigma_b) - f(\sigma)\} \tag{5.5}$$

$$d_b(\sigma) = V_b(\sigma) \exp \{ \sum_{|b \cap A| = 1} h_A \sigma(A)\} \tag{5.6}$$

$$\text{for } F(\sigma) \text{ in } D(X) .$$

In (5.5) the sum is taken in all edges of $\mathbb{Z}^d$ and $V_b(\sigma)$ is the element $V_b(\sigma_Z)$ of $\mathbb{B}_{\text{loc}}$ regarded as an element of $C(X)$.
As we have the decay of potential specified in (3.7) (due to Assumption 4.1), the closure τ of L in $C(X)$ generates a 1 parameter semigroup $S(t)$.
$S(t)$ is a Markov semigroup (positivity preserving with $S(t)\ 1=1$). The proof of this fact can be done precisely in the same line of § 3 of Chapter 1 of [6].

We don't repeat the argument given in [6], but we point out that the following inequality is crucial in the proof.

Lemma 5.1. *Set*

$$\varepsilon = \inf_b \{\inf_b (d_b(\sigma) + d_b(\sigma_b))\} \tag{5.7}$$

$$\gamma(b_1, b_2) = \begin{cases} 0 & \text{if } b_2 = b_1 \quad (5.8a) \\ ||d_{b_1}(\sigma_{b_2}) - d_{b_1}(\sigma)|| & \text{if } b_1 = b_2 \quad (5.8b) \end{cases}$$

where b, b_1 and b_2 are edges of $\mathbb{Z}^d$.
Suppose F is in $D(X)$ and $F - \lambda LF = g$ for some $\lambda Z 0$. Then for any edge b in $\mathbb{Z}^d$ we have

$$\Delta_F(b) \times (1 + \lambda\varepsilon) \le \Delta_g(b) + \lambda \sum_{\substack{x \text{edge} \\ x \neq b}} \gamma(x, b) \Delta_F(x) . \tag{5.9}$$

The following theorem corresponds to Theorem 3.9 of Chapter 1 in [6]. See [6] for the proof.

Theorem 5.2. *Let M be a finite constant determined by*

$$M=\sup_x\left(\sum_b r(x,b)\right)<\infty \tag{5.10}$$

(a) *The closure τ of L is a Markov generator of a Markov semigroup $S(t)$.*
(b) *$D(x)$ is a core for τ.*
(c) *If F is in $D(X)$, so is $S(t)\,F$ for $t\geq 0$ and*

$$|||S(t)F|||\leq\exp\left((M-\varepsilon)\right)|||F|||\,. \tag{5.11}$$

It is possible to show that any Gibbs measure is a reversible measure for $S(t)$. The converse statement is also valid.

Theorem 5.3. *Suppose $M<\varepsilon$ in Theorem 5.2. Then $S(t)$ has the unique invariant measure. In particular, the Gibbs measure for $h(\sigma)$ is unique. We have for $g(\sigma)$ in $D(X)$*

$$||S(t)g(\sigma)-\int g(\sigma)d\mu(\sigma)||\leq\sup_b||d_b(\sigma)||\frac{e^{-(\varepsilon-M)t}}{\varepsilon-M}|||g||| \tag{5.12}$$

where $d\mu(\sigma)$ is the unique Gibbs measure.

If we fix $V_b(\sigma)$ in the following way, Theorem 5.3 implies Theorem 5.4.

$$V_b(\sigma)=\Big[\exp\Big(\sum_{|A\cap b|=1}h_A\sigma(A)\Big)+\exp\Big(-\sum_{|A\cap b|=1}h_A\sigma(A)\Big)\Big]^{-1} \tag{5.13}$$

$$d_b(\sigma)=\Big[1+\exp\Big(-2\sum_{|A\cap b|=1}h_A\sigma(A)\Big)\Big]^{-1} \tag{5.14}$$

Theorem 5.4. (i) *If the following inequality is valid*

$$\sup_b\Big\{\sum_{\substack{b'\\ b'\neq b}}\sup_\sigma|d_b(\sigma)-d_b(\sigma_{b'})|\Big\}<1 \tag{5.15}$$

then the Gibbs measure for $h(\sigma)$ is unique.
(ii) *The following condition implies* (5.15)

$$\sup_b\Big\{\sum_{\substack{A\\ |A\cap b|=1}}(|\partial A|_{\text{edge}}-1)|h_A|\Big\}<\frac{\log 2}{4} \tag{5.16}$$

where $|\partial A|_{\text{edge}}$ is the number of edges 1 *such that $A\cap 1\neq\emptyset$ and $A^c\cap 1\neq\emptyset$.*

For the proof of Theorem 5.3 and 5.4, see Theorems 3.9 and 4.1 of Chapter 1 and Theorem 3.1 of Chapter 4 in [6].

We now return to Quantum Spin Hamiltonians H of (5.1).
By our construction of $H_{A,\text{reg}}$ we have

$$\lim_{A\to\infty} \pi_{\pm}(H_{A,\mathrm{reg}})F(\sigma) = LF(\sigma) \tag{5.17}$$

for $F(\sigma)$ in $D(X)$

Proposition 5.5. *Let $d\mu(\sigma)$ be a Gibbs measure. The selfadjoint extension of L is H_μ, and*

$$e^{-tH_\mu}\pi_{\pm}(F(\sigma_Z))\Omega_\mu = S(t)F(\sigma_Z) \tag{5.18}$$

for $F(\sigma)=F(\sigma_Z)$ in $\mathcal{B}=C(X)$.

We have the sufficient condition of existence of mass gap due to Theorem 5.4.

Theorem 5.6. *Suppose assumptions of Theorem 5.3 is valid. Then the spectrum of H_μ has a gap.*

$$\text{Spectrum} \qquad H_\mu \cap (0, \delta) = \emptyset \tag{5.19}$$

where

$$\delta = (\varepsilon - M) \inf_b \inf_\sigma \left(\frac{V_b(\sigma)}{\tilde{V}_b(\sigma)}\right) > 0 \tag{5.20}$$

$$\tilde{V}_b(\sigma) = [\exp(\sum_{|A\cap b|=1} h_A \sigma(A)) + \exp(-\sum_{|A\cap b|=1} h_A \sigma(A))]^{-1}\,. \tag{5.21}$$

(See Chapter 4, § 4, of [5])

§ 6. Heisenberg Models

In this section, we consider Heisenberg models of spin $\frac{1}{2}$ as examples. First we consider the Hamiltonian

$$H(\delta, \beta) = -\sum_{|j-j'|=1} \{\sigma_x^{(j)}\sigma_x^{(j')} + \delta\sigma_y^{(j)}\sigma_y^{(j')} + \varepsilon(\delta, \beta)\sigma_z^{(j)}\sigma_z^{(j')}\}$$
$$+(1-\delta)\sinh\beta \sum_{Z^d \ni j} \sigma_z^{(j)} \tag{6.1}$$

where $0 \leq |\delta| < 1$ β real and

$$\varepsilon(\delta, \beta) = \delta(\cosh\beta)^2 - (\sinh\beta)^2 \tag{6.2}$$

We now give the explicit form of the regularized Hamiltonian.

$$H(\delta, \beta)_{\mathrm{reg}} = -\sum_{|j-j'|=1} (1-\delta\sigma_z^{(j)}\sigma_z^{(j')})\{\sigma_x^{(j)}\sigma_x^{(j')} - e^{\beta(\sigma_z^{(j)}+\sigma_z^{(j')})}\} \tag{6.3}$$

(6.1) and (6.3) differ in the infinite constant

$$\sum_j \{(\cosh\beta)^2 - \delta(\sinh\beta)^2\}\,. \tag{6.4}$$

For the edge $b=\{j, j'\}$ the operator C_b of § 4 is

$$C_b = (1-\sigma_x^{(j)}\sigma_x^{(j')})e^{\frac{1}{2}\beta(\sigma_z^{(j)}+\sigma_z^{(j')})}. \tag{6.5}$$

The proof of Theorem 4.6 implies that the translationally invariant ground state vector Ω satisfies

$$\pi(\sigma_x^{(j)}\sigma_x^{(j')})\Omega = \pi(e^{\beta(\sigma_z^{(j)}+\sigma_z^{(j')})})\Omega \tag{6.6}$$

where $\pi(\ )$ is the representation of A associated to Ω.

The (classical) potential $h(\sigma)$ is determined by (6.6)

$$h(\sigma) = \beta \sum_{Z^d \ni j} \sigma^{(j)}. \tag{6.7}$$

Thus its Gibbs measure is unique.

Theorem 6.1. *For the Hamiltonian* (6.1) *with β real $|\delta|<1$, there exists two translationally invariant ground states $W^{(+)}_{\cdot,\beta}$ such that*

$$W^{(+)}_{\delta,\beta} = W^{(-)}_{\delta,\beta}\circ\Theta \tag{6.8}$$

$$W^{(+)}_{\delta,\beta}(\sigma_x(A)\sigma_Z(B)) = \int \exp[\beta \sum_{A\ni j}\sigma^{(j)}]\sigma(B)dV_\beta(\sigma) \tag{6.9}$$

where $dV_\beta(\sigma)$ is the translationally invariant product measure of $X=\{1, -1\}^{Z^d}$ such that

$$\int \sigma^{(j)}dV_\beta(\sigma) = -\tanh\beta. \tag{6.10}$$

Any translationally invariant ground state is a convex combination of $W^{(+)}_{\delta,\beta}$ and $W^{(-)}_{\delta,\beta}$.

Next we consider the following Hamiltonian

$$H = -\sum\{\sigma_x^{(j)}\sigma_x^{(j')}+\lambda(\delta\sigma_y^{(j)}\sigma_y^{(j')}+\varepsilon\sigma_z^{(j)}\sigma_z^{(j')})\}. \tag{6.11}$$

Theorem 6.2. *There exists $\lambda_0=\lambda_0$ (δ, ε) such that for λ with $|\lambda|<\lambda_0$. Assumption* 4.1 *and assumptions of Theorem* 5.6 *are satisfied for* (6.11).

Sketch of Proof. The proof can be done by expansion of § 3 of [5], (2.13) is valid for $|A|$ odd. So we have only to solve the equation

$$\sum_{A\supset b}[e^{\frac{1}{2}\{h(\sigma)-h(\sigma_b)\}}(1-\lambda\delta\sigma(b))+\lambda\varepsilon\sigma(b)] = E(\lambda) \tag{6.12}$$

where $E(\lambda)=-E_A^{(+)}=-E_A^{(-)}$, b is the edge.

Note that periodic boundary condition is used and σ is any configuration in $\{1, -1\}^{\Lambda}$ in (6.12).

We expand all the quantities in (6.12) by the power of λ.

$$E(\lambda) = \sum_{k=0}^{\infty} E_k \lambda^k \tag{6.13}$$

$$\chi_b(\sigma) = \frac{1}{2}\{h(\sigma)-h(\sigma_b)\} = \sum_{k=0}^{\infty} \chi_b^{(k)}(\sigma)\lambda^k . \tag{6.14}$$

Then by (6.12), we have

$$\sum_b \{\chi_b^{(k)}(\sigma)+P_k(\chi_b^{(1)}(\sigma)\cdots)-\delta\sigma(b)(\chi_b^{(k-1)}(\sigma)+P_{k-1}(\chi_b^{(1)}(\sigma)\cdots))\}$$

$$= E_k \qquad \text{if } k\geq 2 \tag{6.15a}$$

$$\sum_b \chi_b^{(1)}(\sigma)-(\delta-\varepsilon)\sum_b \sigma(b) = E_1 \tag{6.15b}$$

$$\chi_b^{(0)} = 0 \tag{6.15c}$$

where $P_k(x_1\cdots x_{k-1})$ is defined by

$$\exp(\sum_{k=0}^{\infty} \lambda^k x_k) = 1+\sum_{k=0}^{\infty} (x_k+P_k(x_1\cdots x_{k-1}))\lambda^k \tag{6.16}$$

(6.15) yields

$$||\chi_b^{(k)}||_{00}\leq P_k(||\chi_b^{(1)}||_{00}\cdots)+|\delta|P_{k-1}(||\chi_b^{(1)}||_{00}\cdots) \tag{6.17}$$

where $||\ \ ||_{00}$ is the norm (3.5) with $\delta_1=\delta_2=0$.

Let $F(\lambda)$ be defined implicity

$$(1+\lambda|\delta|)(e^{F(\lambda)}-1)-2F(\lambda)+|\delta-\varepsilon|\lambda = 0 \tag{6.18}$$

then, $F(\lambda)$ is analytic if $|\lambda|\leq\lambda_0(\varepsilon, \delta)$ and

$$F(\lambda) = \sum_{n=1}^{\infty} y_n \lambda^n$$

$$y_1 = |\delta-\varepsilon|$$

$$y_k = P_k(y_1\cdots y_{k-1})+|\delta|(P_{k-1}(y_1\cdots y_{k-2})+y_{k-1}) .$$

We can show $||\chi_b^{(k)}||_{00}\leq y_k$ by induction.
Thus if $|\lambda|\leq\lambda_0\ (\varepsilon, \delta)$

$$\sum_{k=1}^{\infty} ||\chi_b^{(k)}||_{00}|\lambda|^k\leq F(\lambda) \tag{6.19}$$

(6.19) implies the convergence of the above expansion in $||\ \ ||_{00}$.

If we look carefully (6.15), the range of potential $\chi_b^{(k)}$ grows linearly in k and we can prove

$$\|\chi_b^{(k)}\|_{\delta_1\delta_2}\leq e^{ck}\|\chi_b^{(k)}\|_{00}\,. \tag{6.20}$$

This implies Assumption 4.1 and Theorem 5.6. q.e.d.

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