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Neighborhood of a Rational Curve with a Node

By

Tetsuo UEDA*

§ 0. Introduction and Statement of the Result

Let C be an irreducible compact analytic set of dimension 1 in a complex manifold of dimension 2. Let (C^2) denote the self-intersection number of C. It is well known that C has a strongly pseudoconvex neighborhood if and only if $(C^2)<0$; and that C has a fundamental system of strongly pseudoconcave neighborhoods if $(C^2)>0$. When $(C^2)=0$, this topological condition alone is insufficient to derive analytic conclusions. For a smooth curve C with $(C^2)=0$, we obtained some conditions for the existence of a fundamental system of strongly pseudoconcave or pseudoflat neighborhoods in [8] (see also Neeman [5]).

It will be natural to investigate such complex analytic properties in the case where C has singularities. In the present paper we treat one of the simplest cases of such C.

In the sequel, C will always stand for a rational curve with only one node (ordinary double point). To state the result, we note first that the Picard variety of C, i.e., the set of all topologically trivial line bundles over C, is isomorphic to $H^1(C, \mathbf{C}^*)\cong\mathbf{C}^*$ as multiplicative group (see Lemma 1).

Suppose that C is imbedded in a complex manifold S of dimension 2 and $(C^2)=0$. Let $[C]$ denote the line bundle over S associated to the divisor C. The normal bundle N of C is defined to be the restriction $[C]|C$ of $[C]$ to C. By the assumption, N is a topologically trivial line bundle over C. Let $\alpha\in\mathbf{C}^*$ be the number corresponding to N by the above isomorphism. Let $d(p)$ denote the distance of $p\in S$ from the curve C with respect to some Riemannian metric

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* Institute of Mathematics, Yoshida College, Kyoto University, Kyoto, 606 Japan.

on S. The main result of the present paper consists of the following theorems.

Theorem 1. *Suppose* $|\alpha| \neq 1$. *Then there exist a neighborhood* V *of* C *and a strongly plurisubharmonic function* Φ *on* $V-C$ *such that* $\Phi(p) \to +\infty$ *as* $p \to C$. *Moreover, for any real number* $\lambda > 1$, *we can construct* Φ *so that* $\Phi(p) \Big/ \left(\log \dfrac{1}{d(p)}\right)^{2\lambda}$ *is bounded.*

Theorem 2. *Suppose* $|\alpha| \neq 1$. *Let* V' *be a neighborhood of* C *and* $\Psi(p)$ *a plurisubharmonic function on* $V'-C$ *such that* $\Psi(p) \Big/ \left(\log \dfrac{1}{d(p)}\right)^{2\lambda}$ *is bounded for some real number* $\lambda < 1$. *Then there is a neighborhood* V_0 *of* C *such that* $\Psi | (V_0 - C)$ *is constant.*

These results correspond to Theorems 1 and 2 in [8] (for smooth curves of finite type). We note that, in the present case, the neighborhood of C admits plurisubharmonic functions with slower growth than in the case of [8].

Theorems 1 and 2 are proved in § 3, after some preliminaries in § 1 and § 2. We consider, in § 4, such curves in compact complex surfaces. Examples of surfaces of class VII_0 and rational surfaces are given.

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§ 1. Topologically Trivial Line Bundles over C

Let C be a rational curve with only one node p_0. Let $\boldsymbol{P}$ denote a smooth rational curve (Riemann sphere) with inhomogeneous coordinate ζ and let $\iota: \boldsymbol{P} \to C$ be a desingularization map of C such that $\iota(0) = \iota(\infty) = p_0$. We fix ε with $0 < \varepsilon < 1$ and define open subsets U_i ($i=0, 1$) of C by

$$U_0 = U_0^+ \cup U_0^-$$

where

$$U_0^+ = \iota(\{|\zeta| < \varepsilon\}), \quad U_0^- = \iota(\{1/\varepsilon < |\zeta| \leqq \infty\})$$

and

$$U_1 = \iota(\{0 < |\zeta| < \infty\}).$$

Then $\{U_0, U_1\}$ is an open covering of C. The intersection $U_0 \cap U_1$ consists of two connected components

$$U^+ = \iota(\{0<|\zeta|<\varepsilon\}), \quad U^- = \iota(\{(1/\varepsilon<|\zeta|<\infty\}).$$

Every holomorphic line bundle over C is represented by a multiplicative 1-cocycle with respect to the covering $\{U_i\}$. We denote this 1-cocycle by a pair (f^+, f^-) of non-vanishing holomorphic functions $f^\pm$ on $U^\pm$.

Lemma 1. *Every topologically trivial line bundle E over C is represented by a pair $(1, \alpha)$ where α is a complex number $\neq 0$ uniquely determined by E. In other words, the Picard variety of C is isomorphic to $H^1(C, \boldsymbol{C}^*)\cong\boldsymbol{C}^*$.*

Proof. Suppose that E is represented by a pair (f^+, f^-). We regard $f^\pm$ as functions of the variable ζ with $0<|\zeta|<\varepsilon$ and $1/\varepsilon<|\zeta|<\infty$, respectively. Let

$$m^+ = \frac{1}{2\pi i}\int_{|\zeta|=\varepsilon/2}\frac{df^+}{f^+}, \quad m^- = \frac{1}{2\pi i}\int_{|\zeta|=2/\varepsilon}\frac{df^-}{f^-}.$$

Then we can write

$$f^\pm = \zeta^{m^\pm}\exp g^\pm,$$

where g^+, g^- are holomorphic functions on U^+, U^-. By Laurent expansion, they are decomposed as follows:

$$g^+ = a_0^+ + g_+^+ + g_-^+$$
$$g^- = a_0^- + g_+^- + g_-^-$$

with

$$g_+^+(\zeta) = \sum_{k>0} a_k^+\zeta^k, \quad g_-^+(\zeta) = \sum_{k<0} a_k^+\zeta^k;$$
$$g_+^-(\zeta) = \sum_{k>0} a_k^-\zeta^k, \quad g_-^-(\zeta) = \sum_{k<0} a_k^-\zeta^k.$$

The power series g_+^+, g_-^+, g_+^-, g_-^- define holomorphic functions for $|\zeta|<\varepsilon$, $0<|\zeta|\leqq\infty$, $|\zeta|<\infty$, $1/\varepsilon<|\zeta|\leqq\infty$, respectively.

We define non-vanishing holomorphic functions f_i on U_i, $i=0, 1$, by

$$f_0 = \begin{cases} \exp(g_+^+ - g_+^-) & \text{for} \quad |\zeta|<\varepsilon, \\ \exp(-g_-^+ + g_-^-) & \text{for} \quad 1/\varepsilon<|\zeta|\leqq\infty, \end{cases}$$
$$f_1 = \zeta^{-m^+}\exp(-a_0^+ - g_-^+ - g_+^-) \quad \text{for} \quad 0<|\zeta|<\infty.$$

Then we have

$$\text{(1)} \qquad \begin{cases} f^+ f_1/f_0 = 1 & \text{on} \quad U^+\cong\{0<|\zeta|<\varepsilon\}, \\ f^- f_1/f_0 = \alpha\zeta^m & \text{on} \quad U^-\cong\{1/\varepsilon<|\zeta|<\infty\}, \end{cases}$$

where $m=m^{-}-m^{+}$, $\alpha=\exp(a_0^{-}-a_0^{+})$. This shows that the line bundle E is represented by the pair $(1, \alpha\zeta^{m})$.

Now, since E is topologically trivial, the pull-back $\iota^*(E)$ of E by $\iota: \mathbf{P}\to C$ is also topologically trivial. $\iota^*(E)$ is represented by the 1-cocycle $\zeta^m\alpha$ with respect to the covering $\{|\zeta|<\infty\}$, $\{1/\varepsilon<|\zeta|\leqq\infty\}$ of $\mathbf{P}$. Hence we have $m=0$. q.e.d.

Remark 1. This correspondence depends on the choice of the coordinate ζ: If we replace ζ by $1/\zeta$, then α is replaced by $1/\alpha$.

§ 2. Coordinate Systems on the Neighborhood of C

Now suppose that C is imbedded in a complex manifold of dimension 2 and that $(C^2)=0$.

We choose neighborhoods V_0, V_1 of U_0, U_1 such that $C\cap V_i=U_i$, and that $V_0\cap V_1$ consists of two connected components V^+, V^- with $C\cap V^+=U^+$, $C\cap V^-=U^-$. Further we suppose that V_i are Stein open sets. This is possible by a theorem of Siu [7]. In the following consideration, V_0, V_1 will be replaced by smaller ones whenever it is necessary.

Lemma 2. *Suppose that α is not a root of unity. Then, for any integer $\nu\geqq 1$, there exist holomorphic functions w_i on V_i $(i=0, 1)$ such that $w_i=0$ are defining equations of C in V_i and satisfying the conditions*:

(i) *w_1-w_0 vanishes to order $\nu+1$ on U^+,*

(ii) *$w_1-\alpha w_0$ vanishes to order $\nu+1$ on U^-.*

Proof. First we prove the lemma for $\nu=1$. Let w_i, be any holomorphic functions on V_i such that $w_i=0$ are defining equations of C in V_i $(i=0, 1)$. We want to modify w_i and obtain $\tilde{w}_i$ satisfying the conditions (i) and (ii). We write $w_1=F^+w_0$ on V^+ and $w_1=F^-w_0$ on V^- with non-vanishing functions $F^{\pm}$ on $V^{\pm}$. Let $f^{\pm}=F^{\pm}|U^{\pm}$. The pair (f^+, f^-) represents the complex normal bundle N of C. By Lemma 1, there exist non-vanishing holomorphic functions f_i on U_i $(i=0, 1)$ satisfying the equations (1), with $m=0$. We choose non-vanishing holomorphic functions F_i on V_i such that $F_i|U_i=f_i$ and put $\tilde{w}_i=F_iw_i$ $(i=0, 1)$. Then $\tilde{w}_0$, $\tilde{w}_1$ have the required properties for $\nu=1$ as easily verified.

For $\nu\geqq 2$, we proceed by induction. Assume there are holomorphic functions w_i as in the lemma for some integer ν $(\geqq 1)$. Then we can write

$$w_1 = w_0 + H^+ w_0^{\nu+1} \quad \text{on} \quad V^+,$$
$$w_1 = \alpha(w_0 + H^- w_0^{\nu+1}) \quad \text{on} \quad V^-;$$

with holomorphic functions $H^\pm$ on $V^\pm$. Let $h^\pm = H^\pm | U^\pm$. We want to obtain holomorphic functions h_i on U_i ($i=0$, 1) satisfying the equations

$$\text{(2)} \qquad \begin{cases} h_0 - h_1 = h^+ & \text{on} \quad U^+, \\ h_0 - \alpha^\nu h_1 = h^- & \text{on} \quad U^-. \end{cases}$$

To solve (2) we decompose $h^\pm$ as follows:

$$h^+ = b_0^+ + h_+^+ + h_-^+ ,$$
$$h^- = b_0^- + h_+^- + h_-^- ,$$

where

$$h_+^+(\zeta) = \sum_{k>0} b_k^+ \zeta^k , \quad h_-^+(\zeta) = \sum_{k<0} b_k^+ \zeta^k;$$
$$h_+^-(\zeta) = \sum_{k>0} b_k^- \zeta^k , \quad h_-^-(\zeta) = \sum_{k<0} b_k^- \zeta^k .$$

The power series h_+^+, h_-^+, h_+^-, h_-^- define holomorphic functions for $|\zeta| < \varepsilon$, $0 < |\zeta| \leqq \infty$, $|\zeta| < \infty$, $1/\varepsilon < |\zeta| \leqq \infty$, respectively.

We define

$$h_0 = \begin{cases} c_0 + h_+^+ - \alpha^{-\nu} h_+^- & \text{on} \quad U_0^+ \cong \{|\zeta| < \varepsilon\}, \\ c_0 - \alpha^\nu h_-^+ + h_-^- & \text{on} \quad U_0^- \cong \{1/\varepsilon < |\zeta| \leqq \infty\} \end{cases}$$
$$h_1 = c_1 - h_-^+ - \alpha^{-\nu} h_+^- \quad \text{on} \quad U_1 \cong \{0 < |\zeta| \leqq \infty\},$$

where $c_0 = \dfrac{\alpha^\nu b_0^+ - b_0^-}{\alpha^\nu - 1}$, $\quad c_1 = \dfrac{b_0^+ - b_0^-}{\alpha^\nu - 1}$.

Then h_0, h_1 satisfy the equations (2).

Now let H_i be holomorphic functions on V_i such that $H_i | U_i = h_i$ ($i=0$, 1). We define

$$\tilde{w}_0 = w_0 + H_0 w_0^{\nu+1} \quad \text{on} \quad V_0;$$
$$\tilde{w}_1 = w_1 + H_1 w_1^{\nu+1} \quad \text{on} \quad V_1 .$$

Then $\tilde{w}_0$, $\tilde{w}_1$ satisfy the conditions (i), (ii) with $\nu+1$ in place of ν. In fact, on V^+, we have

$$\tilde{w}_1 - \tilde{w}_0 = H^+ w_0^{\nu+1} + H_1 w_1^{\nu+1} - H_0 w_0^{\nu+1}$$
$$= (H^+ + H_1 - H_0) w_0^{\nu+1} + H_1 \{(w_0 + H^+ w_0^{\nu+1})^{\nu+1} - w_0^{\nu+1}\} .$$

Since $H^+ + H_1 - H_0$ vanishes on U^+ by the first equality of (2), $\tilde{w}_1 - \tilde{w}_0$ vanishes to order $\nu+2$ on U^+. On V^-, we have

$$\begin{aligned}\tilde{w}_1-\alpha\tilde{w}_0 &= \alpha H^- w_0^{\nu+1}+H_1 w_1^{\nu+1}-\alpha H_0 w_0^{\nu+1} \\ &= \alpha(H^-+\alpha^\nu H_1-H_0)w_0^{\nu+1}+\alpha^{\nu+1}H_1\{(w_0+H^- w_0^{\nu+1})^{\nu+1}-w_0^{\nu+1}\} .\end{aligned}$$

Since $H^-+\alpha^\nu H_1-H_0$ vanishes on U^- by the second equality of (2), $\tilde{w}_1-\alpha\tilde{w}_0$ vanishes to order $\nu+2$ on U^-. q.e.d.

Remark 2. We can extend the normal bundle N to a complex line bundle F over $V=V_0\cup V_1$ by extending the 1-cocycle $(1, \alpha)$ to V^+, V^- as constant. Then Lemma 2 implies that the bundles $[C]$ and F coincide formally along the curve C.

Now let ζ: $U_1\rightarrow \mathbb{P}$ denote the inverse of the biholomorphic map $\iota|(\mathbb{P}-\{0, \infty\})$: $\mathbb{P}-\{0, \infty\}\rightarrow U_1$.

Lemma 3. *There exist holomorphic functions z_i on V_i $(i=0, 1)$ satisfying the following conditions*:

(i) $z_1|U_1=\zeta$;

(ii) $z_1=B^+z_0$ *on* V^+, *where* B^+ *is a holomorphic function on* V^+ *such that* $B^+|U^+=1$;

(iii) $z_1=B^-z_0/w_0$ *on* V^-, *where* B^- *is a holomorphic function on* V^- *such that* $B^-|U^-$ *is a constant* $\neq 0$.

Proof. We choose and fix as z_1 any holomorphic function on V_1 satisfying the condition (i). Let z_0 be a holomorphic function on V_0 which takes zero of order 1 on U_0^- and such that $z_0|U_0^+=\zeta$. Then z_0, z_1 satisfy the conditions (i), (ii). We wish to modify z_0 and obtain $\tilde{z}_0$ so that the condition (iii) is also satisfied by $\tilde{z}_0$, z_1.

Let $Q=z_0/w_0$. Then Q is a meromorphic function on V_0 which has simple poles on U_0^+ and has no other poles nor zeros. Let $q=Q|U_0^-$ and $r=q/\zeta$. Then r is extended to a non-vanishing holomorphic function on U_0^+. Let $\beta=r(p_0)$ denote the value of the function r at the node p_0. Choose a non-vanishing holomorphic function S on V_0 such that $S|U_0^+=1$ and $S|U_0^-=r/\beta$.

We put $\tilde{z}_0=z_0/S$. Then $\tilde{z}_0$, z_1 satisfy the conditions of the lemma. In fact, on V^+, we have $z_1/\tilde{z}_0=Sz_1/z_0$, whose restriction to U^+ is identically equal to 1. On V^-, we have $z_1w_0/\tilde{z}_0=Sz_1w_0/z_0=Sz_1/Q$, whose restriction to U_0^- is $(r\zeta)/(\beta q)=1/\beta$. q.e.d.

§3. Proofs of the Theorems

Let w_i $(i=0, 1)$ be as in Lemma 2, with $\nu=4$. Then we have $w_1=A^+w_0$

on V^+ and $w_1=\alpha A^- w_0$ on V^-, where $A^\pm$ are holomorphic functions on $V^\pm$ such that $A^\pm-1$ vanishes to order 4 on C. Let z_i $(i=0, 1)$ be as in Lemma 3. Here we assume that $B^-|U^-=1$ in the condition (iii). This is possible since we may multiply w_i by a constant.

We define real-valued functions φ_i on V_i-U_i by

$$\varphi_i = (\log|w_i|)^2 - \log|\alpha|\log|w_i| + 2\log|\alpha|\log|z_i|\,, \qquad i=0, 1\,. \tag{3}$$

The difference $\varphi_1-\varphi_0$ is extended to a function of C^2 class on $V_0 \cap V_1$ which vanishes on $C \cap V_0 \cap V_1$.

In fact, we have $w_1=A^+ w_0$, $z_1=B^+ z_0$ on V^+ and

$$\varphi_1-\varphi_0 = 2\log|A^+|\log|w_0| + (\log|A^+|)^2 - \log|\alpha|\log|A^+| + 2\log|\alpha|\log|B^+|\,.$$

The term $2\log|A^+|\log|w_0|$ is extended to a function of C^2 class by setting to be 0 on C, since $\log|A^+|$ vanishes to order 4 on U^+; and the other terms are real-analytic. This shows the assertion for V^+. Now we write $\hat{w}_0=\alpha w_0$, $\hat{z}_0=z_0/w_0$. Then, by straightforward calculation we have

$$\varphi_0 = (\log|\hat{w}_0|)^2 - \log|\alpha|\log|\hat{w}_0| + 2\log|\alpha|\log|\hat{z}_0|\,.$$

Noting that $w_1=A^-\hat{w}_0$, $z_1=B^-\hat{z}_0$ on V^-, we can verify the assertion similarly for V^-.

Now let V_0' be a neighborhood of the node p_0 which is relatively compact in V_0, and let ρ be a real-valued function of C^∞ class on $V_0 \cup V_1$ such that $0 \leqq \rho(p) \leqq 1$, $\mathrm{supp}(\rho) \subset V_0$, and $\rho|V_0'=1$. We define a function φ of C^∞ class on $V-C$ by

$$\varphi = \rho\,\varphi_0 + (1-\rho)\varphi_1\,.$$

It is clear that $\varphi(p)$ tends to $+\infty$ as p tends to the curve C. We will prove the following assertions:

1. If $\lambda>1$, then there is a sufficiently small neighborhood V of C such that φ^λ is strongly plurisubharmonic on $V-C$.

2. If $0<\lambda<1$, then there is a sufficiently small neighborhood V of C such that the complex Hessian of φ^λ has one positive and one negative eigenvalues on $V-C$.

The assertion 1 implies Theorem 1. Theorem 2 is derived from the assertion 2 in the same manner as in [8, § 3.4].

To prove the assertions, we denote

$$H(\psi)=\begin{bmatrix}\psi_{w\bar{w}} & \psi_{w\bar{z}}\\ \psi_{z\bar{w}} & \psi_{z\bar{z}}\end{bmatrix},\quad G(\psi)=\begin{bmatrix}|\psi_w|^2 & \psi_w\psi_{\bar{z}}\\ \psi_z\psi_{\bar{w}} & |\psi_z|^2\end{bmatrix}$$

for a real-valued function $\psi(w, z)$. Then the complex Hessian $H(\varphi^\lambda)$ of φ^λ is

$$H(\varphi^\lambda)=\lambda\varphi^{\lambda-2}(\varphi H(\varphi)+(\lambda-1)G(\varphi)).$$

First we look at the neighborhood of $U_1=C-\{p_0\}$. We choose a sufficiently small neighborhood V_1 of U_1 so that (w_1, z_1) is a coordinate system on V_1. On V_1, φ has the form

$$\varphi=(\log|w_1|)^2-\log|\alpha|\log|w_1|+\eta,$$

where

$$\eta=2\log|\alpha|\log|z_1|+\rho(\varphi_0-\varphi_1).$$

The function η is of C^2 class on V_1 and $\eta_{z_1}=(\log|\alpha|)/z_1\neq 0$, $\eta_{z_1z_1}=0$ on C. From

$$H(\varphi)=\begin{bmatrix}\dfrac{1}{2|w_1|^2}+\eta_{w_1\bar{w}_1} & \eta_{w_1\bar{z}_1}\\ \eta_{z_1\bar{w}_1} & \eta_{z_1\bar{z}_1}\end{bmatrix},$$

$$G(\varphi)=\begin{pmatrix}\left(\dfrac{\log|w_1|}{|w_1|}\right)^2(1+o(1)) & \dfrac{\log|w_1|}{w_1}(1+o(1))\eta_{\bar{z}_1}\\ \eta_{z_1}\dfrac{\log|w_1|}{\bar{w}_1}(1+o(1)) & |\eta_{z_1}|^2\end{pmatrix}$$

and $\varphi=(\log|w_1|)^2(1+o(1))$, we obtain

$$\frac{1}{\lambda\varphi^{\lambda-2}}H(\varphi^\lambda)=\varphi H(\varphi)+(\lambda-1)G(\varphi)$$

$$=\begin{pmatrix}\left(\lambda-\dfrac{1}{2}\right)\left(\dfrac{\log|w_1|}{|w_1|}\right)^2(1+o(1)) & (\lambda-1)\dfrac{\log|w_1|}{w_1}\eta_{\bar{z}_1}(1+o(1))\\ (\lambda-1)\eta_{z_1}\dfrac{\log|w_1|}{\bar{w}_1}(1+o(1)) & (\lambda-1)|\eta_{z_1}|^2(1+o(1))\end{pmatrix}.$$

Here $o(1)$ denotes the terms which tend to 0 as $p\to C$. The determinant of this matrix is of the form $\dfrac{(\lambda-1)}{2}\left(\dfrac{\log|w_1|}{|w_1|}\right)^2|\eta_{z_1}|^2(1+o(1))$, and hence positive or negative according as $\lambda>1$ or $\lambda<1$ everywhere on V_1-C, if V_1 is sufficiently small.

Now we look at the vicinity of the node p_0. If $V_0''\subset V_0'$ is a sufficiently small neighborhood of p_0, we can regard $(w_0/z_0, z_0)$ as a coordinate system on

V_0''. Hence we can regard (w_0, z_0) as a coordinate system on $V_0''-C$. Noting that $\varphi=\varphi_0$ on V_0', we obtain from (3)

$$\frac{1}{\lambda\varphi^{\lambda-2}}H(\varphi^\lambda) = \varphi H(\varphi)+(\lambda-1)G(\varphi)$$
$$= \begin{bmatrix} \dfrac{\varphi}{2|w_0|^2}+(\lambda-1)|\varphi_{w_0}|^2 & (\lambda-1)\varphi_{w_0}\varphi_{\bar{z}_0} \\ (\lambda-1)\varphi_{z_0}\varphi_{\bar{w}_0} & (\lambda-1)|\varphi_{z_0}|^2 \end{bmatrix},$$

whose determinant $\frac{\varphi}{2|w_0|^2}(\lambda-1)|\varphi_{z_0}|^2$ is positive or negative according as $\lambda>1$ or $\lambda<1$.

Thus the assertions are shown, and Theorems 1 and 2 are proved.

§ 4. Curves on Compact Surfaces

A complex manifold X is said to be 1-convex (or strongly pseudoconvex), if there is an exhaustion function $\Phi: X\to \boldsymbol{R}$ which is strongly plurisubharmonic except on a compact set in X. If X is non-compact and 1-convex, then there are a compact analytic set A, a Stein space $\hat{X}$ and a proper holomorphic mapping $\pi: X\to\hat{X}$ such that $\pi(A)$ is a discrete set and $\pi|X-A$ is a biholomorphic mapping. By Narasimhan [4] we have $H^{2n-1}(\hat{X}, \boldsymbol{Z})=0$ if $n\geqq 2$. Since A is at most real $(2n-2)$-dimensional, we obtain $H^{2n-1}(X, \boldsymbol{Z})=0$.

Lemma 4. *Let S be a compact complex manifold of dimension $n\geqq 2$ and C an analytic set in S. If the complement $S-C$ is 1-convex, then the homomorphism $H_1(C, \boldsymbol{Z})\to H_1(S, \boldsymbol{Z})$ is surjective.*

Proof. We consider the exact sequence

$$H_1(C, \boldsymbol{Z}) \to H_1(S, \boldsymbol{Z}) \to H_1(S, C; \boldsymbol{Z}).$$

Since $S-C$ is 1-convex we have $H^{2n-1}(S-C, \boldsymbol{Z})=0$. Hence $H_1(S, C; \boldsymbol{Z})=0$ by duality. This implies the assertion.

Now suppose that S is a compact complex surface and C a rational curve with a node on S satisfying the condition of Theorem 1. Then $S-C$ is 1-convex and hence, by Lemma 4, the first Betti number $b_1(S)$ is either 0 or 1. We will give examples of both cases:

Example 1. Surfaces of class VII_0 (minimal compact complex surfaces with $b_1(S)=1$) containing divisors $D\neq 0$ with $(D^2)=0$ were determined by Enoki

[1]. By his result, if S is of class VII_0 and contains a rational curve C with a node and with $(C^2)=0$, then $S=S_{1,\alpha,t}$ $(0<|\alpha|<1,\ t\in\boldsymbol{C})$. It is easy to show that the number corresponding to the normal bundle of C, by Lemma 1, is α (or $1/\alpha$, see Remark 1). Hence, by Theorem 1, the complement $S-\mathrm{C}$ is 1-convex.

The complement $S-C$ is described as follows [1]. Let g be the holomorphic automorphism of $\boldsymbol{C}\times\boldsymbol{C}^*$ defined by

$$g\colon (z,\ w)\to(wz+t,\ \alpha w)\,.$$

Then $S-C$ is biholomorphic to the quotient surface $(\boldsymbol{C}\times\boldsymbol{C}^*)/\langle g\rangle$ of $\boldsymbol{C}\times\boldsymbol{C}^*$ by g. We note that, in the case $t=0$, a plurisubharmonic function can be constructed explicity: Since

$$(\log|w|)^2-\log|\alpha|\log|w|+2\log|\alpha|\log|z|\,,\qquad (z,\ w)\in\boldsymbol{C}\times\boldsymbol{C}^*\,,$$

is invariant under g, this defines a function φ on $S-C$. The function φ is plurisubharmonic and increases with the same order as $\left(\log\dfrac{1}{d(p)}\right)^2$, when $p\to C$.

Example 2. Let C_0 be a cubic curve with a node p_0 in the projective plane $\boldsymbol{P}^2$. Let $\iota_0\colon\boldsymbol{P}\to C_0$ be a desingularization map of C_0. We choose an inhomogeneous coordinate ζ on $\boldsymbol{P}$ so that $\iota_0(0)=\iota_0(\infty)=p_0$ and that $p_1=\iota_0(1)$ is one of the three inflexion points of C_0. Then the normal bundle of C_0 is $[C_0]|C_0=9[p_1]$. We choose nine distinct points $\zeta_1,\cdots,\zeta_9\in\boldsymbol{C}^*=\boldsymbol{P}-\{0,\ \infty\}$ and let $q_j=\iota_0(\zeta_j)$. Blow up $\boldsymbol{P}^2$ at the points q_j so that we have a compact surface S with $\pi\colon S\to\boldsymbol{P}^2$. Denote by C the proper transform of C_0. Then the normal bundle N of C is $[C]|C=9[\hat{p}_1]-\sum_{j=1}^{9}[\hat{q}_j]$, where $\hat{p}_1=\pi^{-1}(p_1)\cap C$ and $\hat{q}_j=\pi^{-1}(q_j)\cap C$. Since we have $\alpha(N)=\zeta_1\cdots\zeta_9$, the complement $S-C$ is 1-convex if ζ_j are so chosen that $|\zeta_1\cdots\zeta_9|\neq 1$.

Remark 3. Let F be the complex line bundle over $V_0\cup V_1$ mentioned in Remark 2. In Example 1, we can easily verify that $F=[C]|V_0\cup V_1$. But this is not the case in general. We will show that, in Example 2, $[C]$ and F do not coincide on any small neighborhood of C. To show this, let $C\subset S$ be as in Example 2 and suppose $[C]=F$ on some neighborhood of C. Then we have holomorphic functions w_i on V_i $(i=0,\ 1)$ such that $w_1=w_0$ on V^+ and $w_1=\alpha w_0$ on V^-. Define a holomorphic 1-form on $V_0\cup V_1-C$ by $\omega=dw_i/w_i$ on V_i-U_i $(i=0,\ 1)$. As shown in the Appendix, ω can be extended to a holomorphic

1-form $\tilde{\omega}$ on all of $S-C$. The restriction $\tilde{\omega}|\pi^{-1}(q_j)$ has a pole with residue 1 at $C\cap\pi^{-1}(q_j)$ and holomorphic elsewhere. This contradiction shows the assertion.

Appendix

Theorem 3. *Let X be a 1-convex complex manifold of dimension 2, K a compact set in X, and ω a closed holomorphic 1-form on $X-K$. Then ω can be extended to a 1-form $\tilde{\omega}$ holomorphic on all of X.*

First we show the following

Lemma 5. *Let E be a holomorphic vector bundle over X. Let K be a compact set in X and s a holomorphic section of E over $X-K$. Then s can be extended to a meromorphic section $\tilde{s}$ over all of X. The set of poles of $\tilde{s}$ is a compact analytic set of dimension 1 in X.*

Proof. Let A be the maximal nowhere discrete compact analytic set in X. Let $A=\bigcup_{i=1}^{n} A_i$ be the decomposition of A into irreducible components. By blowing up, we assume that the singularities of A are normal crossings and that the components A_i are non-singular. We can choose positive integers p_i $(i=1,\cdots,n)$ so that the restriction of the line bundle $[D]=\sum_{i=1}^{n} p_i[A_i]$ to A is negative. Hence, for sufficiently large m, we have $H^1(X, \mathcal{O}(K\otimes E^{-1}\otimes[D]^{-m}))=0$, by Ohsawa [6].

Now we consider the exact sequence

$$H^0(X, \mathcal{O}(E\otimes[D]^m)) \xrightarrow{\mu} H^0_\infty(X, \mathcal{O}(E\otimes[D])^m) \to H^1_*(X, \mathcal{O}(E\otimes[D]^m))$$

where the subscript ∞ indicates the cohomology at infinity and $*$ indicates the cohomology with compact support. We have $H^1_*(X, \mathcal{O}(E\otimes[D]^m))=0$, since it is dual to $H^1(X, \mathcal{O}(K\otimes E^{-1}\otimes[D]^{-m}))$ by Serre duality. Hence the mapping μ is surjective.

This shows that every holomorphic section s of E over $X-K$ can be extended to a meromorphic section $\tilde{s}$ over X, whose set of poles is contained in A.

q.e.d.

Proof of the theorem. By the lemma, ω can be extended to a meromorphic 1 form $\tilde{\omega}$ on X. Let A' denote the set of poles of $\tilde{\omega}$. We will show that A' is empty.

Suppose that A' is non-empty. Let $A'=\bigcup_{i=1}^{k} A_i$ be the decomposition of A' into irreducible components. Let $(r_{ij})=(A_i \cdot A_j)$ be the intersection matrix of $\bigcup_{i=1}^{k} A_i$. Since A' is an exceptional set, the matrix (r_{ij}) is negative definite, by a theorem of Grauert [2].

Let α_i denote the 1-cycle which goes around A_i in the positive sense $(i=1, \cdots, k)$. When regarded as elements in $H_1(X-A, \mathbf{Z})$, the 1-cycles α_i are all torsion elements, since the relations

$$\sum_{i=1}^{k} r_{ij}\alpha_j = 0\,, \qquad i = 1, \cdots, k\,,$$

hold by Mumford [3]. Hence we have

$$\int_{\alpha_i} \tilde{\omega} = 0\,, \qquad i = 1, \cdots, k\,. \tag{4}$$

We choose a finite number of simply connected open sets W_λ in X such that $A' \subset \bigcup W_\lambda$. By (4) there are meromorphic functions f_λ on W_λ such that $df_\lambda=\tilde{\omega}$. On $W_\lambda \cap W_\mu$, we have $f_\lambda - f_\mu = c_{\lambda\mu}$ (constant). Let m_i (>0) be the order of poles of f_λ on A_i, and define the divisor $D=\sum_{i=1}^{k} m_i A_i$. Let $[D]$ be the line bundle associated to D and let $s \in \Gamma(X, \mathcal{O}[D])$ be the canonical section: $(s)=D$. The sections $s_\lambda = f_\lambda s \in \Gamma(W_\lambda - A', \mathcal{O}([D]))$ are extended to holomorphic sections $\hat{s}_\lambda \in \Gamma(W_\lambda, \mathcal{O}[D])$. Since $\hat{s}_\lambda - \hat{s}_\mu = c_{\lambda\mu}s$ on $W_\lambda \cap W_\mu$, we can define a holomorphic section $\hat{s} \in \Gamma(A', \mathcal{O}[D]\,|\,A')$ by setting $\hat{s}=\hat{s}_\lambda$ on $A' \cap W_\lambda$. The section $\hat{s}$ does not vanish on any component of A' and

$$0 \leqq \deg(\hat{s}\,|\,A_i) = D \cdot A_i = \sum_{j=1}^{k} r_{ij} m_j\,, \qquad (i = 1, \cdots, k)\,.$$

Hence $\sum_{i,j=1}^{k} r_{ij} m_i m_j \geqq 0$, which contradicts the fact that (r_{ij}) is negative definite. Thus the theorem is proved.

References

[1] Enoki, I., Surfaces of class VII_0 with curves. *Tôhoku Math. J.*, **33** (1981), 453–492.
[2] Grauert, H., Über Modifikationen und exzeptionelle analytische Mengen, *Math. Ann.*, **146** (1962), 331–368.
[3] Mumford, D., The topology of normal singularities of algebraic surfaces and a criterion for simplicity, *Publ. Math. I.H.E.S.*, **9** (1961), 5–22.
[4] Narasimhan, R., On the homology groups of Stein spaces, *Invent. Math.*, **2** (1967), 377–385.
[5] Neeman, A., Ueda Theory: Theorems and problems, *AMS Memoires*, **415** (1989).

[6] Ohsawa, T., Vanishing theorems on complete Kähler manifolds, *Publ. RIMS, Kyoto Univ.*, **20** (1984), 21–38.

[7] Siu, Y.T., Every Stein subvariety admits a Stein neighborhood, *Invent. Math.*, **38** (1976), 89–100.

[8] Ueda, T., On the neighborhood of a compact complex curve with topologically trivial normal bundle, *J. Math. Kyoto Univ.*, **22** (1983), 583–607.