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An Extension of Deligne-Grothendieck-MacPherson's Theory C_* of Chern Classes for Singular Algebraic Varieties

By

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Introduction

Let $\mathcal{V}$ be the category of compact complex algebraic varieties and $\mathcal{Ab}$ be the category of abelian groups. Let $\mathcal{F}:\mathcal{V}\to\mathcal{Ab}$ be the correspondence assigning to any $X\in \mathrm{Obj}(\mathcal{V})$ the abelian group $\mathcal{F}(X)$ of $\mathbf{Z}$-valued constructible functions on X. If we define the pushforward $f_*:=\mathcal{F}(f)$ for any morphism $f:X\to Y$ by $f_*(\mathbf{1}_W)(y):=\chi(f^{-1}(y)\cap W)$, where W is any subvariety of X and $\mathbf{1}_W$ is the characteristic function of X ($\mathbf{1}_W(x)=1$ for $x\in W$ and $\mathbf{1}_W(x)=0$ for $x\notin W$), then the correspondence $\mathcal{F}$ becomes a covariant functor with this "topologically defined" pushforward [6]. Let $H_*(\ ;\mathbf{Z}):\mathcal{V}\to\mathcal{Ab}$ be the usual $\mathbf{Z}$-homology covariant functor. Then Deligne and Grothendieck conjectured and MacPherson [6] proved that *there exists a unique natural transformation* $C_*:\mathcal{F}\to H_*(\ ;\mathbf{Z})$ satisfying the extra condition that $C_*(X)(\mathbf{1}_X)=c(TX)\cap[X]$ for any smooth variety X, where $c:K\to H^*(\ ;\mathbf{Z})$ *is the total Chern class of vector bundles*. This C_* shall be called *DGM-theory of Chern class*.

The total Chern class $c:=\Sigma_{i\geqq 0}c_i:K\to H^*(\ ;\mathbf{Z})$ is a special "value" of the Chern polynomial $c_t:=\Sigma_{i\geqq 0}t^ic_i:K\to H^*(\ ;\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{Z}[t]$, i.e., "evaluating" c_t at $t=1$ gives rise to c. As a matter of fact we can show that the above DGM-theory $C_*:\mathcal{F}\to H_*(\ ;\mathbf{Z})$ is also a special "value" of a natural transformation $C_{t^*}:\mathcal{F}^t\to H_*(\ ;\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{Z}[t]$ such that "evaluating" C_{t^*} at $t=1$ gives rise to the DGM-theory $C_*=C_{1*}$. Here our new functor $\mathcal{F}^t$ (which will be called "twisted" functor) is such that $\mathcal{F}^t(X)=\mathcal{F}(X)\otimes_{\mathbf{Z}}\mathbf{Z}[t]$ for any variety X, i.e., *as a correspondence* F^t is simply a linear extension of the correspondence $\mathcal{F}$ with respect to the polynomial ring $\mathbf{Z}[t]$, but *as a functor* it is *not* simply a linear extension of the functor $\mathcal{F}$,

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but it involves some kind of "twisting". Just like the DGM-theory C_*, our natural transformation C_{t^*} is a unique natural transformation from the twisted functor $\mathcal{F}^t$ to the functor $H_*(\ ;\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{Z}[t]$ satisfying the extra condition that $C_{t^*}(X)(\mathbb{1}_X)=c_t(TX)\cap[X]$ for any smooth variety X. This natural transformation C_{t^*} shall be called a *twisted DGM-theory* of Chern polynomial. It should be remarked that if we take our functor $\mathcal{F}_t$ to be just a linear extension of $\mathcal{F}$ with respect to $\mathbf{Z}[t]$, then there is *no* natural transformation $\tau:\mathcal{F}_t\to H_*(\ ;\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{Z}[t]$ satisfying the extra condition that $\tau(X)(\mathbb{1}_X)=c_t(TX)\cap[X]$ for any smooth variety X (see [10]).

The organization of the paper is as follows. In §1 we define our twisted functor $\mathcal{F}^t$ and construct our twisted DGM-theory C_{t^*}, in analogy with DGM-theory C_*. In §2 we give a certain characterization of the twisted DGM-theory C_{t^*}, and in §3 we give some results related to the twisted DGM-theory. In the final section (§4) we just pose a more general question, motivated by the formulation of DGM-theory C_*, twisted DGM-theory C_{t^*} and Baum-Fulton-MacPherson's theory Td_* of Todd class [1].

At the moment we do not have reasonable applications of our twisted DGM-theory C_{t^*}, but we just remark that Professor M. Kashiwara pointed out that the idea of the twisted DGM-theory might be applicable to the q-analogue of the universal enveloping algebra, which remains to be seen.

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§1. A Twisted Functor $\mathcal{F}^t$ and a Twisted DGM-theory C_{t^*} of Chern Polynomial

Let $\mathcal{Z}(X)$ be the (free) abelian group of algebraic cycles on X for any object $X\in\mathrm{Obj}(\mathcal{V})$. Then obviously there is a trivial isomorphism: for any object $X\in\mathrm{Obj}(\mathcal{V})$,

$$\begin{array}{ccc}\mathcal{Z}(X) & \longrightarrow & \mathcal{F}(X)\\ \cup\!\!\!| & & \cup\!\!\!|\\ \sum_W n_W[W] & \longmapsto & \sum_W n_W\mathbb{1}_W.\end{array}$$

However, there is another striking non-trivial isomorphism ([6, Lemma 2]): *for any object $X\in\mathrm{obj}(\mathcal{V})$ the following is an isomorphism (called "Euler isomorphism")*:

$$\begin{array}{ccc} \mathcal{E}u(X):\mathcal{Z}(X) & \longrightarrow & \mathcal{F}(X) \\ \Cup & & \Cup \\ \sum_W n_W[W] & \longmapsto & \sum_W n_W Eu_W. \end{array}$$

where Eu_W is MacPherson's local Euler obstruction function, which is constructible.

The pushforward $f_*:\mathcal{F}(X) \to \mathcal{F}(Y)$ for $f:X \to Y$ is defined topologically by:

$$(f_* \mathbf{1}_W)(x) = \chi(f^{-1}(x) \cap W).$$

This pushforward satisfies the following property:

If $g:\tilde{W} \twoheadrightarrow W$ is any resolution of singularities, then
$Eu_W(x) = (g_* Eu_{\tilde{W}})(x)$ for any non-singular point x of W,
i.e., $\operatorname{Supp}(Eu_W - g_* Eu_{\tilde{W}}) \subset W_{\text{sing}}$, *the singular part of W.*

Let us call this property "*resolution property*", abusing words. Then, using this "resolution property", by induction of dimension of supports of constructible functions and by the resolution of singularities (due to Hironaka) we can show that for any subvariety W there exist finitely many smooth varieties X_i's and proper maps $g_i:X_i \to W$ and non-zero integers m_i's such that

$$Eu_W = \sum_i m_i g_{i*} Eu_{X_i}.$$

Therefore it is easy to see ([6, Prop. 2] or [3, Prop. 1.3]) that *if there exists a natural transformation $\tau:\mathcal{F} \to H_*(;\mathbf{Z})$ satisfying the extra condition that $\tau(X)(\mathbb{1}_X) = c(TX) \cap [X]$ for any smooth variety X, then it is unique.* (Namely, "resolution property" of $\mathcal{F}$ and the extra condition satisfied by τ imply the uniqueness of such a τ.)

Define a transformation $C_*:\mathcal{F} \to H_*(;\mathbf{Z})$ by

$$C_*(X)\left(\sum_W n_W Eu_W\right) = \sum_W n_W \hat{C}(W),$$

where $\hat{C}(W)$ is the Chern-Mather class of W. Let us call this transformation C_* the "Chern-Mather" transformation, which obviously satisfies the extra condition that $C_*(X)(\mathbb{1}_X) = c(TX) \cap [X]$ for any smooth variety X, because $Eu_X = \mathbb{1}_X$ if X is smooth. R. MacPherson [6] proved by his graph construction method that this "Chern-Mather" transformation C_* is actually *natural*.

Let Λ be a commutative domain with unit. Let $\mathcal{V}ect:\mathcal{TOP} \to \mathcal{ENS}$ be the contravariant functor from the category $\mathcal{TOP}$ of topological spaces to the category $\mathcal{ENS}$ of sets, such that $\mathcal{V}ect(X)$ = the set of isomorphism classes of complex vector bundles over X. Then a usual characteristic class cl (with coefficients in Λ) of complex vector bundles is nothing but a natural transformation cl: $\mathcal{V}ect \to H^*(;\mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda$. If cl satisfies the Whitney product formula, i.e., $cl(E \oplus F) = cl(E)cl(F)$, then $\mathcal{V}ect$ can be replaced by the Grothendieck contravariant functor K. In our earlier paper [10], using "linear independence of Chern numbers" ([8]), we proved the following "characterization" of DGM-theory C_*:

Theorem (1.1). ([10]) *Let Λ be a commutative domain with unit and let the functor $\mathcal{F}_\Lambda$ be the linear extension of the functor $\mathcal{F}$ with respect to Λ. Let cl: $\mathcal{V}ect \to H^*(;\mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda$ be a characteristic class of vector bundles. Then a necessary and sufficient condition for the (unique)* existence of a natural transformation τ: $\mathcal{F}_\Lambda \to H_*(;\mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda$ *satisfying the extra that $\tau(X)(1_X) = \mathrm{cl}(TX) \cap [X]$ for any smooth variety X is that $cl = \lambda c$, a multiple of the total Chern class c by some element λ of Λ, and in which case $\tau = \lambda . C_*$, the multiple of DGM-theory C_* by the element λ.*

So, if we let $\Lambda = \mathbf{Z}[t]$, the polynomial ring, and consider the Chern polynomial $c_t := \Sigma_{i \geqq 0} t^i c_i : K \to H^*(;\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}[t]$, then it follows from Theorem (1.1) that there is no natural transformation τ: $\mathcal{F}_{\mathbf{Z}[t]} \to H_*(;\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}[t]$ satisfying the extra condition that $\tau(X)(1_X) = c_t(TX) \cap [X]$ for any smooth variety X. So, we want to change the above functor $\mathcal{F}_{\mathbf{Z}[t]}$ to another different functor $\mathcal{F}_{\mathbf{Z}[t]}$ by imposing another functoriality on the "correspondence" $\mathcal{F}_{\mathbf{Z}[t]}$ so that we can get a unique natural transformation $\tau : \mathcal{F}_{\mathbf{Z}[t]} \to H_*(;\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}[t]$ satisfying the extra condition that $\tau(X)(1_X) = c_t(TX) \cap [X]$ for any smooth variety X. Let us denote simply $\mathcal{F}_t$ for the "correspondence" $\mathcal{F}_{\mathbf{Z}[t]}$.

Thanks to the fact (Euler isomorphism) that for any variety X $\mathcal{F}(X)$ is freely generated by local Euler obstructions, i.e., $\mathcal{F}(X) = \{\Sigma_W n_W Eu_W | W$ runs through all subvarieties of X, $n_W \in Z\}$, we can impose another non-obvious functoriality on the "correspondence" $\mathcal{F}_t$: $\mathcal{V} \to \mathcal{A}b$:

Theorem (1.2). *Let us define the "twisted" pushforward $f^t_* : \mathcal{F}_t(X) \to \mathcal{F}_t(Y)$ for any $f: X \to Y$ and for any subvariety W of X as follows:*

$$f^t_* Eu_W := \sum_S n_S t^{\dim W - \dim S} Eu_S,$$

provided that under DGM's pushforward $f_* = \mathcal{F}(f)$

$$f_* Eu_W = \sum_S n_S Eu_S,$$

and extend it linearly with respect to $\mathbf{Z}[t]$. Then (i) *the "twisted" pushforward satisfies "resolution property" and* (ii) the correspondence $\mathcal{F}_t$ *becomes a covariant functor with this "twisted" pushforward.* (*Note: each S is a subvariety of $f(W)$* and so, dim $W \geqq$ dim S. *If $t = 1$, then the twisted pushforward f^t_* is nothing but the original pushforward f_*.*)

Proof. It is easy to see (i).
(ii). It suffices to show that for any $X \xrightarrow{f} Y \xrightarrow{g} Z$ and for any subvariety W of X

$$(g^t_* \circ f^t_*) Eu_W = (g \circ f)^t_* Eu_W.$$

Let

$$f_*Eu_W = \sum_S n_S Eu_S \quad \text{and} \quad g_*Eu_S = \sum_Q n_Q Eu_Q.$$

Then, since $\mathcal{F}$ is a functor, we have

$$(1.2.1) \qquad \begin{aligned}(g\circ f)_*Eu_W &= \sum_S n_S g_* Eu_S \\ &= \sum_S n_S\Big(\sum_Q n_Q Eu_Q\Big) \\ &= \sum_S \sum_Q n_S n_Q Eu_Q.\end{aligned}$$

On the other hand, by the definition of the twisted pushforward f_*^t, we have

$$f_*^t Eu_W = \sum_S n_S t^{\dim W - \dim S} Eu_S \quad \text{and} \quad g_*^t Eu_S = \sum_Q n_Q t^{\dim S - \dim Q} Eu_Q.$$

Therefore

$$\begin{aligned}(g_*^t \circ f_*^t) Eu_W &= g_*^t\Big(\sum_S n_S t^{\dim W - \dim S} Eu_S\Big) \\ &= \sum_S n_S t^{\dim W - \dim S}\Big(\sum_Q n_Q t^{\dim S - \dim Q} Eu_Q\Big) \\ &= \sum_S \sum_Q n_S n_Q t^{\dim W - \dim Q} Eu_Q \\ &= (g\circ f)_*^t Eu_W \text{ (by definition and (1.2.1))}\end{aligned}$$

Thus the "correspondence" $\mathcal{F}_t$ equipped with the twisted pushforward is a covariant functor. Q.E.D.

Definition (1.3). Let us denote $\mathcal{F}^t$ (using superscript) for the correspondence $\mathcal{F}_t$ equipped with the above twisted functorial pushforward and this new functor shall be called the *twisted functor*.

With the above twisted functor $\mathcal{F}^t$, we can show the following theorem (announced in [9]):

Theorem (1.4). *Let $c_t := \sum_{i\geqq 0} t^i c_i$: $K \to H^*(\ ;\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}[t]$ be the Chern polynomial and let $\mathcal{F}^t$: $\mathcal{V} \to \mathcal{A}b$ be the twisted functor defined above. Then there exists a unique natural transformation $C_{t*}: \mathcal{F}^t \to H_*(\ ;\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}[t]$ satisfying the extra condition that $C_{t*}(X)(\mathbf{1}_X) = c_t(TX) \cap [X]$ for any smooth variety X, such that if $t=1$, then C_{t*} is nothing but DGM-theory C_*. (This C_{t*} shall be called a twisted DGM-theory of the Chern polynomial c_t.)*

Proof. First, we observe that as in the proof of the uniqueness of DGM-theory C_*, the uniqueness of such a natural transformation C_{t*} follows from the "resolution property" of $\mathcal{F}^t$ and the extra condition satisfied by C_{t*}. Now, in analogy with the transformation C_*, we define the transformation $C_{t*}: \mathcal{F}^t \to H_*(;\mathbf{Z})[t]$ as follows:

For any $X \in \mathrm{Obj}(\mathcal{V})$ and any $\sum_W p_W Eu_W \in \mathcal{F}^t(X)$, where $p_W \in \mathbf{Z}[t]$,

$$C_{t^*}(X)\left(\sum_W p_W Eu_W\right) := \sum_W p_W \hat{C}_t(W).$$

Here $\hat{C}_t(W)$ is defined to be $\Sigma_{i\geqq 0} t^{\dim W-i}\hat{C}_i(W)$. In other words, $\hat{C}_t(W)$ is defined by the Nash blow-up in a similar manner to the definition of the Chern-Mather $\hat{C}(W)$, i.e., $\hat{C}_t(W) := \nu_*(c_t(\widehat{TW}) \cap [\widehat{W}])$, where $\nu:\widehat{W}\to W$ is the Nash blow-up of W and $\widehat{TW}$ is the tautological Nash tangent bundle over $\widehat{W}$. (So, $\hat{C}_t(W)$ could be named the c_t-Mather class of W.) Then we want to show that the transformation C_{t^*} is actually *natural*. For this, it suffices to show that for any $f:X\to Y$ and any subvariety W of X

$$f_* C_{t^*}(Eu_W) = C_{t^*} f^t_*(Eu_W).$$

By the definition of C_{t^*} and the definition of $\hat{C}_t(W)$,

$$\begin{aligned} f_* C_{t^*}(Eu_W) &= f_*(\hat{C}_t(W)) \\ &= \sum_i t^{\dim W-i} f_* \hat{C}_i(W). \end{aligned}$$

By DGM-theory, if $f_* Eu_W = \sum_S n_S Eu_S$, we have

$$f_*\hat{C}(W) = \sum_S n_S \hat{C}(S),$$

whence

$$f_*\hat{C}_i(W) = \sum_S n_S \hat{C}_i(S) \quad \text{for each } i.$$

Therefore

$$\begin{aligned} \sum_i t^{\dim W-i} f_* \hat{C}_i(W) &= \sum_i t^{\dim W-i}\left(\sum_S n_S \hat{C}_i(S)\right) \\ &= \sum_S n_S t^{\dim W-\dim S}\left(\sum_i t^{\dim S-i}\hat{C}_i(S)\right) \\ &= \sum_S n_S t^{\dim W-\dim S}\hat{C}_t(S) \text{ (by the definition of } \hat{C}_t(S)) \\ &= C_{t^*}\left(\sum_S n_S t^{\dim W-\dim S} Eu_S\right) \text{ (by the definition of } C_{t^*}) \\ &= C_{t^*} f^t_*(Eu_W). \end{aligned}$$

Thus C_{t^*} is natural. And now it is clear that when $t=1$ C_{t^*} is nothing but DGM-theory C_*. Q.E.D.

Remark (1.5). Before closing this section we want to remark a possible connection with $\mathcal{D}$-module theory. Let $\mathcal{M}$ be a holonomic $\mathcal{D}$-module on X. Then the (total) Chern class $C(\mathcal{M})$ of the holonomic $\mathcal{D}$-module $\mathcal{M}$ (see [2]) is defined by $C(\mathcal{M}) := C_*(\Sigma_\alpha m_\alpha(-1)^{\operatorname{codim} Z_\alpha} Eu_{Z_\alpha})$, where m_α is the multiplicity of $T^*_{Z_\alpha}X$ in the characteristic variety $Ch(\mathcal{M})$ of $\mathcal{M}$ and C_* is DGM-theory. This constructible function involving "twisting" $(-1)^{\operatorname{codim} Z_\alpha}$ was first considered by M. Kashiwara [4] in his local index theorem for holonomic $\mathcal{D}$-module:

$$\chi_{\mathcal{M}}(x) = \sum_\alpha m_\alpha\, (-1)^{\operatorname{codim} Z_\alpha} Eu_{Z_\alpha}(x).$$

At the moment it is not clear whether or not we can recapture $\chi_{\mathcal{M}}$ via our twisted DGM-theory C_{-1*} of the "dual" total Chern class $c_{-1} = \Sigma_{i \geqq 0}(-1)^i c_i$.

§2. A Characterization of the Twisted DGM-theory C_{t*}

Throughout this section we assume that Λ is a commutative domain with unit.

Suppose we are given the following three data:

Datum 1: Let cl: $\mathcal{V}ect \to H^*(;\mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda$ be a characteristic class of vector bundles.

Datum 2: Let $\mathcal{F}_\Lambda$: $\mathcal{V} \to \mathcal{A}b$ be the "correspondence" such that $\mathcal{F}_\Lambda(X) = \mathcal{F}(X) \otimes_{\mathbf{Z}} \Lambda$.

Datum 3: Let Cl_*: $\mathcal{F}_\Lambda \to H_*(;\mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda$ be the "cl-Mather" transformation defined by:

For any variety X, the homomorphism $Cl_*(X): \mathcal{F}_\Lambda(X) \to H_*(X;\mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda$ is defined by

$$Cl_*\left(\sum_W n_W Eu_W\right) := \sum_W n_W \widehat{Cl}(W),$$

where $\widehat{Cl}(W) := \nu_*(cl(\widehat{TW}) \cap [\widehat{W}])$, the "$cl$-Mather" homology class of W.

Then, motivated by the construction of the twisted DGM-theory C_{t^1}, we want to solve the problem: *Endow the correspondence $\mathcal{F}_\Lambda$ with a reasonable functorial pushforward satisfying "resolution property" so that the above "cl-Mather" transformation Cl_* is natural.* Note that if we can endow the correspondence $\mathcal{F}_\Lambda$ with such a pushforward, then by the same reason as in DGM-theory we can see that the above "cl-Mather" transformation is the unique natural transformation satisfying the extra condition that $Cl_*(X)(\mathbb{1}_X) = cl(TX) \cap [X]$ for any smooth variety X. We call such a theory Cl_* a *"DGM-type" theory of a characteristic class cl*. It seems hard (or perhaps impossible) to define a certain reasonable functorial pushforward $f_*^{cl}: \mathcal{F}_\Lambda(X) \to \mathcal{F}_\Lambda(Y)$ for $f: X \to Y$ *without appeal to* topology or geometry or "something" of the map f. So, motivated by the twisted pushforward f_*^t in Theorem (1.2), we define the following "twisted" pushforward

Definition (2.1). $f_*^{cl} Eu_W := \Sigma_S n_S \alpha_S Eu_S$, *where* $\alpha_S (\in \Lambda)$ depends on dim W and dim S, *provided* that under *DGM's* "topologically defined" pushforward $f_* = \mathcal{F}(f)$,

$$f_* Eu_W = \sum_S n_S Eu_S.$$

Here the twisting coefficients α_S's depend also on the given characteristic class cl and we need to define α_S's so that the above pushforward is functorial and satisfies "resolution property". It turns out that without giving a precise definition of the coefficients α_S's, we can show the following "characterization" theorem of the twisted DGM-theory C_{t^*}:

Theorem (2.2). *With Definition* (2.1), *cl has a DGM-type theory Cl_* if and only if $cl = \eta(\Sigma_{i\geqq 0}\lambda^i c_i)$ for some η, $\lambda \in \Lambda$.*

Proof. "if part": The proof is basically the same as that of Theorem (1.4). In this case, in Definition (2.1), we define

$$\alpha_S := \lambda^{\dim W - \dim S}.$$

Namely, we define the pushforward f_*^{cl} as follows:

$$f_*^{cl} Eu_W := \sum_S n_S \lambda^{\dim W - \dim S} Eu_S,$$

provided that under DGM's "topologically defined" pushforward $f_* = \mathscr{F}(f)$,

$$f_* Eu_W = \sum_S n_S Eu_S.$$

Then it is easy to see that this pushforward satisfies "resolution property" and by the same argument as in the proof of Theorem (1.4) we can show that this pushforward is functorial. The naturality of the "*cl*-Mather" transformation $Cl_*: F_\Lambda \to H_*(;Z) \otimes_Z \Lambda$ is also easy to see. Indeed,

$$
\begin{aligned}
(2.2.1) \qquad f_* Cl_* Eu_W &= f_*(\widehat{Cl}(W)) \text{ (by the definition of } C_*) \\
&= f_*\left(\sum_i \eta\lambda^{\dim W - i}\hat{C}_i(W)\right) \\
&= \sum_i \eta\lambda^{\dim W - i} f_* \hat{C}_i(W) \\
&= \sum_i \eta\lambda^{\dim W - i}\left(\sum_S n_S \hat{C}_i(S)\right) \\
&= \sum_S n_S\left(\sum_i \eta\lambda^{\dim W - i}\hat{C}_i(S)\right) \\
&= \sum_S n_S \lambda^{\dim W - \dim S}\left(\sum_i \eta\lambda^{\dim S - i}\hat{C}_i(S)\right) \\
&= \sum_S n_S \lambda^{\dim W - \dim S}\widehat{Cl}(S) \\
&= Cl_*\left(\sum_S n_S \lambda^{\dim W - \dim S} Eu_S\right) \\
&= Cl_* f_*^{cl} Eu_W.
\end{aligned}
$$

"Only if": First we can prove the following lemma, whose proof is given later.

Lemma (2.3). *If cl has a DGM-type theory Cl_* with Definition* (2.1), *then the characteristic class cl must be a linear form of individual Chern classes, i.e., $cl = \Sigma_{i\geqq 0}\lambda_i c_i$, $\lambda_i \in \Lambda$.*

Then the proof of the "only if" part goes as follows. Let $V^k := \boldsymbol{P}^1 \times \boldsymbol{P}^1 \times \ldots \times \boldsymbol{P}^1$

be the product of k copies of the 1-dimensional projective space P^1. Let m and n be two arbitrary integers $\geqq 1$ and let $m \geqq n$. Let $\pi: V^m \to V^n$ be the projection to the first n factors of V^m. Then by the definition of DGM's topological push-forward π_* and by Definition (2.1), we have

$$\pi_*^{cl} Eu_{V^m} = 2^{m-n} \alpha_{V^n} Eu_{V^n}.$$

Hence we have the following equality:

$$\begin{aligned}(2.2.2) \qquad Cl_* \pi_*^{cl} Eu_{V^m} &= Cl_*(2^{m-n} \alpha_{V^n} Eu_{V^n}). \\ &= 2^{m-n} \alpha_{V^n} \widehat{Cl}(V^n) \\ &= 2^{m-n} \alpha_{V^n} (\lambda_n \hat{C}_0(V^n) + \lambda_{n-1} \hat{C}_1(V^n) + \ldots + \lambda_0 \hat{C}_n(V^n))\end{aligned}$$

On the other hand

$$\begin{aligned}(2.2.3) \quad \pi_* Cl_* Eu_{V^m} &= \pi_*(\widehat{Cl}(V^m)) \\ &= \pi_*\left(\textstyle\sum_i \lambda_{m-i} \hat{C}_i(V^m)\right) \\ &= \textstyle\sum_i \lambda_{m-i} \pi_* \hat{C}_i(V^m) \\ &= 2^{m-n}\left(\textstyle\sum_i \lambda_{m-i} \hat{C}_i(V^n)\right) \text{ (because } \pi_*(\hat{C}(V^m)) = 2^{m-n} \hat{C}(V^n)) \\ &= 2^{m-n}(\lambda_m \hat{C}_0(V^n) + \lambda_{m-1} \hat{C}_1(V^n) + \ldots + \lambda_{m-n} \hat{C}_n(V^n))\end{aligned}$$

Therefore, since $\pi_* Cl_* Eu_{V^n} = Cl_* \pi_*^{cl} Eu_{V^n}$ and our ring Λ is a domain, by looking at the 0- and n-dimensional components of (2.2.2) and (2.2.3), respectively we get

$$(2.2.4) \qquad 2^{m-n} \lambda_m \hat{C}_0(V^n) = 2^{m-n} \alpha_{V^n} \lambda_n \hat{C}_0(V^n), \text{ i.e., } \quad \lambda_m = \alpha_{V^n} \lambda_n,$$

$$(2.2.5) \qquad 2^{m-n} \lambda_{m-n} \hat{C}_n(V^n) = 2^{m-n} \alpha_{V^n} \lambda_0 \hat{C}_n(V^n), \text{ i.e., } \lambda_{m-n} = \alpha_{V^n} \lambda_0.$$

Thus it follows from (2.2.4) and (2.2.5) that if $\lambda_0 = 0$, then all the other coefficients $\lambda_i (i \geqq 1)$ are also zero. If $\lambda_0 \neq 0$, then, from (2.2.4) and (2.2.5) we also get

$$(2.2.6) \qquad \lambda_0 \lambda_m = \lambda_{m-n} \lambda_n.$$

Since Λ is a domain, we take the quotient field $Q(\Lambda)$ of the domain Λ and consider Λ in $Q(\Lambda)$. Then, since m and n are arbitrary integers $\geqq 1$, by induction we get

$$(2.2.7) \qquad \lambda_n = \lambda_0 (\lambda_1 / \lambda_0)^n.$$

So, letting $\eta = \lambda_0$ and $\lambda = \lambda_1/\lambda_0$, we can get the "if part" of the theorem. Here we remark that we require that $\lambda = \lambda_1/\lambda_0$ is in Λ, otherwise we have to extend the coefficient ring Λ to a larger ring. Q.E.D.

Now it remains to prove Lemma (2.3). Since we apply "linear independence of Chern numbers" (see [8]) to prove this lemma, before going to the proof of the lemma we give some preliminary things (A good reference is [8]). Let $I_j(n) = \{r_1, r_2, \ldots, r_j\}$ be a partition of n and let $I(n)$ denote the set of all distinct partitions of n and let $p(n)=|I(n)|$ be the number of all distinct partitions of n. Given a partition $I_j(n)=\{r_1,r_2,\ldots,r_j\}$, the $I_j(n)$-Chern class $c_{I_j(n)}$ is defined to be $c_{r_1}.c_{r_2}.\ldots.c_{r_j}$. If X is a compact complex manifold of dimension m $(m \geqq n)$, then the $2(m-n)$-dimensional homology class $c_{I_j(n)}(X)\cap[X]:=(c_{r_1}(TX).c_{r_2}(TX)\ldots c_{r_j}(TX))\cap[X]$ is simply denoted by $c_{I_j(n)}[X]$. If $m=n$, then $c_{I_j(n)}[X]$ is nothing but the $I_j(n)$-th Chern number of the manifold X. A key fact to use for the proof of Lemma (2.3) is the following fact:

Fact ([8, *Theorem* 16.7 *and a remark right after it*]): *For any partition $I_j(n)=\{r_1,r_2,\ldots,r_j\}$, the $I_j(n)$-projective space $P^{I_j(n)}$ is defined to be $P^{r_1}\times P^{r_2}\times\ldots\times P^{r_j}$. Then the following $p(n)\times p(n)$ matrix M_n whose entries are $I_k(n)$-Chern numbers of $I_j(n)$-projective spaces $P^{I_j(n)}$:*

$$M_n:=(c_{I_k(n)}[P^{I_j(n)}])$$

is a non-singular matrix.

Now we go on to

Proof of Lemma (2.3). Let cl: $\mathcal{V}ect \to H^*(\ ;Z)\otimes_Z A$ be a characteristic class of complex vector bundles. It is well-known (see [8]) that cl can be expressed as $\lambda_0 c_0+\Sigma_{n\geqq 1}P_n(c_1,c_2,\ldots,c_n)$, where $P_n(c_1,c_2,\ldots,c_n)$ is a homogeneous polynomial of degree n with the weight of c_i being i. In other words each polynomial $P_n(c_1,c_2,\ldots,c_n)$ is a linear combination of $I_j(n)$-Chern class $c_{I_j(n)}$,

i.e., $\quad P_n(c_1,c_2,\ldots,c_n)=\sum_{I_k(n)\in I(n)}\lambda_{I_k(n)}\,c_{I_k(n)}$, where $\lambda_{I_k(n)}\in A$.

What we want to claim is that each $P_n(c_1,c_2,\ldots,c_n)=\lambda_n c_n$ for some $\lambda_n\in A$, i.e., $\lambda_{I_k(n)}=0$ for $I_k(n)\neq\{n\}$, where the partition $\{n\}$ is n itself (hence $c_{\{n\}}=c_n$). Thus cl is a linear form of individual Chern classes. For this, we consider the following complex smooth variety: (for any $m\geqq n$)

$$P^{\{m-n,I_j(n)\}}:=P^{m-n}\times P^{I_j(n)},$$

and we let $\pi_{I_j(n)}:P^{\{m-n,I_j(n)\}}\to P^{m-n}$ be the projection to the first factor space. For the sake of simplicity, we just denote $\pi_{I_j(n)}$ by π_j. Then it is not hard to see the following equality (cf.[11])

(2.3.1) $$\pi_{j*}(c_{I_k(n)}[P^{\{m-n,I_j(n)\}}])=(c_{I_k(n)}[P^{I_j(n)}])[P^{m-n}],$$

for any partition $I_k(n)\in I(n)$.

Since $\pi_{j*}Eu_{P^{\{m-n,I_j(n)\}}}=\chi(P^{I_j(n)})Eu_{P^{m-n}}$, if we denote the twisting coefficient appearing in Definition (2.1) simply by α_j, then we have the following

(2.3.2) $$\pi^{cl}_{j*}\,Eu_{P^{\{m-n,I_j(n)\}}}=\chi(P^{I_j(n)})\,\alpha_j Eu_{P^{m-n}}.$$

Here we note that this twisting coefficient α_j is the same for any partition

$I_j(n)$, since the twisting coefficient in Definition (2.1) depends on dim W and dim S and in the case which we now deal with $W = \boldsymbol{P}^{\{m-n, I_j(n)\}}$ and $S = \boldsymbol{P}^{m-n}$, hence dim $W = m$ and dim $S = n$ for any partition $I_j(n)$. So we can denote α_j just by α for any partition $I_j(n)$. Hence by the definition of the *cl*-Mather transformation Cl_* we have

$$\begin{aligned}(2.3.3)\quad & Cl_* \pi_{j*}^{cl} Eu_{\boldsymbol{P}^{\{m-n, I_j(n)\}}} \\ &= \chi(\boldsymbol{P}^{I_j(n)})\, \alpha\, Cl_*(Eu_{\boldsymbol{P}^{m-n}}) \\ &= \chi(\boldsymbol{P}^{I_j(n)})\, \alpha \{\lambda_0[\boldsymbol{P}^{m-n}] + \text{homology classes of degree } < 2(m-n)\}\end{aligned}$$

On the other hand, by the above equality (2.3.1) we have

$$\begin{aligned}(2.3.4)\quad & \pi_{j*} Cl_* Eu_{\boldsymbol{P}^{\{m-n, I_j(n)\}}} \\ &= \pi_{j*}\{\ldots + P_n(c_1, c_2, \ldots, c_n)[\boldsymbol{P}^{\{m-n, I_j(n)\}}] + \ldots\} \\ &= \pi_{j*}\{\ldots + \textstyle\sum_{Ik(n) \in I(n)} \lambda_{Ik(n)} c_{Ik(n)} [\boldsymbol{P}^{\{m-n, I_j(n)\}}] + \ldots\} \\ &= \textstyle\sum_{Ik(n) \in I(n)} \lambda_{Ik(n)} (c_{Ik(n)} [\boldsymbol{P}^{I_j(n)}])\, [\boldsymbol{P}^{m-n}] + \text{homology classes of degree } < 2(m-n)\end{aligned}$$

Now, since $Cl_* \pi_{j\,*}^{cl} Eu_{\boldsymbol{P}^{\{m-n, I_j(n)\}}} = \pi_{j*} Cl_* Eu_{\boldsymbol{P}^{\{m-n, I_j(n)\}}}$ and $H_*(\boldsymbol{P}^{m-n}; \boldsymbol{Z})$ is torsion-free, if we look at the top-degree components of this equality, from (2.3.3) and (2.3.4) we have

$$(2.3.5)\qquad \chi(\boldsymbol{P}^{I_j(n)})\, \alpha\lambda_0 = \textstyle\sum_{Ik(n) \in I(n)} \lambda_{Ik(n)} (c_{Ik(n)} [\boldsymbol{P}^{I_j(n)}]).$$

Since $\chi(\boldsymbol{P}^{I_j(n)}) = c_n[\boldsymbol{P}^{I_j(n)}] = c_{\{n\}}[\boldsymbol{P}^{I_j(n)}]$, where we emphatically denote $\{n\}$ instead of n (since n itself is a partition of n), by considering the projections π_j for all partitions $I_j(n)$ the equality (2.3.5) gives rise to the following system of the $p(n)$ linear equations:

$$(2.3.6)\quad \textstyle\sum_{Ik(n) \in I(n) - \{n\}} \lambda_{Ik(n)} (c_{Ik(n)} [\boldsymbol{P}^{I_j(n)}]) + (\lambda_{\{n\}} - \alpha\lambda_0) c_{\{n\}} [\boldsymbol{P}^{I_j(n)}] = 0.$$

Since the $p(n) \times p(n)$ matrix $M_n = (c_{Ik(n)}[\boldsymbol{P}^{I_j(n)}])$ is non-singular (see FACT above), the above linear system (2.3.6) has only trivial solution, i.e., $\lambda_{Ik(n)} = 0$ for $I_k(n) \in I(n) - \{n\}$ and $\lambda_{\{n\}} - \alpha\lambda_0 = 0$, i.e., $\lambda_{\{n\}} = \alpha\lambda_0$. Therefore each $P_n(c_1, c_2, \ldots, c_n) = \lambda_n c_n$ for some $\lambda_n \in \Lambda$. Q.E.D.

Remark (2.4). One might be tempted to conclude that the twisting coefficient α_S in Definition (2.1) must be always equal to $\lambda^{\dim W - \dim S}$ if $cl = \eta(\Sigma_{i \geqq 0} \lambda^i c_i)$, but the only thing we can say about the twisting coefficients α_S's is sort of "$\alpha_S = \lambda^{\dim W - \dim S}$(modulo Cl_*)", namely

$$Cl_*\Big(\textstyle\sum_S n_S \alpha_S Eu_S\Big) = Cl_*\Big(\textstyle\sum_S n_S \lambda^{\dim W - \dim S} Eu_S\Big).$$

Indeed, $Cl_*\left(\sum_S n_S \alpha_S Eu_S\right) = Cl_*(f^{cl}{}_* Eu_W)$ (by the definition of $f^{cl}{}_*$)
$= f_* Cl_* Eu_W$ (by the naturality of Cl_*)
$= f_*(\widehat{Cl}(W))$ (by the definition of Cl_*)
$= Cl_*\left(\sum_S n_S \lambda^{\dim W - \dim S} Eu_S\right)$ (by the computation in (2.2.1)).

§3. The Twisted Functor $\mathcal{F}^t$ and a Stratified Weighted Euler Characteristic χ^t

The definition of the twisted pushforward defined in Theorem (1.2) is indirect, unlike DGM's "topologically defined" pushforward, which is defined by taking topological Euler characteristic of each fiber. At the moment we do not know a more direct definition of the twisted pushforward. In this section we discuss a little about $f^t_* \mathbb{1}_W$, instead of $f^t_* Eu_W$.

Let $f\colon X \to Y$ be a morphism and W be a subvariety of X. Let $\underline{X}$ ($\underline{Y}$, resp.) be a subvariety of X (Y, resp.) such that $W \subset \underline{X}$ ($f(W) \subset \underline{Y}$, resp.) and $\underline{f}\colon \underline{X} \to \underline{Y}$ be f restricted to $\underline{X}$, then by the definition of DGM's pushforward we have

(3.1) $\qquad f_* \mathbb{1}_W = \underline{f}_* \mathbb{1}_W,$ and also

(3.1)# $\qquad f_* Eu_W = \underline{f}_* Eu_W.$

In particular, if $f|W\colon W \twoheadrightarrow f(W)$ is f restricted to both W and the range $f(W)$, then we have

(3.2) $\qquad f_* \mathbb{1}_W = (f|W)_* \mathbb{1}_W,$ and also

(3.2)# $\qquad f_* Eu_W = (f|W)_* Eu_W.$

It is then not hard to see that (3.1) and (3.2) also hold for the twisted pushforward, i.e.,

(3.3) $\qquad f^t_* \mathbb{1}_W = \underline{f}^t_* \mathbb{1}_W,$ in particular

(3.4) $\qquad f^t_* \mathbb{1}_W = (f|W)^t_* \mathbb{1}_W.$

Therefore, as in DGM's pushforward, to compute $f^t_* \mathbb{1}_W$, it suffices to consider the surjection $f|W\colon W \twoheadrightarrow f(W)$. So, let $f\colon X \twoheadrightarrow Y$ be a surjection. By the Euler isomorphism (see §1) there exists a unique algebraic cycle $\Sigma_W n_W [W]$ (which is called "MacPherson-Schwarz" cycle) such that $\mathbb{1}_X = \Sigma_W n_W Eu_W$. Lê and Teissier [5] gave an inductive method of constructing such a cycle, for the details of which refer to their paper [5]. It turns out that such varieties W's are the closure $\bar{S}$'s of the strata of a certain Whitney stratification $\mathcal{S}_X = \{S\}$ of X (called a "canonical" Whitney stratification in [5, Corollaire (6.1.7)]) with the top stratum being the smooth part $\overset{\circ}{X}$ of X and that the coefficient $n_W = n_{\bar{S}}$ is a certain topologically defined integer $\Theta(S,X)$ (called Dubson-Kashiwara integer) (see

[4], [5] or [2,III]), i.e.,

(3.5) (Dubson's formula) $$\mathbf{1}_X = Eu_X + \sum_{\substack{S\in\mathcal{S}_X\\ S\subset X_{\text{sing}}}} \Theta(S,X)\,Eu_{\bar{S}},$$

Remark (3.5.1). By Lê-Teissier's inductive construction of the stratification $\mathcal{S}_X$ and the integers $\Theta(S,X)$, we can also see the following: Let S be any Whitney stratum of $\mathcal{S}_X$. Then

$$\mathbf{1}_{\bar{S}} = Eu_{\bar{S}} + \sum_{\substack{W\in\mathcal{S}_X\\ W<S}} \Theta(W,\bar{S})\,Eu_{\bar{W}},$$

where $W<S$ means that $\dim W < \dim S$ and $W\subset \bar{S}$.

Thus, $f^t_*\mathbf{1}_X$ can be calculated as follows:

(3.6) $$f^t_*\mathbf{1}_X = f^t_*Eu_X + \sum_{\substack{S\in\mathcal{S}_X\\ S\subset X_{\text{sing}}}} \Theta(S,X)\,f^t_*Eu_{\bar{S}}.$$

In the rest of this section we give some calculations of $f^t_*\mathbf{1}_X$ for certain cases.

If $f:X\to pt$ is a map to the singleton pt, then by definition $f_*\mathbf{1}_X=\chi(X)$, which actually means $\chi(X)\mathbf{1}_{pt}$. On the other hand, in the case of the twisted pushforward, the Euler characteristics of singularities are also involved:

Theorem (3.7). *Let $\mathcal{S}_X$ be such a Whitney stratification of X whose top stratum is the smooth part $\overset{\circ}{X}$ of X and let $f:X\to pt$ be a map. Then we have*

(3.7.1) $$f^t_*\mathbf{1}_X = t^{\dim X}\chi(X) + \sum_{\substack{S\in\mathcal{S}_X\\ S\subset X_{\text{sing}}}} P_S(t)\,\chi(\bar{S}),$$

where each $P_S(t)$ is an integral polynomial of degree $\leqq \dim X$ (see Remark (3.8) below), divisible by $t-1$. To be more precise,

$$P_S(t) = \sum_{i\geqq 0\ S=S_0<S_1<\ldots<S_i<X} (-1)^{i+1}(t^{\dim X}-t^{\dim S_i})\,\Theta(S,S_1,S_2,\ldots,S_i,X),$$

where $\Theta(S,S_1,S_2,\ldots,S_i,X):=\Theta(S,\bar{S}_1)\Theta(S_1,\bar{S}_2)\ldots\Theta(S_{i-1},\bar{S}_i)\Theta(S_i,X)$ and $S_k<S_j$ means that $\dim S_k<\dim S_j$ and $S_k\subset\bar{S}_j$.

Proof. Let S be any Whitney stratum. Then by induction (i.e., by going up step by step from lowest dimensional strata), we can show:

(3.7.2)
$f^t_*Eu_{\bar{S}} = t^{\dim S}f_*Eu_{\bar{S}}$ (by the definition of f^t_* and since $f:X\to pt$.)

$$= t^{\dim S}\Big\{\chi(\bar{S}) + \sum_{\substack{S_k<S_{k+1}<\dots<S_{k+i}<S\\ i\geqq 0, S_k\in\mathcal{S}_X}} (-1)^{i+1}\Theta(S_k,S_{k+1},\dots,S_{k+i},\bar{S})\chi(\bar{S}_k)\Big\}$$

Proof of (3.7.2). Let $n=\dim X$. Let $\mathcal{S}_i (i=0,1,\dots,n)$ be the subset of the total set $\mathcal{S}_X$ consisting of all strata of dimension i. (Note that some $\mathcal{S}_i$ can be empty.)
(i) $S_0\in\mathcal{S}_0$, then $\mathbb{1}_{S_0}=Eu_{S_0}$ (by Remark (3.5.1))

(ii) $S_1\in\mathcal{S}_1$, then $\mathbb{1}_{\bar{S}_1}=Eu_{\bar{S}_1}+\sum_{S_0<S_1}\Theta(S_0,\bar{S}_1)Eu_{S_0}$ (by Remark (3.5.1)). Therefore we have

$$Eu_{\bar{S}_1}=\mathbb{1}_{\bar{S}_1}-\sum_{S_0<S_1}\Theta(S_0,\bar{S}_1)Eu_{S_0}=\mathbb{1}_{\bar{S}_1}-\sum_{S_0<S_1}\Theta(S_0,\bar{S}_1)\mathbb{1}_{S_0}$$

(iii) $S_2\in\mathcal{S}_2$, then $\mathbb{1}_{\bar{S}_2}=Eu_{\bar{S}_2}+\sum_{S_1<S_2}\Theta(S_1,\bar{S}_2)Eu_{\bar{S}_1}+\sum_{S_0<S_2}\Theta(S_0,\bar{S}_2)Eu_{S_0}$ (by Remark (3.5.1)). Therefore we have

$$Eu_{\bar{S}_2}=\mathbb{1}_{\bar{S}_2}-\sum_{S_1<S_2}\Theta(S_1,\bar{S}_2)Eu_{\bar{S}_1}-\sum_{S_0<S_2}\Theta(S_0,\bar{S}_2)Eu_{S_0}.$$

Then, by (i) and (ii) we get the following

(iii-1): $$\begin{aligned}Eu_{\bar{S}_2}&=\mathbb{1}_{\bar{S}_2}-\sum_{S_1<S_2}\Theta(S_1,\bar{S}_2)\Big(\mathbb{1}_{\bar{S}_1}-\sum_{S_0<S_1}\Theta(S_0,\bar{S}_1)\mathbb{1}_{S_0}\Big)-\\&\quad\sum_{S_0<S_2}\Theta(S_0,\bar{S}_2)\mathbb{1}_{S_0}\\&=\mathbb{1}_{\bar{S}_2}-\sum_{S_1<S_2}\Theta(S_1,\bar{S}_2)\mathbb{1}_{\bar{S}_1}-\sum_{S_0<S_2}\Theta(S_0,\bar{S}_2)\mathbb{1}_{S_0}+\\&\quad\sum_{S_0<S_1<S_2}\Theta(S_0,\bar{S}_1)\,\Theta(S_1,\bar{S}_2)\mathbb{1}_{S_0}\end{aligned}$$

Continuing this procedure and by induction we can show the following (its details are left for the reader):

$$Eu_{\bar{S}}=\mathbb{1}_{\bar{S}}+\sum_{\substack{S_k<S_{k+1}<\dots<S_{k+i}<S\\ i\geqq 0, S_k\in\mathcal{S}_X}}(-1)^{i+1}\Theta(S_k,S_{k+1},\dots,S_{k+i},\bar{S})\mathbb{1}_{\bar{S}_k}$$

Hence $$f_*Eu_{\bar{S}}=\chi(\bar{S})+\sum_{\substack{S_k<S_{k+1}<\dots<S_{k+i}<S\\ i\geqq 0, S_k\in\mathcal{S}_X}}(-1)^{i+1}\Theta(S_k,S_{k+1},\dots,S_{k+i},\bar{S})\chi(\bar{S}_k)$$

Thus by the definition of the twisted pushforward we get (3.7.2).

By (3.6) we have

$$f^t_*\mathbb{1}_X=f^t_*Eu_X+\sum_{\substack{S_m\in\mathcal{S}_X\\ S_m\subset X_{\text{sing}}}}\Theta(S_m,X)f^t_*Eu_{\bar{S}_m}$$

Thus, by (3.7.2) we get

(3.7.3)

$$f^t_*\mathbf{1}_X = t^{\dim X}\Big\{\chi(X) + \sum_{\substack{S_j<S_{j+1}<\ldots<\mathring{X}\\ i\geqq 0, S_j\in\mathcal{S}_X}} (-1)^{i+1}\Theta(S_j,S_{j+1},\ldots,S_{j+i},X)\chi(\bar{S}_j)\Big\}$$

$$+ \sum_{\substack{S_m\in\mathcal{S}_X\\ S_m\subset X_{\text{sing}}}} \Theta(S_m,X)t^{\dim S_m}\Big\{\chi(S_m)$$

$$+ \sum_{\substack{S_k<S_{k+1}<\ldots<S_m\\ i\geqq 0, S_k\in\mathcal{S}_X}} (-1)^{i+1}\Theta(S_k,S_{k+1},\ldots,S_{k+i},\bar{S}_m)\chi(\bar{S}_k)\Big\}$$

Hence, if $S(<\mathring{X})$ is a Whitney stratum, say $S=S_j$ for some j, and we look at the coefficient $P_{S_j}(t)$ of $\chi(\bar{S}_j)$ in the messy equality (3.7.3), then we get the following:

$$\sum_{\substack{S_j<S_{j+1}<\ldots<\mathring{X}\\ i\geqq 0}} \Big\{(-1)^{i+1}t^{\dim X}\Theta(S_j,S_{j+1},\ldots,S_{j+i},X)$$

$$+ (-1)^i t^{\dim S_{j+i}}\Theta(S_{j+i},X)\,\Theta(S_j,\ldots,S_{j+i-1},\bar{S}_{j+i})\Big\},$$

which is equal to

$$\sum_{\substack{S_j<S_{j+1}<\ldots<\mathring{X}\\ i\geqq 0}} (-1)^{i+1}(t^{\dim X}-t^{\dim S_{j+i}})\,\Theta(S_j,S_{j+1},\ldots,S_{j+i},X)$$

Q.E.D.

Remark (3.8). The leading coefficient, denoted e_S, of the polynomial $P_S(t)$ is equal to

$$\sum_{\substack{S=S_0<S_1<\ldots<S_i<\mathring{X}\\ i\geqq 0}} (-1)^{i+1}\Theta(S,S_1,\ldots,S_i,X).$$

We have been unable to find an example such that $e_S=0$ for some S. So we conjecture that $\deg P_S(t)=\dim X$. Does this integer e_S have some interesting properties as an invariant of singularities?

Definition (3.9). The *stratified weighted Euler characteristic* of X, denoted by $\chi^t(X)$, is defined to be the right hand side of (3.7.1). (Hence Theorem (3.7) reads that if $f:X\to pt$ is a map, then $f^t_*\mathbf{1}_X=\chi^t(X)$, which is the "twisted version" of DGM's $f_*\mathbf{1}_X=\chi(X)$. Note that $\chi^1(X)=\chi(X)$.)

Remark (3.10). It is not hard to see, by a similar argument as above and using the multiplicativity of local Euler obstruction (i.e., $Eu_{S\times W}=Eu_S\times Eu_W$), that if π: $X\times Y \twoheadrightarrow Y$ is the projection, then

(i) $\pi^t_*(\mathbf{1}_{X\times Y})=\chi^t(X)\mathbf{1}_Y$, and therefore (ii) (multiplicativity of χ^t) $\chi^t(X\times Y)=\chi^t(X)\chi^t(Y)$.

In general, however, unlike DGM's pushforward f_*, we cannot expect $(f^t_*\mathbb{1}_X)(y)$ to be determined only by the fiber $f^{-1}(y)$; in particular, we cannot expect that $(f^t_*\mathbb{1}_X)(y)=\chi^t(f^{-1}(y))$, as we see in some examples below.

Example (3.11) (suggested by K. Miyajima). Let $f:S \longrightarrow\!\!\!\!\rightarrow C$ be Kodaira's elliptic surface with each singular fiber lying over $x_i \in C$. Then, since S is smooth and a generic fiber is smooth and elliptic (so, its Euler characteristic is zero),

$$f_*\mathbb{1}_S(=f_*Eu_S)=\textstyle\sum_i \chi(f^{-1}(x_i))Eu_{x_i},$$

therefore, by definition, $f^t_*\mathbb{1}_S=\sum_i t^2\chi(f^{-1}(x_i))Eu_{x_i}$.

Thus, if $x\neq x_i$, $(f^t_*\mathbb{1}_S)(x)=0(=\chi^t(f^{-1}(x))=t\chi(f^{-1}(x)))$, but $(f^t_*\mathbb{1}_S)(x_i)=t^2\chi(f^{-1}(x_i))\neq\chi^t(f^{-1}(x_i))$.

Examples (3.12). (i) Let C be a smooth plane curve of degree $d(>1)$ and $X(\subset P^3)$ be the projective cone over C, with v denoting the cone point of X. Let $f:\tilde{X}\to X$ be the blow-up of X at the cone point v. Note that $\tilde{X}$ is nonsingular and the fiber $f^{-1}(v)$ of the cone point v is isomorphic to the curve C. Then $f_*\mathbb{1}_{\tilde{X}}=Eu_X+d\,Eu_v$, because $Eu_X(v)=2d-d^2$ (e.g., see [6]) and $\chi(C)=3d-d^2$. Hence, by definition $f^t_*\mathbb{1}_{\tilde{X}}(=f^t_*Eu_{\tilde{X}})=Eu_X+t^2d\,Eu_v$. Thus $(f^t_*\mathbb{1}_{\tilde{X}})(x)=\chi^t(f^{-1}(x))=1$ if $x\neq v$, but $(f^t_*\mathbb{1}_X)(v)=2d-d^2+t^2d\neq\chi^t(f^{-1}(v))=t(3d-d^2)$.
(ii) Let X be the union of distinct n lines in P^2, intersecting at one point x and $f:X\longrightarrow\!\!\!\!\rightarrow P^1$ be a non-generic projection such that $f(x)=v$ and $f^{-1}(v)=$ one of the n lines. Then $f^t_*\mathbb{1}_X=(n-1)Eu_{P^1}+\{t(4-n)-1\}Eu_v$. So $(f^t_*\mathbb{1}_X)(v)=t(4-n)+n-2\neq\chi^t(f^{-1}(v))=\chi^t(P^1)=2t$. (Note: if $f:X\longrightarrow\!\!\!\!\rightarrow P^1$ is a generic projection, then $f_*\mathbb{1}_X=f^t_*\mathbb{1}_X=nEu_{P^1}-(n-1)Eu_v$.)

Remark (3.13). At the moment we do not have a characterization of surjections $f:X\longrightarrow\!\!\!\!\rightarrow Y$ such that $(f^t_*\mathbb{1}_X)(y)=\chi^t(f^{-1}(y))$.

§4. A Naive Question

Before finishing this paper, we cite Baum-Fulton-MacPherson's theory Td_* of Todd class for singular varieties and we pose a more general and naive question, which is motivated by MacPherson's survey article [7].

Let $\mathcal{G}(X)$ be the Grothendieck group of coherent sheaves on X. Then $\mathcal{G}$: $\mathcal{V}\to\mathcal{A}b$ becomes a covariant functor. Baum, Fulton and MacPherson [1] constructed a Riemann-Roch theorem for singular varieties, in which the total Todd class theory of singular varieties is formulated as *a unique natural transformation* $Td_*:\mathcal{G}\to H_*(\ ;Q)=H_*(\ ;Z)\otimes_Z Q$ *satisfying the extra condition that* $Td_*(X)(I_X)=$

$\mathrm{td}(TX) \cap [X]$ *for any smooth variety* X, *where* I_X *is the trivial line bundle over* X *and* $td\colon K \to H^*(\ ;Q) = H^*(\ ;Z) \otimes_Z Q$ *is the classical Todd class of vector bundles.* This Td_* shall be called *BFM-theory of Todd class*. Thus DGM-theory C_* (and now our twisted DGM-theory $C_{t^\perp}$ also) and BFM-theory Td_* are *certain natural transformations from certain covariant functors to the homology functor, and respectively classical Chern and Todd* (*cohomology*) classes are involved in these extra conditions. It is safe to say that this is the content of R. MacPherson's survey article "Characteristic classes for singular varieties" [7]. Since the above extra conditions are about the Poincaré dual of corresponding characteristic classes of the *tangent bundles of smooth varieties*, let us call this extra condition "*smooth condition*", abusing words. So, motivated by the formulations of these three theories C_*, $C_{t^\perp}$ and Td_*, or in line with [7], we pose the following general question:

Question. *Let* $cl\colon K \to H^*(\ ;Z) \otimes_Z \Lambda$ be a *characteristic class of vector bundles*, where Λ is a commutative ring with unit. The question is whether or not one could construct

(i) *a certain covariant functor* $\mathscr{E}\colon \mathcal{V} \to \mathcal{A}b$ (e.g., $\mathscr{F}$ in DGM-theory and $\mathscr{G}$ in BFM-theory) and

(ii) *a unique natural transformation* $Cl_*\colon \mathscr{E} \to H_*(\ ;Z) \otimes_Z \Lambda$ *satisfying "smooth condition"* that $Cl_*(X)(e_X) = cl(TX) \cap [X]$ for any smooth variety X, where e_X is some distinguished element of $\mathscr{E}(X)$ (e.g., $\mathbb{1}_X$ in DGM-theory and I_X in BFM-theory).

In Grothendieck's formulation of Riemann-Roch theorem, the covariant functor $\mathscr{G}$ is the "universal" source and the quest is on the existence of *a natural transformation* τ *from* $\mathscr{G}$ *into a certain covariant functor*. In the above question, the homology functor $H_*(\ ;Z) \otimes_Z \Lambda$ is the "universal" target and the quest is on the existence of *a natural transformation* τ *from a certain covariant functor to the homology functor* $H_*(\ ;Z) \otimes_Z \Lambda$, satisfying a certain extra condition, i.e., "smooth condition." In this sense, the above question is also the quest for a Riemann-Roch type theorem.

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