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# Simple K3 Singularities Which Are Hypersurface Sections of Toric Singularities

By

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## Introduction

Yonemura [9] classified the weights of non-degenerate quasi-homogeneous polynomials on $\boldsymbol{C}^4$ which define simple K3 singularities. On the other hand, to each quasi-homogeneous polynomial $f=\Sigma_{\nu\in(\boldsymbol{Z}_{\geqq 0})^4}c_\nu z^\nu$ there exists an element $u_0$ in $(\boldsymbol{Q}_{>0})^4$ such that $\langle \nu,u_0\rangle=1$ if $c_\nu\neq 0$, where $z^{(m_1,m_2,m_3,m_4)}=z_1^{m_1}z_2^{m_2}z_3^{m_3}z_4^{m_4}$. Then we may regard the point $u_0$ as the weight of $f$. Let $\Delta^*$ be the convex hull of $\{\nu\in(\boldsymbol{Z}_{\geqq 0})^4 \mid \langle \nu,u_0\rangle=1\}$. Then $\dim \Delta^*=3$ and $(1,1,1,1)\in \mathrm{Int}(\Delta^*)$, if $f$ defines a simple K3 singularity (see [9]). As a generalization of this fact, we obtain:

**Theorem.** *Let $f$ be a non-degenerate holomorphic function on the toric singularity $Y=\mathrm{Spec}\,\boldsymbol{C}[\sigma^*\cap(\boldsymbol{Z}^4)^*]$ with $f(y)=0$ and let $X=\{f=0\}$, where $\sigma^*$ is the dual cone of a 4-dimensional strongly convex cone $\sigma$ in $\boldsymbol{R}^4$ generated by primitive elements $u_1, u_2,\ldots$ and $u_s$ in $\boldsymbol{Z}^4$ and $\{y\}=\{x\in Y \mid z^\nu(x)=0$ for any $\nu\in(\sigma^*\cap(\boldsymbol{Z}^4)^*)\setminus\{0\}\}$. If $(X,y)$ is a simple K3 singularily, then the following two conditions are satisfied.*
(1) *$Y$ is Gorenstein, i.e., there exists an element $\nu_0\in(\boldsymbol{Z}^4)^*$ such that $\langle \nu_0,u_i\rangle=1$, if $\boldsymbol{R}_{\geqq 0}u_i$ is a 1-dimensional face of $\sigma$,* for $i=1$ *through $s$.*
(2) *There exists an element $u_0\in\mathrm{Int}(\sigma)$ such that $f=\Sigma_{\nu\in\sigma^*\cap(\boldsymbol{Z}^4)^*}c_\nu z^\nu$ with $c_\nu=0$ if $\langle \nu,u_0\rangle<1$, that $\dim\Delta^*=3$ and that $\nu_0\in\mathrm{Int}(\Delta^*)$, where $\Delta^*$ is the convex hull of $\{\nu\in\sigma^*\cap(\boldsymbol{Z}^4)^* \mid \langle \nu,u_0\rangle=1\}$.*

The purpose of this paper is to show that the pairs $(\sigma,u_0)$ satisfying the conditions of the above theorem are finite modulo GL(4,*Z*). Moreover, all representatives of them are obtained by an algorithm which can be excuted by a computer. (However, the program I wrote spent so much time that I could not wait to the end. The number of the equivalent classes is at least greater than 10000.)

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In §1, we prove the above theorem and show that there exists a partial order on the set of the pairs satisfying the conditions of the above theorem such that for a pair $(\sigma,u_0)$, all the pairs $(\tau,u_0) \geqq (\sigma,u_0)$ are finite and obtained by a simple algorithm (see Proposition 1.6 and its proof).

In §2, we classify "minimal" pairs into some classes.

In §3, we enumerate all pairs belonging to each of the classes and make a list of representatives of them at the end of this paper.

## §1. Toric Singularities and Their Hypersurface Sections

Let $N = \boldsymbol{Z}^{n+1}$ be a free $\boldsymbol{Z}$-module of rank $n+1 \geqq 3$ and let $N^*$ be its dual module with canonical pairing $\langle\ ,\ \rangle: N^* \times N \to \boldsymbol{Z}$. Let $\sigma = \boldsymbol{R}_{\geqq 0}u_1 + \boldsymbol{R}_{\geqq 0}u_2 + \ldots + \boldsymbol{R}_{\geqq 0}u_s$ be an $(n+1)$-dimensional strongly convex rational cone in $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ generated by primitive elements $u_i$ in $N$. Here we may assume that $\boldsymbol{R}_{\geqq 0}u_i$ is a 1-dimensional face of $\sigma$, i.e., there exists an element $v$ in $N^*_{\boldsymbol{R}}$ such that $\{u \in \sigma \mid \langle v,u \rangle = 0\} = \boldsymbol{R}_{\geqq 0}u_i$ for each $i = 1$ through $s$. Let $Y = \operatorname{Spec} \boldsymbol{C}[\sigma^* \cap N^*]$ and let $z^v: Y \to \boldsymbol{C}$ be the character of $v$, which is the natural extension of $v \otimes 1_{\boldsymbol{C}^\times}$: $\operatorname{Spec} \boldsymbol{C}[N^*] \simeq (\boldsymbol{C}^\times)^{n+1} \to \boldsymbol{C}^\times$, for each v in $\sigma^* \cap N^*$. Then the set $\{x \in Y \mid z^v(x) = 0 \text{ for all } v \in (\sigma^* \cap N^*) \setminus \{0\}\}$ consists of only one point $y$ and any holomorphic function $f$ on $Y$ with $f(y) = 0$ is expressed as the power series:

$$f := \Sigma_{v \in (\sigma^* \cap N^*) \setminus \{0\}} c_v z^v.$$

Let $X$ be a hypersurface section of $Y$ containing $y$, i.e., $X = \{f = 0\}$, for a holomorphic function $f$ on $Y$ with $f(y) = 0$. Here we note that if $(X,y)$ is an isolated singularity, then the dimension of the singular locus $\operatorname{Sing}(Y)$ of $Y$ is not greater than 1, i.e., any $(n-1)$-dimensional face of $\sigma$ is generated by a part of a basis of $N$. Assume that $X$ is normal and that $X \setminus \{y\}$ has only rational singularities. Then by [7] and [1], we obtain:

**Proposition 1.1.** *The following three conditions are equivalent.*
(1) *$(X,y)$ is Gorenstein.*
(2) *$(Y,y)$ is Gorenstein.*
(3) *(G) There exists an element $v_0 \in N^*$ such that* $\langle v_0,u_i \rangle = 1$ for $1 \leqq i \leqq s$.
*We denote the above $v_0$, by $v(\sigma)$.*

**Definition 1.2.** The Newton polyhedron $\Gamma_+(f)$ of $f$ is the convex hull of $\cup_{c_v \neq 0} (v + \sigma^*)$ and the Newton boundary $\Gamma(f)$ of $f$ is the union of the compact faces of $\Gamma_+(f)$.

**Definition 1.3.** We call $f$ non-degenerate, if $\partial f_{\Delta^*}/\partial z_1 = \ldots = \partial f_{\Delta^*}/\partial z_{n+1} = 0$ has no solutions in $T := \operatorname{Spec} \boldsymbol{C}[N^*] \subset Y$ for each face $\Delta^*$ of $\Gamma_+(f)$, where $f_{\Delta^*} = \Sigma_{v \in \Delta^* \cap N^*} c_v z^v$ and $(z_1, z_2, \ldots, z_{n+1})$ is a global coordinate of $T$, i.e., $z_i =$

$z^{v_i}$ for a basis $\{v_1,v_2,\ldots,v_{n+1}\}$ of $N^*$.

**Proposition 1.4.** ([6, Theorem 2.2]) *Assume that the condition* (*G*) *in Proposition 1.1 is satisfied and that f is non-degenerate. Then* $(X,y)$ *is purely elliptic if and only if* $v(\sigma)\in\Gamma(f)$. (*See* [8], *for the definition of a purely elliptic singularity.*)

*Remark.* If $v(\sigma)\in\partial\Gamma_+(f)\setminus\Gamma(f)$, then $X\setminus\{y\}$ has irrational singularities.

**Proposition 1.5.** *Under the assumption of Proposition* 1.4, $(X,y)$ *is of* $(0,\ n-1)$*-type if and only if* $\dim\Delta^*=n$, *where* $\Delta^*$ *is the face of* $\Gamma(f)$ *with* $v(\sigma)\in\mathrm{Int}(\Delta^*)$. (*See* [2], *for the definition of* $(0,i)$*-type of a purely elliptic singularity.*)

*Proof.* Let $\Sigma$ be a subdivision of the dual Newton decomposition $\Gamma^*(f)$ of $\Gamma(f)$ consisting of non-singular cones (see [5] and [6], for the definition of $\Gamma^*(f)$). Then $\tilde{Y}:=T_N\mathrm{emb}(\Sigma)$ and $\tilde{X}$ are non-singular, where $\tilde{X}$ is the proper transformation of $X$ under the holomorphic map $P$: $\tilde{Y}\to Y$ obtained by the morphism of r.p.p. decompositions $(N,\Sigma)\to(N,\ \{\text{faces of }\sigma\})$. Let $\Sigma_1$ be the set of the 1-dimensional cones in $\Sigma$ which are not 1-dimensional faces of $\sigma$ and let $E_\tau$ be the intersection of the closure of $\mathrm{orb}(\tau)$ with $\tilde{X}$, for each $\tau$ in $\Sigma_1$. Then $P_{|\tilde{X}}{}^{-1}(y)=\Sigma_{\tau\in\Sigma_0}E_\tau$, where $\Sigma_0=\{\tau\in\Sigma_1\mid\tau\subset\mathrm{Int}(\sigma)\cup\{0\}\}$ and we can express $K_{\tilde{X}}=(P_{|\tilde{X}})^*K_X+\Sigma_{\tau\in\Sigma_1}a_\tau E_\tau$. Here we note that $a_\tau=\langle v(\sigma),u_\tau\rangle-d(u_\tau)-1$, by [6, Lemma 2.1], where $u_\tau$ is the primitive element in $N$ generating $\tau$ and $d(u_\tau)=\min\{\langle v,u_\tau\rangle\mid v\in\Gamma_+(f)\}$. Hence $a_\tau\geqq-1$, for each $\tau$ in $\Sigma_1$. Assume that $\dim\Delta^*=n$. Then there exists only one 1-dimensional cone $\tau$ in $\Sigma_0$ with $a_\tau=-1$ and $E_\tau$ is irreducible. Hence $(X,y)$ is of $(0,\ n-1)$-type. Next, assume that $\dim\Delta^*\leqq n-1$. Then we easily see that there exist at least two 1-dimensional cones $\tau$ in $\Sigma_1$ such that $a_\tau=-1$ and that $E_\tau\neq\emptyset$. Hence $(X,y)$ is not of $(0,\ n-1)$-type. q.e.d.

Assume that $f$ is a non-degenerate holomorphic function on $Y$ with $f(y)=0$ and let $X=\{f=0\}$. When $n=3$, $(X,y)$ is a simple K3 singularity (i.e., $(X,y)$ is Gorenstein purely elliptic of (0,2)-type [3]), if and only if $(Y,y)$ is Gorenstein and $v(\sigma)$ is contained in the interior of a 3-dimensional face of $\Gamma(f)$, by Propositions 1.1, 1.4 and 1.5. Assume that $(X,y)$ is Gorenstein purely elliptic of $(0,\ n-1)$-type. Then there exists the unique element $u_0$ in $\mathrm{Int}(\sigma)$ such that $\langle v,u_0\rangle=1$ for all elements $v$ in the face $\Delta^*$ of $\Gamma(f)$ whose interior contains $v(\sigma)$. Hence $\Delta^*$ is contained in

$$\Delta^*_\sigma(u_0):=\text{convex hull of }\{v\in\sigma^*\cap N^*\mid\langle v,u_0\rangle=1\}.$$

Therefore, the pair $(\sigma,u_0)$ satisfies the following condition:

$$\text{(E)}\quad \dim\Delta^*_\sigma(u_0)=n\ \text{ and }\ v(\sigma)\in\mathrm{Int}\,(\Delta^*_\sigma(u_0)).$$

Thus we obtain the theorem in Introduction. Conversely, assume that $(\sigma,u_0)$

satisfies the conditions (G) and (E), and let $X=\{f=0\}$, where $f=\Sigma_{v\in\Delta^*_\sigma(u_0)\cap N^*}c_v z^v$ + higher terms, for certain non-zero complex numbers $c_v$. Then $(X,y)$ is Gorenstein purely elliptic of $(0, n-1)$-type, if $f$ is non-degenerate and $(X,y)$ is an isolated singularity. Let

$$\tilde{\mathscr{E}}^n=\{(\sigma,u_0)\mid \sigma \text{ is an } (n+1)\text{-dimensional strongly convex rational cone satisfying (G), } u_0\in \operatorname{Int}(\sigma) \text{ and } u_0 \text{ satisfies (E)}\}$$

and let $\mathscr{E}^n=\tilde{\mathscr{E}}^n/\sim$, where $(\sigma,u_0)\sim(\sigma',u_0')$ if and only if there exists an element $g$ in GL(N) such that $g\sigma=\sigma'$ and that $g(u_0)=u_0'$. We define a partial order on $\tilde{\mathscr{E}}^n$ as follows: $(\sigma,u_0)\geqq(\sigma',u_0')$ if and only if $\sigma\supset\sigma'$, $\nu(\sigma)=\nu(\sigma')$ and $u_0=u_0'$. Let

$$\tilde{\mathscr{E}}^n_0=\{(\sigma,u_0)\in\tilde{\mathscr{E}}^n\mid(\sigma,u_0) \text{ is minimal}\}$$

and let $\mathscr{E}^n_0=\tilde{\mathscr{E}}^n_0/\sim$, where we call $(\sigma,u_0)$ minimal, if $(\sigma,u_0)\geqq(\tau,u_0)$ implies $(\sigma,u_0)=(\tau,u_0)$, for any $(\tau,u_0)\in\tilde{\mathscr{E}}^n$.

*Remark.* (1) Assume that $(\sigma, u_0)\in\tilde{\mathscr{E}}^n$. If the cone $\tau$ generated by a subset of $L:=\{u\in\sigma\cap N\mid\langle\nu(\sigma),u\rangle=1\}$ is $(n+1)$-dimensional strongly convex and contains $u_0$ in the interior, then $(\tau,u_0)\in\tilde{\mathscr{E}}^n$, because $\tau^*\supset\sigma^*$.
(2) Since $\#L<+\infty$, for any pair $(\sigma,u_0)$ in $\tilde{\mathscr{E}}^n$, we have $\#\{(\tau,u_0)\in\tilde{\mathscr{E}}^n\mid(\sigma,u_0)\geqq(\tau,u_0)\}<+\infty$. Hence for any pair $(\sigma,u_0)$ in $\tilde{\mathscr{E}}^n$, there exists a pair $(\tau,u_0)$ in $\tilde{\mathscr{E}}^n_0$ with $(\sigma,u_0)\geqq(\tau,u_0)$.

Let $C(\sigma,u_0)=\{(\tau,u_0)\in\tilde{\mathscr{E}}^n\mid(\tau,u_0)\geqq(\sigma,u_0)\}$, for a pair $(\sigma,u_0)$ in $\tilde{\mathscr{E}}^n$. Then by the above remark, we have $\tilde{\mathscr{E}}^n=\cup_{(\sigma,u_0)\in\tilde{\mathscr{E}}^n_0}C(\sigma,u_0)$. Hence if $\mathscr{E}^n_0$ is a finite set, then so is $\mathscr{E}^n$, by the following proposition.

**Proposition 1.6.** *$C(\sigma,u_0)$ is a finite set, for any pair $(\sigma,u_0)$ in $\tilde{\mathscr{E}}^n$.*

*Proof.* Since for any pair $(\tau,u_0)$ in $C(\sigma,u_0)$, $\Delta^*_\tau(u_0)$ is the convex hull of a subset of the finite set $L^*:=\{v\in\sigma^*\cap N^*\mid\langle v,u_0\rangle=1\}$, we have $\#\{\Delta^*_\tau(u_0)\mid(\tau,u_0)\in C(\sigma,u_0)\}<+\infty$. Conversely, let $\Delta^*$ be the convex hull of a subset of $L^*$ such that $\nu(\sigma)\in\operatorname{Int}(\Delta^*)$ and that $\dim\Delta^*=n$. Then $\#\{u\in(\mathbb{R}_{\geqq0}\Delta^*)^*\cap N\mid\langle\nu(\sigma),u\rangle=1\}<+\infty$. Hence $C':=\{(\tau,u_0)\in C(\sigma,u_0)\mid\Delta^*_\tau(u_0)=\Delta^*\}$ is a finite set, because $\tau\subset(\mathbb{R}_{\geqq0}\Delta^*)^*$ for any pair $(\tau,u_0)$ in $C'$. Therefore, $C(\sigma,u_0)$ is a finite set.

q.e.d.

Next, we show that for a cone $\sigma$ satisfying the condition (G), all the elements $u_0$ in $\operatorname{Int}(\sigma)$ satisfying the condition (E) are finite. Let $W_\sigma(v_0)=\{u\in\operatorname{Int}(\sigma)\mid\dim\Delta^+_\sigma(u)=n,\ v_0\in\operatorname{Int}(\Delta^+_\sigma(u))\}$, for an $(n+1)$-dimensional strongly convex rational cone $\sigma$ and for an element $v_0$ in $N^*_{\mathbb{R}}$.

**Theorem 1.7.** *$W_\sigma(v_0)$ is a finite set, for any $v_0\in\operatorname{Int}(\sigma^*)$.*

*Proof.* For $v_1,v_2,\dots,v_j \in \sigma^* \cap N^*$, let $W(v_1,v_2,\dots,v_j)=\{u \in W_\sigma(v_0) \mid \langle v_1,u\rangle = \cdots = \langle v_j,u\rangle = 1\}$. For $u \in N_{\boldsymbol{R}}$, let $W^*(u)=\{v \in \sigma^* \cap N^* \mid \langle v,u\rangle < 1\}$. Here we note that if $u \in \mathrm{Int}(\sigma)$, then $W^*(u)$ is a finite set. First, take an element $u_0 \in \mathrm{Int}(\sigma)$ with $\langle v_0,u_0\rangle = 1$. Then for any element $u$ in $W_\sigma(v_0)$ with $u \neq u_0$, we see that $\{v \in W^*(u_0) \mid \langle v,u\rangle = 1\} \neq \emptyset$. Hence $w_\sigma(v_0) \subset \{u_0\} \cup \bigcup_{v_1 \in W^*(u_0)} W(v_1)$. Here we note that if $W(v_1) \neq \emptyset$, then $v_0$ and $v_1$ are linearly independent. Next, if $W(v_1) \neq \emptyset$, then we can take an element $u_1 \in \mathrm{Int}(\sigma)$ with $\langle v_0,u_1\rangle = \langle v_1,u_1\rangle = 1$, for each $v_1 \in W^*(u_0)$. Then we have $W(v_1) \subset \{u_1\} \cup \bigcup_{v_2 \in W^*(u_1)} W(v_1,v_2)$. Proceeding similarly, we finally obtain $W(v_1,\dots,v_{n-1}) \subset \{u_{n-1}\} \cup \bigcup_{v_n \in W^*(u_{n-1})} W(v_1,\dots,v_n)$. Then $\# W(v_1,\dots,v_n) \leqq 1$, because $v_0,v_1,\dots$ and $v_n$ are linearly independent, if $W(v_1,\dots,v_n) \neq \emptyset$. Hence $\# W(v_1,\dots,v_{n-1}) < +\infty$ and thus $\# W_\sigma(v_0) < +\infty$. q.e.d.

In the next section, the following proposition plays key role.

**Proposition 1.8.** *If $W_\sigma(v_0) \neq \emptyset$ for an element $v_0 \in \mathrm{Int}(\sigma^*)$, then $\# IL_\sigma(v_0) \leqq 1$, where $IL_\sigma(v_0)=\{u \in \mathrm{Int}(\sigma) \cap N \mid \langle v_0,u\rangle = 1\}$. Conversely, if $IL_\sigma(v_0)=\{u_0\}$, then $W_\sigma(v_0) \subset \{u_0\}$.*

*Proof.* If $IL_\sigma(v_0) \neq \emptyset$, then for each element $u_0$ in $IL_\sigma(v_0)$, we have $\langle v,u_0\rangle > 0$ for any $v$ in $\sigma^* \setminus \{0\}$ and hence $\langle v,u_0\rangle \geqq 1$ for any $v$ in $(\sigma^* \setminus \{0\}) \cap N^*$. Therefore, $W_\sigma(v_0) \subset \{u_0\}$, as we see in the proof of Theorem 1.7. Hence if $\# IL_\sigma(v_0) \geqq 2$, then $W_\sigma(v_0) = \emptyset$. q.e.d.

## §2. Classification

We restrict ourselves to the case that $n=3$ and show that $\mathscr{C}_0^3$ is a finite set, in the rest of this paper. For finite elements $u_1,u_2,\dots,u_s$ in $N_{\boldsymbol{R}}$, we denote by $u_1u_2\dots u_s$, the convex hull $\{a_1u_1+a_2u_2+\dots+a_su_s \mid a_i \geqq 0,\ a_1+a_2+\dots+a_s=1\}$ of $\{u_1,u_2,\dots,u_s\}$.

**Theorem 2.1.** *Any pair $(\sigma,u_0)$ in $\tilde{\mathscr{C}}_0^3$ is one of the following.*
(1) *$\sigma$ is generated by four primitive elements $u_1$, $u_2$, $u_3$ and $u_4$ in $N$ and $u_iu_j \cap N = \{u_i,u_j\}$ for each $\{i,j\} \subset \{1,2,3,4\}$.*
(2) *$\sigma$ is generated by five primitive elements $u_1,u_2,\dots$ and $u_5$ in $N$, $u_0 \in \mathrm{Int}(u_1u_2u_3)$ and $u_1u_2u_3 \cap u_4u_5 \neq \emptyset$.*
(3) *$\sigma$ is generated by six primitive elements $u_1,u_2,\dots$ and $u_6$ in* N, *$u_0 \in \mathrm{Int}(u_1u_2)$, $\mathrm{Int}(u_3u_4)$, $\mathrm{Int}(u_5u_6)$ and $u_1u_2\dots u_6 \cap N \subset \{u_0,u_1,\dots,u_6\}$.*

**Proposition 2.2.** *$(\sigma,u_0)$ in (1) of Theorem 2.1 is one of the following.*
(1−1) $u_1u_2u_3u_4 \cap N \subset \{u_0,u_1,u_2,u_3,u_4\}$.
(1−2) $(u_1u_2u_3u_4 \cap N) \setminus \{u_0,u_1,u_2,u_3,u_4\} = \{u_5,\dots,u_s\} \subset u_iu_ju_k$ $(s \geqq 5)$ *and* $u_0 \in u_su_ku_l$, *where* $u_5 = \frac{1}{3}(u_i+u_j+u_6)$, $u_6=\frac{1}{2}(u_5+u_7),\dots$ *and* $u_s=\frac{1}{2}(u_{s-1}+u_k)$, $\{i,j,$

*$k,l\} = \{1,2,3,4\}$ and $u_6 = u_k$, when $s = 5$.*

**Proposition 2.3.** *$(\sigma,u_0)$ in (2) of Theorem 2.1 is one of the following.*
(2−1) $u_1u_2u_3u_4u_5 \cap N \subset \{u_0,u_1,u_2,u_3,u_4,u_5\}$.
(2−2) $(u_1u_2u_3u_4u_5 \cap N)\setminus\{u_0,u_1,u_2,u_3,u_4,u_5\} = \{u_6\}$ *and* $u_0 \in \mathrm{Int}(u_iu_6)$, *where* $u_6 = \frac{1}{2}(u_j + u_k)$ *and* $\{i,j,k\} = \{1,2,3\}$.
(2−2−1) $u_i = \frac{1}{2}(u_4 + u_5)$.
(2−2−2) $u_6 = \frac{1}{2}(u_4 + u_5)$.
(2−2−3) $\tilde{u} \in \mathrm{Int}(u_iu_6)$ *and hence* $u_0 \in u_i\tilde{u}$, *where* $\tilde{u}$ *is the intersection point of* $u_1u_2u_3$ *and* $u_4u_5$.
(2−2−4) $u_j = \frac{1}{2}(u_4 + u_5)$ *and* $u_0 = \frac{1}{2}(u_i + u_6) \in N$.
(2−3) $(u_1u_2u_3u_4u_5 \cap N)\setminus\{u_1,u_2,u_3,u_4,u_5\} = \{u_6,\dots,u_s\} \subset u_iu_j$ $(s \geqq 6)$, $u_i = \frac{1}{2}(u_4 + u_5)$, $u_0 \in u_ju_ku_s$ *and* $u_0 \notin N$, *where* $u_6 = \frac{1}{2}(u_i + u_7),\dots$ *and* $u_s = \frac{1}{2}(u_{s-1} + u_j)$ $(s \geqq 7)$ *or* $u_6 = \frac{1}{2}(u_i + u_j)$ $(s = 6)$.
(2−4) $(u_1u_2u_3u_4u_5 \cap N)\setminus\{u_0,u_1,u_2,u_3,u_4,u_5\} = \{u_6,u_7,u_8\}$, $u_i = \frac{1}{2}(u_4 + u_5)$ *and* $u_0 = \frac{1}{2}(u_k + u_7) \in N$, *where* $u_6 = \frac{1}{3}(2u_i + u_j)$, $u_7 = \frac{1}{3}(u_i + 2u_j)$ *and* $u_8 = \frac{1}{2}(u_i + u_k)$.

*Proof of Theorem* 2.1. Let $\square = \{u \in \sigma \mid \langle \nu(\sigma),u\rangle = 1\}$. Then $\sigma = \mathbf{R}_{\geqq 0}\square$ and $u_0 \in \mathrm{Int}(\square)$. Moreover, $\partial\square$ has the natural polygonal decomposition $\{\tau \cap \square \mid \tau$ are faces of $\sigma$ with $\tau \neq \{0\}$, $\tau \neq \sigma\}$. Since the vertices of $\square$ belongs to $N$, we can take a triangulation $\Delta$ of $\partial\square$ so that $\Delta$ is a subdivision of the polygonal decomposition and that the set of vertices of $\Delta$ coincides with $\partial\square \cap N$. On the other hand, for each point $u$ on $\partial\square$, there exists the unique point $\hat{u}$ on $\partial\square$ with $u_0 \in u\hat{u}$. We denote by $\lambda(u)$, the simplex of $\Delta$ with $\hat{u} \in \mathrm{Int}(\lambda(u))$.
(I) Assume that there exists an element $u_1 \in \partial\square \cap N$ such that $\lambda(u_1)$ is a triangle of $\Delta$ and let $u_2$, $u_3$ and $u_4$ be the vertices of $\lambda(u_1)$. Then $u_0 \in \mathrm{Int}(u_1u_2u_3u_4)$. Hence $\square = u_1u_2u_3u_4$, because $(\sigma,u_0)$ is minimal. When $u_iu_j \cap N = \{u_i,u_j\}$, for any $\{i,j\} \subset \{1,2,3,4\}$, $(\sigma,u_0)$ is in the case of (1). Assume that $u_iu_j \cap N \neq \{u_i,u_j\}$, for a certain $\{i,j\} \subset \{1,2,3,4\}$. Then since $(\sigma,u_0)$ is minimal, we easily see that $u_iu_j \cap N = \{u_i,u_j,u_5\}$ and that $u_0 \in \mathrm{Int}(u_ku_lu_5)$, where $u_5 = \frac{1}{2}(u_i + u_j)$ and $\{i,j,k,l\} = \{1,2,3,4\}$. Hence $(\sigma,u_0)$ is in the case of (2).
(II) Assume that there exists an element $u_1 \in \partial\square \cap N$ such that $\lambda(u_1)$ is an edge of $\Delta$ and let $u_2$ and $u_3$ be the vertices on $\lambda(u_1)$. Then $u_0 \in \mathrm{Int}(u_1u_2u_3)$. Let $H = \mathbf{R}u_1 + \mathbf{R}u_2 + \mathbf{R}u_3 \subset N_{\mathbf{R}}$. Then there exist certain elements $u_4$ and $u_5$ in $\partial\square \cap N$ such that $u_4$, $u_5 \notin H$ and that $u_4u_5 \cap H \neq \phi$, because $\square$ is the convex hull of $\partial\square \cap N$, $\dim\square = 3$ and $u_0 \in \mathrm{Int}(\square)$. Since $(\sigma,u_0)$ is minimal, $\square = u_1u_2u_3u_4u_5$. We denote by $\tilde{u}$, the intersection point of $u_4u_5$ and $H$. Then $\tilde{u} = a_1u_1 + a_2u_2 + a_3u_3 = b_4u_4 + b_5u_5$, for certain real numbers $a_1$, $a_2$, $a_3$ and for certain positive real numbers $b_4$, $b_5$ with $a_1 + a_2 + a_3 = b_4 + b_5 = 1$.

**Lemma.** *One of the following holds.*
(i) $a_1, a_2, a_3 \geqq 0$.
(ii) $a_i < 0$, $a_j > 0$, $a_k > 0$ *and* $u_0 \in \mathrm{Int}(u_i\tilde{u})$, *where* $\{i,j,k\} = \{1,2,3\}$.
(iii) $a_i < 0$, $a_j > 0$ *and* $a_k = 0$, *where* $\{i,j,k\} = \{1,2,3\}$.

*Proof.* Suppose that $a_i$, $a_j < 0$, where $\{i,j,k\} = \{1,2,3\}$. Then $a_k > 0$, $u_k = (1/a_k)\tilde{u} + (-a_i/a_k)u_i + (-a_j/a_k)u_j$ and $1/a_k + (-a_i/a_k) + (-a_j/a_k) = 1$. Hence $u_k \in \mathrm{Int}(\sigma)$. It contradicts the fact that $u_k \in \partial\Box \cap N$. Therefore, it suffices to show that $u_0 \in \mathrm{Int}(u_i\tilde{u})$, if $a_i < 0$, $a_j > 0$, $a_k > 0$, where $\{i,j,k\} = \{1,2,3\}$. Since $u_0 \in \mathrm{Int}(u_1u_2u_3)$, we have $u_0 = c_1u_1 + c_2u_2 + c_3u_3$, for certain positive real numbers $c_1$, $c_2$ and $c_3$ with $c_1 + c_2 + c_3 = 1$. Suppose that $a_j/a_k < c_j/c_k$. Then $u_0 = ((c_ia_k - c_ka_i)/a_k)u_i + ((c_ja_k - c_ka_j)/a_k)u_j + (c_k/a_k)\tilde{u}$, $(c_ia_k - c_ka_i) + (c_ja_k - c_ka_j) + c_k = a_k$, $c_ja_k - c_ka_j > 0$, $c_ia_k - c_ka_i > 0$ and $c_k > 0$. Hence $u_0 \in \mathrm{Int}(u_iu_ju_4u_5)$ and $u_k \notin u_iu_ju_4u_5$. It contradicts the assumption that $(\sigma, u_0) \in \mathscr{C}_0^3$. Therefore, $a_j/a_k = c_j/c_k$. Thus we have $u_0 = ((c_ia_j - c_ja_i)/a_j)u_i + (c_j/a_j)\tilde{u} \in \mathrm{Int}(u_i\tilde{u})$. q.e.d.

*Proof of Theorem* 2.1 *continued.* When (i) in the above lemma holds, $\tilde{u} \in u_1u_2u_3$. Hence $(\sigma, u_0)$ is in the case of (2). When (ii) in the above lemma holds, $u_0 \in \mathrm{Int}(u_iu_4u_5)$ and $u_ju_k \cap u_iu_4u_5 \neq \emptyset$, because $(1/(a_j + a_k))\ (a_ju_j + a_ku_k) = (1/(b_4 + b_5 - a_i))\ (b_4u_4 + b_5u_5 - a_iu_i)$. Hence $(\sigma, u_0)$ is in the case of (2). When (iii) in the above lemma holds, $\tilde{u} = a_iu_i + a_ju_j$. Then $u_j = (1/a_j)\tilde{u} + (-a_i/a_j)u_i \in \tilde{u}u_i$. Hence $\Box = u_iu_ku_4u_5$. We already considered this case in (I).
(III) Assume that $\hat{u} \in N$, for all $u \in \partial\Box \cap N$. There exist certain elements $u_1$, $u_3$ and $u_5 \in \partial\Box \cap N$ such that $u_1 - u_0$, $u_3 - u_0$ and $u_5 - u_0$ are linearly independent, because $\dim \Box = 3$. Let $u_2 = \hat{u}_1$, let $u_4 = \hat{u}_3$ and let $u_6 = \hat{u}_5$. Then $u_0 \in \mathrm{Int}(u_1u_2)$, $\mathrm{Int}(u_3u_4)$, $\mathrm{Int}(u_5u_6)$ and hence $u_0 \in \mathrm{Int}(u_1u_2 \ldots u_6)$. Therefore, $\Box = u_1u_2 \ldots u_6$ and $\Box \cap N \subset \{u_0, u_1, \ldots, u_6\}$, because $(\sigma, u_0)$ is minimal. Then $(\sigma, u_0)$ is in the case of (3). q.e.d.

*Proof of Proposition* 2.2. We may only consider the case that $(u_1u_2u_3u_4 \cap N) \setminus \{u_0, u_1, u_2, u_3, u_4\} \neq \emptyset$.
(I) Assume that there exists an element $u \in \mathrm{Int}(u_1u_2u_3) \cap N$ with $u \neq \hat{u}_4$, where $\hat{u}_4$ is the point in $\partial(u_1u_2u_3u_4)$ with $u_0 \in \mathrm{Int}(u_4\hat{u}_4)$. Then $\hat{u}_4 \in u_1u$, $u_2u$ or $u_3u$, because $(\sigma, u_0)$ is minimal. Hence we may assume that $\hat{u}_4 \in u_3u$. Let $H = \boldsymbol{R}u + \boldsymbol{R}u_3 + \boldsymbol{R}u_4$. Then $(u_1u_2u_3u_4 \cap N) \setminus \{u_1, u_2\} \subset H$, because $u_0 \in \mathrm{Int}(uu_3u_4)$. On the other hand, $u_1u_2$ intersect $H$ at a point, which we denote by $\tilde{u}$. Then $u_1u_2u_3 \cap H = \tilde{u}u_3$. We may assume that $\tilde{u}u_3 \cap N = \{u_3, u_5, \ldots, u_s\}$, where $u_s = \frac{1}{2}(u_3 + u_{s-1})$, $u_{s-1} = \frac{1}{2}(u_s + u_{s-2}), \ldots$ and $u_6 = \frac{1}{2}(u_7 + u_5)$. Then $u_0 \in u_3u_4u_s$. Otherwise, $u_0 \in \mathrm{Int}(u_1u_2u_4u_s)$. Moreover, $u_5 = \frac{1}{3}(u_1 + u_2 + u_6)$, because $u_5 \in \mathrm{Int}(u_1u_2u_6)$ and $u_1u_2u_6 \cap N = \{u_1, u_2, u_5, u_6\}$. Hence $\tilde{u} = \frac{1}{2}(u_1 + u_2)$. In the following, we show that $\tilde{u}u_4 \cap N = \{u_4\}$, i.e., $\partial(u_1u_2u_3u_4) \cap N = \{u_1, u_2, u_3, u_4, \ldots, u_s\}$.
(I–i) Assume that $u_0 \notin N$. Since $u_3u_4u_s \cap N = \{u_3, u_4, u_s\}$, $\{u_4 - u_3,\ u_s - u_3\}$

is a basis of $(\boldsymbol{R}(u_4-u_3)+\boldsymbol{R}(u_s-u_3))\cap N$. On the other hand, any point $u'$ on $\tilde{u}u_4$ is expressed as $u'=au_4+b\tilde{u}$, for certain non-negative real numbers $a$ and $b$ with $a+b=1$. Hence $u'-u_3=a(u_4-u_3)+b(\tilde{u}-u_3)=a(u_4-u_3)+b((s-4+\frac{1}{2})\ (u_s-u_3))$. Therefore, $\tilde{u}u_4\cap N=\{u_4\}$.
(I−ii) Assume that $u_0\in N$. If $u_0\in u_4u_s$ (resp. $u_0\in \mathrm{Int}\,(u_3u_4u_s)$), then $u_0=\frac{1}{2}(u_4+u_s)$ (resp. $u_0=\frac{1}{3}(u_3+u_4+u_s)$). Hence $s\leqq 6$ (resp. $s=5$). Otherwise, $\frac{1}{2}(u_4+u_{s-2})=\frac{1}{2}(2u_0-u_s+3u_s-2u_3)=u_0+u_s-u_3\in N\cap \mathrm{Int}\,(\sigma)$ (resp. $\frac{1}{3}(2u_4+u_{s-1})=\frac{1}{3}(2(3u_0-u_3-u_s)+2u_s-u_3)=2u_0-u_3\in N\cap \mathrm{Int}\,(\sigma)$). Then we easily see that $\tilde{u}u_4\cap N=\{u_4\}$.
(II) Assume that $u_1u_2u_3u_4\cap N\subset\{u_0,u_1,u_2,u_3,u_4,\hat{u}_1,\hat{u}_2,\hat{u}_3,\hat{u}_4\}$ and that $\hat{u}_4\in N$. Then $\hat{u}_4=\frac{1}{3}(u_1+u_2+u_3)$. Suppose that $\hat{u}_3\in N$. Then $\hat{u}_3=\frac{1}{3}(u_1+u_2+u_4)$. Hence $\hat{u}_4-\hat{u}_3=\frac{1}{3}(u_3-u_4)$. Then $u_3+\hat{u}_3-\hat{u}_4=u_3+\frac{1}{3}(u_4-u_3)=\frac{2}{3}u_3+\frac{1}{3}u_4\in \mathrm{Int}\,(u_3u_4)\cap N$. This contradicts the assumption that $u_3u_4\cap N=\{u_3,u_4\}$. Therefore, $u_1u_2u_3u_4\cap N\subset\{u_0,u_1,u_2,u_3,u_4,\hat{u}_4\}$. q.e.d.

*Proof of Proposition* 2.3. We easily see that $(u_1u_2\ldots u_5\cap N)\setminus\{u_4,u_5\}=u_1u_2u_3\cap N$, because $(\sigma,u_0)$ is minimal and $u_0\in \mathrm{Int}\,(u_1u_2u_3)$. We may only consider the case that $\partial(u_1u_2u_3)\cap N\neq\{u_1,u_2,u_3\}$. Let $\tilde{u}$ be the intersection point of $u_1u_2u_3$ and $u_4u_5$.
(I) Assume that $\tilde{u}=u_1$.
(I−i) If $u_2u_3\cap N\neq\{u_2,u_3\}$, then $u_2u_3\cap N=\{u_2,u_3,u_6\}$ and $u_0\in u_1u_6$, where $u_6=\frac{1}{2}(u_2+u_3)$. Hence $\partial u_1u_2u_3\cap N=\{u_1,u_2,u_3,u_6\}$, because $(\sigma,u_0)$ is minimal. Then $(\sigma,u_0)$ is in the case of (2−2−1).
(I−ii) If $u_2u_3\cap N=\{u_2,u_3\}$, then $\mathrm{Int}\,(u_1u_2)\cap N\neq\emptyset$ or $\mathrm{Int}\,(u_1u_3)\cap N\neq\emptyset$. Hence we may assume that $\mathrm{Int}\,(u_1u_2)\cap N=\{u_6,u_7,\ldots,u_s\}$ $(s\geqq 6)$, where $u_6=\frac{1}{2}(u_1+u_7)=\frac{1}{2}(u_6+u_8),\ldots,u_s=\frac{1}{2}(u_{s-1}+u_2)$. Then $u_0\in u_su_2u_3$, because $(\sigma,u_0)$ is minimal.
(I−ii−a) When $u_0\notin N$, $u_2u_3u_s\cap N=\{u_2,u_3,u_s\}$. Hence $\{u_3-u_2,\ u_s-u_2\}$ is a basis of $(\boldsymbol{R}(u_3-u_2)+\boldsymbol{R}(u_s-u_2))\cap N$. On the other hand, any point $u'$ on $u_1u_3$ is expressed as $u'=au_1+bu_3$ for certain non-negative real numbers $a$ and $b$ with $a+b=1$. Hence $u'-u_2=a(u_1-u_2)+b(u_3-u_2)=a(s-4)\ \ (u_s-u_2)+b(u_3-u_2)$. Therefore, $u_1u_3\cap N=\{u_1,u_3\}$. Then $(\sigma,u_0)$ is in the case of (2−3).
(I−ii−b) When $u_0\in N$ and $u_0\in u_3u_s$, we have $s=6$ or $7$. Otherwise $\frac{1}{2}(u_{s-2}+u_3)=\frac{1}{2}(2u_{s-1}-u_s+2u_0-u_s)=u_0+u_{s-1}-u_s\in N\cap \mathrm{Int}\,(\sigma)$. If $s=6$, then $u_6=\frac{1}{2}(u_1+u_2)$ and $u_0=\frac{1}{2}(u_6+u_3)$. Then $(\sigma,u_0)$ is in the case of (2−2−4). If $s=7$, then $u_6=\frac{1}{2}(u_1+u_7)$, $u_7=\frac{1}{2}(u_6+u_2)$, $u_0=\frac{1}{2}(u_7+u_3)$ and $u_8:=\frac{1}{2}(u_1+u_3)=\frac{1}{2}(2u_6-u_7+2u_0-u_7)=u_0+u_6-u_7\in N$. Then $(\sigma,u_0)$ is in the case of (2−4).
(I−ii−c) When $u_0\in N$ and $u_0\in \mathrm{Int}\,(u_2u_3u_s)$, we have $u_0=\frac{1}{3}(u_2+u_3+u_s)$ and $s=6$. Otherwise $\frac{1}{3}(2u_3+u_{s-1})=\frac{1}{3}(2(3u_0-u_2-u_s)+(2u_s-u_2))=2u_0-u_2\in N\cap \mathrm{Int}\,(\sigma)$. Hence $u_6=\frac{1}{2}(u_1+u_2)$, $u_7:=\frac{1}{3}(2u_3+u_1)=2u_0-u_2$, $u_8:=\frac{1}{3}(u_3+2u_1)=u_0+u_6-u_2\in N$ and $u_0=\frac{1}{2}(u_2+u_7)$. Then $(\sigma,u_0)$ is in the case of (2−4).
(II) Assume that $\tilde{u}\in \mathrm{Int}\,(u_2u_3)$. Then $u_0\in u_1\tilde{u}$. Otherwise, $u_0\in \mathrm{Int}\,(u_1u_2u_4u_5)$

or Int $(u_1u_3u_4u_5)$. Moreover, $(u_1u_2u_3 \cap N) \setminus \{u_0,u_1,u_2,u_3\} = \{\tilde{u}\}$, because $(\sigma,u_0)$ is minimal. Therefore, $\tilde{u} = \frac{1}{2}(u_2 + u_3) = \frac{1}{2}(u_4 + u_5) \in N$. Then $(\sigma,u_0)$ is in the case of (2−2−2).

(III) Assume that $\tilde{u} \in$ Int $(u_1u_2u_3)$. If there exists a lattice point $u_6$ on Int$(u_2u_3)$, then $u_0 \in u_6u_1$ and hence $\tilde{u} \in u_6u_1$. Otherwise, $u_1u_2u_4u_5u_6$ or $u_1u_3u_4u_5u_6$ contains $u_0$ in the interior and is strictly contained in $u_1u_2u_3u_4u_5$. Moreover, we see that $u_0 \in u_1\tilde{u}$ and $u_1u_2u_3 \cap N \subset \{u_0,u_1,u_2,u_3,u_6\}$, because $(\sigma,u_0)$ is minimal. Then $(\sigma,u_0)$ is in the case of (2−2−3) q.e.d.

## §3. Representatives of the pairs in $\mathscr{C}_0^3$

In this section, we need the following lemmas, which are easily obtained from the terminal lemma [4] (see also [9, Lemma 3.6]).

**Lemma 3.1.** *Assume that $u_1$, $u_2$, $u_3$ and $u_4$ are elements in $N$ with $u_1u_2u_3u_4 \cap N = \{u_1,u_2,u_3,u_4\}$. Moreover, assume that $u_1$, $u_2$, $u_3$ and $u_4$ are linearly independent and that there exists an element $v_0$ in $N^*$ such that $\langle v_0,u_i\rangle = 1$ for $i = 1$ through 4. Then there exists an element $g$ in $GL(N)$ such that $g(u_1) = (0,0,0,1)$, that $g(u_2) = (1,0,0,1)$, that $g(u_3) = (0,1,0,1)$ and that $g(u_4) = (o,p,q,1)$, where $o, p, q \in Z$, $0 \leqq o, p < q$, $g.c.d.(o,q) = g.c.d.(p,q) = 1$ and $o = 1, p = 1$ or $o + p = q$.*

**Lemma 3.2.** *Under the same assumptions as in Lemma 3.1, there exists an element $g$ in $GL(N)$ such that $g(u_i) = (0,0,0,1)$, that $g(u_j) = (1,0,0,1)$, that $g(u_k) = (0,1,0,1)$ and that $g(u_l) = (1,p,q,1)$, where $\{i,j,k,l\} = \{1,2,3,4\}$, $0 \leqq p \leqq \frac{q}{2}$ and $g.c.d.(p,q) = 1$. Moreover, we may assume that $i = 1$ or $l = 4$.*

Assume that $\sigma = \boldsymbol{R}_{\geqq 0}u_1 + \boldsymbol{R}_{\geqq 0}u_2 + \dots + \boldsymbol{R}_{\geqq 0}u_t$ is a 4-dimensional strongly convex rational cone in $N_{\boldsymbol{R}}$ satisfying the condition (G). Moreover, assume that $\boldsymbol{R}_{\geqq 0}u_i$ is a 1-dimensional face of $\sigma$, for $i = 1$ through $t$.

**Proposition 3.3.** *Assume that $t = 4$, that $\{u \in \sigma \cap N \mid \langle v(\sigma),u\rangle = 1\} = \{u_1,u_2,u_3,u_4\}$ and that $W_\sigma(v(\sigma)) \neq \phi$. Then there exists an element $g$ in $GL(N)$ such that $g(u_i) = (0,0,0,1)$, that $g(u_j) = (1,0,0,1)$, that $g(u_k) = (0,1,0,1)$ and that $g(u_l) = (1,p,q,1)$, where $\{i,j,k,l\} = \{1,2,3,4\}$ and $(p,q) = (0,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,5)$, $(2,7)$, $(3,7)$, $(3,8)$ or $(3,10)$.*

*Proof.* By Lemma 3.2, there exists an element $g$ in $GL(N)$ such that $g(u_i) = (0,0,0,1)$, that $g(u_j) = (1,0,0,1)$, that $g(u_k) = (0,1,0,1)$ and that $g(u_l) = (1,p,q,1)$, where $\{i,j,k,l\} = \{1,2,3,4\}$, $p$, $q \in Z$, $0 \leqq p \leqq q/2$ and $g.c.d.(p,q) = 1$. We may only consider the case that $i = 1$, $j = 2$, $k = 3$ and $l = 4$. When $q \leqq 4$, $(p,q) = (0,1), (1,2), (1,3)$ or $(1,4)$. In these cases, we easily see that $W_\sigma(v(\sigma)) \neq \phi$. In the following, we consider the case that $q > 4$. Then $0 < p < \frac{q}{2}$. Let $v_1 =$

$\frac{1}{q}(0,0,1,0)$, let $v_2=\frac{1}{q}(0,q,-p,0)$, let $v_3=\frac{1}{q}(q,0,-1,0)$, let $v_4=\frac{1}{q}(-q,-q,p,q)$ and let $u_0=\frac{1}{4}(2,p+1,q,4)$. Then $(g\sigma)^*=\boldsymbol{R}_{\geqq 0}v_1+\boldsymbol{R}_{\geqq 0}v_2+\boldsymbol{R}_{\geqq 0}v_3+\boldsymbol{R}_{\geqq 0}v_4$, $u_0\in \mathrm{Int}(g\sigma)$, $\langle v_1,u_0\rangle=\langle v_2,u_0\rangle=\langle v_3,u_0\rangle=\langle v_4,u_0\rangle=\frac{1}{4}$ and $\langle v_0,u_0\rangle=1$, where $v_0=g(v(\sigma))$ $(=(0,0,0,1))$.

*Step* 1. First, we examine when $u_0\in W_{g\sigma}(v_0)=g(W_\sigma(v(\sigma)))$. Assume that $v:=av_1+bv_2+cv_3+dv_4\in N^*$. Then $c-d$, $b-d$, $(a-bp-c+dp)/q$, $d\in \boldsymbol{Z}$. Hence $a$, $b$, $c$, $d\in Z$ and $a-bp-c+dp\equiv 0 \pmod q$. By an easy calculation, we obtain:

**Lemma 3.3.1.** (0) $L_0:=\{v_1+v_2+v_3+v_4,\ 2v_1+2v_3,\ 2v_2+2v_4\}\subset N^*$.
(1) *For* $v\in L_1:=\{2v_1+2v_2,\ 2v_3+2v_4,\ v_1+2v_2+v_4,\ 2v_1+v_2+v_3,\ v_1+2v_3+v_4,\ v_2+v_3+2v_4\}$,
$v\in N^*$ *if and only if* $p=1$.
(2) *For* $v\in L_2:=\{v_1+3v_2,\ v_3+3v_4\}$,
$v\in \mathrm{N}^*$ *if and only if* $3p-1=q$.
(3) *For* $v\in L_3:=\{3v_1+v_2,\ 3v_3+v_4\}$,
$v\in N^*$ *if and only if* $p=3$.
(4) *For* $v\in L_4:=\{3v_1+v_4,\ v_2+3v_3\}$,
$v\in N^*$ *if and only if* $p+3=q$.
(5) *For* $v\in L_5:=\{2v_1+2v_4,\ 2v_2+2v_3\}$,
$v\in N^*$ *if and only if* $2p+2=q$.
(6) *For* $v\in L_6:=\{v_1+3v_4,\ 3v_2+v_3\}$,
$v\in N^*$ *if and only if* $3p+1=q$.
$\{v\in(g\sigma)^*\cap N^*\mid \langle v,u_0\rangle=1\}\subset L_0\cup L_1\cup\ldots\cup L_6$. *Moreover,* $L_0\cup L_i$ *is contained in a sublinear space of* $N_{\boldsymbol{R}}^+$, *for each* $1\leqq i\leqq 6$.

If $u_0\in W_{g\sigma}(v_0)$, then at least two conditions in (1) ~ (6) of the above lemma hold at the same time. When $(p,q)=(2,5)$, the conditions in (2) and (4) hold. When $(p,q)=(3,8)$, the conditions in (2), (3) and (5) hold. When $(p,q)=(3,10)$, the conditions in (3) and (6) hold. In fact, in these cases, $u_0\in W_{g\sigma}(v_0)$. For all pairs $(p,q)$ with g.c.d.$(p,q)=1$ except the above ones, any two conditions in (1) ~ (6) do not hold at the same time.

*Step* 2. Next, we examine when $W_{g\sigma}(v_0)\setminus\{u_0\}\neq\emptyset$. Then there should exist an element $v\in(g\sigma)^*\cap N^*$ such that $\langle v,u_0\rangle<1$ and that $v_0$ and $v$ are linearly independent, as we see in the proof of Theorem 1.7. We easily obtain:

**Lemma 3.3.2.** (0) $L_0':=\{v_1+v_3,\ v_2+v_4\}\subset N^*$.
(1) *For* $v\in L_1':=\{v_1+v_2,\ v_3+v_4\}$,
$v\in N^*$ *if and only if* $p=1$.
(2) *For* $v\in L_2':=\{2v_1+v_2,\ 2v_3+v_4\}$,
$v\in N^*$ *if and only if* $p=2$.
(3) *For* $v\in L_3':=\{v_1+2v_4,\ 2v_2+v_3\}$,

$v \in N^*$ *if and only if* $2p+1=q$.
$\{v \in (g\sigma)^* \cap N^* \mid 0 < \langle v,u_0\rangle < 1\} \subset L'_0 \cup L'_1 \cup L'_2 \cup L'_3$.

If $\langle v_0,u\rangle = \langle v_i+v_j,u\rangle = 1$, for an element $u$ in $N_{\boldsymbol{R}}$, then $\langle v_k+v_l,\ u\rangle = 0$ and hence $u \notin \mathrm{Int}(g\sigma)$, because $v_0 = v_1+v_2+v_3+v_4$, where $\{i,j,k,l\} = \{1,2,3,4\}$. Therefore, if $u \in W_{g\sigma}(v_0) \setminus \{u_0\}$, then $\langle v,u\rangle = 1$, for an element $v$ in $(L'_2 \cup L'_3) \cap N^*$ and hence $p=2$ or $2p+1=q$.

*Step* 2−i. Assume that $p=2$ and that there exists an element $u$ in $W_{g\sigma}(v_0)$ with $\langle 2v_3+v_4,\ u\rangle = 1$. Let $u_1 = \frac{1}{6}(3,3,q,6)$. Then $u_1 \in \mathrm{Int}(g\sigma)$, $\langle v_1,u_1\rangle = \langle v_2,u_1\rangle = \frac{1}{6}$, $\langle v_3,u_1\rangle = \langle v_4,u_1\rangle = \frac{1}{3}$ and hence $\langle v_0,u_1\rangle = \langle 2v_3+v_4,u_1\rangle = 1$. Since $\langle v_0,(0,0,1,0)\rangle = 0$, we see that $u_1 \notin W_{g\sigma}(v_0)$, by the following lemma, which and the next lemma we obtain by an easy calculation.

**Lemma 3.3.3.** $\{v \in (g\sigma)^* \cap N^* \mid \langle v,u_1\rangle = 1\} \subset \{v \in N^*_{\boldsymbol{R}} \mid \langle v,(0,0,1,0)\rangle \leqq 0\}$.

**Lemma 3.3.4.** (0) $M_0 := \{v_2+v_4,\ v_1+v_3,\ 2v_1+v_2\} \subset N^*$.
(1) *For* $v \in M_1 := \{2v_2+v_3,\ v_1+2v_4,\ v_1+3v_2,\ 3v_1+v_4,\ 5v_1,\ 5v_2\}$, $v \in N^*$ *if and only if* $q=5$.
(2) *For* $v \in M_2 := \{v_1+4v_2,\ 3v_2+v_3\}$,
$v \in N^*$ *if and only if* $q=7$.
$\{v \in (g\sigma)^* \cap N^* \mid 0 < \langle v,u_1\rangle < 1\} \subset M_0 \cup M_1 \cup M_2$.

Assume that $\langle 2v_1+v_2,\ u\rangle = \langle 2v_3+v_4,\ u\rangle = \langle v_0,\ u\rangle = 1$, for an element $u$ in $N_{\boldsymbol{R}}$. Then $\langle v_2+v_4,\ u\rangle = 0$. Hence $u \notin \mathrm{Int}(g\sigma)$. Therefore, $q=5$ or $q=7$, by the above lemma.

*Step* 2−ii. Assume that $p=2$ and that there exists an element $u$ in $W_{g\sigma}(v_0)$ such that $\langle 2v_1+v_2,\ u\rangle = 1$. Then we obtain the same results as in Step 2−i, by the same way, letting $u_1 = \frac{1}{6}(3,\ 6,\ 2q,\ 6)$.

*Step* 2−iii. Assume that $q=2p+1$ and that there exists an element $u$ in $W_{g\sigma}(v_0)$ with $\langle 2v_2+v_3,\ u\rangle = 1$. Let $u_1 = \frac{1}{6}(3,\ p+2,\ q,\ 6)$. Then $u_1 \in \mathrm{Int}(g\sigma)$ and $\langle v_0,u_1\rangle = \langle 2v_2+v_3,\ u_1\rangle = 1$. Since $\langle v_0,\ (0,1,2,0)\rangle = 0$, we see that $u_1 \notin W_{g\sigma}(v_0)$, by the following lemma, which and the next lemma we obtain by an easy calculation.

**Lemma 3.3.5.** $\{v \in (g\sigma)^* \cap N^* \mid \langle v,u_1\rangle = 1\} \subset \{v \in N^{+}_{\boldsymbol{R}} \mid \langle v,(0,1,2,0)\rangle \geqq 0\}$.

**Lemma 3.3.6.** (0) $K_0 := \{v_2+v_4,\ v_1+2v_4,\ v_1+v_3\} \subset N^*$.
(1) *For* $v \in K_1 := \{5v_4,\ v_3+3v_4,\ 2v_3+v_4,\ 2v_1+v_2,\ 3v_1+v_4,\ 5v_1\}$, $v \in N^*$ *if and only if* $p=2,\ q=5$.
(2) *For* $v \in K_2 := \{3v_1+v_2,\ 4v_1+v_4\}$,
$v \in \mathrm{N}^*$ *if and only if* $p=3,\ q=7$.
$\{v \in (g\sigma)^* \cap N^* \mid 0 < \langle v,u_1\rangle < 1\} \subset K_0 \cup K_1 \cup K_2$.

Assume that $\langle v_1+2v_4,\ u\rangle=\langle 2v_2+v_3,\ u\rangle=\langle v_0,\ u\rangle=1$, for an element $u$ in $N_{\mathbf{R}}$. Then $\langle v_1+v_3,\ u\rangle=0$. Hence $u\notin \operatorname{Int}(g\sigma)$. Therefore, $(p,q)=(2,5)$ or $(3,7)$, by the above lemma.

*Step* 2–iv. Assume that $q=2p+1$ and that there exists an element $u$ in $W_{g\sigma}(v_0)$ such that $\langle v_1+2v_4,\ u\rangle=1$. Then we obtain the same results as in Step 2–iii, by the same way, letting $u_1=\frac{1}{6}(3,\ q,\ 2q,\ 6)$. q.e.d.

**Proposition 3.4.** *Assume that* $t=4$ *and that* $\{u\in\sigma\cap N \mid \langle v(\sigma),u\rangle=1\}\setminus\{u_1,u_2,u_3,u_4\}$ *consists of only one point* $u_0$ *which is in* $\operatorname{Int}(\sigma)$. *Then there exists an element g in GL(N) such that* $g(u_i)=(1,0,0,1)$, *that* $g(u_j)=(0,1,0,1)$ *and that* $(g(u_k),g(u_l))=((0,0,1,1),(-1,-1,-1,1))$, $((1,1,2,1),(-1,-1,-1.1))$, $((1,1,3,1),(-1,-1,-2,1))$, $((1,2,5,1),(-1,-1,-1,1))$, $((1,2,5,1),(-1,-1,-2,1))$, $((1,2,5,1),(-2,-3,-5,1))$, $((1,2,7,1),(-1,-1,-2,1))$ *or* $((1,3,7,1),(-1,-2,-3,1))$, *where* $\{i,j,k,l\}=\{1,2,3,4\}$.

*Proof.* We may only consider the case that $|\det(u_0,u_1,u_2,u_3)|\geqq|\det(u_0,u_1,u_2,u_4)|$, $|\det(u_0,u_1,u_3,u_4)|$, $|\det(u_0,u_2,u_3,u_4)|$. Since $u_0u_1u_2u_3\cap N=\{u_0,u_1,u_2,u_3\}$, by Lemma 3.2, there exists an element $g$ in GL(N) such that $g(u_0)=(0,0,0,1)$, that $g(u_i)=(1,0,0,1)$, that $g(u_j)=(0,1,0,1)$ and that $g(u_k)=(1,p,q,1)$, where $p,\ q\in\mathbf{Z}, q>0$, $0\leqq p\leqq\frac{q}{2}$ and g.c.d.$(p,q)=1$. Here, we may only consider the case that $i=1$, $j=2$ and $k=3$. Since $u_0\in\operatorname{Int}(u_1u_2u_3u_4)$ and $\sigma$ satisfies the condition *(G)*, $g(u_4)=(-s,-t,-u,1)$, for certain positive integers $s$, $t$ and $u$. Let $\tilde{u}=\dfrac{1}{q+u}(qg(u_4)+ug(u_3))=\dfrac{1}{q+u}(u-sq,pu-tq,0,q+u)$. Since $g(u_0)\in\operatorname{Int}(\tilde{u}g(u_1)g(u_2))$, we have $u-sq<0$, $pu-tq<0$. On the other hand, we have $|\det(u_0,u_1,u_2,u_4)|=u\leqq q$, $|det(u_0,u_2,u_3,u_4)|=|sq-u|\leqq q$ and $|\det(u_0,u_1,u_3,u_4)|=|pu-tq|\leqq q$. Thus we obtain the following inequalities.

(1) $1\leqq u\leqq q$, $\frac{u}{q}<s\leqq\frac{u}{q}+1$, $p\frac{u}{q}<t\leqq p\frac{u}{q}+1$.

When q $=1$, we have $p=0$, $u=1$, $s=2$ and $t=1$. Namely, $g(u_3)=(1,0,1,1)$ and $g(u_4)=(-2,-1,-1,1)$. Next, we consider the case that $q\geqq 2$.

(I) Assume that $u=q$. By the inequalities (1), we have $s=2$ and $t=p+1$. Since

$$\begin{bmatrix}1-q & 0 & 1 & 0\\ -p & 1 & 0 & 0\\ -q & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 1 & -2\\ 0 & 1 & p & -p-1\\ 0 & 0 & q & -q\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & q-2\\ 0 & 1 & 0 & p-1\\ 0 & 0 & 0 & q\\ 1 & 1 & 1 & 1\end{bmatrix}\ \text{and}\ \begin{bmatrix}1-q & 0 & 1 & 0\\ -p & 1 & 0 & 0\\ -q & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\in GL(N),$$

$q-2=1$, $p-1=1$ or $(q-2)+(p-1)=q$. Hence $q=3$, $p=2$ or $p=3$. Since

$$\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & -2\\ 0 & 0 & 1 & -p-1\\ 0 & 0 & 0 & -q\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & q-2\\ 0 & 0 & 1 & q-p-1\\ 0 & 0 & 0 & q\\ 1 & 1 & 1 & 1\end{bmatrix}\ \text{and}\ \begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\in GL(N),$$

$q-2=1$, $q-p-1=1$ or $(q-2)+(q-p-1)=q$. Hence $q=3$, $q=p+2$ or $q=p+3$.

(i) When $q=3$, we have $p=1$ and $t=2$. Then the convex set $g(u_3u_4)$ contains the lattice point $(0,0,1,1)=\frac{2}{3}g(u_3)+\frac{1}{3}g(u_4)$. It contradicts the assumptions that $AL=\emptyset$.

(ii) When $p=2$ and $q=p+2$, we have g.c.d.$(p,q)\neq 1$.

(iii) when $p=2$ and $q=p+3$, we have $q=5$ and $t=3$. Hence $g(u_3)=(1,2,5,1)$ and $g(u_4)=(-2,-3,-5,1)$.

(iv) When $p=3$ and $q=p+2$, we have $p>\frac{q}{2}$.

(v) When $p=3$ and $q=p+3$, we have g.c.d.$(p,q)\neq 1$.

(II) Assume that $u<q$. By the inequalities (1), we have $s=1$ and $0<t\leqq p,u$. Since

$$\begin{bmatrix}1&0&-1&0\\0&1&-1&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&1&0&-1\\0&0&1&-t\\0&0&0&-u\\1&1&1&1\end{bmatrix}=\begin{bmatrix}0&1&0&u-1\\0&0&1&u-t\\0&0&0&u\\1&1&1&1\end{bmatrix}\text{ and }\begin{bmatrix}1&0&-1&0\\0&1&-1&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}\in GL(N),$$

$u=1$, $u-1=1$, $u-t=1$ or $(u-1)+(u-t)=u$. Hence $u=t=1$, $u=2$ or $u=t+1$. Since

$$\begin{bmatrix}1-q&0&1&0\\-p&1&0&0\\-q&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&1&-1\\0&1&P&-t\\0&0&q&-u\\1&1&1&1\end{bmatrix}=\begin{bmatrix}0&0&1&q-u-1\\0&1&0&p-t\\0&0&0&q-u\\1&1&1&1\end{bmatrix},\ \begin{bmatrix}1-q&0&1&0\\-p&1&0&0\\-q&0&1&0\\0&0&0&1\end{bmatrix}\in GL(N)$$

and $0\leqq p-t=p\frac{u}{q}-t+(1-\frac{u}{q})\ p<(1-\frac{u}{q})\ p=(q-u)\ \frac{p}{q}<q-u$, we have $q-u=1$, $q-u-1=1$, $p-t=1$ or $(q-u-1)+(p-t)=q-u$. Hence $q=u+1$, $q=u+2$ or $p=t+1$.

(i) When $u=t=1$ and $q=u+1$, we have $q=2$ and $p=1$. Hence $g(u_3)=(1,1,2,1)$ and $g(u_4)=(-1,-1,-1,1)$.

(ii) When $u=t=1$ and $q=u+2$, we have $q=3$ and $p=1$. Then $g(u_3u_4)$ contains the lattice point $(0,0,1,1)=\frac{1}{2}g(u_3)+\frac{1}{2}g(u_4)$.

(iii) When $u=t=1$ and $p=t+1$, we have $p=2$ and $q\geqq 5$. Since

$$\begin{bmatrix}1&-2&1&0\\0&-2&1&0\\0&-1&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&1&-1&1\\0&0&-1&2\\0&0&-1&q\\1&1&1&1\end{bmatrix}=\begin{bmatrix}0&1&0&q-3\\0&0&1&q-4\\0&0&0&q-2\\1&1&1&1\end{bmatrix}\text{ and }\begin{bmatrix}1&-2&1&0\\0&-2&1&0\\0&-1&1&0\\0&0&0&1\end{bmatrix}\in GL(N),$$

we have $q-2=1$, $q-3=1$, $q-4=1$ or $(q-3)+(q-4)\in(q-2)\boldsymbol{Z}$. Hence $q=5$, because $q\geqq 5$. Therefore, $g(u_3)=(1,2,5,1)$ and $g(u_4)=(-1,-1,-1,1)$.

(iv) when $u=2$ and $q=u+1$, we have $q=3$, $p=1$ and $t=1$, by the third inequality in (1). Hence $g(u_3)=(1,1,3,1)$ and $g(u_4)=(-1,-1,-2,1)$.

(v) When $u=2$ and $q=u+2$, we have $q=4$, $p=1$ and $t=1$, by the third

inequality in (1). Then $g(u_3u_4)$ contains the lattice point $(0,0,1,1)=\frac{1}{2}g(u_3)+\frac{1}{2}g(u_4)$.
(vi) When $u=2$ and $p=t+1$, we have $t=1$ or 2, because $0<t\leqq u$. Suppose that $t=2$. Then $g(u_1u_4)$ contains the lattice point $(0,-1,-1,1)=\frac{1}{2}g(u_1)+\frac{1}{2}g(u_4)$. Hence $t=1$, $g(u_4)=(-1,-1,-2,1)$, $p=2$ and $q\geqq 5$. Since

$$\begin{bmatrix}1 & -3 & 1 & 0\\ 0 & -3 & 1 & 0\\ 0 & -2 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & -1 & 1\\ 0 & 0 & -1 & 2\\ 0 & 0 & -2 & q\\ 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & q-5\\ 0 & 0 & 1 & q-6\\ 0 & 0 & 0 & q-4\\ 1 & 1 & 1 & 1\end{bmatrix} \text{ and } \begin{bmatrix}1 & -3 & 1 & 0\\ 0 & -3 & 1 & 0\\ 0 & -2 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\in GL(N),$$

we have $q-4=1$, $q-5=1$, $q-6=1$ or $(q-5)+(q-6)\in(q-4)\mathbf{Z}$. Hence $q=5$ or $q=7$, because $q\geqq 5$ and g.c.d.$(2,q)=1$. If $q=5$, then $g(u_3)=(1,2,5,1)$. If $q=7$, then $g(u_3)=(1,2,7,1)$.
(vii) When $u=t+1$ and $q=u+1$, we have $q\geqq 3$. By the third inequality in (1), we have $(q-2)q\leqq p(q-1)+q$. Hence $p\geqq q-2q/(q-1)=q-(2+2/(q-1))$. Therefore, $q-p\leqq 3$ with the equality holds only if $q=3$. When $q=3$, we have $u=2$, $t=1$ and $p=1$, because $p\leqq\frac{q}{2}$. Hence $g(u_3)=(1,1,3,1)$ and $g(u_4)=(-1,-1,-2,1)$. When $q>3$, we have $q-p\leqq 2$. Then since $q-2\leqq p\leqq\frac{q}{2}$, we have $q\leqq 4$. Hence $q=4$ and $p=2$. However, then g.c.d.$(p,q)\neq 1$.
(viii) when $u=t+1$ and $q=u+2$, we have $q\geqq 4$. Hence $p-t=1$, because $0\leqq p-t<q-t=3$ and g.c.d.$(p-t,\ q-u)=1$. Then $\frac{q}{2}\geqq p=t+1=q-2$. Hence $q=4$ and $p=2$. However, then g.c.d.$(p,q)\neq 1$.
(ix) When $u=t+1=p$, we have $p\geqq 2$. If $p=2$, then $u=2$. This case was already considered in (vi). Therefore, we may assume that $p\geqq 3$. By the third inequality in (1), we have $(p-1)q\leqq p^2+q$. Hence $p^2\geqq(p-2)q\geqq 2(p-2)p$. Therefore, $p=3$ or $p=4$. When $p=3$, we have $u=3$, $t=2$ and $q=7$ or $q=8$, because g.c.d.$(p,q)=1$, $p\leqq\frac{q}{2}$ and $p^2\geqq(p-2)q$. If $q=8$, then $g(u_0u_1u_3u_4)$ contains the lattice point $(0,0,1,1)=\frac{1}{7}g(u_0)+\frac{1}{7}g(u_1)+\frac{2}{7}g(u_3)+\frac{3}{7}g(u_4)$. If $q=7$, then $g(u_3)=(1,3,7,1)$ and $g(u_4)=(-1,-2,-3,1)$. When $p=4$, we have $q=8$, because $p\leqq\frac{q}{2}$ and $p^2\geqq(p-2)q$. However, then g.c.d.$(p,q)\neq 1$. q.e.d.

For the cases (1−2) through (3) in §2, it is easier to obtain similar propositions as Propositions 3.3 and 3.4, by Lemmas 3.1 and 3.2. Hence we only give a list of representaives of all pairs in $\widetilde{\mathscr{E}}_0^3$ below. We denote by $G(\sigma)$ the minimal set of generators of $\sigma$ which are primitive elements in $N=\mathbf{Z}^4$.

(1−1)
1. $G(\sigma)=\{(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1)\}$. See [9, Table 2.2], for $u_0$.
2. $G(\sigma)=\{(0,0,0,1),\ (1,0,0,1),\ (0,1,0,1),\ (1,1,2,1)\}$ and $u_0=\frac{1}{2}(1,1,1,2)$, $\frac{1}{3}(1,1,1,3)$, $\frac{1}{12}(5,3,4,12)$, $\frac{1}{8}(3,2,2,8)$, $\frac{1}{20}(7,5,4,20)$, $\frac{1}{16}(5,4,2,16)$, $\frac{1}{9}(3,2,1,9)$, $\frac{1}{5}(2,1,1,5)$, $\frac{1}{13}(4,3,1,13)$, $\frac{1}{11}(5,2,3,11)$, $\frac{1}{14}(5,3,2,14)$, $\frac{1}{16}(7,3,4,16)$, $\frac{1}{19}(4,7,3,19)$ or $\frac{1}{24}(5,9,4,24)$.
3. $G(\sigma)=\{(0,0,0,1),\ (1,0,0,1),\ (0,1,0,1),\ (1,1,3,1)\}$ and $u_0=\frac{1}{4}(2,1,2,4)$, $\frac{1}{7}(3,2,1,7)$, $\frac{1}{5}(3,1,2,5)$, $\frac{1}{9}(5,2,3,9)$, $\frac{1}{6}(2,3,3,6)$, $\frac{1}{9}(3,4,3,9)$ or $\frac{1}{2}(1,1,2,2)$.

4. $G(\sigma)=\{(0,0,0,1), (1,0,0,1), (0,1,0,1), (1,1,4,1)\}$ and $u_0=\frac{1}{2}(1,1,2,2)$.
5. $G(\sigma)=\{(0,0,0,1), (1,0,0,1), (0,1,0,1), (1,2,5,1)\}$ and $u_0=\frac{1}{4}(2,3,5,4)$, $\frac{1}{5}(2,3,5,5)$, $\frac{1}{2}(1,1,1,2)$ or $\frac{1}{3}(1,1,1,3)$.
6. $G(\sigma)=\{(0,0,0,1), (1,0,0,1), (0,1,0,1), (1,2,7,1)\}$ and $u_0=\frac{1}{2}(1,1,2,2)$.
7. $G(\sigma)=\{(0,0,0,1), (1,0,0,1), (0,1,0,1), (1,3,7,1)\}$ and $u_0=\frac{1}{2}(1,2,3,2)$.
8. $G(\sigma)=\{(0,0,0,1), (1,0,0,1), (0,1,0,1), (1,3,8,1)\}$ and $u_0=\frac{1}{2}(1,2,4,2)$.
9. $G(\sigma)=\{(0,0,0,1), (1,0,0,1), (0,1,0,1), (1,3,10,1)\}$ and $u_0=\frac{1}{2}(1,2,5,2)$.
10. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (0,0,1,1), (-1,-1,-1,1)\}$ and $u_0=(0,0,0,1)$.
11. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (1,1,2,1), (-1,-1,-1,1)\}$ and $u_0=(0,0,0,1)$.
12. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (1,1,3,1), (-1,-1,-2,1)\}$ and $u_0=(0,0,0,1)$.
13. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (1,2,5,1), (-1,-1,-1,1)\}$ and $u_0=(0,0,0,1)$.
14. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (1,2,5,1), (-1,-1,-2,1)\}$ and $u_0=(0,0,0,1)$.
15. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (1,2,5,1), (-2,-3,-5,1)\}$ and $u_0=(0,0,0,1)$.
16. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (1,2,7,1), (-1,-1,-2,1)\}$ and $u_0=(0,0,0,1)$.
17. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (1,3,7,1), (-1,-2,-3,1)\}$ and $u_0=(0,0,0,1)$.

(1−2)

1. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (-1,-1,0,1), (1,0,1,1)\}$ and $u_0=\frac{1}{2}(0,0,1,2)$, $\frac{1}{3}(-1,-1,1,3)$, $\frac{1}{4}(-1,-1,2,4)$ or $\frac{1}{6}(-1,-1,2,6)$.
2. $G(\sigma)=\{(2,0,0,1), (0,1,0,1), (-1,-1,0,1), (1,0,1,1)\}$ and $u_0=\frac{1}{2}(1,0,1,2)$.
3. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (-1,-1,0,1), (1,2,3,1)\}$ and $u_0=\frac{1}{3}(2,2,3,3)$.
4. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (-1,-1,0,1), (0,0,2,1)\}$ and $u_0=(0,0,1,1)$.
5. $G(\sigma)=\{(2,0,0,1), (0,1,0,1), (-1,-1,0,1), (-1,0,2,1)\}$ and $u_0=(0,0,1,1)$.
6. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (-1,-1,0,1), (-1,0,3,1)\}$ and $u_0=(0,0,1,1)$.

(2−1)

1. $G(\sigma)=\{(0,0,0,1), (1,0,0,1), (0,1,0,1), (1,0,1,1), (-1,0,-1,1)\}$ and $u_0=\frac{1}{3}(1,1,0,3)$, $\frac{1}{4}(1,1,0,4)$ or $\frac{1}{6}(1,2,0,6)$.
2. $G(\sigma)=\{(0,0,0,1), (1,0,0,1), (0,1,0,1), (1,0,1,1), (0,0,-1,1)\}$ and $u_0=\frac{1}{3}(1,1,0,3)$ or $\frac{1}{4}(1,2,0,4)$.
3. $G(\sigma)=\{(0,0,0,1), (1,0,0,1), (0,1,0,1), (1,1,2,1), (0,0,-1,1)\}$ and $u_0=\frac{1}{4}(1,1,0,4)$.
4. $G(\sigma)=\{(0,0,0,1), (1,0,0,1), (0,1,0,1), (1,1,2,1), (-1,-1,-2,1)\}$ and $u_0=\frac{1}{4}(1,1,0,4)$.
5. $G(\sigma)=\{(0,0,0,1), (1,0,0,1), (0,1,0,1), (1,1,3,1), (1,-1,-3,1)\}$ and $u_0=\frac{1}{3}(1,1,0,3)$.
6. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (-1,-1,0,1), (0,0,1,1), (0,0,-1,1)\}$ and $u_0=(0,0,0,1)$.
7. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (-1,-1,0,1), (0,0,1,1), (1,1,-1,1)\}$ and $u_0=(0,0,0,1)$.
8. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (-1,-1,0,1), (0,0,1,1), (2,0,-1,1)\}$ and $u_0=(0,0,0,1)$.
9. $G(\sigma)=\{(1,0,0,1), (0,1,0,1), (-1,-1,0,1), (1,2,3,1), (-1,-2,-3,1)\}$ and $u_0=(0,0,0,1)$.

(2−2−1)

1. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (0,1,1,1), (0,1,-1,1)\}$ and $u_0=\frac{1}{2}(0,1,0,2)$.
2. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (1,1,2,1), (-1,1,-2,1)\}$ and $u_0=\frac{1}{2}(0,1,0,2)$.
3. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (0,2,1,1), (0,2,-1,1)\}$ and $u_0=(0,1,0,1)$.
4. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (1,2,2,1), (-1,2,-2,1)\}$ and $u_0=(0,1,0,1)$.

(2−2−2)

1. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (0,1,0,1), (0,-1,0,1), (0,0,1,1)\}$ and $u_0=\frac{1}{2}(0,0,1,2)$.
2. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (0,1,0,1), (0,-1,0,1), (1,1,2,1)\}$ and $u_0=\frac{1}{2}(1,1,2,2)$.
3. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (0,1,0,1), (0,-1,0,1), (0,0,2,1)\}$ and $u_0=(0,0,1,1)$.

(2−2−3)

1. $G(\sigma)=\{(1,1,0,1), (0,0,1,1), (0,0,-1,1), (0,1,0,1), (1,0,0,1)\}$ and $u_0=\frac{1}{2}(1,1,0,2)$.
2. $G(\sigma)=\{(-1,0,0,1), (1,0,1,1), (1,0,-1,1), (0,1,0,1), (0,-1,0,1)\}$ and $u_0=(0,0,0,1)$.
3. $G(\sigma)=\{(-1,0,0,1), (1,1,2,1), (1,-1,-2,1), (0,1,0,1), (0,-1,0,1)\}$ and $u_0=(0,0,0,1)$.
4. $G(\sigma)=\{(-1,0,0,1), (1,0,1,1), (1,0,-1,1), (0,1,0,1), (1,-1,0,1)\}$ and $u_0=(0,0,0,1)$.

(2−2−4)

1. $G(\sigma)=\{(0,-1,2,1), (0,2,0,1), (1,0,0,1), (-1,0,0,1)\}$ and $u_0=(0,0,1,1)$.

(2−3)

1. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (0,2,0,1), (0,0,1,1)\}$ and $u_0=\frac{1}{2}(0,1,1,2)$, $\frac{1}{3}(0,3,1,3)$ or $\frac{1}{4}(0,4,1,4)$.
2. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (0,3,0,1), (0,0,1,1)\}$ and $u_0=\frac{1}{2}(0,2,1,2)$.
3. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (0,2,0,1), (1,1,2,1)\}$ and $u_0=\frac{1}{2}(1,1,2,2)$.

(2−4)

1. $G(\sigma)=\{(0,0,2,1), (0,3,0,1), (1,0,0,1), (-1,0,0,1)\}$ and $u_0=(0,1,1,1)$.

(3)

1. $G(\sigma)=\{(0,0,0,1), (1,1,0,1), (1,0,0,1), (0,1,0,1), (0,0,1,1), (1,1,-1,1)\}$ and $u_0=\frac{1}{2}(1,1,0,2)$.
2. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (0,1,0,1), (0,-1,0,1), (0,0,1,1), (0,0,-1,1)\}$ and $u_0=(0,0,0,1)$.
3. $G(\sigma)=\{(1,0,0,1), (-1,0,0,1), (0,1,0.1), (0,-1,0,1), (1,1,2,1), (-1,-1,-2,1)\}$ and $u_0=(0,0,0,1)$.

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