Elementary Abelian p Subgroups of Lie Groups

By

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§1. Introduction

In this paper we will study elementary abelian p subgroups

$$V = Z/pZ \times \ldots \times Z/pZ$$

of compact Lie groups. If there are r copies of Z/pZ in the above decomposition then V is said to have rank r. An elementary abelian p group is a $\bmod p$ version of a torus

$$T = S^1 \times \ldots \times S^1.$$

A torus T contains a canonical elementary abelian p subgroup

$$V_T = \{x \in T | \ x^p = 1\}.$$

Given a compact Lie group G an elementary abelian p subgroup is called *toral* if it can be imbedded in a torus of G. Otherwise it is said to be *non toral*. The importance of non toral elementary abelian p subgroups was first pointed out by Borel thirty years ago when he observed the relation between the presence of non toral elementary abelian p subgroups in G and the presence of p torsion in H^*G. In [1] he proved

Theorem 1.1. (Borel) *Let G be a compact connected Lie group and let BG be its classifying space. The following are equivalent.*
(i)   *H^*G has no p torsion.*
(ii)   *H^*BG has no p torsion.*
(iii) *Every elementary abelian p group in G is toral.*
(iv)   *Every elementary abelian p group of rank ≤ 3 in G is toral.*

Our study of elementary abelian p subgroups is motivated by this result and gives some insight into it. When Borel proved his theorem the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) were either trivial or were proved by general arguments. He then proved the implication (iv) $\Rightarrow$ (i) basically via a case by case argument. For every compact connected Lie group G with p torsion Borel located a specific non toral elementary abelian p group of rank ≤ 3 lying in G. The equivalence of (i), (ii), (iii) and (iv) still lacks a general explanation. More precisely there are two implications still lacking such an explanation, namely (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii). This paper will give a conceptual argument explaining (iv) $\Rightarrow$ (iii). It should be observed, however, that, as explained after Theorem 1.2, our argument also depends upon an empirical observation about Lie groups. So it is not a classification free argument. Nevertheless, our argument still seems an advancement.

Before expanding further on the above statements we will state our main results. We will work with $H^*(BG; F_p)$, the mod p cohomology of BG, and develop a cohomological characterization of non toral elementary abelian p subgroups in terms of $H^*(BG; F_p)$ which is valid when p is odd and in certain cases when $p = 2$.

A global relationship between the algebraic structure of $H^*(BG; F_p)$ and the elementary abelian p subgroups of G was worked out by Quillen in [6]. Further extensions of this relationship were obtained in [4] and [7]. The machinery developed in these studies in a key ingredient in the proof of our characterization. Also used are some homotopy theoretic facts obtained by applying the Ω spectrum of connective K-theory.

We begin with the case of p odd. If $V \subset G$ is an elementary abelian p group then

$$H^*(BV; F_p) = E \otimes S$$

where

      E = the exterior algebra on $H^1(BV; F_p)$
      S = the symmetric algebra on $\beta_p H^1(BV; F_p) \subset H^2(BV; F_p)$.

Here $\beta_p H^1(BV; F_p)$ is the isomorphic image of $H^1(BV; F_p)$ under the Bockstein β_p: $H^1(BV; F_p) \rightarrow H^2(BV; F_p)$.

Theorem 1.2. *Let G be a connected, simply connected, compact Lie group. Let p be odd and let α: $V \subset G$ be an elementary abelian p group. Then V is toral if and only if $(B\alpha)^*x \in S \subset H^*(BV; F_p)$ for all $x \in H^4(BG; F_p)$.*

We should remark that this result depends on the Cartan-Killing classification of the compact Lie groups. Namely, in the proof we use the empirical observation that when p is odd and G is 1-connected the module of indecomposables $Q^{even}H^*(G; F_p)$ is concentrated in the degrees $2(p + 1)$ and $2(p^2 + 1)$. The best

theoretical bounds on $Q^{\text{even}}H^*(G; F_p)$ are far cruder. By working at the level of finite H-spaces general theorems have been deduced which severely limit the degrees in which $Q^{\text{even}}H^*(G; F_p)$ can be non trivial. The possible degrees are those of the form $2(p^k + \ldots + p^{i+1} + p^i + p^{i-1} + \ldots + p + 1)$. (See [5]). So there are many possibilities besides $2(p+1)$ or $2(p^2+1)$. However, even for 1 connected finite H-spaces, $Q^{\text{even}}H^*(X; F_p)$ is only known to be non zero in the degrees $2(p+1)$ and $2(p^2+1)$.

When one passes to $p = 2$ one generally modifies statements by replacing even degree indecomposables with squares of odd degree indecomposables. Notably, if V is an elementary abelian 2 subgroup then

$$H^*(BV; F_2) = \text{ the symmetric algebra on } H^1(BV; F_2).$$

For any X we have the Frobenius map

$$\xi: H^*(X; F_2) \to H^*(X; F_2)$$
$$\xi(x) = x^2.$$

In particular ξ is an algebra map and $\xi H^*(X; F_2)$, the image of ξ, is a subalgebra of $H^*(X; F_2)$.

Theorem 1.3. *Let G be a connected, simply connected, compact Lie group where $\xi H^*(G; F_2)$ has indecomposables only in the degrees $2^k + 2$ $(k \geq 2)$. Let α: $V \subset G$ be an elementary abelian 2 subgroup. Then V is toral if and only if $(B\alpha)^*x \in \xi H^*(BV; F_2)$ for all $x \in \dot{H}^4(BG; F_2)$.*

The hypothesis on $\xi H^*(G; F_2)$ may be redundant. However, it is needed for the present type of proof. And it is restrictive. It is not satisfied by the classical compact Lie group $\text{Spin}(n)$ for $n \geq 15$ nor by the exceptional compact Lie group E_8. As we have already alluded to earlier, the preceeding cohomological characterization of toral/non toral elementary subgroup can be used to explain one of the results obtained by Borel.

Corollary 1.4. *Let G be as above. If $V \subset G$ is a non toral elementary abelian p group then there exists a subgroup $V' \subset V$ of rank 3 which is non toral. In particular rank $V \geq 3$.*

The next four sections are devoted to the proof of Theorem 1.2 and 1.3 as well as Corollary 1.4. The theorems are proved in §1.4 and the corollary is proved in §1.5. Sections §1.2 and §1.3 are preliminaries. They develop needed machinery and establish preliminary results.

Assumption. In all that follows we will assume that G is a connected, simply connected, compact Lie group. We will also assume that p *is odd*. The changes to handle the $p = 2$ case are minor and are in line with those mentioned above. We will use $Z_{(p)}$ to denote the integers localized at the prime p and we will use $H^*(\underline{})$ to denote cohomology localized at the prime p.

§2. The Ideals of $H^*(BG; F_p)$

Quillen established in [6] that there is a relation between the elementary abelian p subgroup of G and the ideals of $H^*(BG; F_p)$. This relationship was completely worked out by Rector in [7]. In this section we will sketch the relationship and explain how it leads to a criterion in terms of the ideals of $H^*(BG; F_p)$ for an elementary abelian p subgroup to be non toral.

First of all, one can fix a maximal torus $T \subset G$ and discuss toral/non toral elementary p subgroups in terms of T. For since every torus is contained in a maximal torus and since any maximal torus is conjugate to T it follows that
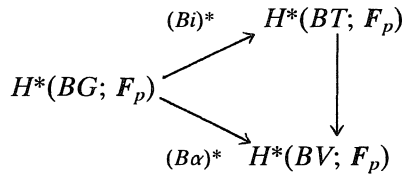
Lemma 2.1.   *An elementary abelian p group $V \subset G$ is total if and only if it can be conjugated into T.*

Consider a fixed maximal torus $i : T \subset G$. Let $\alpha : V \subset G$ be an elementary abelian p subgroup. Let

$$I = \ker\{(Bi)^* : H^*(BG; F_p) \to H^*(BT; F_p)\}$$
$$I_V = \ker\{(B\alpha)^* : H^*(BG; F_p) \to H^*(BV; F_p)\}.$$

If V is toral then we have a commutative diagram

$$
\begin{array}{ccc}
 & \xrightarrow{\;(Bi)^*\;} & H^*(BT; F_p) \\
H^*(BG; F_p) & & \Big\downarrow \\
 & \xrightarrow[\;(B\alpha)^*\;]{} & H^*(BV; F_p)
\end{array}
$$

which gives the inclusion $I \subset I_V$. This inclusion actually characterizes V being toral. Namely

Proposition 2.2.   *V is toral if and only if $I \subset I_V$.*

The rest of this section will be devoted to justifying this result. For the following discussion consult §1 and §2 of [7]. The ideal $I_V \subset H^*(BG; F_p)$, defined above, is a prime ideal invariant under the Steenrod algebra $A^*(p)$. Quillen demonstrated in [6] that the map $V \mapsto I_V$ sets up a one-to-one correspondence between conjugacy classes of elementary abelian p groups and the invariant prime ideals of $H^*(BG; F_p)$. Moreover, the correspondence is

functorial in that, given V_1, V_2 there exists an inner automorphism of G mapping V_1 into V_2 if and only if $I_{V_2} \subset I_{V_1}$.

Now consider an elementary abelian p subgroup V where $I \subset I_V$. As before let

$$V_T = \{x \in T \mid x^p = 1\}.$$

Asserting that V can be conjugated into T is the same as asserting that V can be conjugated into V_T. Moreover, since $H^*(BT; F_p) \to H^*(BV_T; F_p)$ is injective we can also define I as

$$I = \ker\{H^*(BG; F_p) \to H^*(BV_T; F_p)\}.$$

The proposition now follows from the last sentence in the previous paragraph.

§3. The Ideal I

We now give a different description of the ideal $I = \ker\{(Bi)^*: H^*(BG; F_p) \to H^*(BT; F_p)\}$. This new characterization is the key to the proofs of the theorems from §1.

(A) Connective K-Theory

Let bu denote the complex connective K-theory and let $\ell \subset bu_{(p)}$ denote the canonical direct summand where

$$\Pi_* bu = Z[u] \quad \deg u = 2$$
$$\Pi_* \ell = Z_{(p)}[v] \quad \deg v = 2(p - 1).$$

In Brown-Peterson notation $\ell = BP\langle 1 \rangle$ where $BP\langle n \rangle$ is the theory such that $\Pi_* BP\langle n \rangle = Z_{(p)}[V_1, V_2, \ldots, V_n]$. Let $\{\ell_n\}$ be the Ω spectrum associated to ℓ. This Ω spectrum and, more generally, the Ω spectrum of the theories $BP\langle n \rangle$ were studied in [8] and [9]. We will limit ourselves to the spectrum $\{\ell_n\}$.

For each $n \geq 1$ there exists a fibration of the form

$$K(Z_{(p)}, n) \xrightarrow{\Psi} \ell_{n+2p-1} \xrightarrow{\Delta} \ell_{n+1}.$$

For $n \leq 2p + 2$ $H^*\ell_n$ is torsion free and is either an exterior algebra (n odd) or a polynomial algebra (n even). The map Ψ has the property that

Lemma 3.1. Im $\Psi^* =$ *the subalgebra of* $H^*(K(Z_{(p)}, n); F_p)$ *generated over* $A^*(p)$ *by* $Q_1(\iota_n)$.

(B) The Mod p Cohomology of G

Since G is a finite H-space the structure theorems of [5] apply and the module of indecomposables, $Q^{\text{even}} H^*(G; F_p)$ must be trivial except in the degrees $2(p^k + \ldots + p^{i+1} + p^i + p^{i-1} + \ldots + p + 1)$. It is an empirical observation

that, in the case of 1-connected compact Lie groups, $Q^{\text{even}}H^*(G; F_p)$ is trivial except in the degrees $2(p + 1)$ and $2(p^2 + 1)$. The structure theorems of [5] also tell us that the maps

$$(3.2) \qquad Q^3 H^*(G; F_p) \xrightarrow{Q_1} Q^{2p+2} H^*(G; F_p) \xrightarrow{\mathscr{P}^p} Q^{2p^2+2} H^*(G; F_p)$$

are surjective. As another result of [5] there exists a short exact sequence of Hopf algebras over $A^*(p)$.

$$(3.3) \qquad\qquad 0 \longrightarrow \Gamma \longrightarrow H^*(G; F_p) \longrightarrow E \longrightarrow 0$$

where

(i) Γ is a primitively generated Hopf algebra and $Q\Gamma \cong Q^{\text{even}}H^*(G; F_p)$
(ii) E is a primitively generated Hopf algebra and $Q^{\text{odd}}H^*(G; F_p) \cong QE$.

(C) The Space $\widetilde{G}$

We will define and study a space $\widetilde{G}$. We begin by defining a space L and a map $f \colon G \to L$. First of all, any 1-connected mod p finite H-space X is actually 2 connected and $H^3 X$ is torsion free. Choose a map

$$\overline{f} : G \to K = \prod_{i=1}^{k} K(\mathbf{Z}_{(p)}, 3)$$

representing a basis of $H^3 G$. In other words, $\overline{f}^*$ induces an isomorphism on $H^3(\;\;)$. We define f to be the composite

$$f: G \xrightarrow{\overline{f}} K \xrightarrow{\overline{\psi}} L = \prod_{i=1}^{k} \ell_{2p+2}$$

where $\overline{\psi} = \prod_{i=1}^{k} \psi$. We now define

$$\widetilde{G} = \text{ the fibre of } f.$$

The map f is an H-space map. Indeed, as we will shortly point out, it is a loop map. It follows from the discussion in Parts A and B (see, in particular, 3.1 and 3.2) that

$$(3.4) \qquad\qquad\qquad \text{Im} f^* = \Gamma.$$

Since $H^*(L; F_p)$ is a polynomial algebra it is easy to deduce that

Proposition 3.5. $H^*(\widetilde{G}; F_p)$ *is an exterior algebra (on odd degree generators).*

To do this use the Serre spectral sequence associated to the fibration $\Omega L \to \widetilde{G} \to G$. This is a spectral sequence of Hopf algebras. Also $H^*(\widetilde{\Omega L}; F_p) = \bigotimes_{i=1}^{k} H^*(\ell_{2p+1}; F_p)$ is an exterior algebra on odd degree generators. We can use

3.4 to force non trivial differentials. Once we introduce all these non trivial differentials we are left with an exterior algebra. Since the spectral sequence is one of Hopf algebras it collapses from this stage on. (See §1−6 of [3]).

(D) **The Space $B\tilde{G}$**

The space $B\tilde{G}$ is a loop space. We obtain $B\tilde{G}$ by delooping the map f and taking the fibre. We have already observed that G is 2 connected. It follows that BG is 3 connected and

$$H^4 BG \cong H^3 G$$

via the loop map

$$\{BG \xrightarrow{\alpha} K(\mathbf{Z}_{(p)},4)\} \mapsto \{G = \Omega BG \xrightarrow{\Omega\alpha} K(\mathbf{Z}_{(p)},3)\}.$$

So the map $\bar{f}$ given above can be delooped and we can define the delooping Bf of f as the composite

$$Bf\colon BG \xrightarrow{B\bar{f}} BK \xrightarrow{B\overline{\Psi}} BL$$

where

$$BK = \prod_{i=1}^{k} K(\mathbf{Z}_{(p)},4)$$

$$BL = \prod_{i=1}^{k} \ell_{2p+3}$$

If we let $B\tilde{G}$ be the fibre of the map Bf then it follows from Proposition 3.5 that

Corollary 3.6. $H^*(B\tilde{G};\ F_p)$ *is a polynomial algebra* (*on even degree generators*).

In particular, since $H^*(B\tilde{G};\ F_p)$ is concentrated in even degrees H^*BG is torsion free. The definition of $\tilde{G}$ and $B\tilde{G}$ gives maps $j\colon \tilde{G} \to G$ and $Bj\colon B\tilde{G} \to BG$. In the rest of this section we will show

Proposition 3.7. $I = \ker\{(Bj)^*\colon H^*(BG;\ F_p) \to H^*(B\tilde{G};\ F_p)\}$.

The diagram

$$B\tilde{G} \xrightarrow{Bj} BG \xrightarrow{Bf} BL$$
$$Bi \uparrow$$
$$BT$$

can be extended to a diagram of fibrations

$$
\begin{array}{ccccc}
B\tilde{G} & \longrightarrow & BG & \longrightarrow & BL \\
\uparrow & & \uparrow & & \| \\
L \times BT & \longrightarrow & BT & \longrightarrow & BL \\
\uparrow & & \uparrow & & \uparrow \\
G/T & = & G/T & \longrightarrow & *
\end{array}
$$

Regarding the fact that $L \times BT$ is the fibre of the map $BT \to BL$ observe that since $\ell^*(BT)$ is concentrated in even degrees the composite $BT \to BG \to BL = \prod_{i=1}^{k} \ell_{2p+3}$ must be trivial.

Lemma 3.7. *The maps $H^*(B\tilde{G}; F_p) \to H^*(L \times BT; F_p)$ and $H^*(BT; F_p) \to H^*(L \times BT; F_p)$ are injective.*

Proof. Only the first map needs comment. Consider the Serre spectral sequence of the fibration

$$
G/T \to L \times BT \to B\tilde{G}.
$$

Since $H^*(G/T; F_p)$ and $H^*(B\tilde{G}; F_p)$ are concentrated in even degrees the spectral sequence collapses. Q.E.D.

Corollary 3.8. $\operatorname{Ker}(Bi)^* = \operatorname{Ker}(Bj)^*$.

§4.   Proof of Theorem 1.2

Let $\alpha: V \subset G$ be an elementary abelian p subgroup and let $i: T \subset G$ be a fixed maximal torus.

First of all assume that V is toral. As observed in §2 there exists a commutative diagram

$$
\begin{array}{ccc}
& \overset{(Bi)^*}{\nearrow} & H^*(BT; F_p) \\
H^*(BG; F_p) & & \downarrow \\
& \underset{(B\alpha)^*}{\searrow} & H^*(BV; F_p)
\end{array}
$$

If we write $H^*(BV; F_p) = E \otimes S$ as in §1 then $\text{Image}\{H^*(BT; F_p) \to H^*(BV; F_p)\} \subset S$. Consequently, by the above diagram, $(B\alpha)^*x \in S$ for all $x \in H^*(BG; F_p)$.

Conversely, suppose $(B\alpha)^*x \in S$ for all $x \in H^4(BG; F_p)$. We can then set up a homotopy theoretic diagram

$$BG̃ \xrightarrow{Bj} BG \xrightarrow{Bf̄} BK \xrightarrow{B\overline{\Psi}} BL$$

(4.1)
$$Bα \uparrow \qquad \uparrow h$$
$$BV \xrightarrow{g} BT$$

The top maps were defined in §3. In particular $BG̃ \xrightarrow{Bj} BG$ is the fibre of $Bf = B\overline{\Psi} \circ Bf̄$. The map

$$g: BV \to BT$$

is obtained by thinking of BT as the generalized Eilenberg-MacLane space $BT = \prod_{i=1}^{n} K(\mathbf{Z},2)$. It has the property that $g^*: H^*(BT; F_p) \cong S$. The existence of the map

$$h: BT \to BK = \prod_{i=1}^{n} K(\mathbf{Z},4)$$

filling in 4.1 is then equivalent to asserting that $(Bα)^*x \in S$ for all $x \in H^4(BG; F_p)$. Here we also need the fact that $H^4(BV)$ consists of elementary p torsion. This will be proved in §5 in the discussion following Corollary 5.2.

We can use 4.1 to show that $Bα$ factors through $BG̃$. In other words we have a homotopy commutative diagram

(4.2)
$$BG̃ \xrightarrow{Bj} BG$$
$$\diagdown \quad \uparrow Bα$$
$$BV$$

The point is that $BG̃ \xrightarrow{Bj} BG \xrightarrow{Bf} BL$ is a fibration and the composition $BV \xrightarrow{Bα} BG \xrightarrow{Bf} BL$ is trivial. For by (4.1) we can rewrite $Bf \circ Bα$ as

$$BV \xrightarrow{g} BT \xrightarrow{h} BK \xrightarrow{B\overline{\rho}} BL.$$

However, $\ell^*(BT)$, the connective K-theory of BT, is concentrated in even degrees. So any map $BT \to BL = \prod_{i=1}^{k} \ell_{2p+3}$ is trivial.

We now apply §2 and §3. By Theorem 2.1 V will be toral if $I \subset I_V$. By Proposition 3.7 and diagram 4.2 we have $I \subset I_V$.

§5. Proof of Corollary 1.5

Fix $x \in H^4(BG; F_p)$ and assume that $(Bα)^*x \in S \subset H^*(BV; F_p)$. We will use ρ to denote mod p reduction.

Lemma 5.1. $x \in \text{Image } \{\rho: H^*BG \to H^*(BG; F_p)\}$.

Proof. Since G is 1-connected it follows from the structure theorems of [5] that, in degree $<2p+2$, $H^*(G; F_p)$ is an exterior algebra on odd degree generators. It follows from a Bockstein spectral sequence argument that, in degree $<2p+2$, H^*G is torsion free and an exterior algebra on odd degree generators. (see §11-2 of [3]). Consequently, in degree $<2p+3$ H^*BG is torsion free and a polynomial algebra on even degree generators. In particular, ρ: $H^*BG \to H^*(BG; F_p)$ is surjective in degree $\leq 2p$.   Q.E.D.

The commutative diagram

$$
\begin{array}{ccc}
H^*(BG) & \xrightarrow{(B\alpha)^*} & H^*(BV) \\
\rho \downarrow & & \downarrow \rho \\
H^*(BG;\ F_p) & \xrightarrow{(B\alpha)^+} & H^*(BV;\ F_p)
\end{array}
$$

tells one that

Corollary 5.2. $(B\alpha)^*x \in \text{Image } \{\rho: H^*(BV) \to H^*(BV; F_p)\}$.

The image of ρ: $H^4(BV) \to H^4(BV; F_p)$ can be described explicitly. We can write

$$H^*(BV; F_p) = \bigotimes_i A_i$$

where

$$A_i = E(e_i) \otimes S(f_i)$$
$$\beta_p(e_i) = f_i$$

If we calculate homology with respect to β_p we have $H(H^*(BV;\ F_p)) = \otimes H(A_i) = F_p$ (in degree 0). It follows that $H^*(BV)$ consists of elementary p torsion and

$$\text{Image } \{H^*(BV) \to H^*(BV;\ F_p)\} = \text{Image } \beta_p.$$

In degree 4 Image β_p is spanned by the elements $\{\beta_p(e_i e_j e_k)\}$ and $\{f_i f_j\}$. So it follows from Corollary 5.2 that

Lemma 5.3. $(B\alpha)^*x \in S \Leftrightarrow (B\alpha)^*x$ *can be expanded in terms of the elements* $\{f_i f_j\}$.

We now set about proving Corollary 1.4.

Proof of Corollary 1.4. Suppose α: $V \subset G$ is a non toral elementary p subgroup. By Theorem 1.2 there exists $x \in H^4(BG; F_p)$ where $(B\alpha)^*x \notin S$. Write

$$H^*(BV; \, \pmb{F}_p) = E(e_1, \ldots, e_n) \otimes S(f_1, \ldots, f_n).$$

It follows from Lemma 5.3 that when we expand $(B\alpha)^*x$ in terms of $\{f_i f_j\}$ $\amalg \{\beta_p(e_i e_j e_k)\}$ one of the terms $\beta_p(e_i e_j e_k)$ appears as a non trivial summand. We can locate a rank 3 subgroup $V' \subset V$ such that

$$H^*(BV'; \, \pmb{F}_p) = E(e_i, e_j, e_k) \otimes S(f_i, f_j, f_k)$$

and $e_r = f_r = 0$ of $r \neq i, j, k$. So, when we pass to $H^*(BV'; \, \pmb{F}_p)$, $\beta_p(e_i e_j e_k)$ still appears as a non trivial summand of $(B\alpha)^*x$ and $V' \subset G$ is non toral.

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