

Application of Fusion Rules to Classification of Subfactors

By

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Abstract

By the technique of Longo's sector, the following two results are obtained. (1) A proof is given to Ocneanu's announcement about the non-existence of paragroups for Coxeter graphs E_7 and D_{odd} . (2) Subfactors of the type III_1 AFD factor with index 3 are classified.

§1. Introduction and Main Results

V. Jones theory of index $[J]$ for subfactor of type II_1 factors suddenly aroused people's interest on classification of subfactors. A. Ocneanu [O1, O2] has announced complete classification of subfactors of type II_1 hyperfinite factor with index less than 4 in terms of the concept of paragroups associated with Coxeter graphs. While his announcement has not been fully published, some further analysis and classification has been obtained by S. Popa [P1, P2].

While these results are about subfactors of the type II_1 factors, the notion of index by Jones has been extended to subfactors of infinite factors by H. Kosaki [K1] and by a further work of F. Hiai [H].

About 10 years before Jones' first definition of his index, S. Doplicher, R. Haag, and J.E. Roberts [DHR] developed the notion of statistical dimension of sectors in connection with quantum field theory or, more precisely theory of local observables. It turns out that this notion corresponds to the square root of the index of the subfactor, given by the localized morphism for the sector concerned. In recent works of R. Longo [L1, L2] this connection between the statistical dimension and the Jones index has been clarified and a new definition of the index of subfactors of infinite factors has been given. Moreover

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R. Longo introduced the notion of sectors of an infinite factor M , denoted $\text{Sect}(M)$, and an involutive map called conjugation in $\text{Sect}(M)$, both in analogy with the case of quantum field theory. These and their properties, as described in Section 2, will be our basic tools.

Among the announcements of A. Ocneanu, one curious feature is the non-existence of paragroup for Coxeter graphs E_7 and D_{odd} , one of the two main results of the present work is a proof of this non-existence result (Corollary 3.9), based on Longo's theory of $\text{Sect}(M)$. While the non-existence of E_7 paragroup follows from a simple consideration on the statistical dimension of sectors, the non-existence of D_{odd} paragroup follows from the fusion rules of sectors. In Section 3 we calculate fusion rules of sectors associated with Coxeter graphs A_n and D_n . If we assume the existence of D_{odd} paragroups, the calculated fusion rules are found to be inconsistent with the Perron-Frobenius eigenvectors of incidence matrix of the Coxeter graphs, thus proving the result.

The other main result of present work is a complete classification of subfactors of type III_1 AFD factors with index 3 (Theorem 5.1). If M and N are type III_1 AFD factors, N being a subfactor of M with index 3, then we show in Section 5 that there exists a pair of type II_1 AFD factors R and P , P being a subfactor with index 3 and $M \supset N$ is isomorphic to $R \otimes M_0 \supset P \otimes M_0$ for the type III_1 AFD factor M_0 . This has been conjectured for a general case of finite index by H. Kosaki and R. Longo [K2, KL]. Our result implies that the classification of subfactors of the type III_1 AFD factor with index 3 is exactly the same as the classification for the type II_1 AFD factor, namely, if the principal graph is D_4 then N is the fixed point algebra of M under the unique outer action of \mathbb{Z}_3 and, if the principal graph is A_5 then there exist a subfactor L of N and an outer action of S_3 on L satisfying

$$M = L \rtimes S_3 \supset N = L \rtimes S_2 .$$

As a technical tool for the above-described classification, we give in Section 4 a characterization (Theorem 4.1) of fixed point subalgebras of infinite factors under outer actions of finite groups. As a kind of generalization of this result, we give a characterization (Proposition 4.2) of depth 2.

After this work was completed, we received the paper of Y. Kawahigashi [Ka]. He proves the non-existence of E_7 and D_{odd} paragroups and the existence of D_{even} paragroups by the argument of Ocneanu's connections of graphs.

This paper is a revised version of preprint *Some results on classification of subfactors*.

§2. Technical Preliminaries

Throughout this paper we assume that von Neumann algebras have separable pre-duals and Hilbert spaces are separable. Let M be a von Neumann algebra and N be its von Neumann subalgebra. We denote by $\text{End}(M)$ the set of unital normal endomorphisms of M and by $E(M, N)$ the set of faithful normal conditional expectations from M to N . If M and N are factors and $E \in E(M, N)$, $\text{Index } E$ denotes the Kosaki index of E [K1]. If $M \supset N \supset L$ are inclusions of factors and $E_1 \in E(M, N)$, $E_2 \in E(N, L)$, then

$$\text{Index } E_2 \circ E_1 = (\text{Index } E_1) (\text{Index } E_2). \quad (2.1)$$

We denote by $[M: N]_0$ the minimal index of F. Hiai [H], namely the minimum of $\text{Index } E$ for $E \in E(M, N)$, which exists. If $\rho \in \text{End}(M)$ such that $\rho(M) = N$, we define the statistical dimension d_ρ by $d_\rho = ([M: N]_0)^{1/2}$.

2.1. *Sectors and Conjugate Sectors.* Almost all results quoted in this and next subsections are in [L1, L2]. Let M be a type III factor and $\rho_1, \rho_2 \in \text{End}(M)$. ρ_1 and ρ_2 are said to be unitarily equivalent if and only if there exists a unitary $u \in M$ such that

$$\rho_1 = \text{Adu} \circ \rho_2.$$

We denote by $\text{Sect}(M)$ the quotient of $\text{End}(M)$ by unitary equivalence. We call elements in $\text{Sect}(M)$ sectors. If $\rho \in \text{End}(M)$, we denote by $[\rho]$ its class in $\text{Sect}(M)$.

$\text{Sect}(M)$ can be equipped with sum and product which satisfy associativity and distributivity. For given $\rho_1, \rho_2 \in \text{End}(M)$, we define the sum $[\rho_1] \oplus [\rho_2]$ and the product $[\rho_1] [\rho_2]$ as follows. Since we assume M to be a type III factor, there exist non-zero projections $p_1, p_2 \in M$ and isometries $v_1, v_2 \in M$ such that

$$p_1 + p_2 = 1, \quad v_i v_i^* = p_i, \quad i = 1, 2.$$

We define $\rho \in \text{End}(M)$ by

$$\rho(x) \equiv v_1 \rho_1(x) v_1^* + v_2 \rho_2(x) v_2^*$$

and

$$[\rho_1] \oplus [\rho_2] \equiv [\rho], \quad [\rho_1] [\rho_2] \equiv [\rho_1 \rho_2].$$

These sum and product do not depend on the choice of ρ_1, ρ_2 in their classes and on the choice of v_1 and v_2 . They satisfy associativity and distributivity.

We say that $\rho \in \text{End}(M)$ is irreducible if and only if $M \cap \rho(M)' = \mathbf{C}$. If

$\dim M \cap \rho(M)' < \infty$ we can perform irreducible decomposition as follows. Let $\{p_i\}$ be minimal projections in $M \cap \rho(M)'$ with $\sum_i p_i = 1$ and $\{v_i\} \subset M$ be isometries with $v_i v_i^* = p_i$. If we define $\rho_i \in \text{End}(M)$ by

$$\rho_i(x) = v_i^* \rho(x) v_i, \quad x \in M,$$

then $[\rho] = \bigoplus_i [\rho_i]$ and ρ_i is irreducible. This decomposition does not depend on the choice of $\{p_i\}$ and $\{v_i\}$. Note that $\dim M \cap \rho(M)' < \infty$ is always satisfied if $d_p < \infty$.

To define conjugate sector, we need the notion of $M-M$ correspondences [P3].

A (separable) Hilbert space H is called an $M-M$ correspondence if and only if H is an $M-M$ bimodule and left and right actions of M are σ weakly continuous. $M-M$ correspondences H_1 and H_2 are said to be equivalent if and only if there exists a surjective isometry $u: H_1 \rightarrow H_2$ commuting with left and right actions. Let H be an $M-M$ correspondence and \bar{H} be the conjugate Hilbert space of H , i.e. there exists a surjective conjugate isometry from $\xi \in H$ to $\bar{\xi} \in \bar{H}$. We define the conjugate correspondence of H by \bar{H} in which the $M-M$ bimodule action is defined by

$$x \cdot \bar{\xi} \cdot y \equiv \overline{y^* \cdot \xi \cdot x^*} \quad \text{for } \xi \in H, x, y \in M.$$

On the set of unitarily equivalent classes of $M-M$ correspondences, we define the involution by the conjugate correspondence.

Let H be a standard Hilbert space of M and J be a modular conjugation. Let $\rho \in \text{End}(M)$. We define the $M-M$ correspondence H_ρ by

$$x \cdot \xi \cdot y \equiv \rho(x) J y^* J \xi \quad \text{for } \xi \in H, x, y \in M.$$

In [L2, Corollary 2.2], R. Longo shows that $\rho \rightarrow H_\rho$ induces natural bijection between $\text{Sect}(M)$ and the set of unitary equivalence classes of $M-M$ correspondences, and he defines conjugation in $\text{Sect}(M)$ by conjugation in the latter. If $\rho \in \text{End}(M)$, we denote by $[\bar{\rho}]$ the conjugate sector of $[\rho]$ and, for simplicity, by $\bar{\rho}$ one of representatives of $[\bar{\rho}]$. The conjugation defined above satisfies the following relations.

$$\begin{aligned} \overline{\bigoplus_i [\rho_i]} &= \bigoplus_i \overline{[\rho_i]}, \\ \overline{[\rho_1] [\rho_2]} &= \overline{[\rho_2] [\rho_1]}, \\ \overline{[\alpha]} &= [\alpha^{-1}] \quad \text{if } \alpha \in \text{Aut}(M). \end{aligned}$$

We call $\rho \in \text{End}(M)$ self-conjugate if and only if $[\rho] = \overline{[\rho]}$. Note that $\alpha \in \text{Aut}(M)$ is self-conjugate if and only if α^2 is inner.

Remark. If M is a type II_∞ factor, we can define $\text{Sect}(M)$ in the same way, and if $\rho \in \text{End}(M)$ with $d_\rho < \infty$, we can also perform the irreducible decomposition. To see this, it suffices to show that every non-zero projection in $M \cap \rho(M)'$ is infinite. See Appendix for the proof.

2-2. Canonical Endomorphisms and Implementations of Conditional Expectations Let M and N be properly infinite factors, N being a subfactor of M , and Ω be a cyclic and separating vector simultaneously for M and N . (Such a vector always exists). R. Longo defined the canonical endomorphism $\tau_\Omega: M \rightarrow N$ by

$$\tau_\Omega(x) = \Gamma_\Omega x \Gamma_\Omega^*, \quad x \in M$$

where $\Gamma_\Omega = J_N^\Omega J_M^\Omega$ is the product of modular conjugation operators J_M^Ω, J_N^Ω for M and N with respect to Ω . τ_Ω depends on Ω only up to unitaries in N [L3, Theorem 1.1]. He shows in [L2, Theorem 3.1] that if $\gamma: M \rightarrow \rho(M)$ is the canonical endomorphism for $\rho \in \text{End}(M)$, then $[\rho^{-1} \circ \gamma] = \overline{[\rho]}$. This formula gives $\bar{\rho}$ explicitly.

In [L1, Proposition 5.1], R. Longo also shows the following important fact about implementation of conditional expectations by canonical endomorphisms. Let $M \supset N$ be a pair of properly infinite factors, $\tau: M \rightarrow N$ be the canonical endomorphism, and $E \in \text{E}(M, N)$. Then there exists an isometry $u \in N$ satisfying

$$\begin{aligned} ux &= \tau(x)u, \quad x \in N, \\ E(x) &= u^* \tau(x)u, \quad x \in M. \end{aligned}$$

We recall the construction of u . Let φ be a faithful normal state defined by

$$\varphi(x) = \langle E(x)\Omega, \Omega \rangle \quad x \in M.$$

Then there is a unique vector $\xi_0 \in L^2(M, \Omega)_+$ satisfying

$$\varphi(x) = \langle x\xi_0, \xi_0 \rangle \quad x \in M.$$

We define an isometry $u_0 \in N'$ as follows.

$$u_0 x \Omega = x \xi_0 \quad x \in N.$$

Note that the range projection of u_0 is e_N in the sense of Kosaki [K1, page 130]. If we define an isometry u by $u = J_N^\Omega u_0 J_N^\Omega$, then u belongs to $J_N^\Omega N' J_N^\Omega = N$ and

satisfies the above conditions.

If M is isomorphic to N and $\rho \in \text{End}(M)$ with $\rho(M) = N$, then there exists an isometry $v \in M$ satisfying

$$vx = \bar{\rho}\rho(x)v, \quad E(x) = \rho(v^*\bar{\rho}(x)v), \quad x \in M.$$

Let ϕ be a normal, unital, completely positive map on M . We can construct a natural $M-M$ correspondence as follows. Let H be a standard Hilbert space of M and \mathcal{Q} be a cyclic and separating vector for M . Then \mathcal{Q} is a cyclic vector for the opposite algebra $M^0 = JMJ$, i.e.

$$\overline{\mathcal{Q}M} = \overline{JMJ\mathcal{Q}} = H.$$

We define positive semi-definite sesquilinear form on $M \otimes_{\text{alg}} M$ by

$$\left(\sum_i x_i \otimes y_i, \sum_j u_j \otimes v_j \right) \equiv \sum_{i,j} \langle \phi(u_j^* x_i) \mathcal{Q} y_i, \mathcal{Q} v_j \rangle.$$

Like G.N.S. construction, we can construct an $M-M$ correspondence out of this sesquilinear form and, by using the bijection between $\text{Sect}(M)$ and correspondences, we obtain $\rho_\phi \in \text{End}(M)$ and $\xi \in H$ satisfying

$$\langle \rho_\phi(x) \xi y, \xi \rangle = \langle \phi(x) \mathcal{Q} y, \mathcal{Q} \rangle, \quad \overline{\rho_\phi(M) \xi M} = H.$$

If $\phi = E$ and \mathcal{Q} is a cyclic and separating vector for M and N , then

$$\langle \rho_\phi(x) \xi y, \xi \rangle = \langle E(x) \mathcal{Q} y, \mathcal{Q} \rangle = \langle u^* \tau(x) u \mathcal{Q} y, \mathcal{Q} \rangle = \langle \tau(x) u \mathcal{Q} y, u \mathcal{Q} \rangle.$$

So, if $\overline{\tau(M) u \mathcal{Q} M} = H$, we obtain $[\rho_E] = [\tau]$. We prove $\overline{\tau(M) u \mathcal{Q} M} = H$ as follows. Let p be the projection onto $\tau(M) u \mathcal{Q} M^\perp$. Then p belongs to $M \cap \tau(M)'$ and $pu\mathcal{Q} = 0$. Since \mathcal{Q} is a separating vector for M , $pu = 0$ and $puu^* = pJ_N^{\mathcal{Q}} e_N J_N^{\mathcal{Q}} = 0$. So $0 = J_N^{\mathcal{Q}} p J_N^{\mathcal{Q}} e_N \xi_0 = J_N^{\mathcal{Q}} p J_N^{\mathcal{Q}} \xi_0$. Due to $p \in M \cap \tau(M)'$

$$J_N^{\mathcal{Q}} p J_N^{\mathcal{Q}} \in J_N^{\mathcal{Q}} M J_N^{\mathcal{Q}} \cap J_M^{\mathcal{Q}} M' J_M^{\mathcal{Q}} = J_N^{\mathcal{Q}} M J_N^{\mathcal{Q}} \cap M.$$

Since ξ_0 is a separating vector for M , we obtain $p = 0$ and so $\overline{\tau(M) u \mathcal{Q} M} = H$.

Moreover if M is isomorphic to N and $\rho \in \text{End}(M)$ with $\rho(M) = N$, then $[\rho_E] = [\tau] = [\rho\bar{\rho}]$. In Sections 4 and 5, we calculate $[\rho\bar{\rho}]$ from the conditional expectation by the above formula.

2.3. *Minimal Expectations and the Basic Construction.* Let M be a properly infinite factor and $\rho \in \text{End}(M)$ with $d_\rho < \infty$. The tower associated with $M \supset \rho(M)$ is [L2, Corollary 4.14]

$$M \supset \rho(M) \supset \rho\bar{\rho}(M) \supset \rho\bar{\rho}\rho(M) \supset \dots.$$

This formula enable us to analyze the index theory by sectors. By decomposing $[(\rho\bar{\rho})^n]$ and $[(\rho\bar{\rho})^n\rho]$, we get new sectors, which we call descendant sectors.

To calculate statistical dimensions of descendant sectors, we use Hiai's minimal expectation [H].

If $M \supset N$ are a pair of factors with a finite index and $E \in E(M, N)$, the following are equivalent [H] [L1]:

- (i) E is the minimal expectation.
- (ii) $E|_{M \cap N'}$ and $E^{-1}|_{M \cap N'}$ are traces and $E^{-1}|_{M \cap N'} = (\text{Index } E) E|_{M \cap N'}$. (E^{-1} is a map from N' to M' , canonically defined from E [C, Ha].)
- (iii) $\text{Index } E_e = E(e)^2 (\text{index } E)$ for all projections $e \in N' \cap M$ where $E_e = E(e)^{-1} E(\cdot)e|_{M_e}$.

By (iii), if $\{e_i\}$ is a family of mutually orthogonal projections in $M \cap N'$ with $\sum_i e_i = 1$, then

$$\sum_i (\text{Index } E_{e_i})^{1/2} = (\text{Index } E)^{1/2} \sum_i E(e_i) = (\text{Index } E)^{1/2}.$$

This means that if $[\rho] = \bigoplus_i [\rho_i]$, then $d_\rho = \sum_i d_{\rho_i}$ for $\rho, \rho_i \in \text{End}(M)$. This is the reason why we use d_ρ instead of the index.

H. Kosaki and R. Longo show the mutiplicativity of the minimal index in case of the basic construction [KL], and using this, R. Longo shows the multiplicativity in general case [L4]. Let $M \supset N \supset L$ be inclusions of factors with finite indices and $E_1 \in E(M, N), E_2 \in E(N, L)$ be the minimal expectations. Then $E_1 \circ E_2$ is also the minimal expectation from M to L and so $[M: N]_0 [N: L]_0 = [M: L]_0$ by (2.1). This means that if $\rho_1, \rho_2 \in \text{End}(M)$ with $d_{\rho_1}, d_{\rho_2} < \infty$, then $d_{\rho_1 \rho_2} = d_{\rho_1} d_{\rho_2}$.

Let

$$M = N_{-1} \supset N_0 \supset N_1 \supset N_2 \cdots$$

be the tower associated with the inclusion of factors $M \supset N_0$, and $\varepsilon_i \in E(N_{i-1}, N_i)$ be the minimal expectations. If we define E_n by

$$E_n = \varepsilon_n \circ \varepsilon_{n-1} \circ \cdots \circ \varepsilon_0,$$

then E_n turns out to be the minimal expectation from M to N_n , and $[M: N_n]_0 = [M: N_0]_0^{n+1}$.

Since E_n is a trace on $M \cap N'_n$ by (ii),

$$\varphi \equiv \lim_n E_n|_{\bigvee_m M \cap N'_m}$$

is a trace on $\bigvee_m M \cap N'_m$. Moreover, if depth of $M \supset N_0$ is finite, it is well-known

that φ is the unique trace and values of φ on minimal projections of $M \cap N'_\#$ are determined by the Perron-Frobenius eigenvector of the principal graph. [GHJ]

Let $\rho \in \text{End}(M)$ with $\rho(M) = N$ and $[(\rho\bar{\rho})^m] = \bigoplus_i [\rho_i]$ be the irreducible decomposition, with $\{p_i\}$ the corresponding minimal projections in $M \cap (\rho\bar{\rho})^m(M)'$. Then

$$d_{p_i} = (\text{Index } (E_{2m-1})_{\rho_i})^{1/2} = E_{2m-1}(p_i) (\text{Index } E_{2m-1})^{1/2} = \varphi(p_i) (d_\rho)^{2m}.$$

A similar calculation holds for $[(\rho\bar{\rho})^m \rho]$. So we can calculate d_{p_i} from the Perron-Frobenius eigenvector.

2.4. *Characterization of conjugate sectors* The most important tool of this paper is the following theorem.

Theorem 2.1. *Let M be a properly infinite factor and $\rho_1, \rho_2 \in \text{End}(M)$ with $d_{\rho_1}, d_{\rho_2} < \infty$. If ρ_1 and ρ_2 are irreducible, then the following conditions are equivalent:*

- (i) $[\rho_1][\rho_2]$ or $[\rho_2][\rho_1]$ contains the identity sector $[id]$.
- (ii) $[\rho_1] = \overline{[\rho_2]}$.

If ρ_1 and ρ_2 satisfy the above conditions, $[\rho_1][\rho_2]$ and $[\rho_2][\rho_1]$ contain $[id]$ with multiplicity one.

This theorem is essentially due to R. Longo [L2, Theorem 4.1]. The only improvement over his theorem is in the condition (i), in which we demand that one of $[\rho_1][\rho_2]$ and $[\rho_2][\rho_1]$ contains $[id]$, instead of both. For the proof of (i) \Rightarrow (ii), we may assume that $[\rho_1][\rho_2]$ contains $[id]$ and also assume that $d_{\rho_1} \leq d_{\rho_2}$ (if not, we can consider $\overline{[\rho_2]} \overline{[\rho_1]}$ instead). Then his proof for (i) \Rightarrow (ii) applies to our case.

Before closing this section, we would like to pose the following question.

Question 2.2. *Let M be a properly infinite factor and N be a given subfactor of M which is isomorphic to M . Can we find a self-conjugate $\rho \in \text{End}(M)$ such that $\rho(M) = N$?*

In general, the answer is no. We will show an affirmative case in Section 3 and a negative case in Section 4.

§3. Fusion Rules of Sectors

Let $M \supset N$ be a pair of factors with $[M:N] < 4$. Then it is well-known

that the principal graph is one of the Coxeter graphs A_n, D_n, E_6, E_7, E_8 [GHJ, Corollary 4.6.6]. Moreover, A. Ocneanu states in [O1, page 162] that D_{odd} and E_7 do not appear, but no complete proof has been published until now. In this section, we compute the fusion rules of sectors, which is the rules of the irreducible decomposition of sectors, associated with the above graphs and prove the non-appearance of D_{odd} and E_7 .

Let M be a properly infinite factor and $\rho_1 \in \text{End}(M)$ with $d_{\rho_1} < \infty$. Since

$$M \supset \rho_1(M) \supset \rho_1 \bar{\rho}_1(M) \supset \rho_1 \bar{\rho}_1 \rho_1(M) \supset \dots$$

is the tower associated with $M \supset \rho_1(M)$ as quoted in Subsection 2.3, the Bratteli diagram of

$$M \cap M' = \mathcal{C} \subset M \cap \rho_1(M)' \subset M \cap \rho_1 \bar{\rho}_1(M)' \subset M \cap \rho_1 \bar{\rho}_1 \rho_1(M)' \subset \dots$$

is decided by the principal graph of $M \supset \rho_1(M)$. We saw in Subsection 2.3 that minimal projections in $M \cap (\rho_1 \bar{\rho}_1)^n(M)'$ and $M \cap (\rho_1 \bar{\rho}_1)^n \rho_1(M)'$ correspond to descendant sectors. This means that the Bratteli diagram gives fusion rules of descendant sectors of ρ_1 . We show this more precisely.

Let $\rho, \sigma \in \text{End}(M)$ and $[\rho] = \bigoplus_{i \in I} m(i) [\rho_i], [\rho_i] [\sigma] = \bigoplus_{j \in J} n(i, j) [\sigma_j]$ be the irreducible decompositions of $[\rho]$ and $[\rho_i] [\sigma]$, where $m(i)$ and $n(i, j)$ are the multiplicities and $n(i, j)$ may take 0. Then there are isometries $\{S(i)_k\}_{i \in I, k=1, 2, \dots, m_i}$, $\{T(i, j)_l\}_{i \in I, j \in J, l=1, 2, \dots, n(i, j)}$ in M satisfying the following.

$$\sum_{i \in I} \sum_{k=1}^{m(i)} S(i)_k S(i)_k^* = 1, \tag{3.0.1}$$

$$\sum_{j \in J} \sum_{l=1}^{n(i, j)} T(i, j)_l T(i, j)_l^* = 1, \tag{3.0.2}$$

$$\rho(x) = \sum_{i \in I} \sum_{k=1}^{m(i)} S(i)_k \rho_i(x) S(i)_k^*, \tag{3.0.3}$$

$$\rho_i \circ \sigma(x) = \sum_{j \in J} \sum_{l=1}^{n(i, j)} T(i, j)_l \sigma_j(x) T(i, j)_l^*, \tag{3.0.4}$$

$$\rho \circ \sigma(x) = \sum_{j \in J} \left[\sum_{i \in I} \sum_{k=1}^{m(i)} \sum_{l=1}^{n(i, j)} S(i)_k T(i, j)_l \sigma_j(x) T(i, j)_l^* S(i)_k^* \right]. \tag{3.0.5}$$

Due to (3.0.3) and (3.0.5), the central decompositions of $M \cap \rho(M)'$ and $M \cap \rho \circ \sigma(M)'$ are as follows.

$$\begin{aligned} M \cap \rho(M)' &= \bigoplus_{i \in I} A(i), \\ M \cap \rho \circ \sigma(M)' &= \bigoplus_{j \in J} B(j), \\ A(i) &\equiv C^*(S(i)_{k_1} S(i)_{k_2}^*)_{k_1, k_2}, \\ B(j) &\equiv C^*(S(i_1)_{k_1} T(i_1, j)_{l_1} T(i_2, j)_{l_2}^* S(i_2)_{k_2}^*)_{i_1, i_2, k_1, k_2, l_1, l_2}. \end{aligned}$$

Note that $\{S(i)_{k_1} S(i)_{k_2}^*\}_{k_1, k_2}$ and $\{S(i_1)_{k_1} T(i_1, j)_{i_1} T(i_2, j)_{i_2}^* S(i_2)_{k_2}^*\}_{i_1, i_2, k_1, k_2, i_1, i_2}$ are the matrix units of $A(i)$ and $B(j)$. Thanks to (3.0.2), a minimal projection $S(i)_k S(i)_k^*$ in $A(i)$ is also written as follows.

$$S(i)_k S(i)_k^* = \sum_{j \in \mathcal{J}} \left[\sum_{i=1}^{n(i,j)} S(i)_k T(i, j)_i T(i, j)_i^* S(i)_k^* \right].$$

This means that in the Bratteli diagram of $\bigoplus_{i \in \mathcal{I}} A(i) \subset \bigoplus_{j \in \mathcal{J}} B(j)$, there are $n(i, j)$ edges between $A(i)$ and $B(j)$, where $n(i, j)$ is the multiplicity of $[\sigma_j]$ in $[\rho_i] [\sigma]$.

We assume that the principal graph is one of A_n, D_n, E_6, E_7, E_8 and explain the details of computations one by one.

3.1. *The Case of A_{2n}* For A_n , the corresponding subfactors are constructed in [J, Thoerem 4.3.2]. The Bratteli diagram is as in Fig. 1.

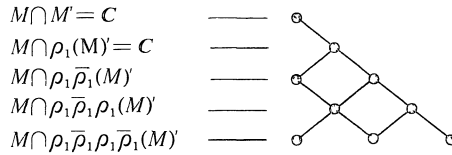


Fig. 1. The Bratteli diagram in the case of A_{2n} .

First we look at the level of $M \cap \rho_1 \bar{\rho}_1(M)' = C \oplus C$. Since there are two points in this level, $[\rho_1 \bar{\rho}_1]$ is decomposed into two sectors. One of these is $[id]$ by the characterization of the conjugate sector in Subsection 2.4 and call the other sector as $[\rho_2]$. The next question is which of two points corresponds to $[id]$. The answer is the left one. Indeed, since

$$[\rho_1 \bar{\rho}_1 \rho_1] = [\rho_1 \bar{\rho}_1] [\rho_1] = ([id] \oplus [\rho_2]) [\rho_1] = [id] [\rho_1] \oplus [\rho_2] [\rho_1]$$

we see from the diagram that one of $[id] [\rho_1]$ and $[\rho_2] [\rho_1]$ is decomposed into two sectors. But $[id] [\rho_1] = [\rho_1]$ is irreducible and then the left one corresponds to $[id]$.

Next we look at the level of $M \cap \rho_1 \bar{\rho}_1 \rho_1(M)' \simeq M_2 \oplus C$. By the above argument, we know that $[\rho_1 \bar{\rho}_1 \rho_1]$ contains $[\rho_1]$ twice and hence $[\rho_2] [\rho_1]$ contains $[\rho_1]$. We call the other irreducible component of $[\rho_2] [\rho_1]$ as $[\rho_3]$. By repeating such an argument, we get Fig. 2, where each sector is mutually different in each row, and the following basic fusion rules. ($[\rho_0] = [id]$ in $k=0$ equation.)

$$[\rho_{2k+1}] [\bar{\rho}_1] = [\rho_{2k}] \oplus [\rho_{2k+2}], \quad k = 0, 1, \dots, n-2, \tag{3.1.1}$$

$$[\rho_{2n-1}] [\bar{\rho}_1] = [\rho_{2n-2}], \tag{3.1.2}$$

$$[\rho_{2k}] [\rho_1] = [\rho_{2k-1}] \oplus [\rho_{2k+1}], \quad k = 1, 2, \dots, n-1. \tag{3.1.3}$$

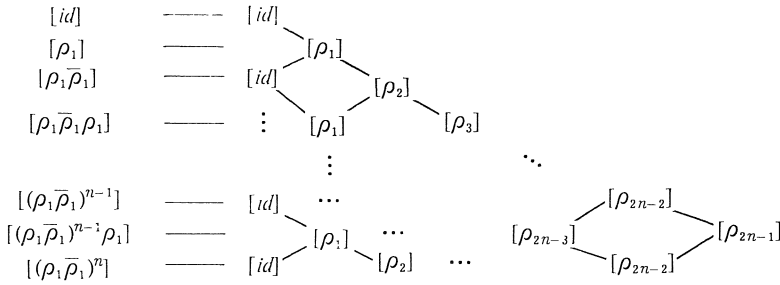


Fig. 2. The diagram of fusion rules in the case of A_{2n} .

We can also compute the products of other pairs:

Lemma 3.1.

$$[\bar{\rho}_{2k+1}] [\rho_2] = [\bar{\rho}_{2k-1}] \oplus [\bar{\rho}_{2k+1}] \oplus [\bar{\rho}_{2k+3}], \quad k = 1, 2, \dots, n-2, \quad (3.1.4)$$

$$[\bar{\rho}_1] [\rho_2] = [\bar{\rho}_1] \oplus [\bar{\rho}_3], \quad (3.1.5)$$

$$[\bar{\rho}_{2n-1}] [\rho_2] = [\bar{\rho}_{2n-3}], \quad (3.1.6)$$

$$[\rho_{2k}] [\rho_2] = [\rho_{2k-2}] \oplus [\rho_{2k}] \oplus [\rho_{2k+2}], \quad k = 1, 2, \dots, n-2, \quad (3.1.7)$$

$$[\rho_{2n-2}] [\rho_2] = [\rho_{2n-4}] \oplus [\rho_{2n-2}]. \quad (3.1.8)$$

Proof. Since $[\rho_1 \bar{\rho}_1] = [id] \oplus [\rho_2]$, we may compute $[\bar{\rho}_{2k+1}] [\rho_1 \bar{\rho}_1]$ and $[\rho_{2k}] [\rho_1 \bar{\rho}_1]$ instead of the above products because of the uniqueness of the irreducible decomposition. $[\rho_{2k}] [\rho_1 \bar{\rho}_1]$ is easily obtained by going down from the level $(\rho_1 \bar{\rho}_1)^k$ to the level $(\rho_1 \bar{\rho}_1)^{k+1}$ (namely by 2 levels), starting from $[\rho_{2k}]$. To compute $[\bar{\rho}_{2k+1}] [\rho_1 \bar{\rho}_1]$ we use fusion rules of descendant sectors of $[\bar{\rho}_1]$. It is easy to see that if $d_{\rho_1} < 2$ the principal graphs of $M \supset \rho_1(M)$ and $M \supset \bar{\rho}_1(M)$ coincide because $M \supset \bar{\rho}_1(M)$ is isomorphic to $M_1 \supset M$ where M_1 is the extension of M by $\rho_1(M)$ [L2, page 296]. (Or more directly, due to $\overline{[(\rho_1 \bar{\rho}_1)^n \rho_1]} = [(\bar{\rho}_1 \rho_1)^n \bar{\rho}_1]$, $\dim Z(M \cap (\rho_1 \bar{\rho}_1)^n \rho_1(M)') = \dim Z(M \cap (\bar{\rho}_1 \rho_1)^n \bar{\rho}_1(M)')$ and so the principal graph of $M \supset \bar{\rho}_1(M)$, which is one of A_n, D_n, E_6, E_7 , and E_8 , must coincide with that of $M \supset \rho_1(M)$). Let $\{[\delta_i]\}_{i=1}^{2n-1} \subset \text{Sect}(M)$ be the descendant sectors of $[\bar{\rho}_1]$. Since $\overline{[(\bar{\rho}_1 \rho_1)^k \bar{\rho}_1]} = [(\rho_1 \bar{\rho}_1)^k \rho_1]$, we get $[\delta_{2k+1}] = [\bar{\rho}_{2k+1}]$. And we can compute $[\bar{\rho}_{2k+1}] [\rho_1 \bar{\rho}_1]$ from the diagram of $\{\delta_i\}_{i=1}^{2n-1}$. Q.E.D.

The Perron-Frobenius eigenvector $(v_j)_{j=1}^{2n}$ of A_{2n} is given by $v_j = (\sin \frac{j\pi}{2n+1}) / (\sin \frac{\pi}{2n+1})$ (see Fig. 3).

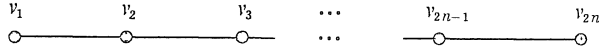


Fig. 3. The Perron-Frobenius eigenvector of A_{2n} .

As we saw in Subsection 2.3, we can compute d_{ρ_i} from the Perron-Frobenius eigenvector:

$$d_{\rho_i} = \frac{\sin \frac{i+1}{2n+1} \pi}{\sin \frac{\pi}{2n+1}}$$

Note that $d_{\rho_{2n-1}}=1$. This means $\rho_{2n-1} \in \text{Aut}(M)$ and we write $\alpha = \rho_{2n-1}$. If θ_1 and θ_2 in $\text{End}(M)$ have the same image $\theta_1(M) = \theta_2(M)$, there is a $\beta \in \text{Aut}(M)$ such that $\theta_1 = \theta_2 \circ \beta$. In other words, if we analyze subfactors by endomorphisms, we have the ambiguity of multiplying automorphisms from right in the choice of the endomorphism representing the subfactor. In the following proposition, we see that we can use this ambiguity to cancel α to make all $[\rho_k]$ self-conjugate. Thus the answer to the Question 2.2 in Section 2 is yes in this case.

Proposition 3.2. *In the case of A_{2n} , a self-conjugate ρ_1 can be chosen such that $[\rho_k] = [\rho_{2n-1-k}]$ and all descendant sectors $[\rho_k]$ are self-conjugate. The diagram of fusion rules is as in Fig. 4.*

Proof. We go back to Fig. 2. It is easy to see that $[\rho_{2k}]$ is self-conjugate by induction because $[id]$ and $[(\rho_1 \bar{\rho}_1)^k]$ are self-conjugate. Since $[\alpha][\bar{\rho}_1] = [\rho_{2n-1}][\bar{\rho}_1] = [\rho_{2n-2}]$ by (3.1.2), we get $[\rho_1][\alpha^{-1}] = [\rho_{2n-2}]$ by conjugation. We now show $[\rho_{2k-1}] = [\rho_{2n-2k}][\alpha]$ or equivalently $[\bar{\rho}_{2k-1}] = [\alpha^{-1}][\rho_{2n-2k}]$ for $k=1, 2, \dots, n$. For $k=1$ we know $[\bar{\rho}_1] = [\alpha^{-1}][\rho_{2n-2}]$ by the above argument. By using Lemma 3.1 and induction, we obtain the desired equality.

Due to $[\rho_1][\alpha^{-1}] = [\rho_{2n-2}]$, we have $\rho_{2n-2}(M) = \rho_1(M)$ so that we can take ρ_{2n-2} instead of ρ_1 . Then $[\rho_1]$ is replaced by $[\rho_1][\alpha^{-1}] = [\rho_1 \alpha^{-1}]$ and hence

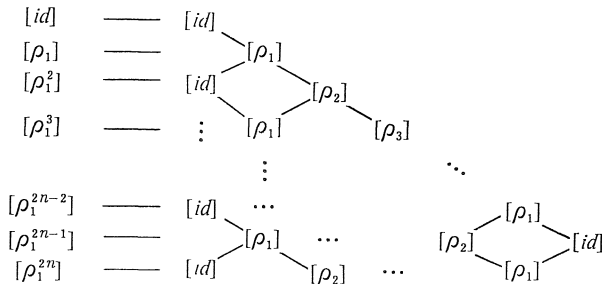


Fig. 4. The diagram of fusion rules in Proposition 3.2.

$[(\rho_1 \bar{\rho}_1)^k \rho_1]$ turns into $[(\rho_1 \bar{\rho}_1)^k \rho_1] [\alpha^{-1}]$, while $[(\rho_1 \bar{\rho}_1)^k]$ does not change. So we complete the proof. Q.E.D.

In principle, we can compute every principal graph of $M \supset \rho_i(M)$ from fusion rules, because

$$M \supset \rho_i(M) \supset \rho_i \bar{\rho}_i(M) \supset \rho_i \bar{\rho}_i \rho_i(M) \supset \dots$$

is the associated tower. For example, due to (3.1.7) and (3.1.8) that of $M \supset \rho_2(M)$ is as in Fig. 5.

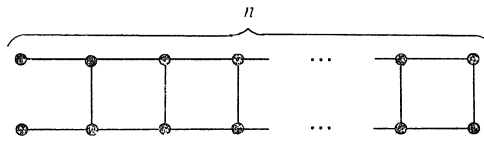


Fig. 5. The principal graph of $M \supset \rho_2(M)$ in the case of A_{2n} .

3.2. *The Case of A_{2n+1}* The diagram of fusion rules is as in Fig. 6 and the basic fusion rules and the statistical dimensions of descendant sectors are as follows.

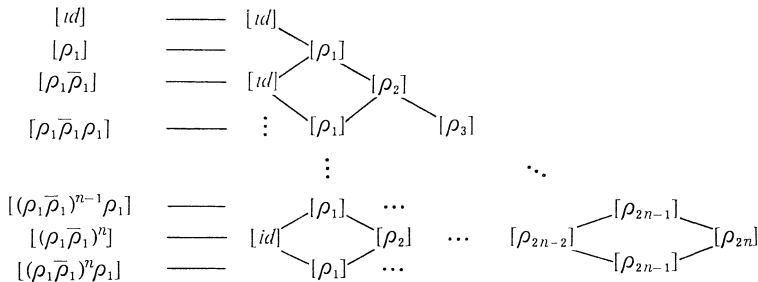


Fig. 6. The diagram of fusion rules in the case of A_{2n+1} .

$$[\rho_{2k+1}] [\bar{\rho}_1] = [\rho_{2k}] \oplus [\rho_{2k+2}] \quad k = 0, 1, \dots, n-1 \tag{3.2.1}$$

$$[\rho_{2k}] [\rho_1] = [\rho_{2k-1}] \oplus [\rho_{2k+1}] \quad k = 1, 2, \dots, n-1 \tag{3.2.2}$$

$$[\rho_{2n}] [\rho_1] = [\rho_{2n-1}] \tag{3.2.3}$$

$$d_{\rho_i} = \frac{\sin \frac{i+1}{2n+2} \pi}{\sin \frac{\pi}{2n+2}}$$

As in Subsection 3.1, $\rho_{2n} \in \text{Aut}(M)$ and we write $\alpha = \rho_{2n}$. Difference from

the case of A_{2n} is that α is self-conjugate and an invariant of the subfactor $\rho_1(M)$. $[\alpha]$ plays essential rôle in Theorem 5.1.

Proposition 1.3. (i) $[\rho_{2n-k}] = [\alpha] [\rho_k]$ and especially $[\rho_n] = [\alpha] [\rho_n]$.

(ii) We can take representatives α and ρ_n such that α is a \mathbb{Z}_2 action and $\rho_n(M) \subset M^\alpha$ where M^α is the fixed point algebra under α .

Proof. (i): This is easily shown by induction. (ii): Since $[\rho_n] = [\alpha] [\rho_n]$, we can take a representative α such that $\rho_n = \alpha \circ \rho_n$. As we mentioned at the end of Subsection 2.1, α^2 is inner and $\alpha^2 \circ \rho_n = \rho_n$. Since ρ_n is irreducible so $M \cap \rho_n(M)' = \mathbb{C}$ and then $\alpha^2 = id$. $\rho_n(M) \subset M^\alpha$ follows from $\alpha \circ \rho_n = \rho_n$.

Q.E.D.

For later use, we show the following lemma.

Lemma 3.4. If M is the type III₁ AFD factor, then we can take ρ_1 satisfying $[\rho_1] [\alpha] = [\alpha] [\rho_1]$.

Proof. By the work of Y. Kawahigashi, C.E. Surtherland and M. Takesaki [KST, Theorem 20], an outer action of \mathbb{Z}_2 on the type III₁ AFD factor is unique as a cocycle conjugacy class. As in the proof of Lemma 3.1, if $\{[\delta_i]\}_{i=1}^{2n}$ are descendant sectors of $\bar{\rho}_1$ then $[\delta_{2i+1}] = [\bar{\rho}_{2i+1}]$ and we may assume that δ_{2n} is a \mathbb{Z}_2 action. So there exists a $\theta \in \text{Aut}(M)$ such that

$$[\delta_{2n}] = [\theta] [\alpha] [\theta^{-1}].$$

Since $[\rho_{2n-1}] = [\alpha] [\rho_1]$ and $[\bar{\rho}_{2n-1}] = [\delta_{2n}] [\bar{\rho}_1]$ from Lemma 3.3, we have

$$[\alpha] [\rho_1] = [\bar{\delta_{2n}}] [\bar{\rho}_1] = [\rho_1] [\delta_{2n}] = [\rho_1] [\theta] [\alpha] [\theta^{-1}]$$

and $[\alpha] [\rho_1 \circ \theta] = [\rho_1 \circ \theta] [\alpha]$. We can take $\rho_1 \circ \theta$ instead of ρ_1 . Then $[\rho_{2k}]$ are unchanged and in particular $[\alpha]$ remains the same. So we have $[\rho_1] [\alpha] = [\alpha] [\rho_1]$ for new ρ_1 .

Q.E.D.

The principal graph of $M \supset \rho_2(M)$ is as in Fig. 7.

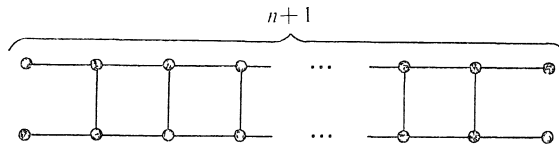


Fig. 7. The principal graph of $M \supset \rho_2(M)$ in the case of A_{2n+1} .

In [W1], subfactors with index $\left(\frac{\sin(k/l)\pi}{\sin\pi/l}\right)^2$ ($l=3, 4, \dots, k=1, 2, \dots$) appear

twice, once in page 360 and once in page 380. Note that the former for $k=3$ and our $M \supset \rho_i(M)$ have the same principal graphs. Since the graphs in Fig. 5 and Fig. 7 do not coincide with the principal graphs of the latter [W2], for $k=3$ the above two subfactors are different. The author would like to thank Y. Kawahigashi who suggested the author the above difference and showed him the note of Wenzl's lecture at the Mittag-Leffler Institute in 1988.

3.3. *The Case of $E_6, E_7,$ and E_8 .* Suppose that the principal graph of $M \supset \rho_1(M)$ is E_7 . The descendant sectors are as in Fig. 8 where we write only the principal part.

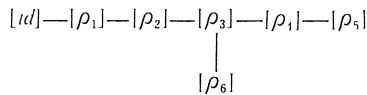


Fig. 8. The descendant sectors in the case of E_7 .

We can compute d_{ρ_i} from the Perron-Frobenius eigenvector as before and we obtain

$$d_{\rho_5} = \frac{\sin \frac{6}{18} \pi}{4 \cos^2 \frac{\pi}{18} \sin \frac{\pi}{18}} = 2 \cos \frac{5}{18} \pi < 2 \cos \frac{\pi}{4}.$$

This value is not admissible as a statistical dimension [J, Theorem 4.3.1] [K1, Theorem 5.4] and hence we have a contradiction.

If we take E_6 and E_8 , we can confirm that there is no problem. Suppose that the principal graph of $M \supset \rho_1(M)$ is E_6 . The descendant sectors are as in Fig. 9.

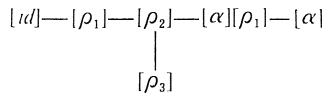


Fig. 9. The descendant sectors in the case of E_6 .

The statistical dimensions are as follows.

$$d_{\rho_1} = 2 \cos \frac{\pi}{12}, \quad d_{\rho_2} = \frac{\sin \frac{3}{12} \pi}{\sin \frac{\pi}{12}}, \quad d_{\rho_3} = 2 \cos \frac{\pi}{4}, \quad d_{\alpha} = 1.$$

As in the case of A_{2n+1} , α is a \mathbb{Z}_2 action. This is a reflection of the \mathbb{Z}_2 symmetry of bicolored labelled E_6 . Fusion rules of descendant sectors are as follows.

$$[\rho_1 \bar{\rho}_1] = [id] \oplus [\rho_2], \tag{3.3.1}$$

$$[\rho_2 \rho_1] = [\rho_1] \oplus [\rho_3] \oplus [\alpha \rho_1], \tag{3.3.2}$$

$$[\rho_2] = [\rho_3 \bar{\rho}_1] = [\rho_1 \bar{\rho}_3], \tag{3.3.3}$$

$$[\rho_2]^2 = [id] \oplus [\alpha] \oplus 2[\rho_2], \tag{3.3.4}$$

$$[\rho_2 \rho_3] = [\rho_1] \oplus [\alpha \rho_1], \tag{3.3.5}$$

$$[\rho_3 \bar{\rho}_3] = [id] \oplus [\alpha]. \tag{3.3.6}$$

(3.3.3) shows the existence of the following inclusions of factors.

$$M \supset \rho_1(M) \supset \rho_2(M),$$

$$[\rho_1(M) : \rho_2(M)]_0 = [M : \bar{\rho}_3(M)]_0 = 2.$$

These inclusions suggest the way to construct an inclusion with the principal graph E_6 . In [I], we construct ρ_2 on the type $\text{III}_{1/(1+\sqrt{3})}$ AFD factor by using Cuntz algebra O_4 . From $[\rho_1(M) : \rho_2(M)]_0=2$, the relation between $\rho_1(M)$ and $\rho_2(M)$ is the crossed product by a \mathbb{Z}_2 action. Using this fact, we also construct an inclusion of factors with the principal graph E_6 . The principal graph of $M \supset \rho_2(M)$ is as in Fig. 10.

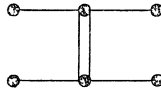


Fig. 10. The principal graph of $M \supset \rho(M)_2$ in the case of E_6 .

Suppose that the principal graph of $M \supset \rho_1(M)$ is E_8 . The descendant sectors are as in Fig. 11.

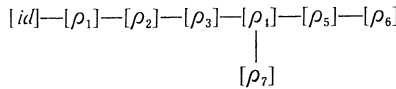


Fig. 11. The descendant sectors in the case of E_8 .

The minimal statistical dimension of the descendant sectors is given by

$$\min \{d_{\rho_i}; i = 1, 2, \dots, 7\} = d_{\rho_6} = 2 \cos \frac{\pi}{5}.$$

Note that $d_{\rho_i} > 2$ for $i \neq 1, 6$. By computing fusion rules of descendant sectors, we have the following relations.

$$[\rho_3] = [\rho_6 \rho_7], \tag{3.3.7}$$

$$[\rho_4] = [\rho_6 \rho_2] = [\rho_7 \bar{\rho}_1], \tag{3.3.8}$$

$$[\rho_5] = [\rho_6 \rho_1]. \tag{3.3.9}$$

From $d_{\rho_6} = 2 \cos \frac{\pi}{5}$, the principal graph of $M \supset \rho_6(M)$ is A_4 . In analogy with the case of E_6 , it is possible to say “ E_8 has A_4 symmetry”. The principal graph of $M \supset \rho_2(M)$ is as in Fig. 12.

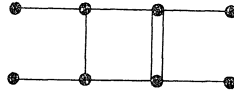


Fig. 12. The principal graph of $M \supset \rho_2(M)$ in the case of E_8 .

3.4. *The Case of D_{2n+3} .* Suppose that the principal graph of $M \supset \rho_1(M)$ is D_{2n+3} , $n \geq 1$. The descendant sectors are as in Fig. 13.

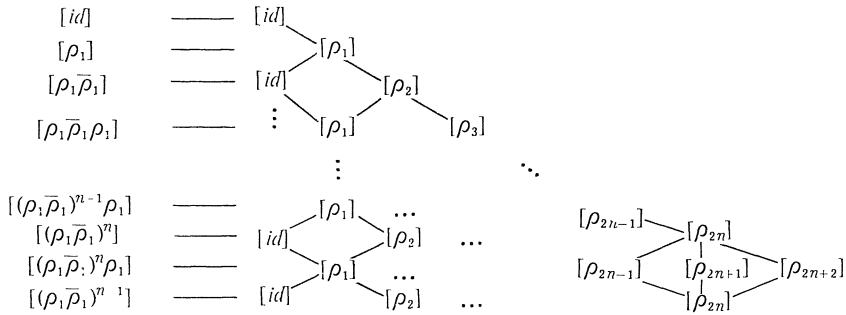


Fig. 13. The diagram of fusion rules in the case of D_{2n+3} .

The basic fusion rules are given by

$$[\rho_{2k+1}] [\bar{\rho}_1] = [\rho_{2k}] \oplus [\rho_{2k+2}], \quad k = 0, 1, \dots, n-1, \tag{3.4.1}$$

$$[\rho_{2n+1}] [\bar{\rho}_1] = [\rho_{2n+2}] [\bar{\rho}_1] = [\rho_{2n}], \tag{3.4.2}$$

$$[\rho_{2k}] [\rho_1] = [\rho_{2k-1}] \oplus [\rho_{2k+1}], \quad k = 1, 2, \dots, n-1, \tag{3.4.3}$$

$$[\rho_{2n}] [\rho_1] = [\rho_{2n-1}] \oplus [\rho_{2n+1}] \oplus [\rho_{2n+2}]. \tag{3.4.4}$$

The statistical dimensions are given by

$$d_{\rho_i} = \frac{\sin \frac{i+1}{4n+4} \pi}{\sin \frac{\pi}{4n+4}}, \quad i = 0, 1, \dots, 2n,$$

$$d_{\rho_{2n+1}} = d_{\rho_{2n+2}} = \frac{1}{2 \sin \frac{\pi}{4n+4}}.$$

We note that $[\rho_i], i=0, 1, \dots, 2n+1, 2n+2$ must be mutually different, for example due to the difference values of their statistical dimensions. $[\rho_{2n+1}]$ and $[\rho_{2n+2}]$ also must be different because if not, the principal graph changes.

Lemma 3.5.

$$d_{\rho_{2n+1} \bar{\rho}_{2n+1}} = d_{\rho_{2n+2} \bar{\rho}_{2n+2}} = d_{\rho_{2n+1} \bar{\rho}_{2n+2}} = d_{\rho_{2n+2} \bar{\rho}_{2n+1}} = \frac{1}{4 \sin^2 \frac{\pi}{4n+4}}.$$

Proof. Due to $d_{\rho_{2n+1}} = d_{\rho_{2n+2}} = \frac{1}{2 \sin \frac{\pi}{4n+4}}$ and the multiplicativity of the

statistical dimension, the equalities hold.

Q.E.D.

Lemma 3.6. (i)

$$[\bar{\rho}_{2k+1}] [\rho_2] = [\bar{\rho}_{2k-1}] \oplus [\bar{\rho}_{2k+1}] \oplus [\bar{\rho}_{2k+3}], \quad k = 1, 2, \dots, n-2, \quad (3.4.5)$$

$$[\bar{\rho}_1] [\rho_2] = [\bar{\rho}_1] \oplus [\bar{\rho}_3], \quad (3.4.6)$$

$$[\bar{\rho}_{2n-1}] [\rho_2] = [\bar{\rho}_{2n-3}] \oplus [\bar{\rho}_{2n-1}] \oplus [\bar{\rho}_{2n+1}] \oplus [\bar{\rho}_{2n+2}], \quad (3.4.7)$$

$$[\bar{\rho}_{2n+1}] [\rho_2] = [\bar{\rho}_{2n-1}] \oplus [\bar{\rho}_{2n+2}], \quad (3.4.8)$$

$$[\bar{\rho}_{2n+2}] [\rho_2] = [\bar{\rho}_{2n-1}] \oplus [\bar{\rho}_{2n+1}], \quad (3.4.9)$$

$$[\rho_{2k}] [\rho_2] = [\rho_{2k-2}] \oplus [\rho_{2k}] \oplus [\rho_{2k+2}], \quad k = 1, 2, \dots, n-1, \quad (3.4.10)$$

$$[\rho_{2n}] [\rho_2] = [\rho_{2n-2}] \oplus 2[\rho_{2n}]. \quad (3.4.11)$$

(ii)

$$\begin{aligned} [\rho_{2n+1}] [\bar{\rho}_{2k+1}] &= [\rho_{2n+2}] [\bar{\rho}_{2k+1}] \\ &= [\rho_{2(n-k)}] \oplus [\rho_{2(n-k+1)}] \oplus \dots \oplus [\rho_{2n}], \quad k = 0, 1, \dots, n-1, \end{aligned} \quad (3.4.12)$$

$$[\rho_{2n+1}] [\bar{\rho}_{2n+1}] \oplus [\rho_{2n+1}] [\bar{\rho}_{2n+2}] = [id] \oplus [\rho_2] \oplus \dots \oplus [\rho_{2n}]. \quad (3.4.13)$$

(iii)

$$[\rho_{2n+1}] [\bar{\rho}_{2n+1}] [\rho_2] = [\rho_2] \oplus [\rho_4] \oplus \dots \oplus [\rho_{2n}] \oplus [\rho_{2n+1}] [\bar{\rho}_{2n+2}], \quad (3.4.14)$$

$$[\rho_{2n+1}] [\bar{\rho}_{2n+2}] [\rho_2] = [\rho_2] \oplus [\rho_4] \oplus \dots \oplus [\rho_{2n}] \oplus [\rho_{2n+1}] [\bar{\rho}_{2n+1}]. \quad (3.4.15)$$

Proof. (i): This can be obtained in the same way as the proof of Lemma 3.1.

(ii): For $k=0$, we already have $[\rho_{2n+1}] [\bar{\rho}_1] = [\rho_{2n+2}] [\bar{\rho}_1] = [\rho_{2n}]$. For $k=1$, the

equality holds from $[\rho_{2n+1}][\bar{\rho}_1][\rho_2]=[\rho_{2n+2}][\bar{\rho}_1][\rho_2]=[\rho_{2n}][\rho_2]$ and (i). In the same way, general case holds by induction.

(iii): From (i) and (ii), we obtain

$$\begin{aligned} [\rho_{2n+1}][\bar{\rho}_{2n+1}][\rho_2] &= [\rho_{2n+1}](\bar{\rho}_{2n-1} \oplus \bar{\rho}_{2n+2}) \\ &= [\rho_2] \oplus [\rho_4] \oplus \cdots \oplus [\rho_{2n}] \oplus [\rho_{2n+1}][\bar{\rho}_{2n+2}]. \end{aligned}$$

Second equality is shown in the same way.

Q.E.D.

Theorem 3.7. *Let $M \supset N$ be a pair of properly infinite factors with $M \simeq N$ and $[M:N] < \infty$. Then the principal graph of $M \supset N$ is neither E_7 nor D_{odd} .*

Proof. In subsection 3.3, we have seen that E_7 can not appear. Suppose that the principal graph of $M \supset N = \rho_1(M)$ is D_{2n+3} . In (3.4.13), $[id], [\rho_2], \dots, [\rho_{2n}]$ divide into two groups. By characterization of the conjugate sector in Subsection 2.4, $[\rho_{2n+1}][\bar{\rho}_{2n+1}]$ contains $[id]$. So, the right hand side of (3.4.15) contains $[id]$. Then, $[\rho_{2n+1}][\bar{\rho}_{2n+2}]$ on the left hand side must contain $[\bar{\rho}_2]=[\rho_2]$.

We now investigate the balance of the $[\rho_2]$ term in (3.4.14). On the right hand side, $[\rho_2]$ appears twice, once as $[\rho_2]$ and once in $[\rho_{2n+1}][\bar{\rho}_{2n+2}]$ as we have seen. On the left hand side, $[\rho_{2n+1}][\bar{\rho}_{2n+1}]$ is a partial sum of the terms on the right hand side of (3.4.13), among which only $[id], [\rho_2]$ and $[\rho_4]$ can produce $[\rho_2]$ after being multiplied by $[\rho_2]$ from right. Since $[\rho_2]$ is in $[\rho_{2n+1}][\bar{\rho}_{2n+2}]$, it can not be in $[\rho_{2n+1}][\bar{\rho}_{2n+1}]$. Therefore $[\rho_{2n+1}][\bar{\rho}_{2n+1}]$ must contain $[\rho_4]$.

By repeating the same type of arguments, we get the following equations.

The case of odd n :

$$\begin{aligned} [\rho_{2n+1}][\bar{\rho}_{2n+1}] &= [id] \oplus [\rho_4] \oplus \cdots \oplus [\rho_{2n-2}], \\ [\rho_{2n+1}][\bar{\rho}_{2n+2}] &= [\rho_2] \oplus [\rho_6] \oplus \cdots \oplus [\rho_{2n}]. \end{aligned}$$

The case of even n :

$$\begin{aligned} [\rho_{2n+1}][\bar{\rho}_{2n+1}] &= [id] \oplus [\rho_4] \oplus \cdots \oplus [\rho_{2n}], \\ [\rho_{2n+1}][\bar{\rho}_{2n+2}] &= [\rho_2] \oplus \cdots \oplus [\rho_{2n-2}]. \end{aligned}$$

By Lemma 3.5, $d_{\rho_{2n+1} \bar{\rho}_{2n+1}} = d_{\rho_{2n+1} \bar{\rho}_{2n+2}}$. But this is impossible because of $d_{\rho_i} = \frac{\sin((i+1)/(4n+4))\pi}{\sin(\pi/(4n+4))}$ $i=1, 2, \dots, 2n$ and the additivity of the statistical dimension.

(Note that $d_{\rho_k} < d_{\rho_l}$ if $k < l \leq 2n$.)

Q.E.D.

To obtain the result in general case, we need the following lemma.

Lemma 3.8. (R. Longo [L4, Lemma 2.3]) *Let $M \supset N$ be a pair of factors.*

Then there exists a factor L such that $M \otimes L$ is isomorphic to $N \otimes L$.

Corollary 3.9. *Let $M \supset N$ be a pair of any factors with $[M : N] < \infty$. Then the principal graph of $M \supset N$ is neither E_7 nor D_{odd} .*

Proof. Since the principal graph dose not change after taking tensor product, we may assume that M and N are properly infinite and M is isomorphic to N due to Lemma 3.8. Then Theorem 3.7 applies. Q.E.D.

3.5 The Case of D_{2n+2} . Suppose that the principal graph of $M \supset \rho_1(M)$ is D_{2n+2} . The diagram of fusion rules is as in Fig. 14 and the basic fusion rules and the statistical dimensions of the descendant sectors are as follows.

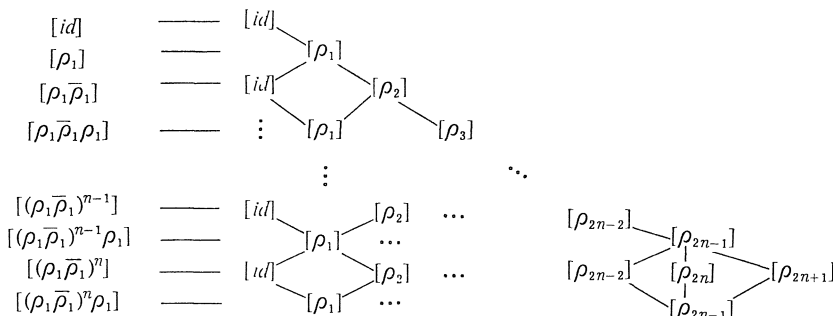


Fig. 14. The diagram of fusion rules in the case of D_{2n+2} .

$$[\rho_{2k+1}] [\bar{\rho}_1] = [\rho_{2k}] \oplus [\rho_{2k+2}], \quad k = 0, 1, \dots, n-2, \tag{3.5.1}$$

$$[\rho_{2n-1}] [\bar{\rho}_1] = [\rho_{2n-2}] \oplus [\rho_{2n}] \oplus [\rho_{2n+1}], \tag{3.5.2}$$

$$[\rho_{2k}] [\rho_1] = [\rho_{2k-1}] \oplus [\rho_{2k+1}], \quad k = 1, 2, \dots, n-1, \tag{3.5.3}$$

$$[\rho_{2n}] [\rho_1] = [\rho_{2n+1}] [\rho_1] = [\rho_{2n-1}]. \tag{3.5.4}$$

$$d_{\rho_i} = \frac{\sin \frac{i+1}{4n+2} \pi}{\sin \frac{\pi}{4n+2}} \quad i = 1, 2, \dots, 2n-1,$$

$$d_{\rho_{2n}} = d_{\rho_{2n+1}} = \frac{1}{2 \sin \frac{\pi}{4n+2}} \begin{cases} = 1 & n = 1 \\ = 2 \cos \frac{\pi}{5} & n = 2 \\ > 2 & n > 2. \end{cases}$$

By a similar argument as in the proof of Lemma 3.6, we get the following fusion rules

$$[\rho_{2n}] [\rho_{2k}] = [\rho_{2n-2k}] \oplus [\rho_{2n-2k+2}] \oplus \cdots \oplus [\rho_{2n-2}] \begin{cases} \oplus [\rho_{2n+1}] & \text{for odd } k, \\ \oplus [\rho_{2n}] & \text{for even } k, \end{cases} \quad (3.5.5)$$

$$[\rho_{2n+1}] [\rho_{2k}] = [\rho_{2n-2k}] \oplus [\rho_{2n-2k+2}] \oplus \cdots \oplus [\rho_{2n-2}] \begin{cases} \oplus [\rho_{2n}] & \text{for odd } k, \\ \oplus [\rho_{2n+1}] & \text{for even } k, \end{cases} \quad (3.5.6)$$

for $1 \leq k \leq n-1$.

$$[\rho_{2n}] [\rho_{2n}] \oplus [\rho_{2n}] [\rho_{2n+1}] = [id] \oplus [\rho_2] \oplus \cdots \oplus [\rho_{2n-2}] \begin{cases} \oplus [\rho_{2n+1}] & \text{for odd } n, \\ \oplus [\rho_{2n}] & \text{for even } n, \end{cases} \quad (3.5.7)$$

$$[\rho_{2n+1}] [\rho_{2n}] \oplus [\rho_{2n+1}] [\rho_{2n+1}] = [id] \oplus [\rho_2] \oplus \cdots \oplus [\rho_{2n-2}] \begin{cases} \oplus [\rho_{2n}] & \text{for odd } n, \\ \oplus [\rho_{2n+1}] & \text{for even } n. \end{cases} \quad (3.5.8)$$

Proposition 3.10. *If n is odd, then $[\rho_{2n+1}]$ is the conjugate sector of $[\rho_{2n}]$ and the following fusion rules hold.*

$$[\rho_{2n}] [\rho_{2n}] = [\rho_2] \oplus [\rho_6] \oplus \cdots \oplus [\rho_{2n-4}] \oplus [\rho_{2n+1}], \quad (3.5.9)$$

$$[\rho_{2n}] [\rho_{2n+1}] = [id] \oplus [\rho_4] \oplus \cdots \oplus [\rho_{2n-6}] \oplus [\rho_{2n-2}]. \quad (3.5.10)$$

If n is even, then $[\rho_{2n}]$ and $[\rho_{2n+1}]$ are self-conjugate and the following fusion rules hold.

$$[\rho_{2n}] [\rho_{2n}] = [id] \oplus [\rho_4] \oplus \cdots \oplus [\rho_{2n-4}] \oplus [\rho_{2n}], \quad (3.5.11)$$

$$[\rho_{2n}] [\rho_{2n+1}] = [\rho_2] \oplus [\rho_6] \oplus \cdots \oplus [\rho_{2n-2}]. \quad (3.5.12)$$

Proof. Suppose that n is odd and $[\rho_{2n}]$ is self-conjugate. As in the proof of Theorem 3.7, the following fusion rules hold.

$$[\rho_{2n}] [\rho_{2n}] = [id] \oplus [\rho_4] \oplus \cdots \oplus [\rho_{2n-6}] \oplus [\rho_{2n-2}],$$

$$[\rho_{2n}] [\rho_{2n+1}] = [\rho_2] \oplus [\rho_6] \oplus \cdots \oplus [\rho_{2n-4}] \oplus [\rho_{2n+1}].$$

Since we assume that $[\rho_{2n}]$ is self-conjugate, $[\rho_{2n+1}]$ is also self-conjugate. By conjugating the second equality, we obtain

$$[\rho_{2n+1}] [\rho_{2n}] = [\rho_2] \oplus [\rho_6] \oplus \cdots \oplus [\rho_{2n-4}] \oplus [\rho_{2n+1}].$$

On the other hand, from the symmetry between $[\rho_{2n}]$ and $[\rho_{2n+1}]$, we also obtain

$$[\rho_{2n}] [\rho_{2n+1}] = [\rho_2] \oplus [\rho_6] \oplus \cdots \oplus [\rho_{2n-4}] \oplus [\rho_{2n}].$$

But this is impossible because $[\rho_{2n}]$ and $[\rho_{2n+1}]$ are different sectors. (If $[\rho_{2n}] = [\rho_{2n+1}]$, the principal graph is not D_{2n+2} .)

Next, suppose that n is even and $[\rho_{2n+1}] = [\bar{\rho}_{2n}]$. In the same way, we have

$$[\rho_{2n}] [\rho_{2n}] = [\rho_2] \oplus [\rho_6] \oplus \cdots \oplus [\rho_{2n-2}],$$

$$[\rho_{2n}] [\rho_{2n+1}] = [id] \oplus [\rho_4] \oplus \cdots \oplus [\rho_{2n-4}] \oplus [\rho_{2n}].$$

By conjugating the second equality, we obtain

$$[\rho_{2n}] [\rho_{2n+1}] = [id] \oplus [\rho_4] \oplus \cdots \oplus [\rho_{2n-4}] \oplus [\rho_{2n+1}].$$

And we get a cotradiction. By the same argument as in the proof of Theorem 3.7, we have the fusion rules. Q.E.D.

The difference between D_{4n} and D_{4n+2} we saw above is stated by A. Ocneanu in [O1, page 157], but no proof has been published until now.

We compute the statistical dimensions of the right hand sides of fusion rules in Proposition 3.10.

The case of odd n:

$$\begin{aligned} \sum_{k=1}^{(n-1)/2} d_{\rho_{4k-2}} + d_{\rho_{2n+1}} &= \sum_{k=1}^{(n-1)/2} \frac{\sin \frac{4k-1}{4n+2} \pi}{\sin \frac{\pi}{4n+2}} + \frac{1}{2 \sin \frac{\pi}{4n+2}} \\ &= \frac{1}{\sin \frac{\pi}{4n+2}} \left[\sum_{k=1}^{(n-1)/2} \sin \frac{4k-1}{4n+2} \pi - \frac{1}{2} \right], \\ \sum_{k=1}^{(n+1)/2} d_{\rho_{4k-4}} &= \sum_{k=1}^{(n+1)/2} \frac{\sin \frac{4k-3}{4n+2} \pi}{\sin \frac{\pi}{4n+2}}. \end{aligned}$$

It is easy to show

$$\sum_{k=1}^{(n+1)/2} \left(\sin \frac{4k-1}{4n+2} \pi - \sin \frac{4k-3}{4n+2} \pi \right) = \frac{1}{2}.$$

The case of even n:

$$\begin{aligned} \sum_{k=1}^{n/2} d_{\rho_{4k-4}} + d_{\rho_{2n+1}} &= \frac{1}{\sin \frac{\pi}{4n+2}} \left[\sum_{k=1}^{n/2} \sin \frac{4k-3}{4n+2} \pi + \frac{1}{2} \right], \\ \sum_{k=1}^{n/2} d_{\rho_{4k-2}} &= \frac{1}{\sin \frac{\pi}{4n+2}} \left[\sum_{k=1}^{n/2} \sin \frac{4k-1}{4n+2} \pi \right]. \end{aligned}$$

It is also easy to show

$$\sum_{k=1}^{n/2} \left(\sin \frac{4k-1}{4n+2} \pi - \sin \frac{4k-3}{4n+2} \pi \right) = \frac{1}{2} .$$

Then there is no contradiction.

The principal graph of $M \supset \rho_2(M)$ is as in Fig. 15.

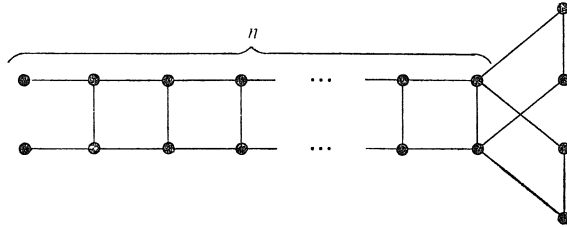


Fig. 15. The principal graph of $M \supset \rho_2(M)$ in the case of D_{2n+2} $n \geq 2$.

§4. Characterization of Fixed Point Algebra and Subfactors with Depth 2

First we prove a theorem of H. Kosaki, A. Ocneanu, M. Pimsner and S. Popa [K3] [PP1, Corollary 1.1.6] by using sectors.

Theorem 4.1. *Let $M \supset N$ be a pair of properly infinite factors with $[M:N] < \infty$ and $M \cap N' = \mathcal{C}$. The following conditions are equivalent.*

- (i) *There exists a finite group G and its outer action α on M such that $N = M^\alpha$ and $|G| = [M:N]_0$.*
- (ii) *If $\tau : M \rightarrow N$ is the canonical endomorphism and $[\tau] = \bigoplus_i [\rho_i]$ is irreducible decomposition, then $d_{\rho_i} = 1$, i.e. $[\tau]$ is decomposed into automorphisms.*

If the above conditions are satisfied, then

$$[\tau] = \bigoplus_{g \in G} [\alpha_g] .$$

Proof. (i) \Rightarrow (ii): We assume (i) and define $E \in E(M, N)$ by

$$E(x) = \frac{1}{|G|} \sum_{g \in G} \alpha_g(x) \quad \text{for } x \in M .$$

In Subsection 2.3, we saw that from E we can construct an $M-M$ correspondence and, by using the bijection between $\text{Sect}(M)$ and the set of unitary equivalence classes of $M-M$ correspondences, we obtain $\rho_E \in \text{End}(M)$ such that $[\rho_E] = [\tau]$. We show $[\rho_E] = \bigoplus_{g \in G} [\alpha_g]$. Let \mathcal{Q} be a cyclic and separating vector for M (and equivalently M') and $\{S_g\}_{g \in G} \subset M$ be isometries satisfying $\sum_{g \in G} S_g S_g^* = 1$. We define $\rho' \in \text{End}(M)$ by

$$\rho'(x) \equiv \sum_{g \in G} S_g \alpha_g(x) S_g^* .$$

Then

$$\langle E(x) \Omega y, \Omega \rangle = \frac{1}{|G|} \sum_{g \in G} \langle \alpha_g(x) \Omega y, \Omega \rangle = \langle \rho'(x) \xi y, \xi \rangle ,$$

$$\text{where } \xi \equiv \frac{1}{\sqrt{|G|}} \sum_{g \in G} S_g \Omega .$$

It suffices to show $\overline{\rho'(M) \xi M} = H$. Since α is an outer action, $M \cap \rho'(M)' = \bigoplus_{g \in G} \mathcal{C} S_g S_g^*$, and the projection p onto $\overline{\rho'(M) \xi M}$ belongs to $\bigoplus_{g \in G} \mathcal{C} S_g S_g^*$. From $S_g S_g^* \xi = \frac{1}{\sqrt{|G|}} S_g \Omega \neq 0$, p can not be orthogonal to $S_g S_g^*$ for any $g \in G$, which implies $p=1$. Hence we obtain $\overline{\rho'(M) \xi M} = H$ and so we have $[\rho_E] = \bigoplus_{g \in G} [\alpha_g]$.

(ii) \Rightarrow (i): First we assume (ii) and $M \simeq N$. We take $\rho, \bar{\rho} \in \text{End}(M)$ such that $\rho(M) = N$. From Subsection 2.2,

$$[\tau] = [\rho \bar{\rho}] = \bigoplus_i [\rho_i] . \tag{4.1.1}$$

Since $[\rho \bar{\rho}]$ is self-conjugate, $\{[\rho_i]\}$ is a self-conjugate set, i.e. $\{[\rho_i]\}$ is invariant under inverse because conjugation and inverse are equivalent on $\text{Out}(M)$. Multiplying both sides of (4.1.1) by $[\rho_j]$ from left, we get

$$[\rho_j \rho] [\bar{\rho}] = \bigoplus_i [\rho_j] [\rho_i]$$

and, from above observation, it follows that the right hand side of this equality contains $[id]$. Since $[\rho_j \rho]$ is irreducible because $[\rho]$ is irreducible and $[\rho_j]$ is an automorphism, this means that $[\rho_j \rho] = [\rho]$ by the characterization of the conjugate sector in Subsection 2.4 and hence

$$\bigoplus_i [\rho_j] [\rho_i] = \bigoplus_i [\rho_i] .$$

Since $[\rho \bar{\rho}]$ contains $[id]$ with multiplicity one, every $[\rho_i]$ is different and $\{[\rho_i]\}_i$ is a subgroup of $\text{Out}(M)$. We write $G = \{[\rho_i]\} = \{[\alpha_g]\}_{g \in G}$.

Once we fix representative $\rho \in \text{End}(M)$, we can take $\alpha_g \in \text{Aut}(M)$ such that $\alpha_g \circ \rho = \rho$. This choice makes $\{\alpha_g\}$ an outer action of G . Indeed, $\alpha_g \circ \alpha_{g'} \circ \alpha_{gg'}^{-1}$ is inner for $g, g' \in G$, and

$$\alpha_g \circ \alpha_{g'} \circ \alpha_{gg'}^{-1} \circ \rho = \alpha_g \circ \alpha_{g'} \circ \rho = \alpha_g \circ \rho = \rho .$$

Since ρ is irreducible, this means $\alpha_g \circ \alpha_{g'} \circ \alpha_{gg'}^{-1} = id$. So $\{\alpha_g\}$ is an action of G and $M^G \supset \rho(M)$. By comparing indices, we get $\rho(M) = M^G$ and have (i).

In a general case (where $M \simeq N$ does not necessarily hold), we define $\beta \in \text{End}(M \otimes N)$ by

$$\beta(x \otimes y) \equiv y \otimes r(x) \quad \text{for } x \in M, y \in N. \tag{4.1.2}$$

It is easy to see that β is irreducible. β is self-conjugate because

$$\hat{\tau} \equiv r \otimes r|_N: M \otimes N \rightarrow N \otimes r(M)$$

is the canonical endomorphism and the equality $\hat{\tau} = \beta^2$ holds. By the characterization of conjugate sector in Subsection 2.4, $[\rho]^2 = [r] \otimes [r|_N] = \bigoplus_i [\rho_i] \otimes [r|_N]$ must contain $[id_M] \otimes [id_N]$ with multiplicity one. Therefore one of $[\rho_i]$ is $[id_M]$ and, $[r|_N]$ contains $[id_N]$ with multiplicity one. By the same argument as in the case of $M \simeq N$, we see that $G \equiv \{[\rho_i] \otimes [id_N]\}$ is a group and we get an outer action $\{\alpha_g\} \subset \text{Aut}(M)$ such that $\{[\alpha_g]\} = \{[\rho_i]\}$ and $(\alpha_g \otimes id_N) \circ \beta = \beta$. Due to (4.1.2), we obtain $N \subset M^G$. By comparing indices, we get $N = M^G$. Q.E.D.

In a sense, the following proposition is a generalization of Theorem 4.1.

Proposition 4.2. *Let M be a properly infinite factor and $\rho \in \text{End}(M)$ with $d_\rho < \infty$ and $M \cap \rho(M)' = \mathbb{C}$. If $[\rho\bar{\rho}] = \bigoplus_i n_i[\alpha_i]$ and $[\bar{\rho}\rho] = \bigoplus_a m_a[\beta_a]$ are irreducible decompositions where n_i and m_a are the multiplicities, then the following conditions are equivalent.*

- (i) *The depth of $M \supset \rho(M)$ is two.*
- (ii) *The depth of $M \supset \bar{\rho}(M)$ is two.*
- (iii) *$M \cap \rho\bar{\rho}\rho(M)'$ is a factor.*
- (iv) *$M \cap \bar{\rho}\rho\bar{\rho}(M)'$ is a factor.*
- (v) *$n_i = d_{\alpha_i}$ for all i .*
- (vi) *$m_a = d_{\beta_a}$ for all a .*
- (vii) *$[\alpha_i][\rho] = d_{\alpha_i}[\rho]$ for all i .*
- (viii) *$[\beta_a][\bar{\rho}] = d_{\beta_a}[\bar{\rho}]$ for all a .*

Proof. (i) \Leftrightarrow (iii) and (ii) \Leftrightarrow (iv) are obvious from the definition of depth 2 and (iii) \Leftrightarrow (iv) follows from $[\overline{\rho\bar{\rho}\rho}] = [\bar{\rho}\rho\bar{\rho}]$. It is enough to show (iii) \Leftrightarrow (vii) \Leftrightarrow (v) because (iv) \Leftrightarrow (viii) \Leftrightarrow (vi) is shown in the same way.

(iii) \Rightarrow (vii): (iii) means that $[\rho\bar{\rho}\rho]$ is a multiple of an irreducible sector. Since $[\rho\bar{\rho}]$ contains $[id]$, $[\rho\bar{\rho}\rho]$ must be a multiple of $[\rho]$. Then

$$[\rho\bar{\rho}\rho] = \bigoplus_i n_i[\alpha_i][\rho] = d_\rho^2[\rho]$$

and hence $[\alpha_i][\rho] = d_{\alpha_i}[\rho]$, where the multiplicity d_{α_i} has been determined by

the relation $d_{\alpha_i \circ \rho} = d_{\alpha_i} d_\rho$ due to the multiplicativity of the statistical dimension. (vii) \Rightarrow (v): Multiplying $[\rho \bar{\rho}] = \bigoplus_i n_i [\alpha_i]$ by $[\alpha_i]$ from left, we have

$$d_{\alpha_j} [\rho] [\bar{\rho}] = [\alpha_j] [\rho] [\bar{\rho}] = \bigoplus_i n_i [\alpha_j] [\alpha_i]$$

because $[\alpha_j]$ must be an element $[\alpha_i]$ of the self-conjugate set $\{[\alpha_i]\}$ and $d_{\alpha_j} = d_{\alpha_j}$. This means that the right hand side contains $[id]$ with multiplicity d_{α_j} and so $n_j = d_{\alpha_j}$.

(v) \Rightarrow (iii): Since $[\alpha_j] [\rho \bar{\rho}] = \bigoplus_i d_{\alpha_i} [\alpha_j] [\alpha_i]$ contains $[id]$ with multiplicity d_{α_j} , $[\alpha_j] [\rho]$ must contain $[\rho]$ with the same multiplicity. But we have $d_{\alpha_j \circ \rho} = d_{\alpha_j} d_\rho$ from the multiplicativity of the statistical dimension. Therefore, $[\alpha_j] [\rho] = d_{\alpha_j} [\rho]$ Q.E.D.

The relation $[\rho_i] [\rho] = [\rho]$ means that $\rho(M)$ is included in M^{ρ_i} for some representative ρ_i of $[\rho_i]$. So if ρ satisfies the conditions of Proposition 4.2, in particular (vii), $\rho(M)$ is a kind of generalized fixed point algebra. In fact, A. Ocneanu announced that depth of $M \supset N$ is 2 if and only if the relation of $M \supset N$ is the crossed product by a Kac algebra action.

For more detailed analysis, we need intertwiners. Let $\rho_1, \rho_2 \in \text{End}(M)$. We define $H(\rho_1, \rho_2)$ as the set of intertwiners from ρ_1 to ρ_2 :

$$H(\rho_1, \rho_2) \equiv \{v \in M; v\rho_1(x) = \rho_2(x)v \text{ for } x \in M\}.$$

If ρ_1 is irreducible, $H(\rho_1, \rho_2)$ is a Hilbert space with $(v_1, v_2) \equiv v_2^* v_1$, $v_1, v_2 \in H(\rho_1, \rho_2)$. Indeed, since $v_2^* \in H(\rho_2, \rho_1)$,

$$v_2^* v_1 \rho_1(x) = v_2^* \rho_2(x) v_1 = \rho_1(x) v_2^* v_1$$

and so $v_2^* v_1 \in \mathbb{C}$. We write $H(\rho) \equiv H(id, \rho)$.

When $[\rho \bar{\rho}] = \bigoplus_{g \in G} [\alpha_g]$ i.e. “ $[\rho \bar{\rho}]$ is decomposed into elements of G ”, it is natural to ask what $[\bar{\rho} \rho]$ is. The answer: “ $[\bar{\rho} \rho]$ is decomposed into elements of \hat{G} ”. We show this more precisely. Since we assume the conditions of Theorem 4.1, ρ satisfies the conditions of Proposition 4.2. So the irreducible decomposition of $[\bar{\rho} \rho]$ is as follows:

$$[\bar{\rho} \rho] = \bigoplus_{\sigma} d_\sigma [\beta_\sigma] \text{ where } d_\sigma \equiv d_{\rho_\sigma}$$

and $[\beta_\sigma] [\bar{\rho}] = d_\sigma [\bar{\rho}]$ is satisfied. By conjugating $[\alpha_g] [\rho] = [\rho]$, we have $[\bar{\rho}] [\alpha_g] = [\bar{\rho}]$ and this means that there is a unitary $\lambda_g \in M$ such that $\lambda_g \in H(\bar{\rho}, \bar{\rho} \circ \alpha_g)$. In fact, a stronger statement holds. From the duality of the fixed point algebra and the crossed product [J, page 7],

$$M_1 = M \rtimes_{\alpha} G$$

where M_1 is the extension of M by $\rho(M)$. On the other hand, there exists $\hat{\rho} \in \text{End}(M_1)$, an extension of $\bar{\rho}$, which satisfies $\hat{\rho}(M_1) = M$ [L2, page 296], and so

$$M = \bar{\rho}(M) \rtimes_{\bar{\rho} \circ \alpha \circ \bar{\rho}^{-1}} G.$$

Then there exists a unitary representation $\{\lambda_g\}_{g \in G} \subset M$ such that

$$\bar{\rho} \circ \alpha_g(x) = (\bar{\rho} \circ \alpha_g \circ \bar{\rho}^{-1})(\bar{\rho}(x)) = \text{Ad} \lambda_g(\bar{\rho}(x))$$

and $(\bar{\rho}(M) \cup \{\lambda_g\}_{g \in G})'' = M$.

Let $\{W(\sigma)_a\}$ be an orthonormal basis of $H(\bar{\rho}, \beta_{\sigma} \circ \bar{\rho})$ and we define matrix elements $\beta_{\sigma}(x)_{ab} \equiv W(\sigma)_a^* \beta_{\sigma}(x) W(\sigma)_b$. By the definition of $H(\bar{\rho}, \beta_{\sigma} \circ \bar{\rho})$, $\beta_{\sigma}(\bar{\rho}(x))_{ab} = \delta_{ab} \bar{\rho}(x)$, i.e. $\bar{\rho}(M)$ is diagonal. Since

$$\begin{aligned} \beta_{\sigma}(\lambda_g)_{ab} \bar{\rho}(x) &= \sum_c \beta_{\sigma}(\lambda_g)_{ac} \delta_{cb} \bar{\rho}(x) = \beta_{\sigma}(\lambda_g \bar{\rho}(x))_{ab} = \beta_{\sigma}(\bar{\rho} \circ \alpha_g(x) \lambda_g)_{ab} \\ &= \sum_c \delta_{ac} \bar{\rho} \circ \alpha_g(x) \beta_{\sigma}(\lambda_g)_{cb} = \bar{\rho} \circ \alpha_g(x) \beta_{\sigma}(\lambda_g)_{ab}, \end{aligned}$$

$\beta_{\sigma}(\lambda_g)_{ab}$ is proportional to λ_g and we write

$$\beta_{\sigma}(\lambda_g)_{ab} = \pi \langle g, \sigma \rangle_{ab} \lambda_g.$$

It is easy to see that the matrices $\pi \langle \cdot, \sigma \rangle$ form a unitary representation of G . Moreover, it is irreducible for each σ and if $\sigma_1 \neq \sigma_2$, then $\pi \langle \cdot, \sigma_1 \rangle$ and $\pi \langle \cdot, \sigma_2 \rangle$ are disjoint. Indeed, if $\pi \langle \cdot, \sigma \rangle$ were reducible, by changing the basis of $H(\bar{\rho}, \beta_{\sigma} \circ \bar{\rho})$, we would have

$$\pi \langle g, \sigma \rangle = \begin{bmatrix} I & J \\ * & \vec{0} \\ 0 & * \end{bmatrix}.$$

By easy computation, we can check

$$\sum_{a \in I} W(\sigma)_a W(\sigma)_a^* \in \langle \beta_{\sigma}(\lambda_G), \beta_{\sigma}(\bar{\rho}(M)) \rangle' \cap M = \beta_{\sigma}(M)' \cap M.$$

Since β_{σ} is irreducible, this implies $\sum_{a \in I} W(\sigma)_a W(\sigma)_a^* = 1$ and J must be empty.

If $\pi \langle \cdot, \sigma_1 \rangle$ and $\pi \langle \cdot, \sigma_2 \rangle$ are equivalent, we may assume that $\pi \langle \cdot, \sigma_1 \rangle = \pi \langle \cdot, \sigma_2 \rangle$ by a change of basis. We define $U = \sum_a W(\sigma_1)_a W(\sigma_2)_a^*$. Then U is unitary and $U \beta_{\sigma_2}(x) = \beta_{\sigma_1}(x) U$, i.e. $[\beta_{\sigma_1}] = [\beta_{\sigma_2}]$.

Note that $\dim \pi \langle \cdot, \sigma \rangle = d_{\sigma}$ is the multiplicity of $[\beta_{\sigma}]$ in $[\bar{\rho}]$ and $\sum_{\sigma} d_{\sigma}^2 = |G|$. This means that $\bigoplus_{\sigma} d_{\sigma} \cdot \pi \langle \cdot, \sigma \rangle$ is equivalent to the regular representation of G .

In fact we see the coaction of G . We write $[\bar{\rho}] = \bigoplus_{\sigma \in \hat{G}} d_\sigma [\beta_\sigma]$.

Proposition 4.3. *Let M be a properly infinite factor and $\rho \in \text{End}(M)$ with $[\rho\bar{\rho}] = \bigoplus_{g \in G} [\alpha_g]$ where α is an outer action of a finite group G . Let β_σ, d_σ and $\pi \langle g, \sigma \rangle$ be as above. Then,*

- (i) *Each $\pi \langle \cdot, \sigma \rangle$ is an irreducible unitary representation.*
- (ii) *If $\sigma_1 \neq \sigma_2$, then $\pi \langle \cdot, \sigma_1 \rangle$ and $\pi \langle \cdot, \sigma_2 \rangle$ are disjoint.*
- (iii) *If $\sigma_1 \otimes \sigma_2 \simeq \bigoplus_i n_i \cdot \sigma_i$ is the irreducible decomposition where n_i is the multiplicity of σ_i , then $[\beta_{\sigma_1}] [\beta_{\sigma_2}] = \bigoplus_i n_i \cdot [\beta_i]$.*

Proof. It suffices to show (iii). Since depth of $M \supset \bar{\rho}(M)$ is 2, the irreducible components of $[\beta_{\sigma_1}] [\beta_{\sigma_2}]$ belong to $\{[\beta_\sigma]\}_{\sigma \in \hat{G}}$. We start from $[\beta_{\sigma_1}] [\beta_{\sigma_2}] = \bigoplus_i n_i [\beta_{\sigma_i}]$ and prove $\sigma_1 \otimes \sigma_2 \simeq \bigoplus_i n_i \cdot \sigma_i$. For simplicity we sometimes omit “ σ ”. Since $[\beta_1] [\beta_2] [\bar{\rho}] = d_1 d_2 [\bar{\rho}]$, $\dim H(\bar{\rho}, \beta_1 \circ \beta_2 \circ \bar{\rho}) = d_1 d_2$. We have two different basis of $H(\bar{\rho}, \beta_1 \circ \beta_2 \circ \bar{\rho})$. Let $\{U(\sigma_i)_{s,t}\}_{s,t}$ be a basis of $H(\beta_i, \beta_1 \circ \beta_2)$. Then $\{U(\sigma_i)_s, W(\sigma_i)_{t,s}\}_{i,s,t}$ is a basis of $H(\bar{\rho}, \beta_1 \circ \beta_2 \circ \bar{\rho})$. On the other hand it is easy to check that $\{\beta_1(W(\sigma_2)_b), W(\sigma_1)_{a,b}\}_{a,b}$ is another basis of $H(\bar{\rho}, \beta_1 \circ \beta_2 \circ \bar{\rho})$. Since

$$\begin{aligned} & W(\sigma_1)_a^* \beta_1(W(\sigma_2)_b^*) \beta_1 \circ \beta_2(\lambda_g) \beta_1(W(\sigma_2)_d) W(\sigma_1)_e \\ &= W(\sigma_1)_a^* \beta_1(W(\sigma_2)_b^*) \beta_2(\lambda_g) W(\sigma_2)_d W(\sigma_1)_e \\ &= W(\sigma_1)_a^* \beta_1(\pi \langle g, \sigma_2 \rangle_{bd} \lambda_g) W(\sigma_1)_e \\ &= \pi \langle g, \sigma_1 \rangle_{ac} \pi \langle g, \sigma_2 \rangle_{bd} \lambda_g \end{aligned}$$

and

$$\begin{aligned} & W(\sigma_i)_e^* U(\sigma_i)_s^* \beta_1 \circ \beta_2(\lambda_g) U(\sigma_j)_t W(\sigma_j)_f \\ &= \delta_{ij} \delta_{st} W(\sigma_i)_e^* \beta_i(\lambda_g) W(\sigma_j)_f \\ &= \delta_{ij} \delta_{st} \pi \langle g, \sigma_i \rangle_{ef} \lambda_g, \end{aligned}$$

we obtain $\sigma_1 \otimes \sigma_2 \simeq \bigoplus_i n_i \cdot \sigma_i$. Q.E.D.

Corollary 4.4. *Let M be a properly infinite factor and N be the fixed point algebra of M under an outer action of a non-commutative finite group G . Then there exists no self-conjugate endomorphism ρ such that $\rho(M) = N$.*

Proof. From Proposition 4.3, we get $[\rho\bar{\rho}] \neq [\bar{\rho}\rho]$ because $d_\sigma > 1$ for some σ if G is non-commutative and hence $[\rho] \neq [\bar{\rho}]$. Q.E.D.

For later use, we generalize Proposition 4.3. Let M be a properly infinite factor and $\rho \in \text{End}(M)$. Assume that ρ satisfies the conditions of Propositions 4.2, i.e. the depth of $M \supset \rho(M)$ is two. Then the irreducible decomposi-

tions of $[\rho\bar{\rho}]$ and $[\bar{\rho}\rho]$ are

$$[\rho\bar{\rho}] = \otimes d_g[\alpha_g], \quad [\bar{\rho}\rho] = \oplus d_\sigma[\beta_\sigma].$$

Let $\{V(g)_i\}_i$ and $\{W(\sigma)_a\}_a$ be basis of $H(\bar{\rho}, \bar{\rho}\circ\alpha_g)$ and $H(\bar{\rho}, \beta_\sigma\circ\bar{\rho})$, and $\beta_\sigma(x)_{ab} \equiv W(\sigma)_a^* \beta_\sigma(x) W(\sigma)_b$ as before. Then $\beta_\sigma(V(g)_i)_{ab} \bar{\rho}(x) = \bar{\rho}\circ\alpha_g(x) \beta_\sigma(V(g)_i)_{ab}$ and hence $\beta_\sigma(V(g)_i)_{ab}$ is a linear combination of $\{V(g)_i\}_i$. We write $\beta_\sigma(V(g)_i)_{ab} = \sum_j \pi_{ab}^{ji} \langle g, \sigma \rangle V(g)_j$. We use the following convention.

$$\begin{aligned} (\pi_{ab}^* \langle g, \sigma \rangle)_{ij} &= \pi_{ab}^{ij} \langle g, \sigma \rangle \in M(d_g). \\ (\pi_{ab}^{*j} \langle g, \sigma \rangle)_{ab} &= \pi_{ab}^{ij} \langle g, \sigma \rangle \in M(d_\sigma). \\ (\pi_{ab}^* \langle g, \sigma \rangle)_{ia, jb} &= \pi_{ab}^{ij} \langle g, \sigma \rangle \in M(d_g d_\sigma). \\ (\pi_{ab}^{*r} \langle g, \sigma \rangle)_{ab} &= \sum_i \pi_{ab}^{ii} \langle g, \sigma \rangle \in M(d_\sigma). \\ (\pi_{ab}^* \langle g, \sigma \rangle)_{ij} &= \sum_a \pi_{aa}^{ij} \langle g, \sigma \rangle \in M(d_g). \end{aligned}$$

Note that $[\alpha_{g_1}][\alpha_{g_2}]$ and $[\beta_{\sigma_1}][\beta_{\sigma_2}]$ are always decomposed into $\{[\alpha_g]\}$ and $\{[\beta_\sigma]\}$.

Proposition 4.5. *Let M be a properly infinite factor and $\rho \in \text{End}(M)$ such that the depth of $M \supset \rho(M)$ is two. If $\{V(g)_i\}_i$, $\{W(\sigma)_a\}_a$ and $\pi_{ab}^{ij} \langle g, \sigma \rangle$ are as above, the following statements hold.*

- (i) $\pi_{ab}^* \langle g, \sigma \rangle$ is unitary.
- (ii) If $[\alpha_{g_1}][\alpha_{g_2}] = \oplus_i n_i [\alpha_{g_i}]$ then $\sum_c \pi_{ac}^* \langle g_1, \sigma \rangle \otimes \pi_{cb}^* \langle g_2, \sigma \rangle$ is unitarily equivalent to $\oplus_i \pi_{ab}^* \langle g_i, \sigma \rangle \otimes 1_{n_i}$. Moreover, there is a unitary which induces the above equivalence simultaneously for every pair (a, b) .
- (iii) If $[\beta_{\sigma_1}][\beta_{\sigma_2}] = \oplus_i m_i [\beta_{\sigma_i}]$, then $\sum_l \pi^{il} \langle g, \sigma_1 \rangle \otimes \pi^{lk} \langle g, \sigma_2 \rangle$ is unitarily equivalent to $\oplus_i \pi^{jk} \langle g, \sigma_i \rangle \otimes 1_{m_i}$. Moreover, there is a unitary which induces the above equivalence simultaneously for every pair (j, k) .
- (iv) For fixed σ , $g \rightarrow \pi_{ab}^{*r} \langle g, \sigma \rangle$ is a representation of the fusion rules of $\{[\alpha_g]\}$, i.e. $[\alpha_g] \rightarrow \pi_{ab}^{*r} \langle g, \sigma \rangle$ preserves the sum and the product.
- (v) For fixed g , $\sigma \rightarrow \pi_{ab}^* \langle g, \sigma \rangle$ is a representation of the fusion rules of $\{[\beta_\sigma]\}$.
- (vi) M is generated by $\bar{\rho}(M)$ and $\{V(g)_i\}_{g,i}$.

Proof. (i): Since $\beta_\sigma(V(g)_i^*) \beta_\sigma(V(g)_j) = \delta_{ij}$

$$\begin{aligned} \delta_{ij} \delta_{ab} &= \sum_c \beta_\sigma(V(g)_i^*)_{ac} \beta_\sigma(V(g)_j)_{cb} \\ &= \sum_c \left(\sum_k \pi_{ca}^{ki} \langle g, \sigma \rangle V(g)_k \right)^* \left(\sum_l \pi_{cb}^{lj} \langle g, \sigma \rangle V(g)_l \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{c,h,l} \overline{\pi_{ca}^{hi} \langle g, \sigma \rangle} \pi_{cb}^{li} \langle g, \sigma \rangle V(g)_k^* V(g)_l \\
 &= \sum_{c,h} \overline{\pi_{ca}^{hi} \langle g, \sigma \rangle} \sigma_{cb}^{hj} \langle g, \sigma \rangle .
 \end{aligned}$$

(ii) and (iii) can be proved in the same way as the proof of Proposition 4.3. (iv) and (v) are direct consequence of (ii) and (iii). (vi): Let N be a von Neumann algebra generated by $\bar{\rho}(M)$ and $\{V(g)_i\}_{g,i}$. Let v be an isometry in $H(\bar{\rho})$ and $f \equiv vv^*$. Then M is generated by $\bar{\rho}(M)$ and f , (f is the conditional expectation from $\bar{\rho}(M)$ to $\bar{\rho}\rho(M)$ [L2]), and it suffices to show that $v \in N$. Let $\{S(g)_i\}$ be a basis of $H(\alpha_g, \rho\bar{\rho})$. By comparing dimension, we see that $\{\bar{\rho}(S(g)_i) V(g)_j\}_{g,i,j}$ is a basis of $H(\bar{\rho}, \bar{\rho}\rho\bar{\rho})$. Since $H(\bar{\rho}\rho) \subset H(\bar{\rho}, \bar{\rho}\rho\bar{\rho})$, we get the desired result.

§5. The Classification of the Subfactors of the Type III₁ AFD Factor with Index 3

Let $R \supset P$ be a pair of type II₁ AFD factors with $[R:P]=3$. The classification of such pairs is well-known [O1, page 161], [P1]. If the principal graph is D_4 , $P=R^{\mathbb{Z}_3}$ and, if the principal graph is A_5 , $R=Q \times S_3 \supset P=Q \times S_2$ where Q is a subfactor of P . The following theorem states that in the case of type III₁ AFD factors the situation is the same.

Theorem 5.1. *Let M be a type III₁ AFD factor and N be a subfactor of M with $[M:N]_0=3$. Then there exists a pair of type II₁ AFD factors $R \supset P$ such that $R \otimes M_0 \supset P \otimes M_0$ is isomorphic to $M \supset N$ where M_0 is a type III₁ AFD factor.*

Proof. By a theorem of P. H. Loi [Li, Corollaire 2.4], N is also a type III₁ AFD factor and there exists $\rho_1 \in \text{End}(M)$ such that $\rho_1(M)=N$. If the principal graph is D_4 , by Theorem 4.1 $N=M^{\mathbb{Z}_3}$ and we get the stated result by the classification of group actions [KST, Theorem 20]. We now assume that the principal graph is A_5 . Our aim is to look for a subfactor with index 6 and an S_3 action. The descendant sectors and fusion rules are as follows (see Fig. 16):

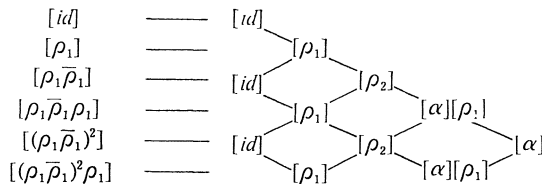


Fig. 16. The diagram of fusion rules in the case of A_5 .

$$\begin{aligned}
[\rho_1][\bar{\rho}_1] &= [id] \oplus [\rho_2], \\
[\rho_2][\rho_1] &= [\rho_1] \oplus [\alpha][\rho_1], \\
[\rho_2]^2 &= [id] \oplus [\alpha] \oplus [\rho_2], \\
[\alpha][\rho_2] &= [\rho_2][\alpha] = [\rho_2], \\
d_{\rho_1} &= \sqrt{3}, d_{\rho_2} = 2, d_{\alpha} = 1.
\end{aligned}$$

By Proposition 3.3, $[\alpha]$ is lifted to \mathbf{Z}_2 action. By Lemma 3.4, we may assume $[\rho_1][\alpha] = [\alpha][\rho_1]$.

We consider the following inclusions:

$$M \supset \rho_1(M) \supset \rho_1(M^\alpha).$$

Let $E \in E(M, \rho_1(M))$. By Subsection 2.2, there exists an isometry $v \in M$ such that $E(x) = \rho_1(v^* \bar{\rho}_1(x) v)$. Since $\rho_1(M^\alpha) = \rho_1(M)^{\rho_1 \circ \alpha \circ \rho_1^{-1}}$, there is a unique $\hat{E} \in E(\rho_1(M), \rho_1(M^\alpha))$ and \hat{E} is given as follows;

$$\hat{E}(x) \equiv \frac{x + \rho_1 \circ \alpha \circ \rho_1^{-1}(x)}{2} \quad \text{for } x \in \rho_1(M).$$

We can define $E_1 \in E(M, \rho_1(M^\alpha))$ by

$$\begin{aligned}
E_1(x) &\equiv \hat{E} \circ E(x) \\
&= \frac{E(x) + \rho_1 \circ \alpha \circ \rho_1^{-1}(E(x))}{2} \\
&= \frac{\rho_1(v^*) \rho_1 \bar{\rho}_1(x) \rho_1(v) + \rho_1(\alpha(v^*)) \rho_1 \circ \alpha \circ \bar{\rho}_1(x) \rho_1(\alpha(v))}{2}.
\end{aligned}$$

Let $\rho_0 \in \text{End}(M)$ with $\rho_0(M) = \rho_1(M^\alpha)$. We saw in Subsection 2.2 that $[\rho_0 \bar{\rho}_0]$ is calculated from E_1 by using an $M-M$ correspondence and the above relation means that $[\rho_0 \bar{\rho}_0]$ is contained in

$$\begin{aligned}
[\rho_1 \bar{\rho}_1] \oplus [\rho_1 \circ \alpha \circ \bar{\rho}_1] &= [\rho_1 \bar{\rho}_1] \oplus [\alpha][\rho_1 \bar{\rho}_1] \\
&= [id] \oplus [\alpha] \oplus 2[\rho_2].
\end{aligned}$$

Since this contains $[id]$ with multiplicity one, ρ_0 is irreducible and E_1 is the unique expectation. Then

$$[M: \rho_0(M)]_0 = \text{Index } E_1 = (\text{Index } \hat{E})(\text{Index } E) = 6.$$

By comparing statistical dimension, we have

$$[\rho_0 \bar{\rho}_0] = [id] \oplus [\alpha] \oplus 2[\rho_2].$$

Due to $d_{id}=1$, $d_{\alpha}=1$, $d_{\rho_2}=2$, ρ_0 satisfies the condition (v) of Proposition 4.2,

and a possible decomposition of $[\bar{\rho}_0\rho_0]$ is one of the following.

$$[\bar{\rho}_0\rho_0] = [id] \oplus [\beta_1] \oplus 2[\beta_2] \quad \text{where } d_{\beta_1} = 1, d_{\beta_2} = 2,$$

or

$$[\bar{\rho}_0\rho_0] = \bigoplus_i [r_i] \quad \text{where } d_{r_i} = 1.$$

In the second case, the relation between M and $\rho_0(M)$ is given by the crossed product by a non-commutative group of order 6 due to Theorem 4.1 and Proposition 4.3, i.e.

$$M = \rho_0(M) \rtimes S_3.$$

To reject the first possibility, we need the following two lemmas.

Lemma 5.2. *Let $\rho \in \text{End}(M)$ with $d_\rho = \sqrt{6}$. If ρ is irreducible and $[\rho\bar{\rho}] = [id] \oplus [\alpha_1] \oplus 2[\alpha_2]$ is the irreducible decomposition such that $d_{\alpha_1} = 1, d_{\alpha_2} = 2$, then α_1 and α_2 are self-conjugate and the fusion rules are the following.*

$$[\alpha_1]^2 = [id], \tag{5.2.1}$$

$$[\alpha_1\alpha_2] = [\alpha_2\alpha_1] = [\alpha_2], \tag{5.2.2}$$

$$[\alpha_2]^2 = [id] \oplus [\alpha_1] \oplus [\alpha_2]. \tag{5.2.3}$$

Proof. Self-conjugacy of $[\alpha_1]$ and $[\alpha_2]$ follows from that of $[\rho\bar{\rho}]$. Since ρ satisfies the conditions of Proposition 4.2, $[\alpha_1][\rho] = [\rho]$ and $[\alpha_2][\rho] = 2[\rho]$ hold. From the first equality, we have

$$[\rho\bar{\rho}] = [\alpha_1][\rho\bar{\rho}] = [\alpha_1][id] \oplus [\alpha_1][\alpha_1] \oplus 2[\alpha_1][\alpha_2].$$

So, we see that $[\alpha_1]^2 = [id]$ and $[\alpha_1][\alpha_2] = [\alpha_2]$, and by conjugation $[\alpha_2][\alpha_1] = [\alpha_2]$. From the second equality, we obtain

$$\begin{aligned} 2[\rho\bar{\rho}] &= [\alpha_2][\rho\bar{\rho}] = [\alpha_2][id] \oplus [\alpha_2][\alpha_1] \oplus 2[\alpha_2][\alpha_2] \\ &= 2[\alpha_2] \oplus 2[\alpha_2]^2. \end{aligned}$$

Therefore, $[\alpha_2]^2 = [id] \oplus [\alpha_1] \oplus [\alpha_2]$. Q.E.D.

Lemma 5.3. *If the assumptions of Lemma 5.2 are satisfied, then $[\bar{\rho}\rho] = \bigoplus_i [r_i]$ such that $d_{r_i} = 1$.*

Proof. Suppose $[\bar{\rho}\rho] = [id] \oplus [\beta_1] \oplus 2[\beta_2]$ such that $d_{\beta_1} = 1, d_{\beta_2} = 2$. Since the depth of $M \supset \rho(M)$ is two, we define

$$\begin{aligned} V(id) &= 1, V(\alpha_1), V(\alpha_2)_1, V(\alpha_2)_2, \\ W(id) &= 1, W(\beta_1), W(\beta_2)_1, W(\beta_2)_2, \end{aligned}$$

$$\pi_{ab}^{ij} \langle \alpha, \beta \rangle,$$

etc. as in Section 4. By definition

$$\pi \langle id, id \rangle = \pi \langle \alpha_1, id \rangle = \pi \langle id, \beta_1 \rangle = 1, \quad (5.3.1)$$

$$\pi^* \langle \alpha_2, id \rangle = \pi^* \langle id, \beta_2 \rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (5.3.2)$$

We may assume $\beta_1 \circ \bar{\rho} = \bar{\rho}$ and $\alpha_1 \circ \alpha_2 = \alpha_2$. Then $\alpha_1^2 = \beta_1^2 = id$ as in the proof of Theorem 4.1. From (5.3.1), (5.3.2), Lemma 5.2 and Proposition 4.5 (iii), we obtain

$$\pi \langle \alpha_1, \beta_1 \rangle^2 = \pi \langle \alpha_1, id \rangle = 1, \quad (5.3.3)$$

$$\pi^* \langle \alpha_2, \beta_1 \rangle^2 = \pi^* \langle \alpha_2, id \rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (5.3.4)$$

By changing the basis $\{V(\alpha_{2i})\}_i$, we may assume that $\pi^* \langle \alpha_2, \beta_1 \rangle$ is diagonal.

First, we show $\pi \langle \alpha_1, \beta_1 \rangle = -1$. If $\pi \langle \alpha_1, \beta_1 \rangle = +1$, from Proposition 4.5 (iv) and Lemma 5.2 we obtain

$$\pi^{tr} \langle \alpha_2, \beta_1 \rangle^2 = \pi \langle id, \beta_1 \rangle + \pi \langle \alpha_1, \beta_1 \rangle + \pi^{tr} \langle \alpha_2, \beta_1 \rangle. \quad (5.3.5)$$

From (5.3.1) and this, we also obtain

$$\pi^{tr} \langle \alpha_2, \beta_1 \rangle^2 = 2 + \pi^{tr} \langle \alpha_2, \beta_1 \rangle.$$

So, $\pi^{tr} \langle \alpha_2, \beta_1 \rangle = -1$ or 2 . Since $\pi^* \langle \alpha_2, \beta_1 \rangle$ is diagonal and unitary, this and (5.3.4) imply

$$\pi^* \langle \alpha_2, \beta_1 \rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This and $\pi \langle \alpha_1, \beta_1 \rangle = +1$ mean

$$\beta_1(V(\alpha_1)) = V(\alpha_1), \beta_1(V(\alpha_{2i})) = V(\alpha_{2i}).$$

We have also taken β_1 out of $[\beta_1]$ such that $\beta_1 \circ \bar{\rho} = \bar{\rho}$. Since $\bar{\rho}(M)$ and $\{V(\alpha_j)\}_j$ generate M by Proposition 4.5 (vi), we obtain $\beta_1 = id$. This is impossible and we conclude $\pi \langle \alpha_1, \beta_1 \rangle = -1$.

Second, we show

$$\pi_{tr}^* \langle \alpha_2, \beta_2 \rangle = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (5.3.6)$$

From eq. (5.3.5), we now have $\pi^{tr} \langle \alpha_2, \beta_1 \rangle^2 = \pi^{tr} \langle \alpha_2, \beta_1 \rangle$ and so $\pi^{tr} \langle \alpha_2, \beta_1 \rangle = 0$

or 1. Since $\pi^\circ \langle \alpha_2, \beta_1 \rangle$ is unitary and diagonal, this and (5.3.4) imply $\pi^{tr} \langle \alpha_2, \beta_1 \rangle = 0$. We may assume

$$\pi^\circ \langle \alpha_2, \beta_1 \rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5.3.7}$$

By Proposition 4.5 (v) and $[\beta_2] [\beta_1] = [\beta_1] [\beta_2] = [\beta_2]$,

$$\pi_{tr}^\circ \langle \alpha_2, \beta_2 \rangle \pi^\circ \langle \alpha_2, \beta_1 \rangle = \pi^\circ \langle \alpha_2, \beta_1 \rangle \pi_{tr}^\circ \langle \alpha_2, \beta_2 \rangle = \pi_{tr}^\circ \langle \alpha_2, \beta_2 \rangle.$$

Therefore,

$$\pi_{tr}^\circ \langle \alpha_2, \beta_2 \rangle = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \quad a \in \mathbb{C}.$$

Also, the relation $[\beta_2]^2 = [id] \oplus [\beta_1] \oplus [\beta_2]$ implies

$$\begin{aligned} \pi_{tr}^\circ \langle \alpha_2, \beta_2 \rangle^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \pi_{tr}^\circ \langle \alpha_2, \beta_2 \rangle \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \pi_{tr}^\circ \langle \alpha_2, \beta_2 \rangle. \end{aligned}$$

So $a=2$ or -1 .

In the same way as $\pi^\circ \langle \alpha_2, \beta_1 \rangle$, we obtain

$$\pi_\circ \langle \alpha_1, \beta_2 \rangle^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By changing basis $\{W(\beta_2)_{af}\}$, we may assume

$$\begin{aligned} \pi_\circ \langle \alpha_1, \beta_2 \rangle &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ \pi_{tr}^\circ \langle \alpha_2, \beta_2 \rangle &= \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{5.3.8}$$

where $b=2$ or -1 , as before. Since $\pi_\circ \langle \alpha_2, \beta_2 \rangle$ is unitary, if a or b is 2, then

$$\pi_\circ \langle \alpha_2, \beta_2 \rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since $\bar{\rho}(M)$ and $\{V(\alpha_j)_{if}\}_{i,j}$ generate M , this and eq. (5.3.8) mean

$$W(\beta_2)_1 W(\beta_2)_1^*, W(\beta_2)_2 W(\beta_2)_2^* \in M \cap \beta_2(M)'.$$

This is impossible because β_2 is irreducible. Therefore we obtain (5.3.6). By the above argument, we may also assume

$$\begin{aligned}\pi_* \langle \alpha_1, \beta_2 \rangle &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ \pi_*^{tr} \langle \alpha_2, \beta_2 \rangle &= \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

But we do not make this assumption because we will make a further change of the basis $\{W(\beta_2)_a\}_{a=1,2}$. Instead of this assumption, we use the information that $\pi_*^{tr} \langle \alpha_2, \beta_2 \rangle$ is Hermitian, $\text{Sp} \pi_*^{tr} \langle \alpha_2, \beta_2 \rangle = \{0, -1\}$ and $\pi_{tr} \langle \alpha_1, \beta_2 \rangle = 0$.

Finally we derive a contradiction. By $[\beta_1][\beta_2] = [\beta_2]$ and Proposition 4.5. (iii), we obtain $\sum_k \pi_*^{ik} \langle \alpha_2, \beta_1 \rangle \otimes \pi_*^{kj} \langle \alpha_2, \beta_2 \rangle = \sum_k \pi_*^{ik} \langle \alpha_2, \beta_1 \rangle \pi_*^{kj} \langle \alpha_2, \beta_2 \rangle$ is unitarily equivalent to $\pi_*^{ij} \langle \alpha_2, \beta_2 \rangle$. Due to (5.3.7), this means that there exists a unitary $U \in M(2)$ such that

$$\begin{aligned}\pi_*^{11} \langle \alpha_2, \beta_2 \rangle U &= U \pi_*^{11} \langle \alpha_2, \beta_2 \rangle, \\ \pi_*^{12} \langle \alpha_2, \beta_2 \rangle U &= U \pi_*^{12} \langle \alpha_2, \beta_2 \rangle, \\ -\pi_*^{21} \langle \alpha_2, \beta_2 \rangle U &= U \pi_*^{21} \langle \alpha_2, \beta_2 \rangle, \\ -\pi_*^{22} \langle \alpha_2, \beta_2 \rangle U &= U \pi_*^{22} \langle \alpha_2, \beta_2 \rangle.\end{aligned}$$

By changing the basis $\{W(\beta_2)_a\}_{a=1,2}$, we may assume

$$U = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} \quad \omega_1 \neq \omega_2,$$

(if $\omega_1 = \omega_2$, $\pi_*^{21} \langle \alpha_2, \beta_2 \rangle = \pi_*^{22} \langle \alpha_2, \beta_2 \rangle = 0$ and $\pi_* \langle \alpha_2, \beta_2 \rangle$ can not be unitary.) and so

$$\begin{aligned}\pi_*^{11} \langle \alpha_2, \beta_2 \rangle &= \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \\ \pi_*^{12} \langle \alpha_2, \beta_2 \rangle &= \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \\ \pi_*^{21} \langle \alpha_2, \beta_2 \rangle &= \begin{bmatrix} 0 & c_1 \\ c_2 & 0 \end{bmatrix}, \\ \pi_*^{22} \langle \alpha_2, \beta_2 \rangle &= \begin{bmatrix} 0 & d_1 \\ d_2 & 0 \end{bmatrix}, \\ \pi_* \langle \alpha_2, \beta_2 \rangle &= \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ 0 & c_1 & 0 & d_1 \\ c_2 & 0 & d_2 & 0 \end{bmatrix},\end{aligned}$$

$$\begin{aligned} \pi_*^{*r} \langle \alpha_2, \beta_2 \rangle &= \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 0 & 0 \end{bmatrix}, \\ \pi_*^{*r} \langle \alpha_2, \beta_2 \rangle &= \begin{bmatrix} a_1 & d_1 \\ d_2 & a_2 \end{bmatrix}. \end{aligned}$$

From (5.3.6), we obtain $a_1+a_2=-1$ and $b_1+b_2=0$. Since $\pi_*^{*r} \langle \alpha_2, \beta_2 \rangle$ is Hermitian and $\text{Sp} \pi_*^{*r} \langle \alpha_2, \beta_2 \rangle = \{0, 1\}$, we obtain

$$a_1, a_2 \in \mathbb{R}, \quad d_1 = \bar{d}_2, \quad a_1 a_2 - |d_1|^2 = 0.$$

We write as follows.

$$\begin{aligned} \pi_* \langle \alpha_2, \beta_2 \rangle &= \begin{bmatrix} a_1 & 0 & b & 0 \\ 0 & a_2 & 0 & -b \\ 0 & c_1 & 0 & d \\ c_2 & 0 & \bar{d} & 0 \end{bmatrix}, \\ a_1, a_2 \in \mathbb{R}, \quad a_1 a_2 &= |d|^2, \quad a_1 + a_2 = -1. \end{aligned}$$

From $[\beta_2]^2 = [id] \oplus [\beta_1] \oplus [\beta_2]$ and Proposition 4.5 (iii), we obtain that

$$\sum_{k=1}^2 \pi_*^{ik} \langle \alpha_2, \beta_2 \rangle \otimes \pi_*^{kj} \langle \alpha_2, \beta_2 \rangle$$

is unitarily equivalent to

$$\pi^{ij} \langle \alpha_2, id \rangle \oplus \pi^{ij} \langle \alpha_2, \beta_1 \rangle \oplus \pi_*^{ij} \langle \alpha_2, \beta_2 \rangle.$$

For $i=1, j=2$

$$\pi_*^{11} \langle \alpha_2, \beta_2 \rangle \otimes \pi_*^{12} \langle \alpha_2, \beta_2 \rangle + \pi_*^{12} \langle \alpha_2, \beta_2 \rangle \otimes \pi_*^{22} \langle \alpha_2, \beta_2 \rangle \simeq 0 \oplus 0 \oplus \pi_*^{12} \langle \alpha_2, \beta_2 \rangle.$$

This means

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \otimes \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix} \otimes \begin{bmatrix} 0 & d \\ \bar{d} & 0 \end{bmatrix} \simeq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -b \end{bmatrix}.$$

If $b \neq 0$, we divide by b and obtain

$$\begin{bmatrix} a_1 & d & 0 & 0 \\ d & -a_1 & 0 & 0 \\ 0 & 0 & a_2 & -d \\ 0 & 0 & -d & -a_2 \end{bmatrix} \simeq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Due to $a_1 a_2 = |d|^2, a_1 + a_2 = -1$,

$$\begin{aligned} \text{Sp} \begin{bmatrix} a_1 & d & 0 & 0 \\ d & -a_1 & 0 & 0 \\ 0 & 0 & a_2 & -d \\ 0 & 0 & -d & -a_2 \end{bmatrix} &= \{ \pm \sqrt{a_1^2 + |d|^2}, \pm \sqrt{a_2^2 + |d|^2} \} \\ &= \{ \pm \sqrt{-a_1}, \pm \sqrt{-a_2} \}. \end{aligned}$$

So a_1 or $a_2=0$, and $d=0$. But this is impossible because $\pi^* \langle \alpha_2, \beta_2 \rangle$ must be unitary and $a_1 + a_2 = -1$. We are now left with the alternative $b=0$. But this is also impossible because, $b=0$ implies $|a_1| = |a_2| = 1$ due to the unitarity of $\pi^* \langle \alpha_2, \beta_2 \rangle$, while a_1 and a_2 are real and $a_1 + a_2 = -1$. Thus we reached a contradiction. Q.E.D.

Completion of proof of Theorem 5.1. By Lemma 5.3, we obtain $[\bar{\rho}_0] [\rho_0] = \bigoplus_{g \in S_3} [\gamma_g]$. By Theorem 4.1 and the remark after Proposition 4.2, we obtain

$$M = \rho_0(M) \rtimes_{\rho_0 \circ \gamma \circ \rho_0^{-1}} S_3.$$

Since $[\rho_1(M) : \rho_0(M)] = 2$, there exists a \mathbb{Z}_2 action θ on M such that

$$\rho_1(M) = \rho_0(M) \rtimes_{\rho_0 \circ \theta \circ \rho_0^{-1}} \mathbb{Z}_2.$$

By the definition of crossed product, there exists a unitary $u \in M$ such that

$$\rho_0 \circ \theta(x) = \rho_0 \circ \theta \circ \rho_0^{-1}(\rho_0(x)) = u \rho_0(x) u^*.$$

This means $[\rho_0 \circ \theta] = [\rho_0]$ and by conjugation, $[\theta] [\bar{\rho}_0] = [\bar{\rho}_0]$. Multiplying both sides of the equality $[\bar{\rho}_0 \rho_0] = \bigoplus [\gamma_g]$ by $[\theta]$ from left, we have

$$[\bar{\rho}_0 \rho_0] = [\theta] [\bar{\rho}_0 \rho_0] = \bigoplus [\theta] [\gamma_g].$$

Therefore $[\theta] \in \{[\gamma_g]\}$ i.e. $\{id, \theta\}$ is cocycle conjugate to a subgroup of $\{\gamma_g\}$. Then

$$M \supset N \simeq \rho_0(M) \rtimes S_3 \supset \rho_0(M) \rtimes S_2.$$

From the classification of group actions on the type III₁ AFD factor [KST, Theorem 20], we obtain the desired conclusion of Theorem 5.1. Q.E.D.

Corollary 5.4. *Let $M \supset N$ be a pair of factors with $[M : N]_0 = 6$ and $M \cap N' = \mathbb{C}$. If the depth of $M \supset N$ is two, the relation between M and N is one of the following.*

- (i) *There exist a group G and its outer action on M such that $N = M^G$.*
- (ii) *There exist a group G and its outer action on N such that $M = N \rtimes G$.*

Proof. Suppose the contrary. Due to Lemma 3.8, we may assume that M and N are properly infinite and M is isomorphic to N , because the principal graph does not change after taking tensor product. (Note that the above two conditions are characterized by the principal graph of $M \supset N$ or the principal graph of $M_1 \subset M$, where M_1 is the extension of M by N [K3] [PP1, Corollary 1.1.6].) Let $\rho \in \text{End}(M)$ with $N = \rho(M)$. Thanks to Theorem 4.1 and Proposition 4.2, $[\rho]$ and $[\bar{\rho}]$ must satisfy the condition of Lemma 5.2. But this is impossible because of Lemma 5.3.

§6. Comments

6.1. In the theory of Ocneanu’s string algebra construction, there is no criterion for the distinguished point in the case of general graphs. (In the case of Coxeter graphs, the following (i) was announced by A. Ocneanu [O1, page 161–162].) By the arguments of Subsection 2.3, we have the following criterions for the distinguished point and the existence of a flat connection.

Theorem 6.1. (i) *The distinguished point must be the point with the minimal component of the Perron-Forbenius eigenvector.*

(ii) *If we normalize the Perron-Frobenius eigenvector such that the minimal component is one, all components must belong to $\{2 \cos \frac{\pi}{N}; N=3, 4, \dots\} \cup [2, \infty)$.*

These criterions essentially follow from the local index formula in [PP2, Corollary 3.2]. D_5, E_7 and the two graphs of U. Haagerup and J.K. Schou [HS] shown in Fig. 17 do not have a point satisfying these criterions and hence can not appear as a principal graph.



Fig. 17. Two graphs in the list of [HS].

6.2. As we saw in Section 3 and just above, the properties of the statistical dimensions such as the restriction of the values, additivity, and multiplicativity, are very powerful for rejecting non-flat graphs. But we can prove Theorem 3.7 solely on the basis of algebraic calculation of fusion rules. We omit the detail.

6.3. In [Ka] Y. Kawahigashi proves the non-existence of D_{odd} and E_7 para-

groups and the existence of D_{even} paragroups by the argument about flatness of graphs. He also shows the numerical evidence for flatness of E_6 and E_8 . His method is constructive and this is the strong point of his method. But it seems very hard to show flatness for general graphs.

The existence of a flat connection of a graph implies indirectly the existence of consistent fusion rules associated with the graph, through the theory of operator algebras. But Lemma 5.3 shows that the converse is not true, i.e. the consistency of the formal calculation of fusion rules does not necessarily imply the existence of a flat connection. The graph in Fig. 18 is also a counter example. Indeed, suppose the principal graph of $M \supset \rho(M)$ is as in Fig. 18. Then we can take ρ_1 such that

$$\rho_1(M) = \rho(M), \quad [\rho_1]^2 = [id] \oplus [\rho_1] \oplus [\rho_1],$$

as in the case of A_{2n} . This implies that we can construct ρ_1 using the Cuntz algebra O_3 [I]. But direct computation shows that this is impossible.

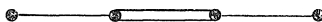


Fig. 18. An example of a non-flat graph which has consistent fusion rules.

We must clarify the relation between flat connections and fusion rules to simplify the axiom of flatness.

6.4. It is not hard to calculate fusion rules for Coxeter graphs with the Perron-Frobenius eigenvalue two such as $D_n^{(1)}, E_6^{(1)}, E_7^{(1)}$ and $E_8^{(1)}$. ($A_n^{(1)}$ is not interesting because the corresponding sector is reducible and is a sum of two automorphisms.) The method in Section 5 is also applicable to the case of $D_6^{(1)}$ and $E_6^{(1)}$. But more complicated calculation is needed for the step corresponding to Lemma 5.3.

6.5. In the theory of the II_1 AFD factor, the counterpart of the notion of the canonical endomorphism is the canonical shift, and that of self-conjugate endomorphism is Choda's roots of the canonical shift [C]. She shows in [C] the following fact.

Fact 6.2. *Let $R \supset P$ be a pair of II_1 factors. If $[R:P] < 4$ or P is the fixed point algebra under an outer action of a finite abelian group, then there exists $\rho \in \text{End}(R)$ such that $\rho(R) = P$ and ρ^2 is the canonical shift.*

In [I], for every finite abelian group G of order n , we construct self-conjugate endomorphism on type $III_{1/n}$ AFD factor, whose image is the fixed point

algebra under an outer action of G , by using Cuntz algebra $O_\#$. Since an outer action of any finite abelian group on the type III_1 AFD factor is unique [KST, Theorem 20], by taking tensor product with the type III_1 factor, we have the following fact.

Fact 6.3. *Let M be the type III_1 AFD factor and N be the fixed point algebra of M under the outer action of a finite abelian group. Then there exists a self-conjugate $\rho \in \text{End}(M)$ such that $\rho(M) = N$.*

In analogy with the case of the II_1 AFD factor we make the following conjecture.

Conjecture 6.4. *Let $M \supset N$ be a pair of type III_1 ASD factors. If $[M : N] < 4$, then there exists a self-conjugate $\rho \in \text{End}(M)$ such that $\rho(M) = N$*

In the case that the principal graph of $M \supset N$ is A_{even} , we saw in Section 3 that this conjecture is true.

§7. Appendix. In the Case of Type II_∞ Factors

Proposition 7.1. *Let $M \supset N$ be a pair of a type II_∞ factors with $[M : N]_0 < \infty$. Then every non-zero projection in $M \cap N'$ is infinite.*

Proof. Let τ_1, τ_2 be the canonical traces on M and N , and let E be the minimal expectation from M to N . We define a faithful normal semi-finite weight by $\tau_2 \circ E$. Then there exists a unique positive self-adjoint operator h affiliated with M such that

$$\tau_2 \circ E = \tau_1(h \cdot).$$

Due to $\sigma_N^{\tau_2 \circ E} = \sigma^{\tau_2} = id_N$, $\sigma^E = id_{M \cup N'}$ and $\sigma_M^{\tau_2 \circ E} = Adh^{it}$, h is affiliated with $Z(M \cap N')$, and in fact h belongs to $Z(M \cap N')$ because of $\dim M \cap N' < \infty$. Let p be a minimal projection in $M \cap N'$. Then $hp = cp$ for some scalar $c > 0$. So

$$\tau_1(p) = \frac{1}{c} \tau_1(hp) = \frac{1}{c} \tau_2(E(p)).$$

Since $E(p)$ is a positive scalar we obtain $\tau_1(p) = \infty$. Q.E.D.

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Note added in proof: After submitting this paper the author received a preprint of V.S. Sunder and A.K. Vijayarajan “On the non-occurrence of the Coxeter graphs E_7 and D_{2n+1} as the principal graph of an inclusion of II_1 factors” on the non-existence of D_{odd} and E_7 para-groups.