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Multiple Solutions for a Class of Non-local Problems for Semilinear Elliptic Equations

By

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Abstract

The purpose of this papers is to investigate the solvability of a class of non-local problems in the sense of Bitsadze-Samarskiĭ (see (N_t)). We prove the existence of multiple solutions under the assumptions of the Ambrosetti-Prodi type on a nonlinear function g.

§1. Introduction

In this paper we investigate the solvability of the following non-local problem for the semilinear elliptic equation

$$(N_t)\qquad \begin{aligned} Lu &= -\sum_{i,j=1}^{n} a_{ij}(x)D_{ij}u + \sum_{i=1}^{n} b_i(x)D_iu + c(x)u \\ &= g(x, u) + t\Phi(x) + f(x) \qquad \text{in } Q, \\ & u(x) - \beta(x)u(\phi(x)) = 0 \qquad \text{on } \partial Q, \end{aligned}$$

in a bounded domain Q with a smooth boundary ∂Q, where $\phi : \partial Q \to Q$ and $\beta : \partial Q \to \boldsymbol{R}$ are given functions and t is real parameter. In the literature the problem of this type is often referred to as the boundary value problem with the Bitsadze-Samarskiĭ condition ([4], [7], [13] and [15]). The most characteristic feature of a non-local problem is that the boundary condition relates values of a solution on the boundary to its values on some parts of the interior of the region. The main purpose of this article is to prove the existence result for the problem (N_t) under the assumption of the Ambrosetti-Prodi type ([1], [4], [5], [10]). It is well known that the Ambrosetti-Prodi conditions played crucial role in the study of the solvability of the Dirichlet problem for the semilinear elliptic equations. This type of assumptions has been widely used in the last decade and we refer to the recent survey article [12].

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In Section 1 we prove the existence of the principal (smallest) eigenvalue of the corresponding linear non-local problem. The proof of Theorem 1 is based on the Krein-Rutman Theorem [11] and we follow here the argument used in papers [2] and [3]. The existence of the principal eigenvalue allows us to study the problem (N_t) under the assumption of the Ambrosetti-Prodi type. In Section 3 we prove the existence of a constant t_0 such that the problem (N_t) has no solution if $t>t_0$, at least one solution if $t=t_0$ and at least two distinct solutions if $t<t_0$ (see Theorem 4). It is obvious that Theorem 4 constitutes an analogue to the corresponding result for the Dirichlet problem for semilinear elliptic equations (see [1], [4], [5], [8], [10] and [11]). The linear non-local problem was first studied by Bitsadze-Samarskii [6] and was subsequently generalized by many authors (see for example [6], [7] and [14]. Finally we mention that the type of the non-local problem considered in this work arises in the physics of plasma [13].

§ 2. Eigenvalue Problem

The main objective of this section is to prove the existence of the principal eigenvalue for the problem

(EVP)
$$Lu=\lambda m(x)u \quad \text{in } Q ,$$
$$u(x)-\beta(x)u(\phi(x))=0 \quad \text{on } \partial Q .$$

We make the following assumptions:

(A) The operator L is uniformly elliptic, that is, there exists $\gamma>0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqq \gamma|\xi|^2$$

for all $x\in Q$ and $\xi\in \boldsymbol{R}_n$.

(B) The coefficients a_{ij}, b_i, c and the derivatives $D_{ij}a_{ij}$ and D_ib_i belong to $C^\alpha(\bar{Q})$ $(0<\alpha<1)$, moreover $c(x)\geqq d>0$ on Q for some constant d. The functions $\beta:\partial Q\to[0, 1]$ and $\phi:\partial Q\to Q$ are continuous. Finally we assume that m is a positive function in $C^\alpha(\bar{Q})$.

Following the terminology from [2] and [3] we provide the Banach space $C(\bar{Q})$ with the natural ordering given by the cone $C^+(\bar{Q})$ of non-negative functions on $\bar{Q}$. We set $\dot{C}^+(\bar{Q})=C^+(\bar{Q})-\{0\}$ and we denote by $\mathring{C}^+(\bar{Q})$ the interior of $C^+(\bar{Q})$.

To proceed further we consider the linear non-local problem

(NL)
$$Lu=f \quad \text{in } Q ,$$
$$u(x)-\beta(x)u(\phi(x))=0 \quad \text{on } \partial Q ,$$

where $f\in C^\alpha(\bar{Q})$. The results of the paper [7] guarantee the existence of a unique solution $u\in C^2(Q)\cap C(\bar{Q})$ of the problem (NL) given by

$$u(x)=\int_{\partial Q}\frac{dG(x,\ y)}{dn_y}v(y)dS_y-\int_Q G(x,\ y)f(y)dy\,, \tag{1}$$

where $v\in C(\partial Q)$ is a solution of the Fredholm integral equation of the second kind

$$v(x)-\int_{\partial Q}\beta(x)\frac{dG(\phi(x),\ y)}{dn_y}v(y)dS_y=\int_Q\beta(x)G(\phi(x),\ y)f(y)dy\,. \tag{2}$$

Here dG/dn_y denotes the conormal derivative of the Green function G associated with the operator L. With the aid of (1) and (2) we define a linear operator $K\colon C^\alpha(\bar{Q})\to C^2(Q)\cap C(\bar{Q})$, given by $u=Kf$, where u denotes a solution of the problem (NL). It follows from Theorem 2 in [7] that

$$|u(x)|=|Kf(x)|\leqq\frac{1}{d}\sup|f(x)|. \tag{3}$$

The estimate (3) has been established in [7] under the assumption that $0\leqq\beta(x)\leqq k<1$ on ∂Q, where k is a constant. An inspection of the proofs of Propositions 1, 2 and Theorem 2 in [7] shows that (3) remains true if $0\leqq\beta(x)\leqq 1$ on ∂Q provided the right hand side of the non-local boundary condition is identically equal to 0. The estimate (3) allows us to extend K by the continuity to the operator $K\colon C(\bar{Q})\to C(\bar{Q})$. Invoking Theorem 9.11 in [9] (p. 235) we have

$$\|u\|_{W^{2,p}(Q')}\leqq C(\|u\|_{L^p(Q)}+\|f\|_{L^p(Q)})$$

for any domain $Q'\Subset Q$ and any $p>1$, where $C=C(Q',\ Q)$ is a constant. Consequently $Kf\in W^{2,p}_{\mathrm{loc}}(Q)\cap C(\bar{Q})$ for each $f\in C(\bar{Q})$ and Kf is a strong solution of the equation $Lu=f$ (see Chapter 9 in [9]).

Let e be the solution of the problem (NL) with $f(x)\equiv 1$ on Q. It follows from the strong maximum principle and Proposition 2 in [7] that $e\in C^+(\bar{Q})$ and we define

$$C_e(\bar{Q})=\{\lambda[-e,\ e],\ \lambda\in\boldsymbol{R}\}$$

equipped with the norm given by the Minkowski functional (see [3] p. 630). It follows from [2] that $C_e^+(Q)=C_e(\bar{Q})\cap C^+(\bar{Q})$ is the positive cone in $C_e(\bar{Q})$ with non empty interior.

Lemma 1. *The operator* $K\colon C(\bar{Q})\to C(\bar{Q})$ *is compact.*

Proof. Let f_m be a bounded sequence in $C(\bar{Q})$. Then $u_m(x)=Kf_m(x)$ is given by (1) with $v_m\in C(\partial Q)$ satisfying the integral equation (2). By Arzela's Theorem the sequence

$$Z_m(x)=\beta(x)\int_Q G(\phi(x),\ y)f_m(y)dy$$

contains a uniformly convergent subsequence on ∂Q, which we relabel again as Z_m. Since $\lambda=1$ is not the eigenvalue of the integral equation (2), we may assume that the sequence v_m is uniformly convergent on ∂Q. Similarly the sequence

$$\int_Q G(x, y) f_m(y) dy$$

also contains a uniformly convergent subsequence on $\bar{Q}$. Therefore the compactness of the operator K follows from the representation formula (1).

We are now in a position to prove the existence of the principal eigenvalue of the (EVP).

Theorem 1. *The (EVP) admits a positive eigenvalue $\Lambda_0(m)$ with a positive eigenfunction Φ_0. Moreover $\Lambda_0(m)$ is the only eigenvalue with the positive eigenfunction.*

Proof. We first assume that $\beta(x)>0$ on ∂Q. Then $Kf \in \mathring{C}^+(\bar{Q})$ for $f \in \dot{C}^+(\bar{Q})$, that is, K is a strongly positive operator. Indeed, if $u(x_0)=0$, then by the strong maximum principle $x_0 \in \partial Q$ and by the non-local boundary condition $u(\phi(x_0))=0$ with $\phi(x_0) \in Q$, which is impossible. The (EVP) is equivalent to the fixed point equation

$$\frac{1}{\lambda} u = K(mu) \tag{4}$$

and the result follows from the Krein-Rutman theorem ([11], see also Theorem 3.2 in [3]). Using the standard regularity theory for the elliptic equations it is easy to see that any solution in $C(\bar{Q})$ of (4) belongs to $C(\bar{Q}) \cap C^2(Q)$ and is a solution of (EVP),

Let us now consider the case when β vanishes at some points of ∂Q. It is evident by the strong maximum principle and the non-local nature of the boundary condition that for every solution u of (NL) with $f \in \dot{C}^+(\bar{Q})$ we have $u(x_0)=0$ if and only if $x_0 \in \beta^{-1}(0)$. We now prove that K is e-positive operator, that is, for each $f \in \dot{C}^+(Q)$ there exist positive constants a and b such that $ae \leqq Kf \leqq be$. The right-hand side of this inequality follows from the positivity of the operator K. To prove the left hand side of this inequality we choose $v_1 \in C^\alpha(\bar{Q})$ such that $v_1 \in \dot{C}^+(\bar{Q})$ and $0 \leqq v_1 \leqq f$ on $\bar{Q}$. Hence $u_1 = Kv_1 \leqq Kf$. By the maximum principle and the above observation $du_1(x)/d\nu < 0$ and $de(x)/d\nu < 0$ for each $x \in \beta^{-1}(0)$, where $d/d\nu$ denotes the directional derivative with respect to the outward normal. By the continuity there exists $\alpha_1 > 0$ such that for all $\alpha \in (0, \alpha_1]$ and all $x \in \beta^{-1}(0)$

$$\frac{d}{d\nu}(u_1(x) - \alpha e(x)) < 0 .$$

Therefore, since $u_1(x)-\alpha e(x)=0$ on $\beta^{-1}(0)$ there exists a neighbourhood U of $\beta^{-1}(0)$ in Q such that $u_1(x)-\alpha e(x)>0$ for all $x\in U-\beta^{-1}(0)$. On the other hand $u_1(x)$ is positive on the set $\bar{Q}-U$, hence $u_1(x)-\alpha e(x)>0$ on $\bar{Q}-U$ for sufficiently small $\alpha>0$. This shows that there exist $a>0$ such that $ae\leqq Kf$ on Q, that is K is e-positive. Since the injection of $C_e(\bar{Q})$ into $C(\bar{Q})$ is strictly positive, the restriction of $K(m\cdot)$ to $C_e(\bar{Q})$ is strongly positive compact linear operator. The result follows from Theorem 3.2 in [3].

Remark 1. Let $\lambda_0(m)$ be the first eigenvalue for the Dirichlet problem

$$Lu=\lambda mu \qquad \text{in } Q\,,$$

$$u=0 \qquad \text{on } \partial Q\,.$$

One can show that $\Lambda_0(m)<\lambda_0(m)$ provided $\beta\not\equiv 0$ on ∂Q. To show this we introduce the solution operator $u=\bar{K}_1 f$ corresponding to the Dirichlet problem $Lu=f$ in Q and $u=0$ on ∂Q (see [3] p. 635). The maximum principle yields that $Kf-K_1 f\in\mathring{C}^+(Q)$. Therefore $\lambda_0(m)>\Lambda_0(m)$.

Remark 2. Let us assume that $\phi(x)=y_0$ with $y_0\in Q$ for all $x\in\partial Q$, that is, the mapping ϕ is constant. Then the second eigenvalue $\Lambda_1(m)$ of (EVP) satisfies the inequality $\lambda_0(m)\leqq\Lambda_1(m)$. In the contrary case $\Lambda_1(m)<\lambda_0(m)$ and the corresponding eigenfunction $v(x)$ must be equal to $\beta(x)v(y_0)$ on ∂Q. Hence by Theorem 4.4 in [3] v must be of constant sign on Q which is impossible. However we were unable to prove this result in a general case.

Remark 3. Theorem 1 remains true if $c(x)\geqq 0$ on Q and $0\leqq\beta(x)\leqq\delta<1$ on ∂Q, where δ is a constant. For this we need only to show that the estimate (3) continues to hold. As in Theorem 3.2 in [9] we introduce the auxiliary function

$$v(x)=u(x)-(e^d-e^{kx_1})\sup_Q|f(x)|,$$

where $d>kb$ and Q is contained in a slab $0<x_1<b$. Thus

$$\begin{aligned} Lv=f-a_{11}k^2e^{kx_1}\sup_Q|f|+b_1ke^{kx_1}\sup_Q|f|\\ -c(e^d-e^{kx_1})\sup_Q|f|\leqq 0 \qquad \text{in } Q\,, \end{aligned}$$

provided k is sufficiently large and $d>bk$, and

$$\begin{aligned} v(x)-\beta(x)v(\phi(x))&=-(e^d-e^{kx_1})\sup_Q|f|+\beta(x)(e^d-e^{k\phi(x_1)})\sup_Q|f|\\ &\leqq e^d(\delta-1)\sup_Q|f|+(e^{kx_1}-\beta(x)e^{k\phi_1(x_1)})\sup_Q|f| \end{aligned}$$

on ∂Q. The last inequality can be achieved by increasing d if necessary. Consequently by Proposition 2 in [7] we have

$$u(x)\leqq(e^{d}-e^{kx_1})\sup_{Q}|f| \qquad \text{on } Q .$$

Using the function

$$w(x)=u(x)+(e^{d}-e^{kx_1})\sup_{Q}|f|$$

we prove that $u(x)\geqq-(e^{d}-e^{kx_1})\sup_{Q}|f|$ on Q.

§ 3. Semilinear Non-local Problem and Multiple Solutions

We commence by investigating the solvability of the non-local problem

$$Lu=f(x, u) \qquad \text{in } Q , \tag{5}$$

$$u(x)-\beta(x)u(\phi(x))=0 \qquad \text{on } \partial Q , \tag{6}$$

using the well known method of sub- and supersolutions.

A function $\Phi\in C^{2}(Q)\cap C(\bar{Q})$ is said to be a subsolution of the problem (5), (6) if $L\Phi\leqq f(x, \Phi)$ in Q and $\Phi(x)-\beta(x)\Phi(\phi(x))\leqq 0$ on ∂Q.

A supersolution is defined by reversing the inequality signs in the above definition.

Theorem 2. *Suppose that* $f\in C^{\alpha}(\bar{Q}\times\boldsymbol{R})$ *and that* $f(x, \xi)-f(x, \eta)\geqq-\omega(\xi-\eta)$ *for some positive constant* ω *and all* (x, ξ) *and* (x, η) *in* $Q\times\boldsymbol{R}$ *with* $\xi>\eta$. *If the problem* (5), (6) *admits a subsolution* Φ *and a supersolution* Ψ *such that* $\Phi(x)\leqq\Psi(x)$ *on* $\bar{Q}$, *then the problem* (5), (6) *has solutions* u *and* v *in* $C(\bar{Q})\cap C^{2}(Q)$ *such that* $\Phi(x)\leqq u(x)\leqq v(x)\leqq\Psi(x)$ *on* $\bar{Q}$. *Moreover any solution* w *of* (5), (6) *satisfying* $\Phi(x)\leqq w(x)\leqq\Psi(x)$ *on* $\bar{Q}$ *is such that*

$$\Phi(x)\leqq u(x)\leqq w(x)\leqq v(x)\leqq\Psi(x) \qquad \text{on } Q .$$

Proof. We define two sequences of solutions of the linear non-local problem

$$Lu_{k+1}=f(x, u_{k}) \qquad \text{in } Q ,$$

$$u_{k+1}(x)-\beta(x)u_{k+1}(\phi(x))=0 \qquad \text{on } \partial Q ,$$

for $k=0, 1, 2, \cdots$, where $u_{0}=\Phi$ and

$$Lv_{k+1}=f(x, v_{k}) \qquad \text{in } Q ,$$

$$v_{k+1}(x)-\beta(x)v_{k+1}(\phi(x))=0 \qquad \text{on } \partial Q ,$$

for $k=0, 1, 2, \cdots$, where $v_{0}=\Psi$. Without loss of generality we may assume that $f(x, u)$ is increasing in u, since otherwise f and L can be replaced by $f(x, u)+\omega u$ and $L+\omega$, respectively. It follows from Proposition 2 in [7] that

$$\Phi(x)\leqq u_{1}(x)\leqq \cdots \leqq u_{k}(x)\leqq v_{k}(x)\leqq \cdots \leqq v_{1}(x)\leqq\Psi(x)$$

on $\bar{Q}$ for each k. Furthermore, by virtue of Theorem 9.11 in [9] we have

$$\|u_{k+1}\|_{W^{2,p}(Q')} \leqq C(\|u_k\|_p + \|f(\circ,\ u_k)\|_p)$$

for each $p>1$ and each $Q' \Subset Q$, where $C=C(Q')>0$ is a constant independent of k. Hence by the Sobolev embedding theorem we may assume that there exists a function $u \in C^1(Q)$ such that $u=\lim_{k\to\infty} u_k$ and $D_i u = \lim_{k\to\infty} D_i u_k$ uniformly on each compact subset of Q. The Schauder interior estimates yield that $u \in C^2(Q)$ and that $D_{ij}u = \lim_{k\to\infty} D_{ij}u_k$ uniformly on each compact subset of Q (see Theorem 6.2 in [9]). Since $u_k(x)-u_l(x)=\beta(x)[u_k(\phi(x))-u_l(\phi(x))]$ on ∂Q for all integers k, l we conclude from the weak maximum principle of Alexandrov (see Theorem 9.1 in [9]), that

$$\sup_{\bar{Q}} |u_k(x)-u_l(x)| \leqq \sup_{\partial Q} |u_k(\phi(x))-u_l(\phi(x))| + C\left\|\frac{1}{D^*}(f(\cdot,\ u_k)-f(\cdot,\ u_l))\right\|_n, \tag{7}$$

where $D^*=\det[a_{ij}]^{1/n}$ and $C>0$ is a constant independent of k and l. Now observe that $\lim_{k\to\infty} u_k(x)=u(x)$ uniformly on $\phi(\partial Q)$ and that $\lim_{k\to\infty}\|f(\cdot,u_k)-f(\cdot,u)\|_n =0$ by the Lebesgue Monotone Convergence Theorem. Combining these two facts with (7) we conclude that $\lim_{k\to\infty} u_k = u$ uniformly on $\bar{Q}$. The remaining part of the proof is standard and therefore is omitted.

To establish the multiplicity result for the problem (N_t) we assume that $m\equiv 1$ on Q. We denote briefly the principal eigenvalue of (EVP) and that of the Dirichlet problem by Λ_0 and λ_0 respectively. Further we assume that $\Phi \in C^\alpha(Q)\cap \dot{C}^+(Q)$, $f \in C^\alpha(Q)$, $(0<\alpha<1)$ and $g \in C^1(\bar{Q}\times \boldsymbol{R})$. Moreover the nonlinearity g satisfies the conditions of the Ambrosetti-Prodi type

$$\lim_{s\to-\infty}\sup \frac{g(x,\ s)}{s} < \Lambda_0 \leqq \lambda_0 < \lim_{s\to\infty}\inf \frac{g(x,\ s)}{s} \tag{8}$$

uniformly in $\bar{Q}$ and

$$\lim_{s\to\infty}\sup \frac{g(x,\ s)}{s} < \infty \tag{9}$$

uniformly in $\bar{Q}$ (the strict inequality $\Lambda_0<\lambda_0$ holds if $\beta \not\equiv 0$ on $\partial\Omega$).

The assumption (8) implies the existence of constants $C>0$ and $\underline{\mu}<\Lambda_0<\lambda_0<\bar{\mu}$ such that

$$g(x,\ s) \geqq \underline{\mu}s - C \quad \text{for all } (x,\ s)\in \bar{Q}\times\boldsymbol{R}, \tag{10}$$

$$g(x,\ s) \geqq \bar{\mu}s - C \quad \text{for all } (x,\ s)\in \bar{Q}\times\boldsymbol{R}. \tag{11}$$

We begin by proving some technical lemmas. The methods used in the proofs are not new and have appeared in several papers (see for example [4], [8] and [10]).

Lemma 2. *There exists a number τ such that for $t<\tau$, the problem (N_t) has no solutions.*

Proof. Let Ψ_0 be the first eigenfunction of the Dirichlet problem for the adjoint operator L^*. Thus

$$\lambda_0\int_Q \Psi_0 u\,dx=\int_Q Lu\Psi_0\,dx=\int_Q g(x,u)\Psi_0\,dx+t\int_Q \Psi_0\Phi\,dx+\int_Q f\Psi_0\,dx.$$

It follows from the inequalities (10) and (11) that

$$\lambda_0\int_Q \Psi_0 u\,dx\geqq\int_Q \underline{\mu} u\Psi_0\,dx-C\int_Q \Psi_0\,dx+t\int_Q \Psi_0\Phi\,dx+\int_Q f\Psi_0\,dx$$

and

$$\lambda_0\int_Q \Psi_0 u\,dx\geqq\int_Q \bar{\mu} u\Psi_0\,dx-C\int_Q \Psi_0\,dx+t\int_Q \Psi_0\Phi\,dx+\int_Q f\Psi_0\,dx.$$

These inequalities imply that

$$t\leqq\left(\int_Q \Psi_0\Phi\,dx\right)^{-1}\left[\int_Q (\lambda_0-\underline{\mu})u\Psi_0\,dx+C\int_Q \Psi_0\,dx-\int_Q f\Psi_0\,dx\right] \tag{12}$$

and

$$t\leqq\left(\int_Q \Psi_0\Phi\,dx\right)^{-1}\left[\int_Q (\lambda_0-\bar{\mu})u\Psi_0\,dx+C\int_Q \Psi_0\,dx-\int_Q f\Psi_0\,dx\right]. \tag{13}$$

If $\int u\Psi_0\,dx\leqq 0$, then (12) yields that

$$t\leqq\left(\int_Q \Psi_0\Phi\,dx\right)^{-1}\left[C\int_Q \Psi_0\,dx-\int_Q f\Psi_0\,dx\right]$$

and if $\int_Q u\Psi_0\,dx\geqq 0$ then if follows from (13) that

$$t\leqq\left(\int_Q \Psi_0\Phi\,dx\right)^{-1}\left[C\int_Q \Psi_0\,dx-\int_Q f\Psi_0\,dx\right].$$

So in both cases the existence of a solution implies that

$$t\leqq\left(\int_Q \Psi_0\Phi\,dx\right)^{-1}\left(C\int_Q \Psi_0\,dx-\int_Q f\Psi_0\,dx\right)\equiv\tau$$

and a solution does not exist if $t>\tau$.

Remark 4. If $\Psi_0\perp f$, then τ is independent of f.

In the following lemma we assume for simplicity that $t=0$.

Lemma 3. *The problem (N_0) admits a subsolution ω satisfying $\omega(x)-\beta(x)\omega(\phi(x))=0$ on ∂Q and such that $\omega<u$ on $\bar{Q}$ for any supersolution u of (N_0) satisfying $u(x)-\beta(x)u(\phi(x))=0$ on ∂Q.*

Proof. Let ω be a unique solution of the non-local problem

$$Lu=\underline{\mu}u-C+f(x) \quad \text{in } Q\,,$$

$$u(x)-\beta(x)u(\phi(x))=0 \quad \text{on } \partial Q\,,$$

where $\underline{\mu}$ and C are constants from (10) and C is chosen in such a way that one has the strict inequality in (10). Let u be a supersolution of the problem (N_0) with $u(x)-\beta(x)u(\phi(x))=0$ on ∂Q. Then we have

$$L(u-\omega)\geqq g(x,\ u)-\underline{\mu}\omega+C\geqq\underline{\mu}(u-v) \quad \text{in } Q$$

and consequently $u>v$ on $\bar{Q}$ since $\underline{\mu}<\Lambda_0$.

Lemma 4. *There exists a $t\in\boldsymbol{R}$ such that the problem (N_t) has a supersolution.*

Proof. For a fixed $N>0$ we set

$$m=\max\{g(x,\ s)+f(x);\ x\in\bar{Q},\ 0\leqq s\leqq N\}.$$

We choose subdomains $Q_1\subset\bar{Q}_1\subset Q_2\subset\bar{Q}_2\subset Q$ with meas$(Q-Q_1)\leqq\delta$, where $\delta>0$ is to be determined. Let $H\in C^{\alpha}(\bar{Q})$ be such that $H(x)=0$ on Q_1, $H(x)=m$ on $\bar{Q}-Q_2$ and $0\leqq H(x)\leqq m$ on $\bar{Q}$. Let v be a solution of the non-local problem

$$Lv=H \quad \text{in } Q$$

$$v(x)-\beta(x)v(\phi(x))=0 \quad \text{on } \partial Q\,.$$

By Proposition 2 in [7] $v>0$ on $\bar{Q}$. The solution v is given by

$$v(x)=\int_{\partial Q}\frac{d}{dn_y}G(x,\ y)w(y)dS_y+\int_Q G(x,\ y)H(y)dy\,,$$

where w is a solution of the integral equation (2) with f replaced by H. The integral $GH(x)=\int_Q G(x,\ y)H(y)dy$ is a solution of the Dirichlet problem $Lu=H$ in Q and $u=0$ on ∂Q, therefore by standard estimates

$$\|GH\|_{W^{2,p}}\leqq M\|H\|_p\leqq Mm\ \text{meas}\,(Q-Q_1)^{1/p}\leqq Mm\delta^{1/p}$$

for each $p>1$, where $M>0$ is a constant. Taking p sufficiently large and applying the Sobolev embedding theorem we obtain

$$\sup_{\bar{Q}}|GH(x)|\leqq M'Mm\delta^{1/p}$$

for some $M'>0$. Since $\lambda=1$ is not eigenvalue for the Fredholm integral equation (2) we get the following estimate for w

$$\|w\|_{L^2}\leqq C_1 m\delta^{1/p}$$

for some $C_1>0$. Finally applying the Hölder inequality we deduce from (2) that

$$|w(x)|\leqq\left(\int_{\partial Q}\left|\frac{d}{dn_y}G(\phi(x),\ y)\right|^2 dS_y\right)^{1/2}\|w\|_{L^2}+MM'm\delta^{1/p}.$$

Now observe that $\operatorname{dist}(\phi(\partial Q),\ \partial Q)>0$ and consequently we derive from the last inequalities that

$$\sup_{\partial Q}|w(x)|\leqq C_2\delta^{1/p} \qquad \text{for some } C_2>0.$$

Now we choose $\delta>0$ so that $C_2\delta^{1/p}\leqq N$. Taking t sufficiently large but negative to ensure that $m+t\Phi\leqq H$ on Q we have

$$Lv=H\geqq m+t\Phi\geqq g(x,\ v)+t\Phi+f,$$

and this completes the proof.

Lemma 5. *Suppose that the problem* (N_0) *has a solution, then the problem*

(14) $\qquad Lu=g(x,\ u)+h$ *in* Q *and* $u(x)-\beta(x)u(\phi(x))=0$ *on* ∂Q

has a solution for each $h\leqq f$ *and* $h\in C^\alpha(\bar{Q})$.

Proof. Since u is a supersolution of the problem (17), by Lemma 3 it has a subsolution. Hence the existence of a solution of (14) follows from Theorem 2.

Corollary. *Suppose that the problem* (N_t) *has a solution for* $t=t_0$, *then it has a solution for any* $t<t_0$.

Theorem 3. *There exists* t_0 *such that the problem* (N_t) *has a solution for* $t<t_0$ *and no solution for* $t>t_0$.

Proof. By Lemma 4 the problem (N_t) has a supersolution for certain t. Lemma 3 implies the existence of a subsolution and consequently Theorem 2 ensures the existence of a solution. To complete the proof we set $t_0=\sup\{t;$ the problem (N_t) admits a solution$\}$. By Lemma 2 t_0 is finite and the above Corollary guarantees the existence of a solution for each $t<t_0$.

Now we are in a position to prove our final result.

Theorem 4. *There exists* t_0 *such that the problem* (N_t) *has at least two solutions for* $t<t_0$ *at least one solution for* $t=t_0$ *and no solution for* $t>t_0$.

Proof. The proof is similar to the proof of the analogous result for the Dirichlet problem presented in the paper [4]. Therefore we only sketch the main idea of the proof. Let $t^*<t_0$. It follows from the proof of Theorem 3 that the problem (N_{t^*}) has a solution u belonging to the interior of the order interval $X=[\underline{u},\ \bar{u}]$, where $\underline{u}$ and $\bar{u}$ are suitably chosen a sub- and supersolutions of the problem (N_{t^*}). We now set

$$\omega_0=\max\{|D_s g(x,\ s)|\ ;\ x\in\bar{Q},\ \underline{u}(x)\leqq s\leqq\bar{u}(x)\}$$

$$\omega=\max(\omega_0+1,\ \|e\|_{C(\bar{Q})})$$

and denote by K the solution operator for the non-local problem for the operator $L+\omega$. One can assume that u is the only fixed point of the operator $G=KF(\cdot, t^*)$ in Int X, where $F(u, t^*)=g(x, u)+t^*\Phi+f+\omega u$. It is easy to see that the Leray-Schauder degree $\deg(id-G, u_0+\varepsilon B, 0)=1$ for small $\varepsilon>0$, where B denotes a unit ball in $E=\{u; u\in C(\bar{Q}), u(x)=\beta(x)u(\phi(x))$ on $\partial Q\}$. The second key idea is to show that $\deg(id-G, \rho B, 0)=0$ for some $\rho>0$ such that $u_0+\varepsilon B \subset \rho B$. Thus, the degree on $\rho B-(u_0+\varepsilon\bar{B})$ is -1, which implies that there is a fixed point of G in $\rho B-(u_0+\varepsilon\bar{B})$. The existence of a solution for the problem (N_{t_0}) can be obtained as a limit of the sequence of solutions of problems (N_{t_j}) with $t_j<t_0$ and $t_j\to t_0$ (for details see [4], p. 150.

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