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On the Whitney-Schwartz Theorem

By

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Let F be a closed set in R^n. Then, according to L. Schwartz [6], F is called regular if for each $x \in F$ there are numbers $d(>0)$, $\omega(\geqq 0)$ and $q(\geqq 1)$ such that any two points y, z of F with $r_{xy} \leqq d$ and $r_{xz} \leqq d$, are joined by a rectifiable curve in F, of length not greater than $\omega r_{yz}^{1/q}$ where r_{xy} is the distance between x and y. This definition is a generalization of "Property (P) local" by H. Whitney ([9]). Schwartz stated in [6] the following theorem without proof.

Theorem (Whitney-Schwartz). *Let T be a distribution in R^n of order m whose support is contained in a compact regular set F. Then*

(A) $\langle T, \varphi_j \rangle \to 0$ *provided $\varphi_j \in C^\infty(R^n)$ and their derivatives of order not greater than m' converge to zero uniformly on F, where m' is any integer $\geqq q(F)m$ and $q(F)$ is a number $\geqq 1$, depending on F.*

(B) *T is represented by a finite sum of derivatives of measures whose supports are contained in F.*

A similar result to the part (A) of Theorem was given for a general compact set F by G. Glaeser, in such a sense that it has an advantage not making the behavior of φ_j interfered in a 'neighborhood' of F (see Proposition II, Chap III in [2]). We shall give an elementary proof of Theorem for a distribution in an open set Ω of R^n. For the proof we make use the reproduction of Whiteny's extension theorem by L. Hörmander ([4]). The key lemma is the following:

Lemma. *Let u be a distribution of order m in Ω with support in a compact regular set $F \subset \Omega$. Then there is a constant C depending on m' and F such that for any $\varphi \in C^\infty(\Omega)$*

$$|\langle u, \varphi \rangle| \leqq C \|\varphi\|_{m', F} \tag{1}$$

where $q = q(F)$ is a positive number depending on F, m' is any integer $\geqq qm$, and

$$\|\varphi\|_{k, F} = \sum_{|\alpha| \leqq k} \sup_{x \in F} |(\partial/\partial x)^\alpha \varphi(x)|.$$

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In case F is a closed ball, a simple proof of (1) was given by S. Mizohata in proving that evolution equations with finite propagation speed should be of kowalevskian type (see [5]). Now we shall restate the Whitney-Schwartz theorem in our form.

Theorem. *Let u be a distribution in Ω of order m with support in a compact regular set $F\subset\Omega$. Then*

(A) $\langle u, \varphi_j\rangle\to 0$ *provided $\varphi_j\in C^\infty(\Omega)$ and their derivatives of order not greater than m' converge to zero uniformly on F, where m' is any integer $\geqq q(F)m$.*

(B) *u is represented by a finite sum of derivatives of measures in Ω whose supports are in F.*

Proof of (A) is immediate from Lemma. For proving (B) we can apply the Hahn-Banach theorem to the inequality (1) through the well-known method. We omit the details (see [6]).

Before proceeding to prove our lemma, we shall give a sketch of the partition of unity by Whitney in [8], following the reproduction by Hörmander.

Let A be a closed set in R^n. The partition of unity is constructed as follows. First, divide R^n into n-cubes of side 1, and let K_0 be the set of all those cubes whose distance from A are at least $\sqrt{n}$. Next, divide the remaining cubes into 2^n cubes of side 1/2, and let K_1 be the set of those distant from A at least $\sqrt{n}/2$. Repeating such a division process, we have a series of the sets $\{K_0, K_1, \cdots\}$ where the union of all cubes of them is $R^n\setminus A$. Arrange all cubes in order of a series $Q_1, Q_2, \cdots$; the center and side of each Q_j are denoted by y^j and s_j, respectively. Now take $\chi_0\in C_0^\infty$ being equal to 1 on the cube

$$|x_i|\leqq 1/2, \qquad i=1, \cdots, n$$

and vanishing outside the cube

$$|x_i|\leqq 1/2+1/8, \qquad i=1, \cdots, n\,.$$

Then define $\chi_j\in C_0^\infty(R^n)$ by

$$\chi_j(x)=\chi_0\Big(\frac{x-y^j}{s_j}\Big)\Big/\sum_{k=1}^\infty\chi_0\Big(\frac{x-y^k}{s_k}\Big), \qquad j=1, 2, \cdots.$$

As for the denominator, it is verified

$$1\leqq\sum_{i=1}^\infty\chi_0\Big(\frac{x-y^k}{s_k}\Big)\leqq 4^n\,.$$

The sequence χ_j in $C_0^\infty(R^n)$ is locally finite in $R^n\setminus A$ and has the properties:

(i) $\chi_j\geqq 0$; $\sum_{j=1}^\infty\chi_j(x)=1$ for $x\in R^n\setminus A$

(ii) for each α, there is a constant C_α such that

$$\sum_{j=1}^\infty|D^\alpha\chi_j(x)|\leqq C_\alpha(d(x, A)^{-|\alpha|}+1)$$

for $x\in R^n\setminus A$ where $D=\partial/\partial x$

(iii) (the diameter of $\operatorname{supp}\chi_j)\leqq Cd(\operatorname{supp}\chi_j,\ A)$, $\quad j=1, 2, \cdots$

for some constant C.

In the following we quote each of (i), (ii), (iii) as the property of χ_j. Let $x\in\operatorname{supp}\chi_j$. Then it can be easily verified $d(x, A)>1$, provided $s_j=1$ (see [4]). So we note $d(x, A)\leqq 1$ implies $s_j<1$.

Proof of Lemma. Since F is regular, to each $x\in F$ there corresponds an open ball $B_d(x)$ of center x, with radius d in Ω such that any two points y, z of $F\cap\overline{B_d(x)}$ can be joined by a rectifiable curve in F. Here we note the radius d depends on x. Since F is compact, we can choose a finite family $\{B_{d_1}(x_1), \cdots, B_{d_m}(x_m)\}$ from the open cover $\{B_d(x)|x\in F\}$ of F so that

$$F\subset B_{d_1}(x_1)\cup\cdots\cup B_{d_m}(x_m).$$

Take a partition of unity ϕ_j subordinate to the finite open cover $\{B_{d_j}(x_j)\}$. Then u is represented in the form

$$u=\phi_1 u+\cdots+\phi_m u=u_1+\cdots+u_m \tag{2}$$

where $\operatorname{supp}u_j=\operatorname{supp}\phi_j u\subset F_j=F\cap\overline{B_{d_j}(x_j)}$, F_j being also compact and regular. Suppose the estimate (1) is valid for each u_j, F_j, q_j instead of u, F, q. Then for $\varphi\in C^\infty(\Omega)$, there is a constant C_j such that

$$|\langle u_j,\ \varphi\rangle|\leqq C_j\|\varphi\|_{m_j, F_j}. \tag{3}$$

where m_j is any integer $\geqq mq_j$, q_j being a number ($\geqq 1$) related to the regularity of F_j. Clearly, Lemma is a consequence of (2) and (3), with $q=q(F)=\sup_{1\leqq j\leqq m}q_j$ and $C=\sup_{1\leqq j\leqq m}C_j$. So it suffices to derive (3) for each $\varphi\in C^\infty(\Omega)$, in which we write $F=F_j$ and $q=q_j$, dropping the subscript j for simplicity of notation.

Take φ in $C_0^\infty(\Omega)$ and extend it to a function in $C_0^\infty(R^n)$ by setting zero outside Ω, which we denote φ again. We shall give a function $\phi\in C^m(\Omega)$ so that $\phi^{(\alpha)}(x)=\varphi^{(\alpha)}(x)$ in F when $|\alpha|\leqq m$ where $D^\alpha f=f^{(\alpha)}$. This can be carried out by the method of Whitney's extension theorem, as follows. Making use of the partition of unity χ_j in $R^n\setminus F$ just constructed above, we define a function ϕ by

$$\phi(x)=\begin{cases}\varphi(x) & \text{for}\quad x\in F\\ \sum_j'\chi_j(x)\varphi_m(x\,;\ y^j) & \text{for}\quad x\in R^n\setminus F\end{cases} \tag{4}$$

where $y^j\in F$ is taken so that

$$d(\operatorname{supp}\chi_j,\ F)=d(\operatorname{supp}\chi_j,\ y^j)$$

$$\varphi_m(x\,;\ y)=\sum_{|\alpha|\leqq m}\frac{1}{\alpha\,!}\varphi^{(\alpha)}(y)(x-y)^\alpha$$

and $\sum_j'$ stands for the sum with $s_j<1$. Then $\phi\in C_0^m(R^n)$ and satisfies

$$D^\alpha \psi = D^\alpha \varphi \quad \text{in} \quad F$$

when $|\alpha| \leqq m$ ([6]). What we are going to obtain is the estimate

$$\|\psi\|_{m,\Omega} \leqq C\|\varphi\|_{m',F} \tag{5}$$

where C is a constant depending only on m, q and F. Let $x \in \Omega \setminus F$ be fixed. Then $d(x, F) > 0$. To derive (5) we divide the case into 1) $d(x, F) > 1$ and 2) $d(x, F) \leqq 1$. It is enough to show for the case where $m \geqq 1$.

1) $d(x, F) > 1$.

Differentiation of ψ in (4) gives, by Leibniz's formula,

$$\psi^{(\alpha)}(x) = \sum_{\beta+\gamma=\alpha} \sum_j{}' \chi_j^{(\beta)}(x) \varphi_m^{(\gamma)}(x\,;\, y^j).$$

Since $\sum_j' |\chi_j^{(\beta)}(x)| \leqq 2C_\beta$ by the property (ii) of $\{\chi_j\}$, we have

$$|\psi^{(\alpha)}(x)| \leqq C'_\alpha \|\varphi\|_{m,F}$$

for a constant C'_α when $|\alpha| \leqq m$, which implies (5).

2) $0 < d(x, F) \leqq 1$.

In this case, as we noted before, $s_j < 1$ provided $x \in \operatorname{supp} \chi_j$. Hence we have $\sum_j' \chi_j(x) = \sum_j \chi_j(x) = 1$, which gives

$$\psi(x) = \varphi_m(x\,;\, y) + \sum_j \chi_j(x)[\varphi_m(x\,;\, y^j) - \varphi_m(x\,;\, y)] \tag{6}$$

where $y \in F$ is so chosen as $d(x, F) = |x - y|$. Further we take $x^j \in \operatorname{supp} \chi_j$ so as to satisfy $d(\operatorname{supp} \chi_j, F) = |x^j - y^j|$. Then

$$|x - x^j| \leqq \operatorname{diam}(\operatorname{supp} \chi_j) \leqq C d(\operatorname{supp} \chi_j, F) \leqq C d(x, F)$$

for some constant C where we used the property (iii) of $\{\chi_j\}$. Thus in view of the definitions of y, x^j and y^j, we get the inequalities

$$\begin{aligned} |x - y^j| &\leqq |x - x^j| + |x^j - y^j| \leqq (C+1) d(x, F) \\ |y - y^j| &\leqq |y - x| + |x - y^j| \leqq (C+2) d(x, F) \end{aligned} \tag{7}$$

which will be needed later. Denoting by $R_m(x\,;\, y)$ the remainder term of Taylor's formula at y, we have

$$\varphi(x) = \varphi_m(x\,;\, y) + R_m(x\,;\, y). \tag{8}$$

Our basic concern is to estimate the derivatives of the difference $\varphi_m(x\,;\, y^j) - \varphi_m(x\,;\, y)$ in (6). The Taylor polynomial of $\varphi^{(\gamma)}(z'')$

$$\varphi_m^{(\gamma)}(z''\,;\, z') = \sum_{|\beta| \leqq m - |\gamma|} \frac{1}{\beta!} \varphi^{(\beta+\gamma)}(z')(z'' - z')^\beta$$

combined with the formula for any z, z', z''

$$\varphi^{(\beta+\gamma)}(z') = \varphi_m^{(\beta+\gamma)}(z'\,;\, z) + R_m^{(\beta+\gamma)}(z'\,;\, z)$$

obtained by differentiating (8), gives

$$\varphi_m^{(\gamma)}(z'';\ z') = \sum_{|\delta| \leqq m-|\gamma|} \frac{1}{\delta\,!} [\varphi_m^{(\gamma+\delta)}(z';\ z) + R_m^{(\gamma+\delta)}(z';\ z)](z''-z')^{\delta}. \tag{9}$$

On the other hand

$$\varphi_m^{(\gamma)}(z'';\ z) = \sum_{|\delta| \leqq m-|\gamma|} \frac{1}{\delta\,!} \varphi_m^{(\gamma+\delta)}(z';\ z)(z''-z')^{\delta}. \tag{10}$$

Thus the subtraction of (10) from (9) yields

$$\varphi_m^{(\gamma)}(z'';\ z') - \varphi_m^{(\gamma)}(z'';\ z) = \sum_{|\delta| \leqq m-|\gamma|} \frac{1}{\delta\,!} R_m^{(\gamma+\eta)}(z';\ z)(z''-z')^{\delta}, \tag{11}$$

so that changing z, z', z'' to y, y^j, x gives

$$\begin{aligned} |\varphi_m^{(\gamma)}(x;\ y^j) - \varphi_m^{(\gamma)}(x;\ y)| &\leqq \sum_{|\eta| \leqq m-|\gamma|} |R_m^{(\gamma+\eta)}(y^j;\ y)(x-y^j)^{\eta}| \\ &\leqq C_0\|\varphi\|_{m',F} + \sum_{|\eta| \leqq m'-|\gamma|} |R_{m'}^{(\gamma+\eta)}(y^j;\ y)(x-y^j)^{\eta}| \end{aligned} \tag{12}$$

since $R_m(x;\ y) = \varphi_{m'}(x;\ y) - \varphi_m(x;\ y) + R_{m'}(x;\ y)$, where m' is any integer $\geqq mq$. So we are left with estimation of $R_{m'}^{(\gamma+\eta)}(y^j;\ y)$. This will be worked out by a technical modification of [7]. As y, $y^j \in F$, there is a rectifiable curve C in F of length, say L, joining y and y^j. Let $\Delta : y = z^0, z^1, \cdots, z^p = y^j$ be a subdivision of C in F and let $|\Delta| = \sup_{1 \leqq i \leqq p} |z^i - z^{i-1}|$. Note that

$$\varphi_{m'}^{(\gamma)}(z'';\ z') - \varphi_{m'}^{(\gamma)}(z'';\ z) = R_{m'}^{(\gamma)}(z'';\ z) - R_{m'}^{(\gamma)}(z'';\ z')$$

since

$$\begin{aligned} \varphi^{(\gamma)}(z'') &= \varphi_{m'}^{(\gamma)}(z'';\ z) + R_{m'}^{(\gamma)}(z'';\ z) \\ &= \varphi_{m'}^{(\gamma)}(z'';\ z') + R_{m'}^{(\gamma)}(z'';\ z'). \end{aligned}$$

Thus we get by (11)

$$R_{m'}^{(\gamma+\eta)}(z'';\ z) - R_{m'}^{(\gamma+\eta)}(z'';\ z') = \sum_{|\delta| \leqq m'-|\gamma+\eta|} \frac{1}{\delta\,!} R_{m'}^{(\gamma+\eta+\delta)}(z';\ z)(z''-z')^{\delta}.$$

Changing z, z', z'' to z^{i-1}, z^i, y^j in this equation, summing over i and noting $R_{m'}^{(\kappa)}(y^j;\ y^j) = 0$ when $|\kappa| \leqq m'$, we consequently have

$$R_{m'}^{(\gamma+\eta)}(y^j;\ y) = \sum_{i=1}^{p} \sum_{|\delta| \leqq m'-|\gamma+\eta|} \frac{1}{\delta\,!} R_{m'}^{(\gamma+\delta+\eta)}(z^i;\ z^{i-1})(y^j - z^i)^{\delta}. \tag{13}$$

Note that by the classical formula for the remainder term

$$|R_{m'}^{(\gamma+\delta+\eta)}(z^i;\ z^{i-1})| \leqq |z^i - z^{i-1}|^{m'-|\gamma+\delta+\eta|} \varepsilon(|z^i - z^{i-1}|) \tag{14}$$

where $\varepsilon(h) \to 0$ when $h \to 0$.

Now split the sum for δ in (13) into the sums for $|\delta| < m' - |\gamma+\eta|$ and for $|\delta| = m' - |\gamma+\eta|$, and then denote the former by I_Δ and the latter by J_Δ, respectively. Since $|z^i-z^{i-1}|\leqq L$ and $|y^j-z^i|\leqq L$, in view of (14) we have

$$|I_\Delta|\leqq\sum_{|\delta|<m'-|\gamma+\eta|}\frac{1}{\delta!}L^{m'-|\gamma+\eta|}\sum_{i=1}^{p}|z^i-z^{i-1}|\varepsilon(|\Delta|)$$

$$\leqq\sum_{|\delta|<m'-|\gamma+\eta|}\frac{1}{\delta!}L^{m'+1-|\gamma+\eta|}\varepsilon(|\Delta|)$$

which tends to 0 when $|\Delta|\to 0$.

On the other hand

$$J_\Delta=\sum_{i=1}^{p}\sum_{|\delta|=m'-|\gamma+\eta|}\frac{1}{\delta!}[\varphi_{m'}^{(\gamma+\varepsilon+\eta)}(z^i)-\varphi_{m'}^{(\gamma+\delta+\eta)}(z^{i-1})](y^j-z^i)^\delta$$

since for $|\delta|=m'-|\gamma+\eta|$,

$$\varphi_{m'}^{(\gamma+\delta+\eta)}(z^i\,;\ z^{i-1})=\varphi^{(\gamma+\delta+\eta)}(z^{i-1})$$

and so

$$\varphi^{(\gamma+\delta+\eta)}(z^i)=\varphi^{(\gamma+\delta+\eta)}(z^{i-1})+R_{m'}^{(\gamma+\delta+\eta)}(z^i\,;\ z^{i-1}).$$

Now for each fixed δ we have

$$\sum_{i=1}^{p}[\varphi^{(\gamma+\delta+\eta)}(z^i)-\varphi^{(\gamma+\delta+\eta)}(z^{i-1})](y^j-z^i)^\delta$$

$$=-\sum_{i=1}^{p-1}[\varphi^{(\gamma+\delta+\eta)}(z^i)-\varphi^{(\gamma+\delta+\eta)}(z^0)][(y^j-z^{i+1})^\delta-(y^j-z^i)^\delta],$$

which tends to a Stieltjes integral

$$-\int_0^L[\varphi^{(\gamma+\delta+\eta)}(z(s))-\varphi^{(\gamma+\delta+\eta)}(z^0)]d(y^j-z(s))^\delta \tag{15}$$

when $|\Delta|\to 0$, where $z(s)$ denotes the point on the curve C of length s along C from y.

After the differentiation in the integral, (15) becomes

$$\sum_{|\kappa|=1}\frac{\delta!}{(\delta-\kappa)!}\int_y^{y^j}[\varphi^{(\gamma+\delta+\eta)}(z)-\varphi^{(\gamma+\delta+\eta)}(y)](y^j-z)^{\delta-\kappa}(dz)^\kappa. \tag{16}$$

Denote the sum of integrals (16) by $I_{\gamma,\delta,\eta}$. Then we have

$$R_{m'}^{(\gamma+\eta)}(y^j\,;\ y)=\sum_{|\delta|=m'-|\gamma+\eta|}\frac{1}{\delta!}I_{\gamma,\delta,\eta}=\lim_{|\Delta|\to 0}J_\Delta.$$

Hence, taking the regularity of F into consideration, we have the estimates

$$|R_{m'}^{(\gamma+\eta)}(y^j\,;\ y)|\leqq C_1L^{m'-|\gamma+\eta|}\|\varphi\|_{m',F}$$

$$\leqq C_1d(x,\ F)^{(m'-|\gamma+\eta|)/q}\|\varphi\|_{m',F}$$

for some constant C_1 and for any γ and η with $|\gamma+\eta|\leqq m'$. The last estimate combined with (7) and (12) implies

$$|\varphi_{m'}^{(\gamma)}(x\,;\,y^j)-\varphi_{m'}^{(\gamma)}(x\,;\,y)|\leqq C_2\|\varphi\|_{m',F}\sum_{|\eta|\leqq m'-|\gamma|} d(x,F)^{(m'-|\gamma+\eta|)/q}d(x,F)^{|\eta|}$$

where C_2 is a constant depending only on m' and F. Now, the differentiation of ϕ in (6) with respect to x gives

$$\phi^{(\alpha)}(x)=\varphi_m^{(\alpha)}(x\,;\,y)+\sum_{\beta+\gamma=\alpha}\sum_{j=1}^{\infty}\chi_j^{(\beta)}(x)[\varphi_m^{(\gamma)}(x\,;\,y^j)-\varphi_m^{(\gamma)}(x\,;\,y)].$$

Thus in view of the property (ii) of $\{\chi_j\}$,

$$|\phi^{(\alpha)}(x)|\leqq\|\varphi\|_{m,F}$$
$$+\,C_2\|\varphi\|_{m',F}\sum_{\beta+\gamma=\alpha}(d(x,F)^{-|\beta|}+1)\sum_{|\eta|\leqq m'-|\gamma|}d(x,F)^{((m'-|\gamma+\eta|)/q)+|\eta|}.$$

As for the exponent of $d(x,F)$, if $|\alpha|\leqq m$

$$((m'-|\gamma+\eta|)/q)+|\eta|-|\beta|\geqq\frac{1}{q}\{mq-|\alpha-\beta|+(q-1)|\eta|-q|\beta|\}$$
$$=\frac{1}{q}\{q(m-|\beta|+|\eta|)-(|\alpha|-|\beta|+|\eta|)\}$$
$$\geqq\frac{q-1}{q}(m-|\beta|+|\eta|)\geqq 0.$$

Since $d(x,F)\leqq 1$, we finally have

$$|\phi^{(\alpha)}(x)|\leqq C\|\varphi\|_{m',F}$$

when $|\alpha|\leqq m$, where C is a constant depending only on m' and F. Collecting the results obtained so far, we consequently proved the estimate (5). Recall a property of distributions with compact support that if $\chi\in C_0^\infty(\Omega)$ and its derivatives of order up to m vanish on F, then $\langle u,\chi\rangle=0$ (cf. [6]). Suppose $\eta\in C_0^\infty(\Omega)$ is equal to 1 on a neighborhood of F. Then $\eta\phi$ can be regarded as a function in $C_0^\infty(\Omega)$ and $(\eta\varphi)^{(\alpha)}=(\eta\phi)^{(\alpha)}$ on F when $|\alpha|\leqq m$. Thus we get

$$\langle u,\eta\varphi\rangle=\langle u,\eta\phi\rangle,$$

and so

$$|\langle u,\varphi\rangle|=|\langle u,\eta\varphi\rangle|=|\langle u,\eta\phi\rangle|$$
$$\leqq C_1\|\phi\|_{m,\Omega}$$
$$\leqq C_2\|\varphi\|_{m',F}\qquad\text{(by (5))}$$

for any integer $m'\geqq mq$ where C_1, C_2 are constants depending only on m' and F which completes the proof of Lemma.

Remark. A typical example of regular set is a convex set, where $q=1$. In this particular case, the proof of Lemma is carried out much more readily than the above, since it is enough to use the classical formula for $R_m^{(\gamma+\eta)}(y^j\,;\,y)$ in (12). Today, we know a large family of regular sets, that is, compact subanalytic sets in R^n (or in real analytic manifolds) (see [1], [3]).

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