

Quantum Deformation of Classical Groups

By

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Abstract

We construct coordinate algebras of quantum orthogonal, special orthogonal and symplectic groups using M. Jimbo's solutions of the Yang-Baxter equation and determine their Peter-Weyl decompositions. To do this, we study some class of bialgebras and their group-like elements (quantum determinants). A new realization of the universal R -matrix is also given.

Introduction

Recently some interesting classes of Hopf algebras, referred as quantum groups, are discovered. They are quantum deformations of function algebras of Lie groups. Let $A(G)$ be the coordinate algebra of a Lie group G . A quantum deformation $A(G_q)$ of $A(G)$ is a one-parameter family of Hopf algebras whose representation theories (or coalgebra structures) are the same as those of $A(G)$. S.L. Woronowicz gave a real form of the first example $A(SL_q(N))$.

For orthogonal and symplectic case, some families of Hopf algebras were constructed by Faddeev, Reshetikhin and Takhtajan [5] and independently by Takeuchi [22]. In this paper, we will show that their Hopf algebras are indeed quantum deformations of $A(O(N))$ and $A(Sp(N))$ in the above sense. We will construct also a quantum deformation of $A(SO(N))$. For this purpose, we investigate a class of bialgebras which we call *quantum matrix bialgebras*. Quantum matrix bialgebras are defined by means of Yang Baxter operators, i. e., solutions of the (constant) Yang-Baxter equation. We define Hopf algebras $A(G_q)$ as quotients of quantum matrix bialgebras corresponding to the M. Jimbo's solutions of type $X_i=A_i, B_i, C_i$ or D_i [10]. Those quantum matrix bialgebras are completely determined as direct sums of dual coalgebras of simple algebras.

In §1, §2, we develop a general theory of quantum matrix bialgebras and their graded dual notion called *Schur algebras*. By their connection with the algebraic structure of Yang-Baxter equation, we show $L \otimes M \simeq M \otimes L$ for any

Communicated by M. Kashiwara, June 15, 1990. Revised June 21, 1991.

1991 Mathematics Subject Classifications: 16W30.

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comodules L, M of a quantum matrix bialgebra. In §3, we give a construction of Hopf algebras from quantum matrix bialgebras, which is based on *cofactor matrices* [22] and the “*Laplace expansion*”. In §4, we give a structure theorem of quantum matrix bialgebras corresponding to Jimbo’s Yang-Baxter operators. Section 5 is devoted to study group-like elements of these quantum matrix bialgebras. For $X_i = B_i, C_i, D_i$, there exist two important group-like elements which we denote by $\det_q(X_i)$ and $\text{quad}_q(X_i)$, such that every other group-like element is a monomial of these two elements. Further, we completely determine their relations, for example, $\det_q(B_i)^2 = \text{quad}_q(B_i)^{2l+1}$. Also, we show the existence of cofactor matrices with respect to these elements, which enables us to construct various Hopf algebras. In §5, together with Peter-Weyl theorem, we prove that the coordinate algebras of $SO_q(N)$ and $Sp_q(N)$ are sub Hopf algebras of the dual of the Drinfeld-Jimbo’s algebras $U_q(\mathfrak{so}(N))$ and $U_q(\mathfrak{sp}(N))$ respectively. In §7, we give a useful criterion of the semisimplicity of Schur algebras.

We work over any field K in §1–§3, and over the complex number field \mathbb{C} in §4–§7, unless otherwise noted.

The author would like to thank Professors K. Hasegawa, M. Hashimoto, Y. Kanie and A. Tsuchiya for valuable discussions. He also wishes to thank Professors E. Date, M. Jimbo, T. Miwa and M. Okado for their useful information and kind hospitality during his stay in Kyoto.

Notation. Throughout this paper, $\Delta: C \rightarrow C \otimes C$ (resp. $m: A \otimes A \rightarrow A$) denotes the coproduct (resp. product) of a coalgebra C (resp. algebra A), ε denotes the counit of C , and $\omega_L: L \rightarrow L \otimes C$ (or $\omega_L: L \rightarrow C \otimes L$) denotes the structure map of a right (or left) C -comodule L . Let A (resp. U) be another coalgebra (resp. an algebra) and $f: C \rightarrow A$ (resp. $\langle \cdot, \cdot \rangle: U \otimes C \rightarrow C$) be a coalgebra map (resp. a bilinear pairing such that $\langle x \otimes y, a \rangle = \langle xy, \Delta(a) \rangle$, $\langle 1, a \rangle = \varepsilon(a)$ ($x, y \in U, a \in C$)). Then, each right C -comodule L becomes a right A -comodule with the structure map $u \mapsto (\text{id}_L \otimes f)(\omega(u))$ (resp. a left U -module with the action $xu := \text{id} \otimes \langle x, \cdot \rangle \omega_L(u)$ ($x \in U, u \in L$)). We denote this A -comodule (resp. U -module) by L_A (resp. L_U). The antipode of a Hopf algebra H is denoted by S . For a finite dimensional left H -module L , *L denotes a linear dual of L equipped with a left H -action defined by $\langle xv, u \rangle = \langle v, S(x)u \rangle$. For finite dimensional K -vector spaces V, W , we identify $V^* \otimes W^*$ with $(V \otimes W)^*$ by the pairing $\langle f \otimes g, v \otimes w \rangle := \langle f, v \rangle \langle g, w \rangle$ ($f \in V^*, g \in W^*, v \in V, w \in W$). We denote by $\tau_{VW}: V \otimes W \rightarrow W \otimes V$ a linear map defined by $\tau_{VW}(v \otimes w) = w \otimes v$ ($v \in V, w \in W$). For a vector space with a fixed basis $\{u_i\}$, we denote by E_{ij} the matrix units $u_k \mapsto \delta_{jk} u_i$.

§ 1. Schur Algebras and Yang-Baxter Equation

Let V be a vector space over a field K . We call an endomorphism β_V on $V \otimes V$ a *Yang-Baxter operator* (or (V, β_V) is a *YB-pair*) if it satisfies the following Yang-Baxter equation :

$$(\beta_V)_1 \circ (\beta_V)_2 \circ (\beta_V)_1 = (\beta_V)_2 \circ (\beta_V)_1 \circ (\beta_V)_2. \tag{1.1}$$

Here $(\beta_V)_1, (\beta_V)_2$ denote elements of $\text{End}(V \otimes V \otimes V)$ defined by $(\beta_V)_1 := \beta_V \otimes \text{id}_V$, $(\beta_V)_2 := \text{id}_V \otimes \beta_V$. For each element σ of the r -th symmetric group \mathfrak{S}_r , we can uniquely define $\beta_V(\sigma) \in \text{End}(V^{\otimes r})$ by the following two conditions (see [17]) :

$$\begin{aligned} \beta_V((i, i+1)) &= (\beta_V)_i := \text{id}_V^{\otimes i-1} \otimes \beta_V \otimes \text{id}_V^{\otimes r-i-1} \quad (1 \leq i \leq r-1), \\ \beta_V(\sigma_1 \sigma_2) &= \beta_V(\sigma_1) \beta_V(\sigma_2) \quad (\sigma_1, \sigma_2 \in \mathfrak{S}_r, l(\sigma_1 \sigma_2) = l(\sigma_1) + l(\sigma_2)). \end{aligned} \tag{1.2}$$

Here $l(\sigma)$ denotes the *length* $\text{card}\{(i, j) \mid 1 \leq i < j \leq r, \sigma(i) > \sigma(j)\}$ of $\sigma \in \mathfrak{S}_r$.

Let $V = (V, \beta_V)$ be a *YB-pair* such that $\dim V < \infty$. We define the *Schur algebra* $\text{Sch}(V)$ of V by

$$\text{Sch}(V) = \bigoplus_{r \geq 0} \text{Sch}_r(V),$$

$$\text{Sch}_0(V) = K, \quad \text{Sch}_1(V) = \text{End}(V), \quad \text{Sch}_r(V) = \text{End}_{B(r)}(V^{\otimes r}) \quad (r \geq 2).$$

Here $B(r)$ denotes the subalgebra of $\text{End}(V^{\otimes r})$ generated by $(\beta_V)_1, \dots, (\beta_V)_{r-1}$. Since $B(r) \otimes B(s) \subset B(r+s)$ under the identification $\text{End}(V^{\otimes r}) \otimes \text{End}(V^{\otimes s}) = \text{End}(V^{\otimes r+s})$, there exists algebra inclusion

$$\Delta^{r,s} : \text{Sch}_{r+s}(V) \subset \text{End}_{B(r) \otimes B(s)}(V^{\otimes r} \otimes V^{\otimes s}) \simeq \text{Sch}_r(V) \otimes \text{Sch}_s(V).$$

Let $\Delta : \text{Sch}(V) \rightarrow \text{Sch}(V) \otimes \text{Sch}(V)$ be the direct sum $\bigoplus_{r,s \geq 0} \Delta^{r,s}$. Then, clearly, we have

Proposition 1.1. *The Schur algebra becomes a (non-unital) bialgebra with the coproduct Δ and a counit ε defined by $\varepsilon(\sum_r a_r) = a_0 \in K$ ($a_r \in \text{Sch}_r(V)$).*

Example. Let V be a K -vector space and $\tau_V \in \text{End}(V^{\otimes 2})$ be a linear map defined by $\tau_V(x \otimes y) = y \otimes x$ ($x, y \in V$). It is easy to see that (V, τ_V) is a *YB-pair* and that $\sigma \mapsto \tau_V(\sigma)$ defines a representation of \mathfrak{S}_r on $V^{\otimes r}$. It was proved by Schur [21] that $\text{Sch}_r(V)$ coincides with the linear span of the image of the representation $GL(V) \rightarrow \text{End}(V^{\otimes r})$ if $\text{ch } K = 0$.

Let $\overline{\text{Sch}}(V)$ be the completion of $\text{Sch}(V)$ with respect to the fundamental neighbourhood system $\{\bigoplus_{r,s \geq 0} \text{Sch}_r(V) \mid s \geq 0\}$ at 0. As an algebra, $\overline{\text{Sch}}(V)$ is isomorphic to the direct product of $\text{Sch}_r(V)$'s ($r \geq 0$). Moreover $\overline{\text{Sch}}(V)$ becomes a topological bialgebra with a coproduct defined by

$$\Delta := \prod_{r,s \geq 0} \Delta^{r,s} : \overline{\text{Sch}}(V) \longrightarrow \overline{\text{Sch}}(V) \otimes \overline{\text{Sch}}(V) = \prod_{r,s \geq 0} \text{Sch}_r(V) \otimes \text{Sch}_s(V).$$

Let $\rho^{r,s} = \rho_V^{r,s} \in \text{End } V^{\otimes r+s}$ be the composite map $\tau_V(\chi_{s,r}) \circ \beta_V(\chi_{r,s})$, where $\chi_{r,s} \in \mathfrak{S}_{r+s}$ is defined by

$$\chi_{r,s} = \begin{pmatrix} 1 & 2 & \cdots & r & r+1 & r+2 & \cdots & r+s \\ s+1 & s+2 & \cdots & s+r & 1 & 2 & \cdots & s \end{pmatrix}. \quad (1.3)$$

Since $\beta_V(\chi_{r,s}) \circ (\beta_V)_t = (\beta_V)_{t+s} \circ \beta_V(\chi_{r,s})$ ($1 \leq t < r$) and $\beta_V(\chi_{r,s}) \circ (\beta_V)_t = (\beta_V)_{t-r} \circ \beta_V(\chi_{r,s})$ ($r+1 \leq t < r+s$), we get $\rho^{r,s} \circ (\beta_V)_t = (\beta_V)_t \circ \rho^{r,s}$ for $1 \leq t < r+s$, $t \neq r$. Hence $\rho^{r,s}$ defines an element of $\text{Sch}_r(V) \otimes \text{Sch}_s(V) \cong \text{End}_{B(r) \otimes B(s)}(V^{\otimes r+s})$. Let $\rho = \rho_V \in \overline{\text{Sch}}(V) \otimes \overline{\text{Sch}}(V)$ be the sum $\sum_{r,s \geq 0} \rho^{r,s}$. For $i, j \geq 1$, we set $\rho_{ij} = \sum_k 1 \otimes \cdots \otimes \overset{j}{\rho}_k \otimes 1 \otimes \cdots \otimes \overset{i}{\rho}_k \otimes 1 \otimes \cdots \otimes 1$, where $\rho = \sum_k \rho_k \otimes \rho'_k$. The following shows that ρ is a sort of so-called “universal R -matrix.”

Proposition 1.2. *We have the following identities:*

$$(1) \quad \tau_{\overline{\text{Sch}}(V)}(\Delta(a))\rho = \rho\Delta(a) \quad (a \in \overline{\text{Sch}}(V)), \quad (1.4)$$

$$(2) \quad \Delta \otimes \text{id}(\rho) = \rho_{13}\rho_{23}, \quad \text{id} \otimes \Delta(\rho) = \rho_{13}\rho_{12}, \quad (1.5)$$

$$(3) \quad \rho_{12}\rho_{13}\rho_{23} = \rho_{23}\rho_{13}\rho_{12}. \quad (1.6)$$

Proof. (1) It is sufficient to show that $\tau(\Delta(a))\rho^{r,s}$ and $\rho^{r,s}(\Delta(a))$ ($a \in \text{Sch}_{r+s}(V)$) define a same operator on $V^{\otimes r+s}$. Since $\Delta(a)$ commutes with $\beta_V(\chi_{r,s}) \in B(r+s)$, we get $\rho_{r,s}\Delta(a) = \tau_V(\chi_{s,r})(\Delta(a))\beta_V(\chi_{r,s})$. Hence (a) follows from $\tau_V(\chi_{s,r})(\Delta(a))\tau_V(\chi_{s,r})^{-1} = \tau(\Delta(a))$.

(2) For $w \in V^{\otimes r+s+t}$, we have $\rho_{13}(w) = \tau_V(\chi_{r,s} \times 1_t)^{-1} \rho_{23} \tau_V(\chi_{r,s} \times 1_t)(w)$, where $\chi_{r,s} \times 1_t \in \mathfrak{S}_{r+s+t}$ is defined by $\chi_{r,s} \times 1_t(i) = \chi_{r,s}(i)$ ($1 \leq i \leq r+s$) and $\chi_{r,s} \times 1_t(i) = i$ ($r+s < i \leq r+s+t$). Hence we have

$$\begin{aligned} \rho_{13}\rho_{23}(w) &= \tau_V(\chi_{s+t,r})(\beta_V)_2 \tau_V(\chi_{r+t,s})(\beta_V)_2(w) \\ &= \tau_V(\chi_{s+t,r}) \tau_V(\chi_{t+r,s})(\beta_V)_1(\beta_V)_2(w) \\ &= \tau_V(\chi_{t,r+s}) \beta_V(\chi_{r+s,t})(w) \\ &= (\Delta \otimes \text{id}(\rho))(w). \end{aligned}$$

Part (3) follows easily from (1) and (2). \square

Definition 1.3. For left $\text{Sch}(V)$ -modules L, M , we define a map $\beta_{LM} : L \otimes M \rightarrow M \otimes L$ by

$$\beta_{LM}(u \otimes v) = \tau_{LM}(\rho(u \otimes v)) \quad (u \otimes v \in L \otimes M). \quad (1.7)$$

Theorem 1.4. (1) *The map β_{LM} is a $\text{Sch}(V)$ -module homomorphism.*

(2) *For $\text{Sch}(V)$ -modules L, M and N , the following “Yang-Baxter equation” holds:*

$$(\beta_{MN} \otimes \text{id}_L)(\text{id}_M \otimes \beta_{LN})(\beta_{LM} \otimes \text{id}_N) = (\text{id}_N \otimes \beta_{LM})(\beta_{LN} \otimes \text{id}_M)(\text{id}_L \otimes \beta_{MN}). \quad (1.8)$$

In particular, (L, β_{LL}) is a YB-pair.

(3) For $\text{Sch}(V)$ -module maps $f: L \rightarrow L'$ and $g: M \rightarrow M'$, we have $\beta_{L'M'} \circ f \otimes g = g \otimes f \circ \beta_{LM}$.

(4) If β_V is invertible, then $L \otimes M \simeq M \otimes L$ as $\text{Sch}(V)$ -modules.

Proof. Part (1), (2) and (3) follow immediately from (1.4), (1.6) and (1.7) respectively. Part (4) follows from the existence of $\rho^{-1} = \sum_{r,s \geq 0} (\beta_V^{-1})(\chi_{rs}) \tau_V(\chi_{rs})$. \square

Let (V, β_V) be a YB-pair such that $\dim V < \infty$ and β_V is invertible. For right $\text{Sch}(V)$ -modules L, M , we define $\beta_{LM}: L \otimes M \rightarrow M \otimes L$ by $\beta_{LM}(u \otimes v) = \tau_{LM}((u \otimes v)\rho^{-1})$. Then this map satisfies properties similar to the above theorem. Moreover we have

$$\langle \beta_{L^*M^*}(u \otimes u'), v \otimes v' \rangle = \langle u \otimes u', \beta_{LM}^{-1}(v \otimes v') \rangle \quad (1.9)$$

for $u \in L^*, u' \in M^*, v \in L$ and $v' \in M$.

§2. A Commutator Formula for Quantum Matrix Bialgebras

We begin by recalling some notations and results of [5, 8]. For YB-pairs $V=(V, \beta_V)$ and $W=(W, \beta_W)$, we define the product $V \times W$ as a YB-pair defined by

$$V \times W := (V \otimes W, (\text{id}_V \otimes \tau_{VW} \otimes \text{id}_W) \circ (\beta_V \otimes \beta_W) \circ (\text{id}_V \otimes \tau_{VW} \otimes \text{id}_W)). \quad (2.1)$$

If $\dim V < \infty$ and β_V is invertible, we define the dual V^\sim of V by $V^\sim := (V^*, ({}^t\beta_V)^{-1})$, where ${}^t\beta_V \in \text{End}(V^* \otimes V^*)$ is defined by $\langle \beta_V(u \otimes u'), v \otimes v' \rangle = \langle u \otimes u', {}^t\beta_V(v \otimes v') \rangle$, $(u, u' \in V, v, v' \in V^*)$. We call a YB-pair (E, β_E) of the form $E := V^\sim \times V$ the quantum matrix of V .

Let $T(V)$ be the free non-commutative algebra generated by V . We define the symmetric algebra $S(V, \beta_V) = S(V) = \bigoplus_{r \geq 0} S_r(V)$ on V as the quotient graded algebra $T(V)/(\text{Im}(\text{id}_{V \otimes V} - \beta_V))$ of $T(V)$. It is known that the symmetric algebra $S(E)$ on a quantum matrix $E = V^\sim \times V$ becomes a bialgebra whose coproduct and counit are defined by $\Delta(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$ and $\varepsilon(x_{ij}) = \delta_{ij}$, where u_i, v_j and x_{ij} ($1 \leq i, j \leq N$) are bases of V, V^* and E satisfying $\langle u_i, v_j \rangle = \delta_{ij}$, $x_{ij} = v_i \otimes u_j$ (cf. [5, 8]). We define a right (resp. left) $S(E)$ -comodule structure ω_V (resp. ω_{V^*}) on V (resp. V^*) by $\omega_V(u_j) = \sum_i u_i \otimes x_{ij}$ (resp. $\omega_{V^*}(v_i) = \sum_j x_{ij} \otimes v_j$). Then β_V (resp. ${}^t\beta_V$) is an $S(E)$ -comodule endomorphism on $V \otimes V$ (resp. $V^* \otimes V^*$) (see [5, 8]). The following observation essentially due to [8] plays an essential role in this paper.

Proposition 2.1. *There exists a non degenerate bilinear pairing $\langle \cdot, \cdot \rangle: \text{Sch}(V) \otimes S(E) \rightarrow \mathbf{K}$ satisfying the following conditions.*

$$(a) \quad \langle \text{Sch}_r(V), S_t(E) \rangle = 0 \quad \text{if } r \neq t.$$

$$(b) \quad \langle \Delta(a), x \otimes y \rangle = \langle a, xy \rangle, \quad \langle a \otimes b, \Delta(x) \rangle = \langle ab, x \rangle \quad (a, b \in \text{Sch}(V), x, y \in S(E)).$$

In particular, the category of finite dimensional left $\text{Sch}_r(V)$ -modules is equivalent to that of finite dimensional right $S_r(E)$ -comodules. For a right $S(E)$ -comodule L , the left action of $\text{Sch}_r(V)$ is given by

$$au = \text{id}_L \otimes \langle a, \rangle (\omega_L(u)) \quad (a \in \text{Sch}(V), u \in L). \quad (2.2)$$

Proof. As is observed in [8, §5], the dual of the projection $E^{\otimes r} \rightarrow S_r(E)$ is naturally identified with the inclusion $\text{Sch}_r(V) \hookrightarrow \text{End}(V^{\otimes r})$. It is rather easy to verify that, under this identification, the graded dual of the product and the coproduct of $S(E)$ coincides with the coproduct and the product of $\text{Sch}(V)$ respectively. \square

Noting $S_r(V)$ is a quotient $\text{Sch}(V)$ -module of $V^{\otimes r}$, we will define a YB operator $\beta_{S(V)}$ over $S(V)$ by $\beta_{S(V)} = \coprod_{r, s \geq 0} \beta_{S_r(V)S_s(V)}$.

Proposition 2.2 *Let $m : S(V) \otimes S(V) \rightarrow S(V)$ be the product of $S(V)$. Then,*

$$(1) \quad m \circ \beta_{S(V)} = m, \quad (2.3)$$

$$(2) \quad \beta_{S(V)} \circ (m \otimes \text{id}_{S(V)}) = (\text{id}_{S(V)} \otimes m) \circ (\beta_{S(V)})_1 \circ (\beta_{S(V)})_2, \quad (2.4)$$

$$\beta_{S(V)} \circ (\text{id}_{S(V)} \otimes m) = (m \otimes \text{id}_{S(V)}) \circ (\beta_{S(V)})_2 \circ (\beta_{S(V)})_1. \quad (2.5)$$

Proof. By $\beta_{S_r(V)S_s(V)} = \beta_V(\chi_{rs})$ and Theorem 1.4 (3), $\beta_{S(V)}$ coincides with the map $\phi_{S(V)}$ defined in [8, §4]. Hence this is nothing but (4.14) and (4.13) of [8]. \square

Example. Let $V = (V, \tau_V)$ be as the example of §1 such that $\dim V < \infty$. Then, as algebras, $S(V)$ and $S(E)$ coincide with the polynomial algebras generated by elements of V and E respectively. Since ρ is the unit of the algebra $\text{Sch}(V) \hat{\otimes} \text{Sch}(V)$, $\beta_{LM} = \tau_{LM}$ for any $\text{Sch}(V)$ -modules L, M . Hence the equality (1) of the above proposition is nothing but $xy = yx$ for $x, y \in S(V)$.

The above example seems to suggest the map $\beta_{S(V)}$ express ‘‘commutativity’’ of $S(V)$. Unfortunately, explicit form of $\beta_{S(V)}$ is not so simple in general.

Example. Let V be a complex vector space with a basis $\{u_i | 1 \leq i \leq N\}$. For $0 \neq q \in \mathbf{C}$, we define a YB operator $\beta_q(A_{N-1})$ by

$$\beta_q(A_{N-1}) = \sum_{i=1}^N E_{ii} \otimes E_{ii} + q \sum_{i \neq j} E_{ij} \otimes E_{ji} + (1-q^2) \sum_{i>j} E_{ii} \otimes E_{jj}. \quad (2.6)$$

We call $\beta_q(A_{N-1})$ Jimbo’s YB operator of type A_{N-1} and denote the corresponding YB-pair by $V_q(A_{N-1})$. The symmetric algebra $S(V_q(A_{N-1}))$ is an algebra

with generators u_1, \dots, u_N and relations $u_i u_j = q u_j u_i$ ($i < j$). Hence $\{u_{i_1} \cdots u_{i_r} \mid 1 \leq i_1 \leq \dots \leq i_r \leq N\}$ is a basis of $S_r(V)$. Since $S(V)$ is generated by V , we can calculate $\beta_{S(V)}$ by using (2.4), (2.5) and $\beta_{S(V)}|_{V \otimes V} = \beta_V$. If $N=2$, the result is as follows:

$$\begin{aligned} \beta_q(A_{N-1})(u_1^i u_2^j \otimes u_1^m u_2^n) &= \sum_{r \geq 0} (1-q^2)^r q^{(3r-1)r/2 + (i-j-m+n)r + in+jm} \\ & [r]!_q \begin{bmatrix} j \\ r \end{bmatrix}_q \begin{bmatrix} m \\ r \end{bmatrix}_q u_1^{m-r} u_2^{n+r} \otimes u_1^{i+r} u_2^{j-r}. \end{aligned} \quad (2.7)$$

Here $[r]!_q$ and $\begin{bmatrix} j \\ r \end{bmatrix}_q$ are defined by

$$\begin{aligned} \begin{bmatrix} j \\ r \end{bmatrix}_q &= \frac{[j]!_q}{[r]!_q [j-r]!_q} \quad (0 \leq r \leq j), \quad 0 \quad (\text{otherwise}), \\ [r]!_q &= [r]_q [r-1]_q \cdots [1]_q, \quad [r]_q = \frac{q^r - q^{-r}}{q - q^{-1}}. \end{aligned} \quad (2.8)$$

The rest of this section is devoted to study a *YB* operator $\beta_{S(E)}$ on a quantum matrix bialgebra $S(E)$. First we show a relation between this operator and $\rho \in \overline{\text{Sch}}(V) \widehat{\otimes} \overline{\text{Sch}}(V)$. Define a linear map $\phi \in \text{End}(\text{Sch}(V)^{\otimes 2})$ by $\phi(a) = \tau_{\text{Sch}(V)}(\rho a \rho^{-1})$. Then,

Proposition 2.3. *Let E be the quantum matrix on V . Then,*

$$\langle \beta_{S(E)}(x), a \rangle = \langle x, \phi(a) \rangle \quad (x \in S(E)^{\otimes 2}, a \in \text{Sch}(V)^{\otimes 2}).$$

Proof. The map $\beta_{S(V)}$ is uniquely characterized as a *YB* operator on $S(V)$ satisfying the equation of (2.4), (2.5) and $\beta_{S(V)}|_{V \otimes V} = \beta_V$. Hence it is enough to show that ϕ satisfies (1) $\phi_1 \circ \phi_2 \circ \phi_1 = \phi_2 \circ \phi_1 \circ \phi_2$, (2) $\phi_1 \circ \phi_2 \circ \Delta \otimes \text{id} = \text{id} \otimes \Delta \circ \phi$, $\phi_2 \circ \phi_1 \circ \text{id} \otimes \Delta = \Delta \otimes \text{id} \circ \phi$ and (3) $\phi|_{\text{Sch}_1(V) \otimes \text{Sch}_1(V)} = \beta_E$.

For $a, b, c \in \text{Sch}(V)$, we have

$$\begin{aligned} \phi_1 \circ \phi_2 \circ \phi_1(a \otimes b \otimes c) &= (\rho_{21} \rho_{31} \rho_{32})(c \otimes b \otimes a)(\rho_{21} \rho_{31} \rho_{32})^{-1}, \\ \phi_2 \circ \phi_1 \circ \phi_2(a \otimes b \otimes c) &= (\rho_{32} \rho_{31} \rho_{21})(c \otimes b \otimes a)(\rho_{32} \rho_{31} \rho_{21})^{-1}. \end{aligned}$$

By (1.6), $\rho_{21} \rho_{31} \rho_{32} = \rho_{32} \rho_{31} \rho_{21}$, hence (1) holds. By (1.5), we get $\text{id} \otimes \Delta(\rho_{21}) = \rho_{21} \rho_{31}$ and $\text{id} \otimes \Delta((\rho^{-1})_{21}) = (\rho^{-1})_{13} (\rho^{-1})_{12}$. Using this, we get $\text{id} \otimes \Delta \circ \phi(a \otimes b) = (\rho_{21} \rho_{31})(b \otimes \Delta(a))(\rho_{21} \rho_{31})^{-1}$. It is easy to see this coincides with $\phi_1 \otimes \phi_2 \circ \Delta \otimes \text{id}$. Part (3) follows from direct calculation. \square

Lemma 2.4. *For finite dimensional left $\text{Sch}(V)$ -modules L and M , the following diagrams are commutative:*

$$\begin{array}{ccc}
\text{Sch}(V) \otimes L \otimes \text{Sch}(V) \otimes M & \longrightarrow & L \otimes M & \quad & \text{Sch}(V) \otimes \text{Sch}(V) & \longrightarrow & L \otimes L^* \otimes M \otimes M^* \\
\downarrow \psi \times \beta_{LM} & & \downarrow \beta_{LM} & & \downarrow \psi & & \downarrow \beta_{LM} \times \beta_{L^*M^*} \\
\text{Sch}(V) \otimes M \otimes \text{Sch}(V) \otimes L & \longrightarrow & M \otimes L & \quad & \text{Sch}(V) \otimes \text{Sch}(V) & \longrightarrow & M \otimes M^* \otimes L \otimes L^* .
\end{array} \tag{2.9}$$

Here $\text{Sch}(V) \otimes L \rightarrow L$ is the action of $\text{Sch}(V)$, $\text{Sch}(V) \rightarrow L \otimes L^* \cong \text{End}(L)$ is the corresponding representation and the maps $\psi \times \beta_{LM}$, $\beta_{LM} \times \beta_{L^*M^*}$ are defined by

$$\begin{aligned}
\psi \times \beta_{LM} &= 1 \otimes \tau \otimes 1 \circ \psi \otimes \beta_{LM} \circ 1 \otimes \tau \otimes 1, \\
\beta_{LM} \times \beta_{L^*M^*} &= 1 \otimes \tau \otimes 1 \circ \beta_{LM} \otimes \beta_{L^*M^*} \circ 1 \otimes \tau \otimes 1.
\end{aligned} \tag{2.10}$$

Proof. The first diagram follows from direct computation. The second diagram follows easily from the first diagram using similar argument of [8, § 1, § 3]. \square

Let L be a right $S(E)$ -comodule. By definition of the left action of $S(E)$ on L^* , the following diagram is commutative:

$$\begin{array}{ccc}
L^* \otimes L & \xrightarrow{\omega_L \otimes \text{id}} & S(E) \otimes L^* \otimes L \\
\downarrow \text{id} \otimes \omega_L & & \downarrow \text{id} \otimes \langle , \rangle \\
L^* \otimes L \otimes S(E) & \xrightarrow{\langle , \rangle \otimes \text{id}} & S(E)
\end{array} . \tag{2.11}$$

We define the *coefficient map* $cf_L: L^* \otimes L \rightarrow S(E)$ by this diagram. It is easy to see that cf_L is a coalgebra map from the dual coalgebra $L^* \otimes L \simeq \text{End}(L)^*$. This means that the coefficient map is a dual notion of the representation map. By the above lemma, we get the following.

Theorem 2.5 (*commutator formula for $S(E)$*). *For right $S(E)$ -comodules L, M , the following diagram is commutative:*

$$\begin{array}{ccc}
L^* \otimes L \otimes M^* \otimes M & \xrightarrow{cf_L \otimes cf_M} & S(E) \otimes S(E) \\
\downarrow \beta_{L^*M^*} \times \beta_{LM} & & \downarrow \beta_{S(E)} \\
M^* \otimes M \otimes L^* \otimes L & \xrightarrow{cf_M \otimes cf_L} & S(E) \otimes S(E) .
\end{array} \tag{2.12}$$

§ 3. Inverse of Quantum Matrices

In this section, we investigate a “linear algebraic” method of constructing Hopf algebras from quantum matrix bialgebras. Let $V, E, \{u_i\}, \{v_j\}, \{x_{ij}\}$ be as in § 2. We call an element $0 \neq g \in S_\tau(E)$ *group-like* if $\Delta(g) = g \otimes g$. We note that there exists one to one correspondence between group-like elements of $S(E)$ and the isomorphism classes of one dimensional left (resp. right) $S(E)$ -comodules. It is given by $g \mapsto [Kg]$.

Example. (cf. [5, 16, 25]) Let $V = V_q(A_{N-1})$ be as in § 2. We denote the quantum matrix on $V_q(A_{N-1})$ by $E = E_q(A_{N-1})$. As an algebra, $S(E)$ is generated by $\{x_{ij} | 1 \leq i, j \leq N\}$ with the following defining relations :

$$\begin{aligned} qx_{il}x_{ik} &= x_{ik}x_{il}, & qx_{jk}x_{ik} &= x_{ik}x_{jk}, \\ x_{jk}x_{il} &= x_{il}x_{jk}, & x_{ik}x_{jl} - x_{jl}x_{ik} - (q - q^{-1})x_{jk}x_{il} &= 0 \\ & & (1 \leq i < j \leq N, 1 \leq k < l \leq N). & \end{aligned} \tag{3.1}$$

Define an element $\det_q = \det_q(A_{N-1}) \in S(E)$ by

$$\det_q = \sum_{\sigma \in \mathfrak{S}_N} (-q)^{l(\sigma)} x_{1 \sigma(1)} x_{2 \sigma(2)} \cdots x_{N \sigma(N)}. \tag{3.2}$$

Then \det_q is a central group-like element of $S(E)$. Moreover $A(GL_q(N)) := S(E)[\det_q^{-1}]$ and $A(SL_q(N)) := S(E)/(\det_q - 1)$ are Hopf algebras (i.e. have an antipode). We call them the *coordinate algebra of the quantum general linear group* and the *quantum special linear group* respectively.

Let $g \in S(E)$ be a group-like element. We say that elements $y_{ij} (1 \leq i, j \leq N)$ (resp. $z_{ij} (1 \leq i, j \leq N)$) form a *left* (resp. *right*) *cofactor with respect to g* if they satisfy the equation

$$\sum_{k=1}^N y_{ik}x_{kj} = \delta_{ij}g, \quad \sum_{k=1}^N x_{ik}z_{kj} = \delta_{ij}g. \tag{3.3}$$

If $y_{ij} = z_{ij}$, then we say that y_{ij} form a cofactor (see [22]).

Proposition 3.1. (1) *Let G be a set of group-like elements of $S(E)$. If there exist both left and right cofactor with respect to an element g_0 of G , then the quotient $S(E)/\sum_{g \in G} (g - 1)$ is a Hopf algebra whose antipode is given by $S(\bar{x}_{ij}) = \bar{y}_{ij} = \bar{z}_{ij}$, where $\bar{} : S(E) \rightarrow S(E)/\sum_{g \in G} (g - 1)$ is the projection.*

(2) *If, in addition, each element of G is central, then the localization $S(E)[g^{-1}]_{g \in G}$ have an antipode which sends x_{ij} to $g_0^{-1}y_{ij}$.*

Proof. Quite similar to those of [22, Propositions 1.3, 3.4]. □

Lemma 3.2. (Laplace expansion) *Let L_1, L_2, L_3 (resp. M_1, M_2, M_3) be right*

(resp. left) $S(E)$ -comodules and $\mu: L_1 \otimes L_2 \rightarrow L_3$ (resp. $\nu: M_1 \otimes M_2 \rightarrow M_3$) be a $S(E)$ -comodule map. Then the following diagram (3.4) (resp. (3.5)) is commutative:

$$\begin{array}{ccc}
 L_3^* \otimes L_1 \otimes L_2 & \xrightarrow{\mu^* \otimes 1} & L_1^* \otimes L_2^* \otimes L_1 \otimes L_2 \\
 1 \otimes \mu \downarrow & & \downarrow cf_{L_1 \otimes L_2} \\
 L_3^* \otimes L_3 & \xrightarrow{cf_{L_3}} & S(E)
 \end{array} \quad (3.4)$$

$$\begin{array}{ccc}
 M_1 \otimes M_2 \otimes M_3^* & \xrightarrow{1 \otimes \nu^*} & M_1 \otimes M_2 \otimes M_1^* \otimes M_2^* \\
 \nu \otimes 1 \downarrow & & \downarrow cf_{M_1^* \otimes M_2^*} \\
 M_3 \otimes M_3^* & \xrightarrow{cf_{M_3^*}} & S(E)
 \end{array} \quad (3.5)$$

Proof. The commutativity of (3.4) follows immediately from those of the following:

$$\begin{array}{ccccc}
 L_3^* \otimes L_1 \otimes L_2 & \xrightarrow{\quad} & L_1^* \otimes L_2^* \otimes L_1 \otimes L_2 & & \\
 \downarrow 1 \otimes \mu & \searrow \omega \otimes 1 & \downarrow 1 \otimes \mu^* & \searrow & \\
 L_3^* \otimes L_3 & & S(E) \otimes L_3^* \otimes L_1 \otimes L_2 & \xrightarrow{1 \otimes \mu^* \otimes 1} & S(E) \otimes L_1^* \otimes L_2^* \otimes L_1 \otimes L_2 \\
 & \searrow \omega \otimes 1 & \downarrow 1 \otimes \mu & & \downarrow 1 \otimes \langle, \rangle \\
 & & S(E) \otimes L_3^* \otimes L_3 & \xrightarrow{1 \otimes \langle, \rangle} & S(E).
 \end{array} \quad \square$$

By the above lemma, we get a sufficient condition of the existence of cofactor matrices. For a group-like element $g \in S(E)$, we define a condition (*) (resp. (*))' as follows.

(*) (resp. (*))' There exist right (resp. left) comodules $L_1, \mathbf{K}\bar{u}$ (resp. $M_2, \mathbf{K}\bar{v}$), a comodule map $\mu: L_1 \otimes V \rightarrow \mathbf{K}\bar{u}$ (resp. $\nu: V^* \otimes M_2 \rightarrow \mathbf{K}\bar{v}$) and a basis $\{{}^c u_i \mid 1 \leq i \leq N'\} \subset L_1$ (resp. $\{v_i^c \mid 1 \leq i \leq N'\} \subset M_2$) such that $\mathbf{K}\bar{u} \simeq \mathbf{K}g$ (resp. $\mathbf{K}\bar{v} \simeq \mathbf{K}g$), $N' \geq N$ and $\mu({}^c u_i \otimes u_j) = \delta_{ij} \bar{u}$ (resp. $\nu(v_j \otimes v_i^c) = \delta_{ij} \bar{v}$) for $1 \leq i \leq N', 1 \leq j \leq N$.

Theorem 3.3. *Suppose g satisfies the above condition (*) (resp. (*))'. Then there exists a left cofactor y_{ij} (resp. right cofactor z_{ij}) with respect to g . It is given by the following formula:*

$$y_{ij} = cf_{L_1}({}^c v_j \otimes {}^c u_i), \quad z_{ij} = cf_{M_2}(v_j^c \otimes u_i^c) \quad (1 \leq i, j \leq N). \quad (3.6)$$

Here $\{{}^c v_i\}$ and $\{u_i^c\}$ denote the dual bases of $\{{}^c u_i\}$ and $\{v_i^c\}$ respectively.

Proof. Set $L_2 = V$, $L_3 = \mathbf{K}\bar{u}$ and define $\bar{u}^* \in (\mathbf{K}\bar{u})^*$ by $\langle \bar{u}^*, \bar{u} \rangle = 1$. Comparing the images of $\bar{u}^* \otimes {}^c u_i \otimes u_j$ by (3.4), we get

$$\sum_{k=1}^N cf_{L_1}({}^c v_k \otimes {}^c u_i) x_{kj} = \delta_{ij} g .$$

Hence y_{ij} form a left cofactor. □

Example. Let p, q be non-zero complex numbers such that $1, -p^2, -q^2, p^2q^2$ are distinct. We define a *YB-pair* by $(W, \gamma_W) := V_p(A_1) \times V_q(A_1)$. Then γ_W is a diagonalizable matrix which has 1 (resp. $-p^2, -q^2, p^2q^2$) as an eigenvalue of multiplicity 9 (resp. 3, 3, 1). Hence $\text{Sch}_2(V)$ is a semisimple algebra isomorphic to $\text{Mat}(9, \mathbf{C}) \oplus \text{Mat}(3, \mathbf{C}) \oplus \text{Mat}(3, \mathbf{C}) \oplus \mathbf{C}$. In particular, $S_2(W^\vee \times W)$ has a unique group-like element g . Applying the above theorem to the projections $\mu: V \otimes V \rightarrow \text{Ker}(\gamma_W - p^2q^2), \nu: V^* \otimes V^* \rightarrow \text{Ker}(\check{\gamma}_W - p^{-2}q^{-2})$, we get the following formula of the cofactor matrix with respect to g :

$$y_{\langle ij \rangle \langle mn \rangle} = q^{\delta_{m1} - \delta_{i1}} r^{\delta_{n1} - \delta_{j1}} x_{\langle m' n' \rangle \langle i' j' \rangle} \quad (i, j, m, n=1, 2).$$

Here $i'=3-i$ and the basis $\{x_{\langle ij \rangle \langle mn \rangle}\}$ of $W^\vee \times W$ is defined by $x_{\langle ij \rangle \langle mn \rangle} = (v_i \otimes v_j) \otimes (u_m \otimes u_n)$.

§ 4. Brauer-Schur-Weyl Reciprocity

Now, we will begin to study some important examples of *YB-pairs* obtained by Jimbo. Let X_l be the Cartan matrix B_l, C_l or D_l . Define integers N, ν by

$$N = \begin{cases} 2l+1 & (X=B) \\ 2l & (X=C), \\ 2l & (X=D) \end{cases}, \quad \nu = \begin{cases} -1 & (X=B) \\ 1 & (X=C) \\ -1 & (X=D) \end{cases}. \quad (4.1)$$

For $1 \leq i \leq N$, we set $i' = N+1-i$ and

$$\bar{i} = \begin{cases} i - \nu/2 & (1 \leq i < (N+1)/2) \\ i & (i = (N+1)/2) \\ i + \nu/2 & ((N+1)/2 < i \leq N) \end{cases}, \quad \varepsilon(i) = \begin{cases} 1 & (1 \leq i \leq (N+1)/2) \\ -\nu & ((N+1)/2 \leq i \leq N) \end{cases}. \quad (4.2)$$

We define a *YB operator* $\beta_q = \beta_q(X_l)$ on $V := \bigoplus_{1 \leq i \leq N} \mathbf{C} u_i$ by the following formula and call it *Jimbo's YB operator of type X_l* :

$$\begin{aligned} \beta_q(X_l) = & \sum_{i^i i'} (E_{ii} \otimes E_{ii} + q^2 E_{i'i'} \otimes E_{i'i}) + q \sum_{i^i i'} E_{ii} \otimes E_{ii} + q \sum_{i^i j, j'} E_{ij} \otimes E_{ji} \\ & + (1 - q^2) \sum_{i > j} (E_{ii} \otimes E_{jj} - \varepsilon(i) \varepsilon(j) q^{\bar{i}-\bar{j}} E_{ij'} \otimes E_{i'j}). \end{aligned} \quad (4.3)$$

Here for $X=C, D$ (resp. $X=B$), q (resp. $q^{1/2}$) denotes a non-zero complex number. Besides the Yang-Baxter equation (1.1), it satisfies the following relation

$$q^{-1} \beta_q - q \beta_q^{-1} = (q - q^{-1})(\iota_q - 1), \quad (4.4)$$

where $\iota_q \in \text{End}(V^{\otimes 2})$ is defined by

$$\iota_q = \sum_{i,j=1}^N \varepsilon(i)\varepsilon(j)q^{\bar{i}-\bar{j}}E_{ij'} \otimes E_{i'j}. \quad (4.5)$$

For $q^2 \neq 1$, we denote the *YB pair* $(V, \beta_q(X_l))$ and the corresponding quantum matrix by $V(X_l) = V_q(X_l)$ and $E(X_l) = E_q(X_l)$ respectively.

As is pointed out by Jimbo, these *YB operators* have deep connection with quantum enveloping algebras. Let $X_l = [a_{ij}]_{1 \leq i, j \leq l}$ be a Cartan matrix and d_i ($1 \leq i \leq l$) be positive integers such that $d_i a_{ij} = d_j a_{ji}$ and the greatest common divisor of d_i 's is 1. Let $q \neq 0$ be a complex number such that $q_i := q^{d_i} \neq \pm 1$. The *quantum enveloping algebra* $U_q(X_l)$ is a \mathbb{C} -Hopf algebra with unit 1 and generators $e_i, f_i, k_i^{\pm 1}$ ($1 \leq i \leq l$) satisfying the following relations:

$$\begin{aligned} k_i k_i^{-1} &= k_i^{-1} k_i = 1, & k_i k_j &= k_j k_i, \\ k_i e_j k_i^{-1} &= q_i^{a_{ij}} e_j, & k_i f_j k_i^{-1} &= q_i^{-a_{ij}} f_j, \end{aligned} \quad (4.7)$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \quad (4.8)$$

$$\sum_{0 \leq n \leq 1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} e_i^{1-a_{ij}-n} e_j e_i^n = 0 \quad (i \neq j), \quad (4.9)$$

$$\sum_{0 \leq n \leq 1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} f_i^{1-a_{ij}-n} f_j f_i^n = 0 \quad (i \neq j), \quad (4.10)$$

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \quad \Delta(k_i) = k_i \otimes k_i, \quad (4.11)$$

$$\varepsilon(e_i) = 0, \quad \varepsilon(f_i) = 0, \quad \varepsilon(k_i) = 1, \quad (4.12)$$

$$S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i, \quad S(k_i) = k_i^{-1}. \quad (4.13)$$

From now on, until the end of this section, we shall assume that $q \in \mathbb{C}$ is *transcendental over \mathbb{Q}* , unless otherwise noticed. Then every finite dimensional $U_q(X_l)$ -module is complete reducible (see [20]). For a $U_q(X_l)$ -module L and $\mathbf{n} = (n_i) \in \mathbb{Z}^l$, we set $L_{\mathbf{n}} = \{u \in L \mid k_i u = q_i^{n_i} u\}$. For each $\mathbf{n} \in \mathbb{Z}_{\geq 0}^l$, there exists the unique irreducible finite dimensional module $L(\mathbf{n}) \neq 0$ such that $L(\mathbf{n}) = U_q(X_l)L(\mathbf{n})_{\mathbf{n}}$ and $e_i L(\mathbf{n})_{\mathbf{n}} = 0$ ($1 \leq i \leq l$). We call $L(\mathbf{n})$ the irreducible finite dimensional $U_q(X_l)$ -module with highest weight \mathbf{n} . The module $L(\mathbf{n})$ has a *weight space decomposition* $L(\mathbf{n}) = \bigoplus_{m \in \mathbb{Z}^l} L(\mathbf{n})_m$. Moreover the dimension of $L(\mathbf{n})_m$ is given by the Weyl character formula (cf. [13]). In particular, $L(\mathbf{m}) \otimes L(\mathbf{n})$ has the unique decomposition of the form $\bigoplus_k c_{mn}^k L(\mathbf{k})$, and the branching coefficient c_{mn}^k is the same as those of the corresponding modules of simple Lie algebra of type X_l . For each finite dimensional irreducible module L there exist the unique numbers $\mathbf{n} \in \mathbb{Z}_{\geq 0}^l$ and $(\zeta_i) \in \{\pm 1\}^l$ such that the algebra automorphism $e_i \mapsto \zeta_i e_i, f_i \mapsto \zeta_i f_i, k_i \mapsto \zeta_i k_i$ carries L to a module isomorphic to $L(\mathbf{n})$.

For $X_l=A_l, B_l, C_l, D_l$, we define algebras $U=U_X$ and $\check{U}=\check{U}_X$ by

$$U_X = \begin{cases} U_q(X_l) & (X=A, C, D) \\ U_{q^{1/2}}(B_l) & (X=B) \end{cases}, \quad \check{U}_X = \begin{cases} U_X & (X=A, C) \\ \mathbf{C}[\langle\sigma\rangle] \rtimes U_X & (X=B, D) \end{cases}, \quad (4.14)$$

where $\langle\sigma\rangle$ denotes an order 2 finite group acting on U_B or U_D by $\sigma=\text{id}$ ($X=B$) or

$$\begin{aligned} \sigma(e_i) &= e_{\sigma(i)}, & \sigma(f_i) &= f_{\sigma(i)}, & \sigma(k_i) &= k_{\sigma(i)}, \\ \sigma(i) &= i + \delta_{i, l-1} - \delta_{i, l} & (X=D). \end{aligned} \quad (4.15)$$

It is easy to see that \check{U}_X ($X=B, D$) is a Hopf algebra with a coproduct defined by (4.11) and $\Delta(\sigma)=\sigma\otimes\sigma$. For \check{U}_D -modules, we have the following lemma.

Lemma 4.1. (1) *Every finite dimensional \check{U}_D -module is completely reducible.*
 (2) *For $\mathbf{n}=(n_i)\in\mathbf{Z}_{\geq 0}^l$ such that $n_{l-1}=n_l$, up to isomorphism, there exists the unique irreducible \check{U}_D -modules $\check{L}(\mathbf{n})$ and $\check{L}(\mathbf{n})^t$ such that $\check{L}(\mathbf{n})\simeq\check{L}(\mathbf{n})^t\simeq L(\mathbf{n})$ as U_D -modules and that $\sigma v_n=v_n$ (resp. $\sigma v_n^t=-v_n^t$) for $0\neq v_n\in\check{L}(\mathbf{n})_n$ (resp. $0\neq v_n^t\in\check{L}(\mathbf{n})_n^t$).* (3) *For $\mathbf{n}=(n_i)\in\mathbf{Z}_{\geq 0}^l$ such that $n_{l-1}\neq n_l$, up to isomorphism, there exists the unique irreducible \check{U}_D -module $\check{L}(\mathbf{n})$ such that $\check{L}(\mathbf{n})\simeq L((n_i))\oplus L((n_{\sigma(i)}))$ as U_D -modules.* (4) *Up to the algebra automorphisms of the form $e_i\rightarrow\zeta_i e_i, f_i\rightarrow\zeta_i f_i, k_i\rightarrow\zeta_i k_i, \sigma\rightarrow\sigma, (\{\zeta_i\}\in\{\pm 1\}^l)$, each irreducible finite dimensional \check{U}_D -module is isomorphic to a module of the form mentioned above.*

Proof. The existence of the above modules is easily shown using Verma modules. For a \check{U}_D -module M , there exists a one to one correspondence between all submodules of M and all sub $\mathbf{C}[\langle\sigma, k_1, \dots, k_l\rangle]$ -modules of $\{v\in M \mid e_i v=0\}$. Hence the problem is reduced to the study of $\mathbf{C}[\langle\sigma, k_i\rangle]$ -modules which decompose into direct sum of one-dimensional $\mathbf{C}[\langle k_i\rangle]$ -modules. \square

There exists a representation π_X of \check{U}_X on $V(X_l)$ such that $\beta_q(X_l)_i\in\text{End}_{\check{U}_D}(V^{\otimes r})$ ($1\leq i < r$). These are defined by the following formulas:

$$\begin{aligned} \pi_A(e_i) &= E_{i, i+1}, & \pi_A(f_i) &= {}^t(\pi_X(e_i)), & \pi_A(k_i) &= T_i T_{i+1}^{-1} \quad (1\leq i\leq l), \\ \pi_X(e_i) &= E_{i, i-1} - E_{(i+1)', i'}, & \pi_X(f_i) &= {}^t(\pi_X(e_i)), \\ \pi_X(k_i) &= T_i T_{i+1}^{-1} T_{(i+1)'} T_{i'}^{-1} \quad (X=B, C, D, 1\leq i\leq l-1), \\ \pi_B(e_l) &= q^{1/2} E_{l, l+1} - E_{l+1, l+2}, & \pi_B(f_l) &= q^{-1/2} E_{l-1, l} - E_{l+2, l+1} \\ \pi_B(k_l) &= T_l T_{l+2}^{-1}, & \pi_B(\sigma) &= -\text{id}_V, \\ \pi_C(e_l) &= E_{l, l+1}, & \pi_C(f_l) &= {}^t(\pi_C(e_l)), & \pi_C(k_l) &= T_l^2 T_{l+1}^{-2}, \\ \pi_D(e_l) &= E_{l-1, l+1} - E_{l, l+2}, & \pi_D(f_l) &= {}^t(\pi_D(e_l)), \end{aligned}$$

$$\begin{aligned}
\pi_D(k_i) &= T_{i-1} T_i T_{i+1}^{-1} T_{i+2}^{-1}, \\
\pi_D(\sigma) &= \sum_{j \neq i, i+1} E_{jj} + E_{i \ i+1} + E_{i+1 \ i}, \\
T_i &= \sum_j q^{\delta_{ij}} E_{jj}.
\end{aligned} \tag{4.16}$$

Remarks. Our definition of A_i, B_i, C_i and D_i is the transpose of those of [7].

To discuss the tensor product module $V^{\otimes r}$, let us recall some notations of the Young diagrams. An element $\lambda = (\lambda_1, \dots, \lambda_k)$ of \mathbf{Z}^k is called a *partition* (or *Young diagram*) if $\lambda_1 \geq \dots \geq \lambda_k \geq 0$. A partition $\lambda = (\lambda_1, \dots, \lambda_k)$ is identified with $(\lambda_1, \dots, \lambda_k, 0, \dots, 0)$. We denote by \mathcal{P} the set of all partitions. For a partition $\lambda = (\lambda_i)$, the *transpose* $\lambda' = (\lambda'_i, \dots, \lambda'_j)$ is defined by $\lambda'_i = \text{card}\{\lambda_n \mid \lambda_n \geq i\}$. For $\lambda \in \mathcal{P}$, we set $|\lambda| := \sum_i \lambda_i$. We define sets $\mathcal{P}(X_i)$ and $\mathcal{P}_r(X_i)$ of partitions as follows (cf. [24]):

$$\begin{aligned}
\mathcal{P}(X_i) &= \begin{cases} \{\lambda \in \mathcal{P} \mid \lambda'_1 \leq l+1\} & (X=A) \\ \{\lambda \in \mathcal{P} \mid \lambda'_1 + \lambda'_2 \leq 2l+1\} & (X=B) \\ \{\lambda \in \mathcal{P} \mid \lambda'_1 \leq l\} & (X=C) \\ \{\lambda \in \mathcal{P} \mid \lambda'_1 + \lambda'_2 \leq 2l\} & (X=D) \end{cases} \\
\mathcal{P}_r(X_i) &= \begin{cases} \{\lambda \in \mathcal{P}(A_i) \mid |\lambda| = r\} & (X=A) \\ \{\lambda \in \mathcal{P}(X_i) \mid |\lambda| \leq r, |\lambda| \equiv r \pmod{2}\} & (X=B, C, D). \end{cases}
\end{aligned} \tag{4.17}$$

For $\lambda \in \mathcal{P}(B_i)$ or $\mathcal{P}(D_i)$, we set $\lambda^t = (N - \lambda'_1, \lambda'_2, \lambda'_3, \dots)'$. We note that $\lambda^{t^t} = \lambda$ and that $\lambda^t = \lambda$ if and only if $X=D$ and $\lambda'_1 = l$. For $\lambda \in \mathcal{P}(X_i)$, we define $\mathbf{n}(\lambda) \in \mathbf{Z}_{\geq 0}^l$ as follows.

- (a) If $X=A, C$, then $\mathbf{n}(\lambda) = (\lambda_1 - \lambda_2, \dots, \lambda_{l-1} - \lambda_l, \lambda_l - \lambda_{l+1})$.
- (b) If $X=B$ and $\lambda'_1 \leq l$, then, $\mathbf{n}(\lambda) = (\lambda_1 - \lambda_2, \dots, \lambda_{l-1} - \lambda_l, 2\lambda_l)$.
- (c) If $X=D$ and $\lambda'_1 \leq l$, then, $\mathbf{n}(\lambda) = (\lambda_1 - \lambda_2, \dots, \lambda_{l-1} - \lambda_l, \lambda_{l-1} + \lambda_l)$.
- (d) If $X=B, D$ and $\lambda'_1 > l$, then, $\mathbf{n}(\lambda) := \mathbf{n}(\lambda^t)$.

Let λ be an element of $\mathcal{P}(X_i)$. We define an irreducible \tilde{U}_X -module $\tilde{L}(\lambda)$ as follows. For $X=A, B, C$, we set $\tilde{L}(\lambda) := L(\mathbf{n}(\lambda))$. If $X=B$, define the action of $\sigma \in \tilde{U}_B$ on $\tilde{L}(\lambda)$ by $\sigma \mapsto (-1)^{\lambda_1}$. For $X=D$, we set $\tilde{L}(\lambda) := \tilde{L}(\mathbf{n}(\lambda))$ if $\lambda'_1 \leq l$, and $\tilde{L}(\lambda) := \tilde{L}(\mathbf{n}(\lambda))^t$ if $\lambda'_1 > l$, where $\tilde{L}(\mathbf{n})$ is as in Lemma 4.1. By definition, $\tilde{L}(\lambda)$ is U_X -irreducible unless $X=D, \lambda'_1 = l$. If $X=D$ and $\lambda'_1 = l$, $\tilde{L}(\lambda)$ has two U_D -irreducible components. We note that if $\lambda \neq \mu$ then $\tilde{L}(\lambda) \neq \tilde{L}(\mu)$ unless $X=A$. For $\lambda, \mu \in \mathcal{P}(X_i)$, we say $\lambda \sim \mu$, if $X=A$ and $\mu_i = \lambda_i + \delta_{ij}$, or $X=B, C, D$ and $\lambda_i = \mu_i \pm \delta_{ij}$ for some j .

Proposition 4.2. (1) For $\lambda \in \mathcal{P}(X_i)$, $\tilde{L}(\lambda) \otimes V \simeq \bigoplus_{\lambda \sim \mu \in \mathcal{P}(X_i)} \tilde{L}(\mu)$.
(2) For $r > 0$, $V^{\otimes r} \simeq \bigoplus_{\lambda \in \mathcal{P}_r(X_i)} m_\lambda \tilde{L}(\lambda)$, where the multiplicity $m_\lambda > 0$ is given

by

$$m_\lambda = \text{card}\{(\lambda^0, \dots, \lambda^r) \mid \lambda^i \in \mathcal{P}_i(X_l), \lambda^i \sim \lambda^{i+1}, \lambda^0 = (0), \lambda^r = \lambda\}.$$

Proof. It suffices to prove part (1). For $X=A, B, C$, this follows immediately from the general theory of quantum enveloping algebras. Let $X_l=D_l$. Then $\tilde{L}(\lambda) \otimes V \simeq \bigoplus_{\lambda \sim \mu \in \mathcal{P}(D)} \tilde{L}(\mu)$ as U_D -modules. Hence by Lemma 4.1, it suffices to determine the multiplicity of $\tilde{L}(\mu)$ and $\tilde{L}(\mu^\dagger)$ in $\tilde{L}(\lambda) \otimes V$ for $\mu \in \mathcal{P}(D_l)$ such that $\lambda \sim \mu$ and $\mu'_i \neq l$. Suppose $\lambda'_i \neq l$. Then, up to constant, there is a unique vector $0 \neq v_\mu \in (L(\lambda) \otimes V)_{n(\mu)}$ such that $e_i v_\mu = 0$. It is easy to see that v_μ is of the form $v_\mu = v_\lambda \otimes u_i + \sum_{j>i} w_j \otimes u_j$, for some $1 \leq i < l, 0 \neq v_\lambda \in \tilde{L}(\lambda)_{n(\lambda)}$ and $w_{i+1}, \dots, w_N \in \tilde{L}(\lambda)$. Hence $\sigma v_\mu = v_\mu$ if and only if $\sigma v_\lambda = v_\lambda$. Thus we get $\tilde{L}(\lambda) \otimes V \supset \tilde{U}_D v_\mu \simeq \tilde{L}(\mu)$. Next suppose $\lambda'_i = l$. Since $\tilde{L}((0)^\dagger) \otimes \tilde{L}(\lambda) \simeq \tilde{L}(\lambda)$ and $L((0)^\dagger) \otimes \tilde{L}(\mu) \simeq L(\mu^\dagger)$, we get

$$\begin{aligned} \text{Hom}_{\tilde{U}_D}(\tilde{L}(\lambda) \otimes V, \tilde{L}(\mu)) &\simeq \text{Hom}_{\tilde{U}_D}(\tilde{L}((0)^\dagger) \otimes (\tilde{L}(\lambda) \otimes V), \tilde{L}((0)^\dagger) \otimes \tilde{L}(\mu)) \\ &\simeq \text{Hom}_{\tilde{U}_D}(\tilde{L}(\lambda) \otimes V, \tilde{L}(\mu^\dagger)). \end{aligned}$$

On the other hand, $\dim \text{Hom}_{\tilde{U}_D}(\tilde{L}(\lambda) \otimes V, L(n(\mu))) = 2$. Hence the \tilde{U}_D -module $\tilde{L}(\lambda) \otimes V$ has both $\tilde{L}(\mu)$ and $\tilde{L}(\mu^\dagger)$ as multiplicity one irreducible components. This completes the proof of the proposition.

Theorem 4.3. (*q-analogue of Brauer-Schur-Weyl reciprocity*) Let $\pi_{V^{\otimes r}}$ be the representation $\pi_X^{\otimes r} \circ \Delta : \tilde{U}_X \rightarrow \text{End}(V^{\otimes r})$ and $B(r)$ be as in §1. If the parameter q is transcendental over \mathbf{Q} , then we have,

$$\begin{aligned} \text{Im}(\pi_{V^{\otimes r}}) &= \text{Sch}_r(V(X)) := \text{End}_{B(r)}(V^{\otimes r}) \simeq \bigoplus_{\lambda \in \mathcal{P}_r(X_l)} \text{End}(\tilde{L}(\lambda)), \\ \text{End}_{\tilde{U}_X}(V^{\otimes r}) &= B(r) \simeq \bigoplus_{\lambda \in \mathcal{P}_r(X_l)} \text{Mat}(m_\lambda; \mathbf{C}). \end{aligned}$$

Proof. We give a proof for $X=B, C, D$. The case $X=A$ is quite similar. We set $G(B)=G(D)=O(N, \mathbf{C})$ and $G(C)=Sp(N, \mathbf{C})$. It was proved by Brauer [3] that the algebra $\text{End}_{G(X)}(V^{\otimes r})$ is generated by $(\beta_i)_i$ and $(\iota_i)_i := \text{id}_{V^{\otimes i-1}} \otimes \iota_1 \otimes \text{id}_{V^{\otimes r-i-1}}$ ($1 \leq i \leq r-1$). Since $\dim \text{End}_{G(X)}(V^{\otimes r})$ is $k_0 := \sum_{\lambda \in \mathcal{P}_r(X_l)} (\dim \tilde{L}(\lambda))^2$, there exist vectors $\{\xi_k(q) \mid 1 \leq k \leq k_0\}$ ($q \in \mathbf{C}^\times$) of $\text{End} V^{\otimes r}$ satisfying the following two conditions.

- (i) For each k , $\xi_k(q)$ is a monomial of operators $\{(\beta_q)_i, (\iota_q)_i \mid 1 \leq i < r\}$.
- (ii) The vectors $\{\xi_k(1) \mid 1 \leq k \leq k_0\}$ are linearly independent. We identify $\text{End} V^{\otimes r}$ with $\mathbf{C}^{N^{2r}}$ by means of the basis $\{E_{i_1 j_1} \otimes \dots \otimes E_{i_r j_r}\}$. By (i) and (4.3), (4.5), each component of a vector $\xi_k(q)$ is a Laurent polynomial of the parameter q with coefficients in \mathbf{Z} . Hence $\{\xi_k(q) \mid 1 \leq k \leq k_0\}$ are linearly independent if q is transcendental over \mathbf{Q} . Since $B(r) \subset \text{End}_{\tilde{U}_X} V^{\otimes r}$ and $\dim \text{End}_{\tilde{U}_X} V^{\otimes r} = k_0$, we get $B(r) = \text{End}_{\tilde{U}_X} V^{\otimes r}$. On the other hand, by Proposition 4.2, $\text{Im}(\pi_{V^{\otimes r}})$ is isomorphic to the semi-simple algebra $\bigoplus_{\lambda \in \mathcal{P}_r(X_l)} \text{End} \tilde{L}(\lambda)$. Hence the rest part of this theorem is a consequence of the general theory of semisimple algebras. □

Let λ be an element of $\mathcal{P}_r(X_l)$ ($r \geq 0$). By the above theorem and Proposition 2.1, up to isomorphism, there exists the unique irreducible right $S_r(E_q(X_l))$ -comodule L such that $L_{\bar{v}} \simeq \tilde{L}(\lambda)$. We denote this comodule by $\tilde{L}(\lambda; r)$. Each $S(E_q(X_l))$ -comodule is isomorphic to a direct sum of comodules of this form. We can rewrite Theorem 4.3 as follows.

Corollary 4.4. (1) *As an algebra (resp. coalgebra), $\text{Sch}(V_q(X_l))$ (resp. $S(E_q(X_l))$) decomposes as follows:*

$$\text{Sch}(V_q(X_l)) = \bigoplus_{r \geq 0} \bigoplus_{\lambda \in \mathcal{P}_r(X_l)} \text{End}(\tilde{L}(\lambda; r)),$$

$$S(E_q(X_l)) = \bigoplus_{r \geq 0} \bigoplus_{\lambda \in \mathcal{P}_r(X_l)} \text{Im } \text{cf}_{L(\lambda; r)}.$$

(2) *An $S(E_q(X_l))$ -comodule L is irreducible if and only if $L_{\bar{v}}$ is irreducible.*

(3) *For each $S_r(E_q(X_l))$ -comodules L, M ($r \geq 0$), $L \simeq M$ if and only if $L_{\bar{v}} \simeq M_{\bar{v}}$.*

§ 5. Quantum Determinants

Let V be a $U_q(X_l)$ -module. We call that V is a type 1 module if $V = \bigoplus_{n \in \mathbb{Z}^l} V_n$. Let $\beta_V \in \text{End}_{U_q(X_l)}(V \otimes V)$ be a YB operator on V . Denote by E the quantum matrix on V .

Theorem 5.1. *Let (V, β_V) be as above. If V is an irreducible type 1 module, then each group-like element of $S(E)$ is central.*

Proof. We define a left action of $\text{Sch}(V)$ on $S(E)$ by $a(x) = \text{id}_{S(E)} \otimes \langle a, \rangle(\Delta(x))$ ($a \in \text{Sch}(V)$, $x \in S(E)$). Combining this with the algebra map $U_q(X_l) \rightarrow \text{Sch}(V)$, we get a left action of $U_q(X_l)$ on $S(E)$. Since tensor products and composition factors of type 1 modules are also type 1, Cg is a type 1 $U_q(X_l)$ -module. Hence the action is given by $e_i g = f_i g = 0$, $k_i g = g$. Therefore $V \otimes Cg$ and $Cg \otimes V$ are both irreducible and the map $u \otimes g \mapsto g \otimes u$ ($u \in V$) gives an isomorphism. By Schur's Lemma, $\beta_{V, Cg}(u_i \otimes g) = cg \otimes u_i$ for some non-zero constant c , where u_i and x_{ij} be as in § 2. Comparing the images of $u_i \otimes g$ by the maps $\omega_{Cg \otimes V} \circ \beta_{V, Cg} = \beta_{V, Cg} \otimes \text{id} \circ \omega_{V \otimes Cg} : V \otimes Cg \rightarrow (Cg \otimes V) \otimes S(E)$, we get $gx_i = x_{ij}g$. \square

Proposition 5.2. *Let V and E be as above and let $g \in S_r(E)$ be a group-like element. Suppose there is a right $S(E)$ -comodule map $0 \neq \mu : V^{\otimes r} \rightarrow Cg$, then there exist both left and right cofactors with respect to g .*

Proof. Using (4.11), (4.13), one can verify $\text{Hom}_{U_q(X_l)}(L \otimes M, N) \simeq \text{Hom}_{U_q(X_l)}(L, N \otimes^* M)$ for finite dimensional $U_q(X_l)$ -modules L, M, N . Hence

there exists a $U_q(X_l)$ -module map $0 \neq \mu' : V^{\otimes r-1} \rightarrow \mathcal{C}g \otimes^* V$. Since $\mathcal{C}g \otimes^* V$ is irreducible, μ' is surjective. Hence there exist bases $\{{}^c u_i\}$, $\{u_i\}$ of $L_1 := V^{\otimes r-1}$, V satisfy the condition (*) in §3. \square

Now we will return to the study of the examples of §4. Let X_l be A_l , B_l , C_l or D_l and $q \in \mathcal{C}$ be transcendental over \mathcal{Q} . Following Manin [16] and Takeuchi [22], we introduce the following graded algebras $\Omega(V)$, $\Omega(V^\vee)$ in order to study some group-like elements :

$$\Omega(V) = \begin{cases} T(V)/(\text{Ker}(\text{id}_{V \otimes V} - \beta)) & (X=A, C) \\ T(V)/(\text{Ker}(\text{id}_{V \otimes V} - \beta), \text{Im } \iota_q) & (X=B, D). \end{cases}$$

$$\Omega(V^\vee) = \begin{cases} T(V^\vee)/(\text{Ker}(\text{id}_{V^\vee \otimes V^\vee} - \beta^\vee)) & (X=A, C) \\ T(V^\vee)/(\text{Ker}(\text{id}_{V^\vee \otimes V^\vee} - \beta^\vee), \text{Im } \iota_q^*) & (X=B, D). \end{cases} \quad (5.1)$$

The defining relations of $\Omega(V_q(X_l))$ is as follows :

$$\begin{aligned} \Omega(V_q(A)) &= \langle u_i \ (1 \leq i \leq N) \mid u_i^2 = 0, \quad qu_i u_j + u_j u_i = 0 \ (i < j) \rangle, \\ \Omega(V_q(B)) &= \langle u_i \ (1 \leq i \leq N) \mid u_i^2 = 0 \ (i \neq l+1), \quad qu_i u_j + u_j u_i = 0 \ (i < j, i \neq j'), \\ &\quad u_i \cdot u_i + u_i u_{i'} - (q - q^{-1}) \sum_{1 \leq j < i} q^{j-i+1} u_j u_{j'} = 0 \ (1 \leq i \leq l), \\ &\quad u_{l+1} u_{l+1} - (q^{1/2} - q^{-1/2}) \sum_{1 \leq j \leq l} q^{j-1} u_j u_{j'} = 0 \ (1 \leq i \leq l) \rangle, \\ \Omega(V_q(C)) &= \langle u_i \ (1 \leq i \leq N) \mid u_i^2 = 0 \ (1 \leq i \leq N), \quad qu_i u_j + u_j u_i = 0 \ (i < j, i \neq j'), \\ &\quad u_i \cdot u_i + q^2 u_i u_{i'} + (q - q^{-1}) \sum_{i < j \leq l} q^{j-i+1} u_j u_{j'} = 0 \ (1 \leq i \leq l) \rangle, \\ \Omega(V_q(D)) &= \langle u_i \ (1 \leq i \leq N) \mid u_i^2 = 0 \ (1 \leq i \leq N), \quad qu_i u_j + u_j u_i = 0 \ (i < j, i \neq j'), \\ &\quad u_i \cdot u_i + u_i u_{i'} - (q - q^{-1}) \sum_{1 \leq j < i} q^{j-i+1} u_j u_{j'} = 0 \ (1 \leq i \leq l) \rangle. \end{aligned} \quad (5.2)$$

Since $\beta_q(X_l)$ is a symmetric matrix, the defining relation of $\Omega(V_q(X_l)^\vee)$ with respect to the generators $\{v_i\}$ is the same as those of $\Omega(V_q(X_l))$ with respect to $\{u_i\}$. In particular, $\Omega(V_q(X_l)) \simeq \Omega(V_q(X_l)^\vee)$ as algebras. As an application of the diamond lemma [2], we have the following (cf. [16, 22]).

Lemma 5.3. *The products $u_{i_1} u_{i_2} \cdots u_{i_r}$ (resp. $v_{i_1} v_{i_2} \cdots v_{i_r}$) ($1 \leq i_1 < i_2 < \cdots < i_r \leq N$) form a linear basis of $\Omega_r(V)$ (resp. $\Omega_r(V^\vee)$). In particular, $\Omega_N(V)$ and $\Omega_N(V^\vee)$ (resp. $\Omega_{N-1}(V)$ and $\Omega_{N-1}(V^\vee)$) is one dimensional (resp. N -dimensional).*

We define a bilinear pairing $\langle \cdot, \cdot \rangle : \Omega_{N-1}(V) \otimes \Omega_{N-1}(V^\vee) \rightarrow \mathcal{C}$ and $\langle \cdot, \cdot \rangle : \Omega_N(V) \otimes \Omega_N(V^\vee) \rightarrow \mathcal{C}$ by

$$\langle \check{v}_i, \check{u}_j \rangle = \delta_{ij}, \quad \langle \check{v}, \check{u} \rangle = 1, \quad (5.3)$$

where \check{v}_i, \check{u}_i ($1 \leq i \leq N$) and \bar{u}, \bar{v} are defined by

$$\begin{aligned} \check{u}_i &= u_1 u_2 \cdots u_{i-1} u_{i+1} \cdots u_N, & \check{v}_i &= v_1 v_2 \cdots v_{i-1} v_{i+1} \cdots v_N, \\ \bar{u} &= u_1 u_2 \cdots u_N, & \bar{v} &= v_1 v_2 \cdots v_N. \end{aligned} \quad (5.4)$$

Then, we have

$$\langle \check{v}_i x, \check{u}_j \rangle = \langle \check{v}_i, x \check{u}_j \rangle, \quad \langle \bar{v} x, \bar{u} \rangle = \langle \bar{v}, x \bar{u} \rangle \quad (x \in \check{U}_X). \quad (5.5)$$

We will prove these formulas for $X=B$. Since $e_i \check{u}_j$ is a weight vector of $\mathcal{Q}_{N-1}(V)$, we have $e_i \check{u}_j = 0$ unless $i < l, j = i, (i+1)'$ or $i = l, j = l, l+1$. On the other hand, we have

$$\begin{aligned} e_i \check{u}_i &= k_i (u_1 \cdots u_{i-1}) e_i (u_{i+1}) u_{i+2} \cdots u_N \\ &\quad + k_i (u_1 \cdots \check{u}_i \cdots u_{(i+1)'}) e_i (u_{i'}) (u_{(i-1)'}, \cdots u_N) \\ &= \check{u}_{i+1} + 0 \quad (i < l), \\ e_i \check{u}_{(i+1)'} &= -\check{u}_i, \quad (i < l), \quad e_i \check{u}_l = -q^{1/2} \check{u}_{l+1}, \quad e_l \check{u}_{l+1} = -q \check{u}_{l+2}. \end{aligned}$$

Similarly, we obtain the explicit formulas for the action of e_i, f_i, k_i and σ on the modules $\mathcal{Q}_{N-1}(V), \mathcal{Q}_{N-1}(V^\vee), \mathcal{Q}_N(V)$ and $\mathcal{Q}_N(V^\vee)$. The formula (5.5) follows easily from these formulas.

By (5.5) and Corollary 4.4 (3), we can identify left $S(E)$ -comodules $\mathcal{Q}_{N-1}(V^\vee)$ and $\mathcal{Q}_N(V^\vee)$ with $\mathcal{Q}_{N-1}(V)^*$ and $\mathcal{Q}_N(V)^*$ respectively. We define a group-like element of $S(E)$ by $\det_q(X_l) := cf_{\mathcal{Q}_N(V)}(\bar{v} \otimes \bar{u})$ and call it *quantum determinant* of $S(E)$. For $X=A$, this definition agrees with that of in §3 (see e.g. [16]).

Proposition 5.4. *The following $y_{i,j}$ form a cofactor of $S(E_q(X_l))$ with respect to $\det_q(X_l)$ (cf. [22, Problem 5.6 a]):*

$$y_{i,j} = (-q)^{i-j} q^{Y_{\geq 0}(i-N_0) - Y_{> 0}(j-N_0)} cf_{\mathcal{Q}_{N-1}}(\check{v}_j \otimes \check{u}_i), \quad (5.6)$$

where $N_0 = (N+1)/2$ and functions $Y_{\geq 0}, Y_{> 0}$ are defined by

$$Y_{\geq 0}(i) = \begin{cases} 0 & (i < 0) \\ 1 & (i \geq 0) \end{cases}, \quad Y_{> 0}(i) = \begin{cases} 0 & (i \geq 0) \\ 1 & (i > 0) \end{cases}. \quad (5.7)$$

Proof. Set $V = V_q(B_l), L_1 = \mathcal{Q}_{N-1}(V_q(B_l))$, and

$${}^c u_i = (-1)^{i-1} q^{-2l+i-Y_{\geq 0}(i-N_0)} \check{u}_i, \quad {}^c v_j = (-1)^{j-1} q^{2l-j+Y_{> 0}(i-N_0)} \check{v}_j.$$

Denote by $\mu: L_1 \otimes V \rightarrow C\bar{u}$ the restriction of the product of $\mathcal{Q}(V)$. Then, by (5.2), (5.3), we have $\mu({}^c u_i \otimes u_j) = \delta_{i,j} \bar{u}$ and $\langle {}^c v_j, {}^c u_i \rangle = \delta_{i,j}$. Hence by Theorem 3.3, $y_{i,j}$ form a left cofactor with respect to $\det_q(B_l)$. Other cases are similar. \square

If $X_l = B_l, C_l$ or D_l , there exists another important group-like element of

$S(E_q(X_l))$. Let $\text{quad}_q(X_l)$ be the group-like element corresponding to the one-dimensional $S(E)$ -comodule $\text{Im } \epsilon_q(X_l) = \mathbf{C} \sum_i \epsilon(i) q^{i+1/2} u_i \otimes u_{i'}$. We call it the *quadratic group-like element* of $S(E_q(X_l))$. The next proposition was obtained by [5] and [22].

Proposition 5.5. *The following elements $y_{i,j}$ form a cofactor of $S(E_q(X_l))$ with respect to $\text{quad}_q(X_l)$:*

$$y_{i,j} = \epsilon(i)\epsilon(j)q^{i-j}x_{j,i'}. \tag{5.8}$$

Lemma 5.6. *Let q be transcendental over \mathbf{Q} . Then, we have the following isomorphisms of $S(E_q(X_l))$ -comodules:*

$$\begin{aligned} \tilde{L}(\lambda; r) \otimes \mathbf{C} \det_q &\simeq \begin{cases} \tilde{L}(\lambda + (N)'; r + N) & (X=A) \\ \tilde{L}(\lambda; r + N) & (X=C) \\ \tilde{L}(\lambda^t; r + N) & (X=B, D), \end{cases} \\ \tilde{L}(\lambda; r) \otimes \mathbf{C} \text{quad}_q &\simeq \tilde{L}(\lambda; r + 2) \quad (X=B, C, D). \end{aligned}$$

In particular, for $X=B, D$, $\mathbf{C} \text{quad}_q \simeq L((0); 2)$ and $\mathbf{C} \det_q \simeq L((0)'; N)$.

Proof. Let $X=D$. Since $\sigma(\det_q) = -\det_q$ and $\sigma(\text{quad}_q) = \text{quad}_q$, we have

$$(\tilde{L}(\lambda; r) \otimes \det_q)_{\bar{v}} \simeq \tilde{L}(\lambda^t; r + N)_{\bar{v}}, \quad (\tilde{L}(\lambda; r) \otimes \text{quad}_q)_{\bar{v}} \simeq \tilde{L}(\lambda; r + 2)_{\bar{v}}$$

by Lemma 4.1. By Corollary 4.4.(3), this proves the lemma for $X=D$. The proof of other cases are similar. \square

Theorem 5.7. *Let q be a complex number transcendental over \mathbf{Q} .*

- (1) *Each group-like element of $S(E_q(X_l))$ is central, and is not a zero-divisor.*
- (2) *For each group-like element $g \in S(E_q(X_l))$, there exists a cofactor with respect to g .*
- (3) *We have the following identities (cf. [22]).*

$$\begin{aligned} \det_q(B_l)^2 &= \text{quad}_q(B_l)^{2l+1}, & \det_q(C_l) &= \text{quad}_q(C_l)^l \\ \det_q(D_l)^2 &= \text{quad}_q(D_l)^{2l}. \end{aligned}$$

- (4) *The set \mathcal{G} of all group-like elements of $S(E_q(X_l))$ is given as follows:*

$$\mathcal{G} = \begin{cases} \{\det_q^r \mid r \geq 0\} & (X=A) \\ \{\text{quad}_q^r \mid r \geq 0\} \amalg \{\det_q \text{quad}_q^r \mid r \geq 0\} & (X=B, D) \\ \{\text{quad}_q^r \mid r \geq 0\} & (X=C). \end{cases}$$

Proof. We denote by \mathcal{G}_r the set of all group-like elements of $S_r(E_q(X_l))$. We prove (3), (4) only for the case $X=D$. The proof of other cases are similar and relatively easy. By the above lemma, we have

$$C \det_q \text{quad}_q^{l+t} \simeq \tilde{L}((0)^+; 2l+2t) \neq \tilde{L}((0); 2l+2t) \simeq C \text{quad}_q^{l+t}.$$

Hence quad_q^{l+t} and $\det_q \text{quad}_q^l$ are distinct in \mathcal{G}_{2l+2t} . On the other hand, by Corollary 4.4 (1), we have $\text{card } \mathcal{G}_r = 1$ if $r \in 2\mathbb{Z}$ and $|r| \leq 2l-2$, $\text{card } \mathcal{G}_r = 2$ if $r \in 2\mathbb{Z}$ and $|r| \geq 2l$ and $\text{card } \mathcal{G}_r = 0$ if otherwise. Thus we get (4). Since \det_q^2 is an element of \mathcal{G}_{4l} different from $\det_q \text{quad}_q^l$, it must coincide with quad_q^{2l} . This proves (3). Let $\{u_i(\lambda; r)\}$ be a linear basis of $\tilde{L}(\lambda; r)$ and $\{x_{i_j}(\lambda; r)\}$ be a linear basis of $\text{Im } cf_{\tilde{L}(\lambda; r)}$ defined by $\omega(u_j(\lambda; r)) = \sum_i u_i(\lambda; r) \otimes x_{i_j}(\lambda; r)$. By part (4) and the above lemma, for each $g \in \mathcal{G}_s$, $\{g x_{i_j}(\lambda; r)\}$ is a linear basis of either $\text{Im } cf_{\tilde{L}(\lambda; r+s)}$ or $\text{Im } cf_{\tilde{L}(\lambda^+; r+s)}$, because

$$\omega(g \otimes u_j(\lambda; r)) = \sum_i (g \otimes u_i(\lambda; r)) \otimes g x_{i_j}(\lambda; r).$$

Hence $S(E) \rightarrow S(E)$; $x \mapsto gx$ defines a linear isomorphism from $\text{Im } cf_{\tilde{L}(\lambda; r)}$ onto $\text{Im } cf_{\tilde{L}(\lambda; r+s)}$ or onto $\text{Im } cf_{\tilde{L}(\lambda^+; r+s)}$. By Corollary 4.4 (1), this proves that $x \mapsto gx$ is injective. If y_{i_j} form a cofactor with respect to $g \in \mathcal{G}$, then $g' y_{i_j}$ form a cofactor with respect to $gg' \in \mathcal{G}$. Hence (2) follows from Propositions 5.4, 5.5. \square

§ 6. Peter-Weyl Decomposition

Let q be a complex number transcendental over \mathbb{Q} . For $X = X_l = B_l, C_l, D_l$, define Hopf algebras \tilde{A}_X, A_X by

$$\begin{aligned} \tilde{A}_X &:= S(E_q(X_l)) / (\text{quad}_q(X_l) - 1), \\ A_X &:= S(E_q(X_l)) / (\text{quad}_q(X_l) - 1, \det_q(X_l) - 1). \end{aligned} \quad (6.1)$$

For $X_l = A_l$, we set $\tilde{A}_X = A_X := A(SL_q(l+1))$. By Theorem 5.7 (3), we have $A_C = \tilde{A}_C$.

Definition 6.1. For $X_l = B_l, C_l, D_l$, we denote \tilde{A}_X by $A(O_q(2l+1))$, $A(Sp_q(2l))$, $A(O_q(2l))$ respectively. For $X_l = B_l, D_l$, we denote A_X by $A(SO_q(2l+1))$, $A(SO_q(2l))$ respectively.

Note. The Hopf algebras $A(O_q(N))$ and $A_q(Sp_q(N))$ were first introduced by Faddeev, Reshetikhin and Takhtajan [5] and independently by Takeuchi [22]. In [5] (resp. [22]), the quantum groups $O_q(N)$ and $Sp_q(N)$ are introduced under the notation $SO_{1/q}(N)$ and $Sp_{1/q}(N)$ (resp. $O_{1/q}(N)$ and $Sp_{1/q}(N)$) respectively.

Combining the representation maps $\tilde{U}_X \rightarrow \text{Sch}_\tau(V)$ with the pairing of Proposition 2.1, we get a bialgebra pairing $\langle \cdot, \cdot \rangle$ between \tilde{U}_X (or U_X) and $S(E_q(X_l))$. Since $\langle a, (\text{quad}_q - 1) \rangle = 0$ ($a \in \tilde{U}_X$) and $\langle a, (\det_q - 1) \rangle = 0$ ($a \in U_X$), we get Hopf algebra pairings $\langle \cdot, \cdot \rangle: \tilde{U}_X \otimes \tilde{A}_X \rightarrow \mathbb{C}$ and $U_X \otimes A_X \rightarrow \mathbb{C}$.

Lemma 6.2. *Let L, M be right $S(E)$ -comodules. Then,*

- (1) *If $L_{\tilde{\sigma}}$ (resp. L_U) is irreducible, then $L_{\tilde{\chi}}$ (resp. L_A) is irreducible.*
- (2) *If a linear map $f : L \rightarrow M$ gives an isomorphism $L_{\tilde{\chi}} \simeq M_{\tilde{\chi}}$ (resp. $L_A \simeq M_A$), it also gives an isomorphism $L_{\tilde{\sigma}} \simeq M_{\tilde{\sigma}}$ (resp. $L_U \simeq M_U$).*
- (3) *We have the following isomorphisms of comodules:*

$$\tilde{L}(\lambda; r)_{\tilde{\chi}} \simeq \tilde{L}(\lambda; s)_{\tilde{\chi}}, \quad \tilde{L}(\lambda; r)_A \simeq \tilde{L}(\lambda; s)_A \quad (\lambda \in \mathcal{P}_r(X_l) \cap \mathcal{P}_s(X_l)),$$

$$\tilde{L}(\lambda; r)_A \simeq \tilde{L}(\lambda^t; s)_A \quad (\lambda \in \mathcal{P}_r(X_l), \lambda^t \in \mathcal{P}_s(X_l)),$$

$$L(\lambda; r)_A \simeq L(\lambda + (N)'; r + N)_A \quad (\lambda \in \mathcal{P}_r(A_l)).$$

- (4) *If $\mathbf{n}(\lambda) = \mathbf{n}(\mu)$ for $\lambda \in \mathcal{P}_r(X_l)$ and $\mu \in \mathcal{P}_s(X_l)$, then $\tilde{L}(\lambda; r)_A \simeq \tilde{L}(\mu; s)_A$.*
- (5) *If $X = D$ and $\lambda'_1 = l$, then $\tilde{L}(\lambda; r)_A$ has the irreducible decomposition $L' \oplus L''$ such that $L'_U \simeq L(\mathbf{n}(\lambda))$ and $L''_U \simeq L(\mathbf{n}(\lambda)^\sigma)$, where we set $\mathbf{n}^\sigma := (n_{\sigma(i)})$ for $\mathbf{n} = (n_i)$. If otherwise, $\tilde{L}(\lambda; r)_A$ is irreducible.*

Proof. Part (1) and (2) follow immediately from the definition of $L_{\tilde{\chi}}$ and L_A (see the notation before §1). Since $(\mathbf{C} \text{quad}_q)_{\tilde{\chi}}$, $(\mathbf{C} \text{quad}_q)_A$ and $(\mathbf{C} \text{det}_q)_A$ are trivial comodules, part (3) follows from Lemma 5.6. If $\mathbf{n}(\lambda) = \mathbf{n}(\mu)$, then we have either $\lambda = \mu$, $\lambda = \mu \pm (N, \dots, N)'$ ($X = A$) or $\lambda = \mu^t$ ($X = B, D$). Hence part (4) follows from part (3). Let $X = D$ and $\lambda'_1 = l$. By Lemma 5.6, there exists an $S_{r+2l}(E)$ -comodule isomorphism $\varphi : \tilde{L}(\lambda; r) \otimes \mathbf{C} \text{quad}_q^l \simeq \tilde{L}(\lambda; r) \otimes \mathbf{C} \text{det}_q$. Let v be a non-zero vector of $L(\mathbf{n}(\lambda))_{\mathbf{n}(\lambda)} \hookrightarrow \tilde{L}(\lambda; r)_U$. Since φ is also an \tilde{U}_D -module isomorphism, we may assume $\varphi(v \otimes \text{quad}_q^l) = v \otimes \text{det}_q$. Since $\sigma(\text{quad}_q^l) = \text{quad}_q^l$ and $\sigma \text{det}_q = -\text{det}_q$, we have $\varphi(\sigma v \otimes \text{quad}_q^l) = -\sigma v \otimes \text{det}_q$. On the other hand, from the definition of A_D , $\varphi' : u \otimes \text{quad}_q^l \rightarrow u \otimes \text{det}_q$ ($u \in \tilde{L}(\lambda; r)$) gives an isomorphism $(\tilde{L}(\lambda; r) \otimes \mathbf{C} \text{quad}_q^l)_A \simeq (\tilde{L}(\lambda; r) \otimes \mathbf{C} \text{det}_q)_A$. Define A_D -comodule isomorphisms by $\varphi_\pm := \varphi' \pm \varphi$. Then we have

$$\begin{aligned} \text{Im } \varphi_+ &= \varphi_+(U(v \otimes \text{quad}_q^l) + U(\sigma v \otimes \text{quad}_q^l)) \\ &= U\varphi_+(v \otimes \text{quad}_q^l) + U\varphi_+(\sigma v \otimes \text{quad}_q^l) \\ &= Uv \otimes \text{det}_q, \end{aligned}$$

$$\text{Im } \varphi_- = U\sigma v \otimes \text{det}_q.$$

Hence $\text{Im } \varphi_+$ and $\text{Im } \varphi_-$ are distinct irreducible component of $(\tilde{L}(\lambda; r) \otimes \mathbf{C} \text{det}_q)_U$. This proves part (5) for $X = D$ $\lambda'_1 = l$. The rest case of part (5) follows from part (1). \square

We introduce the following sets:

$$\begin{aligned} P_A &= PSL(l+1) = P_C = PSp(2l) = \mathbf{Z}_{\geq 0}^l, \\ P_B &= PSO(2l+1) = \{\mathbf{n} = (n_i) \in \mathbf{Z}_{\geq 0}^l \mid n_l \in 2\mathbf{Z}\}, \\ P_D &= PSO(2l) = \{\mathbf{n} = (n_i) \in \mathbf{Z}_{\geq 0}^l \mid n_{l-1} + n_l \in 2\mathbf{Z}\}. \end{aligned} \tag{6.2}$$

For $X=A, B, C$, P_X is the image of the map $\mathcal{P}(X_l) \rightarrow \mathcal{Z}_{\neq 0}^l$; $\lambda \mapsto \mathbf{n}(\lambda)$. While for $X=D$, $P_D = \{\mathbf{n}(\lambda), \mathbf{n}(\lambda)^\sigma \mid \lambda \in \mathcal{P}(D_l)\}$. Let $\mathbf{n} = (n_i)$ be an element of P_X . We will define an A_X -comodule $L_A(\mathbf{n})$ as follows. Suppose $X \neq D$ or $n_{l-1} = n_l$. Noting Lemma 6.2 (4), we set $L_A(\mathbf{n}) = \tilde{L}(\lambda; r)_A$, where λ denotes an element of $\mathcal{P}_r(X_l)$ such that $\mathbf{n}(\lambda) = \mathbf{n}$. Suppose $X_l = D_l$ and $n_{l-1} < n_l$. Then there exists the unique $\lambda \in \mathcal{P}(D_l)$ such that $\mathbf{n} = \mathbf{n}(\lambda)$ (and that $\lambda'_i = l$). We define $L_A(\mathbf{n}) := L'$ and $L_A(\mathbf{n}^\sigma) := L''$, where L' and L'' be as in Lemma 6.2 (5). From the above lemma, we have

- Proposition 6.3.** (1) For each $\mathbf{n} \in P_X$, the A_X -comodule $L_A(\mathbf{n})$ is irreducible.
(2) If $\mathbf{n} \neq \mathbf{n}'$, then $L_A(\mathbf{n}) \neq L_A(\mathbf{n}')$.
(3) We have the following irreducible decomposition :

$$\tilde{L}(\lambda; r)_A \simeq \begin{cases} L_A(\mathbf{n}(\lambda)) \oplus L_A(\mathbf{n}(\lambda)^\sigma) & (X=D, \lambda'_i=l) \\ L_A(\mathbf{n}(\lambda)) & (\text{otherwise}). \end{cases}$$

(4) In particular, the irreducible decomposition of $\tilde{L}(\lambda; r)_A$ is the same as those of $\hat{L}(\lambda; r)_U$.

Theorem 6.4. (1) (Peter-Weyl decomposition) We have the following coalgebra isomorphisms :

$$\begin{aligned} A(O_q(N)) &= \bigoplus_{\lambda \in \mathcal{P}O(N)} (\text{End } \tilde{L}(\lambda))^*, \\ A(SL_q(N)) &= \bigoplus_{\mathbf{n} \in \mathcal{P}SL(N)} (\text{End } L(\mathbf{n}))^*, \\ A(SO_q(N)) &= \bigoplus_{\mathbf{n} \in \mathcal{P}SO(N)} (\text{End } L(\mathbf{n}))^*, \\ A(Sp_q(N)) &= \bigoplus_{\mathbf{n} \in \mathcal{P}Sp(N)} (\text{End } L(\mathbf{n}))^*, \end{aligned}$$

where we set $\mathcal{P}O(2l+1) = \mathcal{P}(B_l)$, $\mathcal{P}O(2l) = \mathcal{P}(D_l)$.

(2) The pairings $\langle, \rangle : \tilde{U}_X \otimes \tilde{A}_X \rightarrow \mathbf{C}$, $U_X \otimes A_X \rightarrow \mathbf{C}$ define Hopf algebra injections $\tilde{A}_X \rightarrow \tilde{U}_X^\circ$, $A_X \rightarrow U_X^\circ$.

Proof. We will prove this theorem for $A(SO_q(N))$. Other cases are similar. Let A_s be the image of $\bigoplus_{0 \leq r \leq s} S_r(E)$ by the projection $S(E) \rightarrow A_D$.

Step 1. We will determine the structure of the comodule $\tilde{L}(\lambda; r)_{A_s}$ for each $r \leq s$ and $\lambda \in \mathcal{P}_r(D_l)$. Let L be a subcomodule of $\tilde{L}(\lambda; r)_A$. Since $\omega(L) \subset L \otimes A$ and

$$\omega(L) \subset \omega(\hat{L}(\lambda; r)_A) \subset \hat{L}(\lambda; r) \otimes A_r \subset \hat{L}(\lambda; r) \otimes A_s,$$

we have $\omega(L) \subset L \otimes A_s$. Hence L becomes a subcomodule of $\tilde{L}(\lambda; r)_{A_s}$. On the other hand, since A_s is a subcoalgebra of A , each subcomodule of $\tilde{L}(\lambda; r)_{A_s}$ naturally becomes a subcomodule of $\tilde{L}(\lambda; r)_A$. Thus, the comodule $\tilde{L}(\lambda; r)_{A_s}$ is

completely reducible, and its irreducible decomposition is the same as those of $\tilde{L}(\lambda; r)_A$.

Step 2. We will show that A_s^* is a multi-matrix algebra, that is, it is isomorphic to a direct sum of matrix algebras. Since the ground field \mathcal{C} is algebraically closed, it suffices to show that the Jacobson radical J of A_s^* is 0. Let Φ be the representation

$$\Phi : A_s^* \hookrightarrow \bigoplus_{r=0}^s \text{Sch}_r(V) \hookrightarrow \text{End} \left(\bigoplus_{r=0}^s V^{\otimes r} \right).$$

By Corollary 4.4 (1) and step 1, the A_s^* -module $\bigoplus_{0 \leq r \leq s} V^{\otimes r}$ is completely reducible. Hence we have $\Phi(J)=0$. Since Φ is injective, this implies $J=0$.

Step 3. Let M and N be irreducible components of $(\bigoplus_{0 \leq r \leq s} V^{\otimes r})_{A_s}$. Since A_s is a subcoalgebra of A , $M \simeq N$ if and only if $M_A \simeq N_A$. Hence the irreducible decomposition of $(\bigoplus_{0 \leq r \leq s} V^{\otimes r})_{A_s}$ is the same as those of $(\bigoplus_{0 \leq r \leq s} V^{\otimes r})_A$. On the other hand, since A_s^* is a sub multi-matrix algebra of $\text{End}(\bigoplus_{0 \leq r \leq s} V^{\otimes r})$, each of its irreducible modules is isomorphic to a submodule of $\bigoplus_{0 \leq r \leq s} V^{\otimes r}$. Thus we get $A_s^* \simeq \bigoplus_{n \in P\langle s \rangle} (\text{End } L_A(\mathbf{n}))$, where $P\langle s \rangle = \{ \mathbf{n}(\lambda), \mathbf{n}(\lambda)^\sigma \mid \lambda \in \mathcal{P}_r(D_i) \ (0 \leq r \leq s) \}$. Since $A_D = \varinjlim A_s$ as a coalgebra, we obtain the decomposition of part (1).

Step 4. Let $\Psi : U \rightarrow A_s^*$ be the map induced by the pairing $\langle \cdot, \cdot \rangle : U \otimes A_s \hookrightarrow U \otimes A \rightarrow \mathcal{C}$. It is easy to see that the following diagram is commutative:

$$\begin{array}{ccc} & U & \\ \Psi \swarrow & & \searrow \\ A_s^* & \xrightarrow{\quad} & \text{End} \left(\bigoplus_{r=0}^s V^{\otimes r} \right). \end{array}$$

From the structure of $(\bigoplus_{0 \leq r \leq s} V^{\otimes r})_U$, it follows that the image of the representation $U \rightarrow \text{End}(\bigoplus_{0 \leq r \leq s} V^{\otimes r})$ is isomorphic to $\bigoplus_{n \in P\langle s \rangle} (\text{End } L_A(\mathbf{n}))$. Therefore the map Ψ is surjective.

Step 5. Let $a \in A_D$ be an element of the kernel of the map $A_D \rightarrow U_D \hookrightarrow U_D^*$. Let s be an integer such that $a \in A_s$. By step 4 and $\langle U_D, a \rangle = 0$, we have $a=0$. Thus we get (2). □

Note. Let K be an arbitrary field and q ($X=C, D$) or $q^{1/2}$ ($X=B$) be a non-zero element of K such that $q^2+1 \neq 0$. Let $E_q(X_i)$ ($X=B, C, D$) be the quantum matrix on a YB -pair defined again by (2.6) and (4.3). Then the quantum matrix $S(E_q(X_i))$ has the group-like elements quad_q and det_q corresponding to the one-dimensional comodules $\text{Im } \iota_q$ and $\mathcal{Q}(V_q(X_i))$. Since the elements $y_{i,j}$ defined by (5.7) still form a cofactor with respect to quad_q , we can define Hopf algebras A_X and \tilde{A}_X again by (6.1) (see Proposition 3.1. (1)). It is

unclear whether these are proper q -analogues of the function algebras of classical groups.

§7. Symmetric and Hermitian Yang-Baxter Operators

Let (V, β_V) be a YB -pair and $E, \{u_i\}, E_{ij}$ etc. be as before. We define an antialgebra automorphism J on $\text{End}(V^{\otimes r})$ by

$$J(E_{i_1j_1} \otimes E_{i_2j_2} \otimes \cdots \otimes E_{i_rj_r}) = E_{j_1i_1} \otimes E_{j_2i_2} \otimes \cdots \otimes E_{j_ri_r}. \tag{7.1}$$

We say that (V, β_V) is *symmetric* if $J(\beta_V) = \beta_V$. It is easy to see that the YB -pair $V_q(X_i)$ is symmetric for each $q \in \mathbb{C}^\times$ and $X = A, B, C, D$. Let (V, β_V) be a symmetric YB -pair. Since $J(a)\beta_i = J(\beta_i a) = \beta_i J(a)$ ($1 \leq i < r$) for $a \in \text{Sch}_r(V)$, the map J defines an antialgebra automorphism on $\text{Sch}(V)$, which we denote again by J (cf. [6, Chapter 2.7]). We define $J : S(E) \rightarrow S(E)$ by $\langle J(a), x \rangle = \langle a, J(x) \rangle$ ($a \in \text{Sch}(V), x \in S(E)$). This map is characterized as the algebra anticoalgebra automorphism of $S(E)$ satisfying $J(x_{ij}) = x_{ji}$. The following result on quantum determinants corresponds to the fact $\det [a_{ij}] = \det [a_{ji}]$.

Proposition 7.1. *Let g be a group-like element of $S(E_q(X_i))$ ($X = A, B, C, D$) and suppose $q \in \mathbb{C}^\times$ be transcendental over \mathbb{Q} . Then we have $J(g) = g$.*

Proof. For each $r \geq 0$, J induces a permutation of the group-like elements of $S_r(E)$. Hence this proposition follows easily from Theorem 5.6 (4). □

Let (V, β_V) be a YB -pair on \mathbb{C} . We call (V, β_V) *Hermitian* if $\beta_V^* = \beta_V$, where for $r \geq 0$, $*$ denotes an antilinear antialgebra automorphism on $\text{End}(V^{\otimes r})$ defined by (7.1). The YB -pair $V_q(X_i)$ is Hermitian if $q \in \mathbb{R}$. Similarly to a symmetric YB -pair, for a Hermitian YB -pair V , we have an antilinear antialgebra coalgebra automorphism $*$ on $\text{Sch}(V)$.

Proposition 7.2. *If (V, β_V) is a Hermitian YB -pair, then $\text{Sch}_r(V)$ ($r \geq 0$) is a semisimple algebra.*

Proof. Let $(\cdot | \cdot)$ be a Hermitian form on $V^{\otimes r}$ such that $\{u_{i_1} \otimes \cdots \otimes u_{i_r}\}$ is an orthonormal basis. Since $(V^{\otimes r}, (\cdot | \cdot))$ is a unitary representation of $\text{Sch}_r(V)$, it is completely reducible. Since $\text{Sch}_r(V) \subset \text{End}(V^{\otimes r})$, the Jacobson radical of $\text{Sch}_r(V)$ must be 0. □

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