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Invariants of 3-Manifolds Associated with Quantum Groups and Verlinde's Formula

By

Toshie TAKATA*

Abstract

We obtain a projectively linear representation of $SL(2, \boldsymbol{Z})$ from invariants defined by Reshetikhin and Turaev and prove 'Verlinde's formula' for $SU(2)$ based on the computation of invariants. Using an algebra associated with 'Ising model', we constract invariants of links and 3-manifolds.

Introduction

In [16], Witten obtained new topological invariants of closed 3-manifolds and links in 3-manifolds from the quantum field theory. Shortly afterwards, in [13], Reshetikhin and Turaev defined related invariants of closed oriented 3-manifolds and links in such 3-manifolds, by means of representations of quantum groups. More precisely, they use quantized universal enveloping algebra $U_q(sl(2, \boldsymbol{C}))$, which is a q-deformation of the universal enveloping algebra of $sl(2, \boldsymbol{C})$ discovered independently by Drinfeld [1] and Jimbo ([3], [4]). The algebra $U_q(sl(2, \boldsymbol{C}))$ has a structure of a Hopf algebra. Reshetikhin and Turaev introduced the additional structure in the case $q=\exp(2m\pi\sqrt{-1}/r)$ called a 'modular' Hopf algebra to define invariants of 3-manifolds. They obtain invariants of 3-manifolds as a combinational formula using invariants of framed link associated with the algebra $U_q(sl(2, \boldsymbol{C}))$. This is based on the fact that any closed connected oriented 3-manifold is obtained by Dehn surgery [12] of S^3 along a framed link [9].

As an application of the invariants, we construct a projectively linear representation of $SL(2, \boldsymbol{Z})$. Let $Z(T^2)$ be an $(r-1)$-dimensional vector space over $\boldsymbol{C}$ and $\{e_i\}_{i=0}^{r-2}$ a basis of the vector space $Z(T^2)$ and we associate to a basis element e_i a solid torus U_i which has a link in the interior. Gluing such two solid tori U_i and U_j by an element X of the mapping class group of the torus T^2, we obtain a closed 3-manifold M_X with a link. We denote the invariant of the resulting manifold M_X by X_{ij}. We define an action ρ of $SL(2, \boldsymbol{Z})$

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* Department of Mathematics, Kyushu University 33, Fukuoka 812, Japan.

on the vector space $Z(T^2)$ by the formula

$$\rho(X)e_j=\sum_{i=0}^{r-2}X_{ij}e_i \quad (j=0, \cdots, r-2).$$

For generators S and T of $SL(2, \boldsymbol{Z})$, we obtain the equations

$$S_{ij}=\sqrt{\frac{2}{r}}\sin\frac{m(i+1)(j+1)\pi}{r},$$

$$T_{ij}=q^{i(i+2)/4}\delta_{ij}.$$

This matrix (S_{ij}) is the unitary matrix and the representation of $SL(2, \boldsymbol{Z})$ by means of the matrices above was discovered by Kac and Peterson [5] to discribe the modular property of the character of the affine Lie algebra and was also used by Kohno [7] to define invariants of 3-manifolds. The above representation

$$\rho: SL(2, \boldsymbol{Z}) \longrightarrow GL(Z(T^2))/\langle C\rangle,$$

where $\langle C\rangle$ is the cyclic group generated by a root of unity $C=\exp(\sqrt{-1}(-\varphi+(3\pi m/2r)-(\pi/2)))$, is a projectively linear representation. Here φ is determined from the following Gauss sum:

$$\sqrt{2r}\exp(\sqrt{-1}\varphi)=\sum_{k=0}^{2r-1}\exp(\sqrt{-1}\pi k^2m/2r).$$

As an application, we prove 'Verlinde's Formula' for $SU(2)$ [15]. This is given by the following formula:

$$\frac{S_{ij}S_{ik}}{S_{i0}}=\sum_{l=0}^{r-2}S_{il}N_{ljk},$$

where

$$N_{ijk}=\begin{cases}1 & \text{if} \quad |i-j|\leqq k\leqq i+j,\ i+j+k\in 2\boldsymbol{Z},\ i+j+k\leqq 2(r-2)\\ 0 & \text{otherwise.}\end{cases}$$

We verify it by computing the invariant of $S^2\times S^1$ with a link in two ways. The proof is similar to that by Witten [16], but our approach is based on representations of $U_q(sl(2, \boldsymbol{C}))$ with $q=\exp(2m\pi\sqrt{-1}/r)$.

Finally, instead of the above modular Hopf algebra, we consider an algebra associated with 'Ising model'. Recently, it has been discovered that this algebra is related for example to the conformal field theory (see for example [2], [10]), the representation theory of the infinite dimensional Lie algebras. This algebra is an associative algebra with 3 generators 1, σ, ψ whose relations are $\psi\cdot\psi=1$, $\psi\cdot\sigma=\sigma\cdot\psi=\sigma$, $\sigma\cdot\sigma=1+\psi$, and has the conformal dimensions $\Delta_1=0$, $\Delta_\psi=1/2$, $\Delta_\sigma=1/16$. Using this algebra, we define $\boldsymbol{C}$-linear operators for tangle diagrams and construct invariants of framed links. Then, by the same way as in [13], one can obtain invariants of closed oriented 3-manifolds. The topological invariance follows from the invariance under Kirby moves [6].

The paper is organized as follows. In § 1, we review some of the results in [13]. We explain a representation of a modular Hopf algebra and define invariants of links and 3-manifolds derived by Reshetikhin and Turaev. In § 2, using the invariants derived in § 1, we establish a representation of $SL(2, \boldsymbol{Z})$. The action of generators S and T on the vector space $Z(T^2)$ is represented by matrices and it is shown that they satisfy their relations. In § 3, a proof of 'Verlinde's formula' for $SU(2)$ is presented. To compute the invariants, we make use of the idea in § 2. In § 4, an algebra associated with 'Ising model' is described. Based on the algebra, we define invariants of framed links and obtain invariants of 3-manifolds by means of the link invariants by a similar way as in § 1.

§ 1. Review

1.1 Modular Hopf Algebra U_t

In [13], Reshetikhin and Turaev give U_t as an example of 'modular' Hopf algebra. In this paper, we consider the definition of topological invariants of 3-manifolds for this modular Hopf algebra U_t. We explain this modular Hopf algebra U_t. For a non zero $q \in \boldsymbol{C}$, $U_q(sl(2, \boldsymbol{C}))$ is the Hopf algebra which is a q-deformation of the universal enveloping algebra of Lie algebra $sl(2, \boldsymbol{C})$. Let us recall the definition of U_t due to Reshetikhin and Turaev. Let q be a root of unity and $t=\exp(\pi\sqrt{-1}m/2r)$ where m and r are mutually prime integers with odd m, $2r-1 \geqq m \geqq 1$, $r \geqq 2$ and $q=t^4$. We fix an integer r satisfying $r \geqq 2$. We define U_t to be the associative algebra with unit over the cyclotomic field $\boldsymbol{Q}(t)$ with 4 generators K, K^{-1}, X, Y satisfying the following relations:

$$XY-YX=\frac{K^2-K^{-2}}{t^2-t^{-2}} \tag{1.1.1}$$

$$XK=t^{-2}KX,\ YK=t^2KY \tag{1.1.2}$$

$$K^{4r}=1,\ X^r=Y^r=0\,. \tag{1.1.3}$$

The relations (1.1.1), (1.1.2) define the algebra $U_q(sl(2, \boldsymbol{C}))$. The structure of Hopf algebra in $U_q(sl(2, \boldsymbol{C}))$ induces a structure of a Hopf algebra in U_t. The action of comultiplication Δ, counit ε, antipode γ are given on the generators by the following formulas.

$$\Delta(X)=X\otimes K+K^{-1}\otimes X \tag{1.1.4}$$

$$\Delta(X)=Y\otimes K+K^{-1}\otimes Y \tag{1.1.5}$$

$$\Delta(K)=K\otimes K \tag{1.1.6}$$

$$\varepsilon(X)=\varepsilon(Y)=0,\ \varepsilon(K)=1 \tag{1.1.7}$$

$$\gamma(X)=-t^2X,\ \gamma(Y)=-t^{-2}Y,\ \gamma(K)=K^{-1}\,. \tag{1.1.8}$$

The structure of the ribbon Hopf algebra in $U_q(sl(2, \boldsymbol{C}))$ induces a structure of the ribbon Hopf algebra in U_t. Thus U_t has the universal R-matrix $R \in U_t \otimes U_t$ due to Drinfel'd [1] which satisfies Yang Baxter equation, $u \in U_t$ defined from R, and $v \in U_t$ which is a central element of U_t. If we write as sum $R = \sum_i \alpha_i \otimes \beta_i$, then $u = \sum_i \gamma(\beta_i)\alpha_i$ and $v = uK^{-2}$. Moreover, U_t satisfies six axioms (see [13, § 3]) and has a structure of modular Hopf algebra. We describe the representation of modular Hopf algebra U_t. Let I be a finite set of integers $\{0, 1, \cdots, r-2\}$. For an integer $i \in I$, V_i denotes $(i+1)$-dimensional irreducible representation of U_t. It is an $(i+1)$-dimensional U_t-module. The action ρ of the generator K of U_t on V_i has the following matrix representation:

$$(1.1.9) \qquad \rho(K) \longmapsto \begin{pmatrix} t^i & & & \mathbf{0} \\ & t^{i-2} & & \\ & & \ddots & \\ \mathbf{0} & & & t^{-i} \end{pmatrix}.$$

For any U_t-module V_i we provide the dual linear space $V_i^\vee = \mathrm{Hom}_{\boldsymbol{C}}(V, \boldsymbol{C})$ with the action of U_t:

$$\rho_{V_i^\vee}(a) = (\rho_{V_i}(\gamma(a)))^* \in \mathrm{End}\, V_i^\vee .$$

The matrix representation of this action is given by the following matrix:

$$(1.1.10) \qquad \rho_{V_i^\vee}(K) \longmapsto \begin{pmatrix} t^{-i} & & & \mathbf{0} \\ & t^{-i+2} & & \\ & & \ddots & \\ \mathbf{0} & & & t^{i} \end{pmatrix}.$$

Let V_i, V_j be U_t-modules and ρ_{V_i} (resp. ρ_{V_j}) the action of U_t on V_i (resp. V_j). Their tensor product is the U_t-module $V_i \otimes V_j$ equipped with the action of U_t defined by the formula for $a \in U_t$:

$$\rho_{V_i \otimes V_j}(a) = (\rho_{V_i} \otimes \rho_{V_j})(\Delta(a)) .$$

Here Δ is the comultiplication of U_t. One may consider the category $Rep\, U_t$ of finite dimensional linear representations of U_t. The objects of $Rep\, U_t$ are left U_t-modules

$$V_{i_1}^{\varepsilon_1} \otimes \cdots \otimes V_{i_k}^{\varepsilon_k}$$

where $i_l \in I$, $\varepsilon_l \in \{\pm 1\}$, $V_{i_l}^{+1} = V_{i_l}$, $V_{i_l}^{-1} = V_{i_l}^\vee$, $1 \leq l \leq k$. The morphisms of $Rep\, U_t$ are U_t-linear homomorphisms.

Definition 1.1. Let V be an object of $Rep\, U_t$. For any linear operator $f : V \to V$, we define its quantum trace $\mathrm{tr}_q f$ to be the ordinary trace over $\boldsymbol{C}$ of the linear operator

$$f' : V \longrightarrow V, \; f'(x) = \rho(u^{-1}v) f(x) .$$

In particular, if f is the identity map id_V, then we denote $\mathrm{tr}_q id_V$ by $\dim_q V$ and call it the quantum dimension of V. Note that if $V=V_j$, for $j\in I$, then using $v=uK^2$ and (1.1.9), we get

$$\text{(1.1.11)} \qquad \dim_q V_j=\mathrm{tr}_q(id_{V_j})=\mathrm{Tr}(\rho_{V_j}(K^2)id_{V_j}) = \sum_{n=0}^{j} t^{j-2n}=\frac{t^{2j+2}-t^{-2j-2}}{t^2-t^{-2}}=[j+1]$$

where $[n]=(t^{2n}-t^{-2n}/t^2-t^{-2})=(\sin(\pi mn/r)/\sin(\pi m/r))$.

In [13], Reshetikhin and Turaev proved the following theorem.

Theorem 1.2 (Reshetikhin-Turaev). *Let $V_i(i\in I)$ be an irreducible representation of U_t. There exists a decomposition*

$$\text{(1.1.12)} \qquad V_i\otimes V_j=(\oplus_k V_k)\oplus Z_{ij}$$

as a U_t-module, where k satisfies the following conditions

$$\text{(1.1.13)} \qquad |i-j|\leqq k\leqq i+j,\ i+j+k\in 2\boldsymbol{Z},$$

$$\text{(1.1.14)} \qquad i+j+k\leqq 2(r-2).$$

Moreover Z_{ij} is a U_t-module with the following property. For any integers $i, j\in I$ and any U_t-linear homomorphism $f: Z_{ij}\to Z_{ij}$, the quantum trace of f is equal to zero:

$$\text{(1.1.15)} \qquad \mathrm{tr}_q f=0.$$

1.2 Ribbon Graph

An oriented, directed, homogeneous ribbon tangle is a collection of ribbons and annuli as illustrated in Fig. 1 ([13], [14]).

A ribbon (annulus) is oriented if it has an orientation as a surface in $\boldsymbol{R}^3$. By the shaded regions, we express that the tangle is oriented (Fig. 1). A

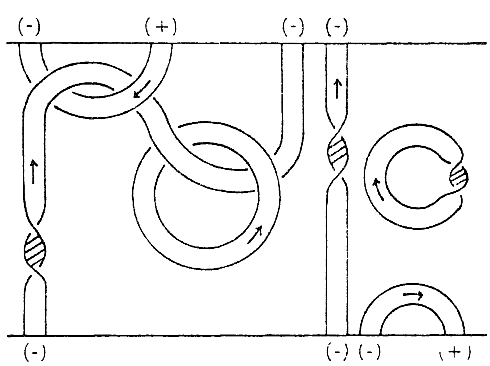


Fig. 1

tangle is homogeneous if each twist of all ribbons and annuli in the tangle is a full twist. A ribbon tangle is directed if the cores of its ribbons and annuli are provided with directions. For each ribbon tangle we assign to each component a finite dimensional irreducible representation V_i of U_t, where i is called its colour. The procedure is called colouring and we denote it by λ. In Fig. 2, elementary coloured ribbon tangles are sketched. We consider ribbons which are called coupons. A small neighborhood of each coupon Q is depicted in Fig. 3, where the rectangle illustrates the coupon. A colour of each coupon is a $\boldsymbol{C}$-linear homomorphism defined from the colours and directions of the ribbons gluing to it. We add coupons to the tangle.

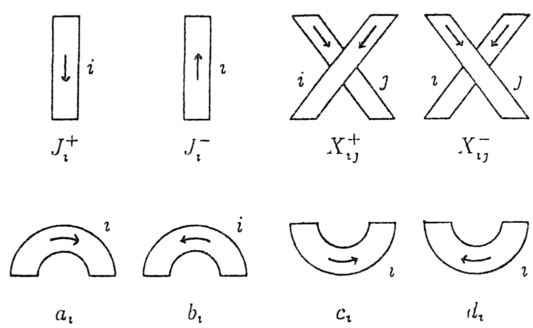


Fig. 2

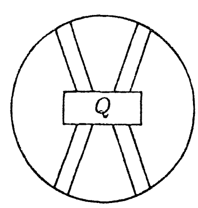


Fig. 3

Let us introduce the category $\mathcal{H}$ of ribbon graphs. The objects of $\mathcal{H}$ are sequences

$$\eta=((i_1, \varepsilon_1), \cdots, (i_k, \varepsilon_k)) \qquad (i_1, \cdots, i_k \in I, \varepsilon_1, \cdots, \varepsilon_k \in \{1, -1\}).$$

We denote the set of such sequences by N. If $\eta, \eta' \in N$, then a morphism $\eta \to \eta'$ is a coloured ribbon graph (considered up to isotopy) such that the sequence of colours and directions of the bottom (resp. top) ribbons is equal to η (resp. η'). The composition $\Gamma' \circ \Gamma$ of such two morphisms $\Gamma: \eta \to \eta'$, $\Gamma': \eta' \to \eta''$ is the ribbon graph obtained by gluing the bottom ends of Γ' with the corresponding top ends of Γ. The tensor product of objects η, η' is their juxtaposition η, η' (see Fig. 4).

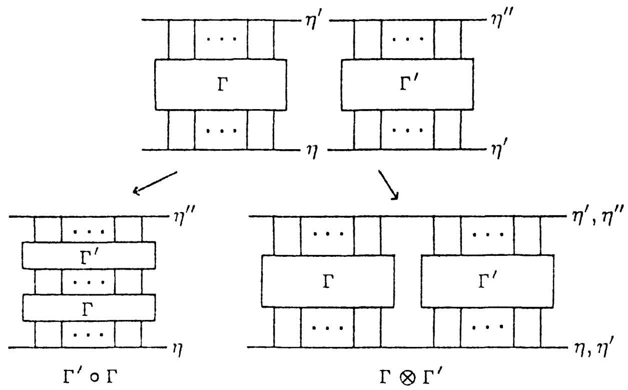


Fig. 4

1.3 Invariants of Closed 3-Manifolds

Reshetikhin and Turaev show that there exists a unique covariant functor from $\mathscr{H}$ to $Rep\, U_t$ with five properties (see § 2.5 in [13]). They define U_t-linear homomorphisms corresponding to elementary coloured ribbon graphs pictured in Fig. 2 and graphs pictured in Fig. 5.

Since the graphs J_i^+, J_i^-, X_{ij}^+, X_{ij}^-, a_i, b_i, c_i, d_i generated the category $\mathscr{H}$, the compositions and tensor products of the corresponding homomorphisms determine $F(\Gamma)$ for a coloured ribbon tangle Γ. In particular, a coloured (0, 0)-ribbon tangle Γ defines $\boldsymbol{C}$-linear homomorphism $\boldsymbol{C}\to\boldsymbol{C}$, i.e. a multiplication by a certain element of $\boldsymbol{C}$. The element is a regular isotopy invariant of Γ. It is also denoted by $F(\Gamma)$.

Example 1.3. Let Γ be a coloured (0, 0)-ribbon tangle in Fig. 6. Then $F(\Gamma)=F(b_i)\circ F(c_i)$ and an easy computation shows $F(\Gamma)=\dim_q V_i$.

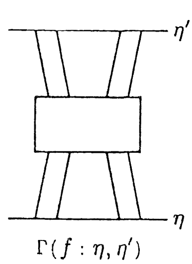


Fig. 5

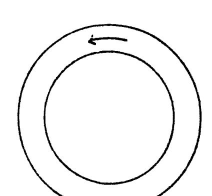


Fig. 6

Let us recall that $\dim_q V_i$ is equal to the quantum trace of identity homomorphism. The following lemma generalizes this computation.

Lemma 1.4. *Let Γ be a coloured (k, k)-ribbon graph which corresponds to an endomorphism of a certain sequence $\eta\in N$. Let L be the coloured* (0, 0)*-ribbon*

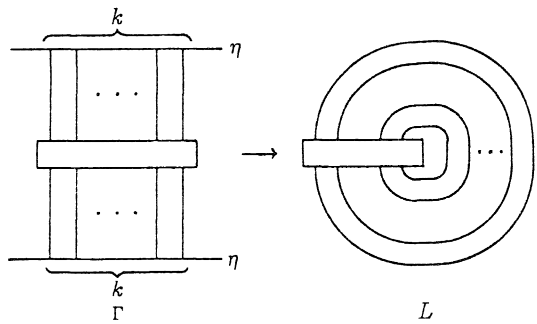


Fig. 7

tangle obtained by closing Γ (see Fig. 7). Then $F(L)=\mathrm{tr}_q F(\Gamma)$.

We introduce the presentation of closed 3-manifolds via framed links. A framed link in the 3-sphere is a finite collection L of disjoint smoothly embedded circles $L_1, \cdots, L_l$ in S^3, each component L_k of L is provided with a framing which is an integer n_k. Let ω be an orientation of L. We may regard each component L_k of the annulus with n_k full twists. This identification gives us a (0, 0)-ribbon tangle $\Gamma(L, \omega)$. The notation ω may be thought of as the directions of the annuli. Let λ be a colouring of $\Gamma(L, \omega)$. Then $F(\Gamma(L, \omega, \lambda))$ is a regular isotopy invariant of coloured (0, 0)-ribbon tangle $\Gamma(L, \omega, \lambda)$. By means of the above results, we define invariants of closed 3-manifolds. The idea of their construction is reduced to the following theorem which relates framed links to closed 3-manifolds.

Theorem 1.5 (**Lickorish** [9]). *Each closed connected oriented 3-manifold can be obtained by Dehn surgery on S^3 along a certain framed link.*

Let M be a closed connected oriented 3-manifold and L a framed link in S^3 with components $L_1, \cdots, L_l$ and framing $n_1, \cdots, n_l$ which can be related to M by the above theorem. Dehn surgery is the following process. We remove an open tubular neighborhood of each L_k and on the resulting total boundary we glue l solid tori such that their meridians are identified with the curves on the boundaries. We consider such a pair (M, L). Let ω be an orientation of the framed link L. By $\mathrm{col}(L)$ we denote the set of colourings of the (0, 0)-ribbon tangle $\Gamma(L, \omega)$. Put

$$F(M, L)=C^{\sigma(L)} \sum_{\lambda\in \mathrm{col}(L)} \prod_{k=1}^{l} d_{\lambda(L_k)} F(\Gamma(L, \omega, \lambda)) \in \boldsymbol{C}. \tag{1.3.1}$$

Here C, $d_k (k=0, \cdots, r-2)$ are constants contained in the data of the modular Hopf algebra U_t and given by the following formulas:

$$C=\exp(-\sqrt{-1}d), \tag{1.3.2}$$

(1.3.3) $$d_k=\sqrt{\frac{2}{r}}\sin\frac{m(k+1)\pi}{r},$$

where

(1.3.4) $$d=\varphi-\frac{3\pi m}{2r}+\frac{\pi}{2},$$

the number φ being determined from the following Gauss sum

(1.3.5) $$\sqrt{2r}\exp(\sqrt{-1}\varphi)=\sum_{k=0}^{2r-1}\exp(\sqrt{-1}\pi k^2 m/2r).$$

The notation $\sigma(L)$ stands for the signature of the linking matrix of the framed link L. We remark that the normalization coincides with that in [8].

Theorem 1.6 (Reshetikhin-Turaev). *For a closed connected oriented 3-manifold M, $F(M, L)$ is a topological invariant of M.*

We may denote $F(M, L)$ by $F(M)$. The invariant is multiplicative with respect to a connected sum:

(1.3.6) $$F(M_1\# M_2)=F(M_1)F(M_2).$$

We have the following relations between invariants with opposite orientations

$$F(M)=\overline{F(-M)},$$

where the bar is the complex conjugation.

Example 1.7. The formula (1.3.6) implies that $F(S^3)=1$.

Since $S^2\times S^1$ is obtained by Dehn surgery on S^3 along an unknotted circle with framing 0, we have

(1.3.7) $$F(S^2\times S^1)=\sum_{i=0}^{r-2} d_i \dim_q V_i = \sqrt{\frac{r}{2}}\left(\sin\frac{m\pi}{r}\right)^{-1}.$$

Here we used the equation $\dim_q V_i=\sin(m(i+1)\pi/r)/\sin(m\pi/r)$. In the case $m=1$, $F(S^2\times S^1)$ is equal to Kohno's invariant $\phi_K(S^2\times S^1)$ with $K=r+2$.

Let M be a closed connected oriented 3-manifold and T be a coloured (0, 0)-ribbon tangle in M. As above, let us present M as the result of surgery on S^3 along a framed link L with components $L_1, \cdots, L_l$. The ribbon tangle $T\cup\Gamma(L, \omega, \lambda)$ may be thought of as a coloured (0, 0)-ribbon tangle in S^3. We put

(1.3.8) $$F(M, T)=C^{\sigma(L)}\sum_{\lambda\in col(L)}\prod_{k=1}^{l} d_{\lambda(L_k)}F(T\cup\Gamma(L, \omega, \lambda)).$$

In particular, we have $F(S^3, T)=F(T)$.

§2. A Representation of $SL(2, \boldsymbol{Z})$

Using the invariants defined in §1, we establish a projectively linear representation of $SL(2, \boldsymbol{Z})$. Let M_1 be the mapping class group of torus T^2. We fix a basis a, b in $H_1(T^2) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$ as depicted in Fig. 8.

The group M_1 may be canonically identified with $SL(2, \boldsymbol{Z})$. A presentation of $SL(2, \boldsymbol{Z})$ is given by

$$SL(2, \boldsymbol{Z}) = \langle S, T\,;\ S^4 = I,\ (ST)^3 = S^2 \rangle, \tag{2.1}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $Z(T^2)$ be an $(r-1)$-dimensional vector space over $\boldsymbol{C}$ and $\{e_0, e_1, \cdots, e_{r-2}\}$ a basis of the vector space. We associate to each e_i a solid torus U_i with an annulus T_i in the interior, depicted in Fig. 9. We suppose that the colour of annulus T_i is $i \in \{0, \cdots, r-2\}$ and the direction as in Fig. 9. We construct a projectively linear representation

$$\rho:\ SL(2, \boldsymbol{Z}) \longrightarrow GL(Z(T^2))/\langle C \rangle,$$

where C is given by (1.3.2) and $\langle C \rangle$ means the cyclic group generated by $C \cdot I$, when I denotes the identity matrix.

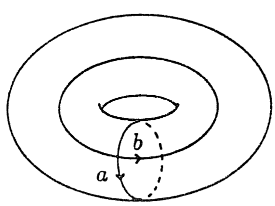


Fig. 8

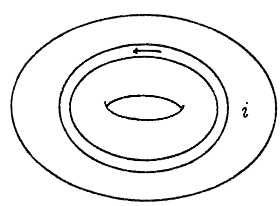


Fig. 9

For any element X of $SL(2, \boldsymbol{Z})$, put

$$\rho(X) e_j = \sum_{i=0}^{r-2} X_{ij} e_i. \tag{2.2}$$

Let $[h]$ be an isotopy class in M_1 corresponding to X. The map h is a degree 1 homeomorphism $T^2 \to T^2$. We identify ∂U_i and ∂U_j using h. The resulting closed connected 3-manifold with the (0, 0)-ribbon tangle consisting of two annuli T_i, T_j is denoted by M_X. Then X_{ij} in (2.2) is defined by the following formula:

$$X_{ij} = F(M_X, T_i \cup T_j)/F(S^2 \times S^1). \tag{2.3}$$

Clearly, it follows from the definition that X_{ij} does not depend on the choice of the representative element of the isotopy class.

Theorem 2.1. *The following homomorphism constructed above is a projectively linear representation.*

$$\rho:\ SL(2, \boldsymbol{Z}) \longrightarrow GL(Z(T^2))/\langle C \rangle,$$

where $\langle C\rangle$ *means the cyclic group generated by* $C\cdot I$ *in* $GL(Z(T^2))$ *with* C *given by* (1.3.2). *The values of* S_{ij}, I_{ij} *and* T_{ij} *are given by the following formulas*:

$$S_{ij}=\sqrt{\frac{2}{r}}\sin\frac{m(i+1)(j+1)\pi}{r},$$

$$I_{ij}=\delta_{ij},$$

$$T_{ij}=t^{i(i+2)}\delta_{ij}.$$

Proof. Let C_h be the mapping cylinder of the homeomorphism $h: T^2\to T^2$. We parametrize $T^2\times\{0\}$ via the identity and $T^2\times\{1\}$ via h. Using the parametrization, we glue solid tori U_i and U_j to C_h. Then we obtain a closed 3-manifold M_X with two annuli T_i, T_j. If isotopy classes $[h]$, $[g]\in M_1$, then the cylinder $C_{h\circ g}$ splits into a composition of C_g and C_h. Let X (resp. Y) be an element $SL(2, C)$ corresponding to the isotopy class $[h]$ (resp. $[g]$). Gluing U_i and U_j to the composition C_hC_g, we obtain a closed 3-manifold M_{XY}. Let L_X (resp. L_Y) be a framed link in S^3 which we obtain M_X (resp. M_Y) by Dehn surgery along. We consider $(M_X, T_i\cup T_k)$, $(M_Y, T_k\cup T_j)$ and $(M_{XY}, T_i\cup T_j)$. Connecting the annulus T_k in $L_X\cup T_k\cup T_i$ in S^3 with the annulus T_k in $L_Y\cup T_k\cup T_j$, a new ribbon tangle $L_X\cup L_Y\cup L_0\cup T_i\cup T_j$ is constructed, where L_0 is a circle determined by T_k. We denote it by L. We can get $(M_{XY}, T_i\cup T_j)$ by Dehn surgery along the framed link L in S^3. From the definition of d_k and a homomorphism corresponding to the composition of tangles, it follows that

$$F(M_{XY}, T_i\cup T_j)=C^n\sum_{k=0}^{r-2}F(M_X, T_i\cup T_k)F(M_Y, T_k\cup T_j)$$

with $n=\sigma(L)-\sigma(L_X)-\sigma(L_Y)$. This shows that $\rho(XY)=C^n\rho(X)\rho(Y)$.

Let us compute S_{ij}, I_{ij}, and T_{ij}.

(1) the case $X=S$

M_S is the 3-sphere S^3. Two annuli T_i, T_j are linked in M_S and make up the Hopf link (see Fig. 10).

Therefore we get $F(M_S, T_i\cup T_j)=F(T_i\cup T_j)$. One computes

$$F(T_i\cup T_j)=\sin\frac{m(i+1)(j+1)\pi}{r}\Big/\sin\frac{m\pi}{r}. \tag{2.4}$$

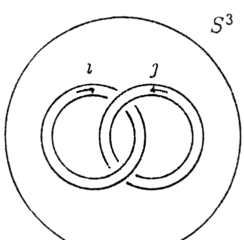


Fig. 10

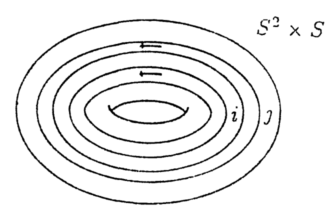


Fig. 11

Applying (2.3) with (1.3.7) and (2.4), we get

$$S_{ij}=\sqrt{\frac{2}{r}}\sin\frac{m(i+1)(j+1)\pi}{r}. \tag{2.5}$$

(2) the case $X=I$

M_I is $S^2\times S^1$. In M_I, T_i and T_j are unlinked unknotted annuli with no twists (see Fig. 11). Let us consider S^3 with the above annuli and the unknotted circle L that links a pair of the annuli and that has the zero framing as illustrated in Fig. 12a.

The Dehn surgery on S^3 along L produces $S^2\times S^1$ with T_i and T_j depicted in Fig. 11. To calculate $F(T_i\cup T_j\cup\Gamma(L, \omega, \lambda))$, we can use the formula (1.1.2)

$$V_i\otimes V_j=(\oplus_k V_k)\oplus Z_{ij}.$$

Let us replace T_i and T_j with a unknotted annulus T_k which runs parallel to T_i and T_j (Fig. 12b). We assume that T_k has a colour k and the same direction as two annuli. Then $T_k\cup\Gamma(L, \omega, \lambda)$ is a (0, 0)-ribbon tangle in S^3.

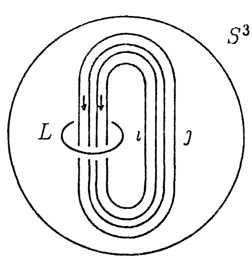


Fig. 12a

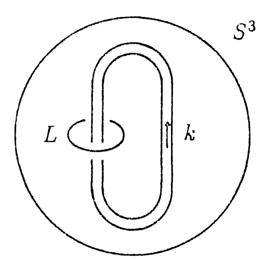


Fig. 12b

The property (1.1.15) of the U_t-module Z_{ij} ensures the equation

$$F(T_i\cup T_j\cup\Gamma(L, \omega, \lambda))=\sum_k F(T_k\cup\Gamma(L, \omega, \lambda)), \tag{2.6}$$

where the summation runs over k satisfying (1.1.13) and (1.1.14). As $T_k\cup \Gamma(L, \omega, \lambda)$ is the Hopf link, we can apply (2.4) to the computation of $F(T_k\cup \Gamma(L, \omega, \lambda))$. If $\lambda(L)=l$, then we obtain

$$F(T_k\cup\Gamma(L, \omega, \lambda))=F(S^2\times S^1)\sqrt{\frac{2}{r}}\sin\frac{m(k+1)(l+1)\pi}{r}. \tag{2.7}$$

Thus, we get

$$I_{ij}=\frac{1}{F(S^2\times S^1)}\sum_{l=0}^{r-2} d_l\left(\sum_k F(S^2\times S^1)\sqrt{\frac{2}{r}}\sin\frac{m(k+1)(l+1)\pi}{r}\right),$$

where k satisfies the conditions (1.1.13) and (1.1.14). We have the following formula:

$$\sum_{l=0}^{r-2}\sin\frac{m(i+1)(l+1)\pi}{r}\sin\frac{m(l+1)(j+1)\pi}{r}=\frac{r}{2}\delta_{ij}. \tag{2.8}$$

Using (2.8), we show the formula:

$$I_{ij}=\frac{2}{r}\sum_{k}\frac{r}{2}\delta_{0k}.$$

The condition (1.1.13) of k asserts that k is equal to zero if and only if $i=j$. Therefore we get

$$I_{ij}=\delta_{ij}. \tag{2.9}$$

(3) the case $X=T$

M_T is also $S^2\times S^1$. But the unknotted annulus T_i with no twists links the unknotted annulus T_j with one full twist (Fig. 13). To obtain $(M_T, T_i\cup T_j)$, we start from S^3 with the two above annuli T_i and T_j and with an unknotted circle L which has the zero framing and which links them (Fig. 14a). Carrying out the Dehn surgery on S^3 along the circle L turns S^3 into $M_T\cong S^2\times S^1$.

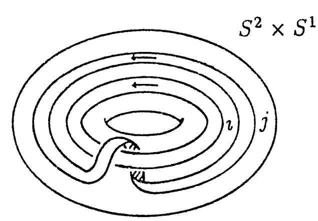


Fig. 13

One claims that we can use of the idea of the case $X=I$ to calculate $F(T_i\cup T_j\cup\Gamma(L, \omega, \lambda))$. We deform the annulus T_i adding the same twist as the annulus T_j. One denotes the resulting annulus by T_i'. The computaion in [13, the proof of Lemma 7.1] implies

$$F(T_i'\cup T_j\cup\Gamma(L, \omega, \lambda))=(v_i)^{-1}F(T_i\cup T_j\cup\Gamma(L, \omega, \lambda)),$$

where $v_i=t^{i(i+2)}$. A full twist can be expressed by a curl (Fig. 14b). It follows from it that we can turn $T_i'\cup T_j$ into two parallel annuli with no twists (Fig. 14c).

Let T_k be an annulus of colour k provided with the same twist and direction as two annuli. We replace two annuli by T_k (Fig. 14d).

Then, applying Theorem 1.2, one gets the following equation

$$F(T_i'\cup T_j\cup\Gamma(L, \omega, \lambda))=\sum_{\substack{k\\ |i-j|\leq k\leq i+j\\ i+j+k\in 2\mathbf{Z}\\ i+j+k\leq 2(r-2)}} F(T_k\cup\Gamma(L, \omega, \lambda)),$$

Thus

$$T_{ij}=\frac{1}{F(S^2\times S^1)}\sum_{l=0}^{r-2} d_l v_i \sum_k F(T_k\cup\Gamma(L, \omega, \lambda))$$

here $\lambda(L)=l$. Substituting $v_i=t^{i(i+2)}$, we obtain

$$T_{ij}=t^{i(i+2)}\delta_{ij}. \tag{2.10}$$

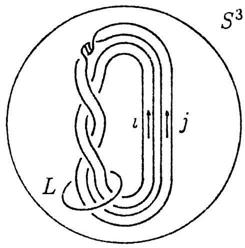


Fig. 14a

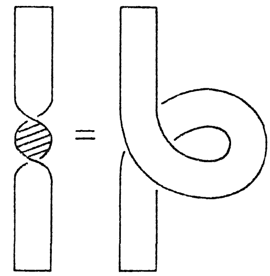


Fig. 14b

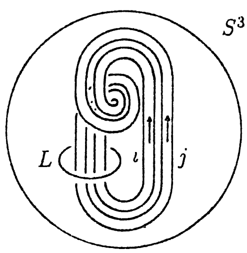


Fig. 14c

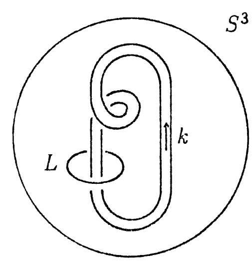


Fig. 14d

We put $I_{id}=(I_{ij})$, $S=(S_{ij})$ and $T=(T_{ij})$. They are $(r-1)\times(r-1)$ matrices.

We show the following:

$$S^4=I_{id} \quad \mathrm{mod}\ C\cdot I \tag{2.11}$$

$$(ST)^3=S^2 \quad \mathrm{mod}\ C\cdot I\,. \tag{2.12}$$

One easily computes

$$S^2=I_{id}\,. \tag{2.13}$$

Note that the equation $(ST)^3=S^2$ is equivalent to the equation $STS=T^{-1}ST^{-1}$. It is easy to compute that an (i, j)-entry of $T^{-1}ST^{-1}$ is

$$\sqrt{\frac{2}{r}}\,t^{-i(i+2)-j(j+2)}\sin\frac{m(i+1)(j+1)\pi}{r}\,. \tag{2.14}$$

Using $t=\exp(\pi\sqrt{-1}m/2r)$ and Gauss sum (1.3.5), an (i, j)-entry of STS is

$$C\sqrt{\frac{2}{r}}\,t^{-i(i+2)-j(j+2)}\sin\frac{m(i+1)(j+1)\pi}{r}\,. \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$STS=T^{-1}ST^{-1}\cdot CI_{id}\,. \tag{2.16}$$

(2.13) implies (2.11) and (2.16) implies (2.12). □

§ 3. Proof of Verlinde's Formula

As another application of the invariants given in § 1, we prove 'Verlinde's formula' (see [15]). It is given by the following formula.

$$\frac{S_{ij}S_{ik}}{S_{i0}}=\sum_{l=0}^{r-2}S_{il}N_{ljk} \tag{3.1}$$

where m and r are mutually prime integers with odd m, $1\leqq m\leqq 2r-1$, $2\leqq r$, and

$$S_{ij}=\sqrt{\frac{2}{r}}\sin\frac{m(i+1)(j+1)\pi}{r}, \tag{3.2}$$

$$N_{ijk}=\begin{cases}1 & \text{if } |i-j|\leqq k\leqq i+j,\ i+j+k\in 2\boldsymbol{Z},\ i+j+k\leqq 2(r-2)\\ 0 & \text{otherwise.}\end{cases}$$

Proof of Verlinde's formula. Let us consider $S^2\times S^1$ with three parallel non-twisted annuli T_l, T_j, T_k in the interior (see Fig. 15). The directions of them is as in Fig. 15 and the colour of T_l (resp. T_j, T_k) is l (resp. j, k).

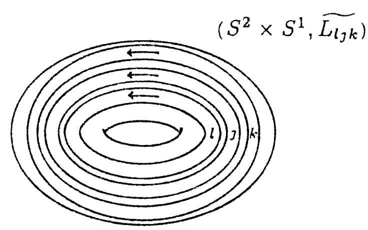


Fig. 15

We call this configuration of three annuli $\widetilde{L_{ljk}}$. The idea of the proof is to evaluate $F(S^2\times S^1, \widetilde{L_{ljk}})$ in two ways.

Let us begin with the surgery representation of $(S^2\times S^1, \widetilde{L_{ljk}})$. Let L be an unknotted circle with the zero framing which links $\widetilde{L_{ljk}}$ in S^3 (Fig. 16a). The Dehn surgery on S^3 along the circle L produces $(S^2\times S^1, \widetilde{L_{ljk}})$.

In the first evaluation, we use an analogue of the computation of I_{ij} and T_{ij} in § 2. We replace T_j and T_k by an unknotted non-twisted annulus T_p with colour p and the same direction as them (Fig. 16b). Then applying Theorem 1.2 with i replaced by l, we obtain the following equation:

$$F(\widetilde{L_{ljk}}\cup\Gamma(L, \omega, \lambda))=\sum_p F(T_l\cup T_p\cup\Gamma(L, \omega, \lambda)).$$

Here p satisfies the conditions (1.1.13) and (1.1.14) replaced i by p.

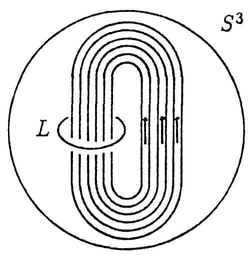


Fig. 16a

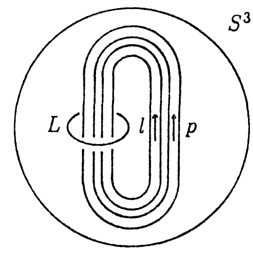


Fig. 16b

Then we can apply the formula (2.9) to the computation. Thus we get

$$F(S^2\times S^1,\ \widetilde{L_{ljk}})=\sum_{t=0}^{r-2} d_t\Big(\sum_p F(T_l\cup T_p\cup\Gamma(L,\ \omega,\ \lambda))\Big)$$

$$=F(S^2\times S^1)\sum_{\substack{p\\ |l-j|\leq p\leq j+k\\ p+j+k\in 2\mathbf{Z}\\ p+j+k\leq 2(r-2)}}\delta_{l,p}\,.$$

It follows from the condition of p that

$$F(S^2\times S^1,\ \widetilde{L_{ljk}})=F(S^2\times S^1)N_{ljk}\,. \tag{3.3}$$

To evaluate $F(S^2\times S^1,\ \widetilde{L_{ljk}})$ in the second way, we rotate the (0, 0)-ribbon tangle $\widetilde{L_{ljk}}\cup\Gamma(L)$ in S^3 (Fig. 17a). The result may be thought of as the closure of the (1, 1)-ribbon tangle B^t_{ljk} illustrated in Fig. 17b. $F(B^t_{ljk})$ is the homomorphism $V_t\to V_t$. Moreover, it may be thought of as the composition of three homomorphisms determined by (1, 1)-ribbon tangles τ^t_l, τ^t_j, τ^t_k illustrated in Fig. 17c.

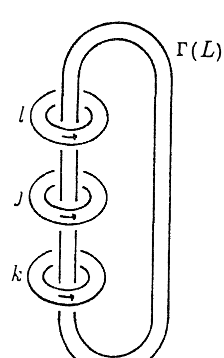


Fig. 17a

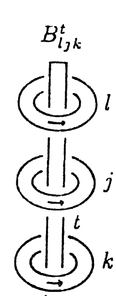


Fig. 17b

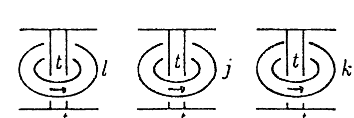


Fig. 17c

The map $F(\tau^t_l)$ is a $\mathbf{C}$-linear homomorphism $V_t\to V_t$ and V_t is irreducible, by Schur's lemma, it is a multiplication by an element of $\mathbf{C}$. We denote this element by b^t_l. Similarly, $F(\tau^t_j)$ (resp. $F(\tau^t_k)$) is a multiplication by an element

b_j^t (resp. b_k^t) of $\boldsymbol{C}$. The closure of the (1, 1)-ribbon tangle τ_l^t makes up the Hopf link. We denote this invariant by s_{tl}. Analogously, the invariant which corresponds to τ_j^t (resp. τ_k^t) is denoted by s_{tj} (resp. s_{tk}). Using (2.4), we derive

$$s_{t\mu}=\sin\frac{m(t+1)(\mu+1)\pi}{r}\Big/\sin\frac{m\pi}{r},$$

where $\mu\in\{l, j, k\}$. Note that $s_{t0}=\dim_q V_t$. Then Lemma 1.5 shows that

$$s_{t\mu}=b_\mu^t\dim_q V_t=b_\mu^t s_{t0}\,. \tag{3.4}$$

The above discussion and (3.6) imply that

$$\begin{aligned} F(B_{ljk}^t)&=\mathrm{tr}_q(F(\tau_l^t)\circ F(\tau_j^t)\circ F(\tau_k^t)) \\ &=b_l^t b_j^t b_k^t \dim_q V_t\,. \end{aligned} \tag{3.5}$$

Using (3.4) and (3.5),

$$\begin{aligned} F(S^2\times S^1,\ \widetilde{L_{ljk}})&=\sum_{t=0}^{r-2} d_t F(B_{ljk}^t)\dim_q V_t \\ &=\sum_{t=0}^{r-2} d_t\frac{s_{tl}s_{tj}s_{tk}}{(s_{t0})^2}\,. \end{aligned} \tag{3.6}$$

Multiplying (3.3) and (3.6) by s_{il} and summing up over $l=0, \cdots, r-2$, we get

$$\sum_{l=0}^{r-2} s_{il}F(S^2\times S^1)N_{ljk}=d_i\Big(\sin\frac{m\pi}{r}\Big)^{-2}\frac{r}{2}\frac{s_{ij}s_{ik}}{(s_{i0})^2}\,. \tag{3.7}$$

We remark that

$$\begin{aligned} d_i&=\sqrt{\frac{2}{r}}\sin\frac{m(i+1)\pi}{r} \\ &=\sqrt{\frac{2}{r}}\,s_{i0}\sin\frac{m\pi}{r}\,. \end{aligned} \tag{3.8}$$

Substituting (3.8) in (3.7), we obtain

$$\sum_{l=0}^{r-2} s_{il}F(S^2\times S^1)N_{ljk}=F(S^2\times S^1)\frac{s_{ij}s_{ik}}{s_{i0}}\,. \tag{3.9}$$

The value S_{ij} is related to s_{ij} by the formula

$$s_{ij}=\sqrt{\frac{r}{2}}\Big(\sin\frac{m\pi}{r}\Big)^{-1}S_{ij}\,.$$

Thus (3.9) implies (3.1). □

§4. Ising model

Instead of the modular Hopf algebra U_t in §1, we consider a fusion algebra A over $\boldsymbol{C}$ corresponding to 'Ising model' [10]. It has generators 1, σ, ψ and their relations are

$$\phi\cdot\phi=1, \quad \phi\cdot\sigma=\sigma\cdot\phi=\sigma, \quad \sigma\cdot\sigma=1+\phi\,. \tag{4.1}$$

It is known that this algebra describes the fusion rule for the critical Ising model (see for example [2]). The algebra A has the conformal dimensions:

$$\Delta_1=0, \quad \Delta_\sigma=\frac{1}{16}, \quad \Delta_\phi=\frac{1}{2}. \tag{4.2}$$

It is analogous to the case $m=1$ and $r=4$ in the algebra U_t. But the element in U_t corresponding to Δ_σ has a different value. Using the algebra A, we construct invariants of links and 3-manifolds. As an application, we obtain a projectively linear representation of $SL(2, \boldsymbol{Z})$ and an equation similar to Verlinde's formula.

Let L be a framed link in S^3 with m components $L_1, \cdots, L_m$ and $\Gamma(L)$ a diagram of L. We assign to each component L_i of L one of generators of A. We denote the assignment by λ, which gives a colouring

$$\{L_1, \cdots, L_m\} \longrightarrow \{1, \sigma, \phi\}.$$

Next, we assign an element (or colour) of the set $\{1, \sigma, \phi\}$ to each region of $\Gamma(L)$. This assignment follows the fusion rule (4.1) in the following sense. We assign 1 to the unbounded region. Let A_1 and A_2 be adjacent regions and the component of L between A_1 and A_2 have colour $j\in\{1, \sigma, \phi\}$. A colour a_i of the region A_i for $i=1, 2$ satisfies the following equation:

$$a_2=\begin{cases} 1 \text{ or } \phi & \text{if } a_1=j=\sigma\,, \\ a_1\cdot j & \text{otherwise.}\end{cases}$$

When a colour of the link L in Fig. 18 is σ, all the colourings of regions are given in Fig. 19.

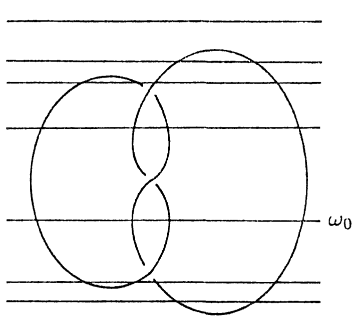


Fig. 18

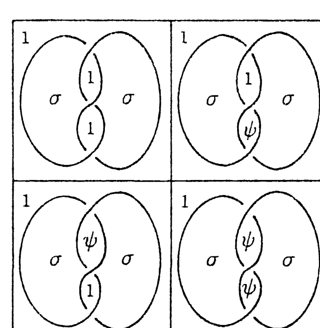


Fig. 19

Any horizontal line ω which avoids crossings and extreme (maximum or minimum) points hits $\Gamma(L)$ in a finite number of points as pictured in Fig. 18.

We assume that the critical points occur in distinct levels. We may then decompose the diagram $\Gamma(L)$ level by level as the composite of a number of elementary diagrams, which each diagram contains just one critical point. In each elementary diagram, one of the four diagrams (b)-(d) shown in Fig. 20 exists, while the rest of the diagrams consists of the strings passing from the top to the bottom without crossings as (a) shown in Fig. 20.

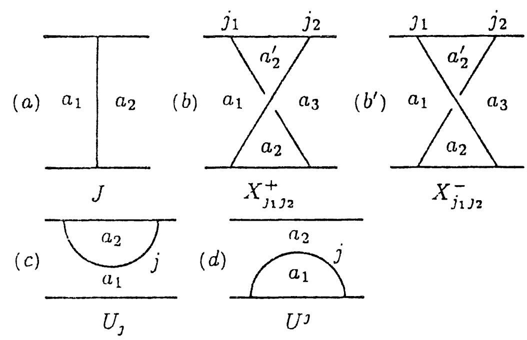


Fig. 20

We associate to any horizontal line ω a vector space V_ω. If the colours of the regions which the line ω passes through are in the order 1, $a_1, \cdots, a_l$, 1, the vector space V_ω has a base element $e_1 \otimes e_{a_1} \otimes \cdots \otimes e_{a_l} \otimes e_1$. If the colour of the link L in Fig. 18 is σ, the vector space V_{ω_0} corresponding to the line ω_0 has a basis

$$\{e_1 \otimes e_\sigma \otimes e_1 \otimes e_\sigma \otimes e_1,\ e_1 \otimes e_\sigma \otimes e_\psi \otimes e_\sigma \otimes e_1\}.$$

Finally, we associate a $\boldsymbol{C}$-linear homomorphism $\boldsymbol{C} \to \boldsymbol{C}$ to the diagram $\Gamma(L)$. Let us denote a linear operator over $\boldsymbol{C}$ for a diagram T by F_T. The operators for elementary diagrams shown in Fig. 20 are determined by the following formulas. The notation a_i (resp. j_i) represents a colour of a region (resp. a string) in Fig. 20.

(a) F_J is the identity homomorphism $F_J: e_{a_1} \otimes e_{a_2} \mapsto e_{a_1} \otimes e_{a_2}$.

(b) Let us consider a tangle diagram pictured in Fig. 21.

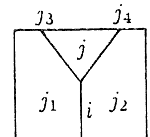


Fig. 21

The operator for this diagram is a multiplication by $F_{ij}\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix}$. The matrix $F\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix}=\left(F_{ij}\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix}\right)_{ij}$ is called 'fusing matrix'. Explicitly, we have

$$F\begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad F\begin{bmatrix} \sigma & \psi \\ \psi & \sigma \end{bmatrix}=F\begin{bmatrix} \psi & \sigma \\ \sigma & \psi \end{bmatrix}=-1$$

and the other fusing matrices $F\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix}=1$. The operator for (b) in Fig. 20 is determined from the fusing matrix and the conformal dimensions Δ_1, Δ_σ, Δ_ψ, and we denote the entry of the matrix for the operator by $B_{ij}\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix}$. The matrix $B\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix}=\left(B_{ij}\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix}\right)_{ij}$ is called 'braiding matrix' and related to the fusing matrix and Δ_1, Δ_σ, Δ_ψ by the following equation:

$$B\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix}=F^{-1}\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix}\varepsilon\, diag_k(\exp \pi\sqrt{-1}(\Delta_k-\Delta_{j_2}-\Delta_{j_3}))F\begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix},$$

where

$$k=\begin{cases} 1 \text{ or } \psi & \text{if } j_2=j_3=\sigma, \\ j_2\cdot j_3 & \text{otherwise,} \end{cases}$$

and

$$\varepsilon=\begin{cases} -1 & \text{if } \{j_2, j_3\}=\{\sigma, \psi\},\ \psi\in\{j_1, j_4\}, \\ 1 & \text{otherwise.} \end{cases}$$

These fusing matrices and braiding matrices satisfy the pentagon relation and the hexagon relation (see [10]).

(1) the case $j_1\neq\sigma$ or $j_2\neq\sigma$

$F_{X^+_{j_1j_2}}$ is a multiplication by scalar $B\begin{bmatrix} j_2 & j_2 \\ a_1 & a_3 \end{bmatrix}=\varepsilon \exp \pi\sqrt{-1}(\Delta_k-\Delta_{j_1}-\Delta_{j_2})$, where $k=j_1\cdot j_2$ and

$$\varepsilon=\begin{cases} -1 & \text{if } \{j_1, j_2\}=\{\sigma, \psi\},\ \psi\in\{a_1, a_3\}, \\ 1 & \text{otherwise.} \end{cases}$$

(2) the case $j_1=j_2=\sigma$

$$B\begin{bmatrix} \sigma & \sigma \\ \sigma & \sigma \end{bmatrix}=\frac{\gamma}{\sqrt{2}}\exp\frac{\pi\sqrt{-1}}{4}\begin{pmatrix} 1 & -\sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix},$$

$$B\begin{bmatrix} \sigma & \sigma \\ 1 & 1 \end{bmatrix}=B\begin{bmatrix} \sigma & \sigma \\ \psi & \psi \end{bmatrix}=\gamma, \qquad B\begin{bmatrix} \sigma & \sigma \\ 1 & \psi \end{bmatrix}=B\begin{bmatrix} \sigma & \sigma \\ \psi & 1 \end{bmatrix}=\gamma\sqrt{-1}.$$

Here $\gamma^2=\exp\left(-\frac{\pi\sqrt{-1}}{4}\right)$.

(b′) $F_{X^{-}_{j_1 j_2}}$ is the inverse map of $F_{X^{+}_{j_1 j_2}}$.

(c)

$$F_{U_j}(e_{a_1})=\begin{cases} e_{a_1}\otimes e_{a_2}\otimes e_{a_1} & \text{if } j\in\{1, \psi\}, \\ \alpha(e_{a_1}\otimes e_{\sigma}\otimes e_{a_1}) & \text{if } j=\sigma,\ a_1\in\{1, \psi\}, \\ \dfrac{\alpha}{\sqrt{2}}(e_{\sigma}\otimes e_1\otimes e_{\sigma}+e_{\sigma}\otimes e_{\psi}\otimes e_{\sigma}) & \text{if } j=a_1=\sigma. \end{cases}$$

where $\alpha^2=\sqrt{2}$.

(d)

$$F_{U^j}(e_{a_1}\otimes e_{a_2}\otimes e_{a_1})=\begin{cases} e_{a_1} & \text{if } j\in\{1, \psi\}, \\ \alpha e_{a_1} & \text{if } j=\sigma,\ a_1\in\{1, \psi\}, \\ \dfrac{\alpha}{\sqrt{2}}e_{a_1} & \text{if } j=a_1=\sigma. \end{cases}$$

We make a list of the above operators in Fig. 22.

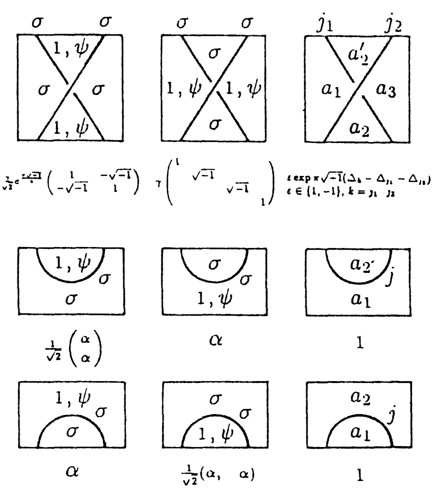


Fig. 22

A homomorphism for any link diagram is a composition of homomorphisms obtained by combining the operator for only one critical point with identity operators for strings passing from the top to the bottom without crossings. By the definition of the operators, for any link diagram $\Gamma(L)$, we obtain a homomorphism $F_{\Gamma(L)}: \boldsymbol{C}\to\boldsymbol{C}$. Thus $F_{\Gamma(L)}$ is a multiplication by some complex number μ. We put $\tau_{L,\lambda}=\mu$.

Theorem 4.1. *For any coloured framed link L and any colouring map λ, $\tau_{L,\lambda}$ is invariant under regular isotopy for the diagram $\Gamma(L)$. Let T_j be a dia-*

gram pictured in Fig. 23. *The homomorphism* F_{T_j} *is a multiplication by* $\exp 2\pi\sqrt{-1}\Delta_j = v_j$.

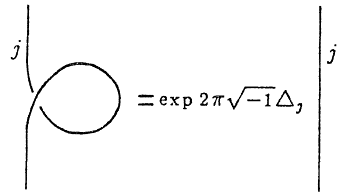


Fig. 23

Proof. By the definition of the operators $F_{X^{\pm}_{jj'}}$, F_{U_j}, F_{U^j}, we can show that F_{T_j} is a multiplication by the $\exp 2\pi\sqrt{-1}\Delta_j$. To prove invariance under regular isotopy, it is enough to show that the homomorphisms defined by the isotopic diagrams in Fig. 24 are equal, for any colouring λ ([8], [11], [17]). We can verify it by an easy calculation. □

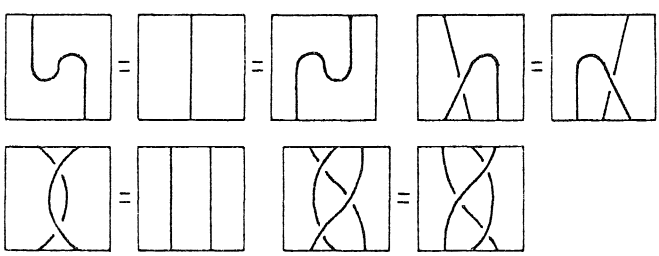


Fig. 24

We give some examples.

Example 4.2. Let L be an unknotted circle with zero framing. If $\lambda(L)=i$, which is an element i of the set $\{1, \sigma, \phi\}$, then we denote $\tau_{L,\lambda}$ by δ_i.

Then we deduce: $\delta_1=\delta_\phi=1$, $\delta_\sigma=\sqrt{2}$.

Example 4.3. Let H be the Hopf link which have zero framings. We assume that one component is assigned with i and another with j. Then we write the invariant of $\Gamma(H)$ by $\tilde{S}_{ij}$. Let us put

$$S_{ij}=\frac{1}{2}\tilde{S}_{ij}$$

and

$$S=\begin{pmatrix} S_{11} & S_{1\sigma} & S_{1\phi} \\ S_{\sigma 1} & S_{\sigma\sigma} & S_{\sigma\phi} \\ S_{\phi 1} & S_{\phi\sigma} & S_{\phi\phi} \end{pmatrix}.$$

We deduce

$$S=\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}.$$

In the same way as in [13], we construct invariants of 3-manifolds, using the above invariants of link diagrams.

Let M be a closed connected oriented 3-manifold and L a framed link with components $L_1, \cdots, L_m$ such that M is homeomorphic to 3-manifold M_L obtained by Dehn surgery of S^3 along L. We denote the set of colourings of $\Gamma(L)$ by $col(L)$ and the signature of the linking matrix of L by $\sigma(L)$. Then put

$$\tau(M;\, L)=C^{\sigma(L)} \sum_{\lambda\in col(L)} \prod_{i=1}^{m} d_{\lambda(L_i)} \tau_{L,\lambda}, \tag{4.2}$$

where C and d_i for $i\in\{1, \sigma, \phi\}$ are complex numbers characterized by

$$\sum_{i\in\{1,\sigma,\phi\}} C d_i \tilde{S}_{ij} v_i = v_j^{-1}\delta_j, \tag{4.3}$$

for any $j\in\{1, \sigma, \phi\}$ and

$$C=\sum_{i\in\{1,\sigma,\phi\}} v_i^{-1}\delta_i d_i. \tag{4.4}$$

Here δ_i is the same as in Example 4.2. More explicitly, we have

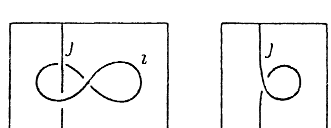
$$d_1=d_\phi=\frac{1}{2}, \qquad d_\sigma=\frac{\sqrt{2}}{2},$$

$$C=\sum_{i\in\{1,\sigma,\phi\}} v_i^{-1}\delta_i d_i=\exp\frac{-\pi\sqrt{-1}}{8}.$$

Theorem 4.4. *Let M be a closed connected oriented 3-manifold and let L be a framed link so that M is the result of surgery along L on S^3. Then $\tau(M;\, L)$ is a topological invariant of M.*

The proof of this theorem is similar to that of Theorem 1.6 (see [13]). To prove Theorem 4.4, we need the following lemma.

Lemma 4.5. *Let $\Gamma_1(i, j)$, $\Gamma_2(j)$ be the cloured (1, 1)-tangles shown in Fig. 25.*



$\Gamma_1(i,j)$ $\qquad$ $\Gamma_2(j)$

Fig. 25

Then

$$\sum_{i\in\{1,\sigma,\psi\}} Cd_i\tau_{\Gamma_1(i,j)}=\tau_{\Gamma_2(j)}\in\mathrm{End}_C(V\otimes V).$$

Proof. From the fact that d_i satisfies the following equation (4.3)

$$\sum_{i\in\{1,\sigma,\psi\}} Cd_i\tilde{S}_{ij}v_i=v_j^{-1}\delta_j,$$

we can obtain this lemma. □

Let us introduce new tangle diagrams and operators corresponding to them. We consider tangle diagrams shown in Fig. 26. The corresponding operators are defined from fusing matrices.

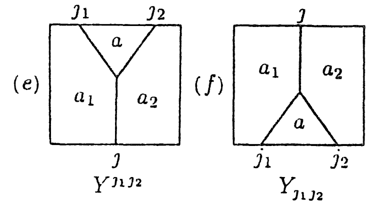


Fig. 26

In Fig. 26, j is equal to 1 or ψ if $j_1=j_2=\sigma$ and $j_1\cdot j_2$ otherwise.

(e)

$$F_{Y^{\sigma\sigma}}(e_\sigma\otimes e_\sigma)=\sum_{a\in\{1,\psi\}} F_{ja}\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}(e_\sigma\otimes e_a\otimes e_\sigma)$$

In other cases,

$$F_{Y^{j_1j_2}}(e_{a_1}\otimes e_{a_2})=F_{ja}\begin{bmatrix}j_1 & j_2\\ a_1 & a_2\end{bmatrix}(e_{a_1}\otimes e_a\otimes e_2)$$

where a is uniquely determined from a_1, a_2, j_1, j_2.

(f) $F_{Y_{j_1j_2}}$ is a multiplication by a scalar.

$$F_{Y_{j_1j_2}}(e_{a_1}\otimes e_a\otimes e_{a_2})=F_{aj}\begin{bmatrix}a_1 & a_2\\ j_1 & j_2\end{bmatrix}(e_{a_1}\otimes e_{a_2}).$$

In terms of the above operators, we obtain the following lemma.

Lemma 4.6. *Suppose $k\geqq 2$. Let Γ_i, $\tilde{\Gamma}_{i,p}$, Γ_2 and $\tilde{\Gamma}_{2,p}$ be tangle diagrams pictured in Fig. 27, where the indices present colours assigned to strings, i.e., elements of the $\{1, \sigma, \psi\}$. Then*

$$F_{\Gamma_i}=\begin{cases} F_{\tilde{\Gamma}_{i\ p}} & \text{if } p=j_1\cdot j_2,\ (j_1, j_2)\neq(\sigma, \sigma)\\ \sum_{p=1,\psi}F_{\tilde{\Gamma}_{i,p}} & \text{if } j_1=j_2=\sigma,\end{cases}$$

$$F_{\Gamma_2}=\begin{cases} F_{\tilde{\Gamma}_{2,p}} & \text{if } p=j_1\cdot j_2,\ (j_1, j_2)\neq(\sigma, \sigma) \\ \sum_{p=1,\phi} F_{\tilde{\Gamma}_{2,p}} & \text{if } j_1=j_2=\sigma. \end{cases}$$

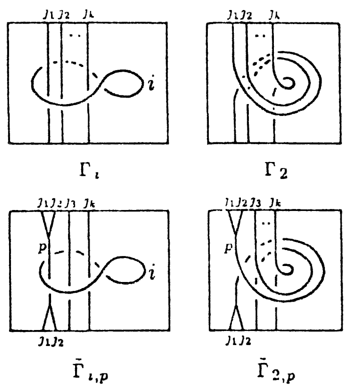


Fig. 27

Proof. The diagram $\tilde{\Gamma}_{i,p}$ is gained by applying the moves depicted in Fig. 28 to Γ_i.

Moreover, in addition to the above moves, we can obtain $\tilde{\Gamma}_{2,p}$ by applying the moves shown in Fig. 28 to Γ_2. Thus it is enough to prove simply invariance of the homomorphism under these moves.

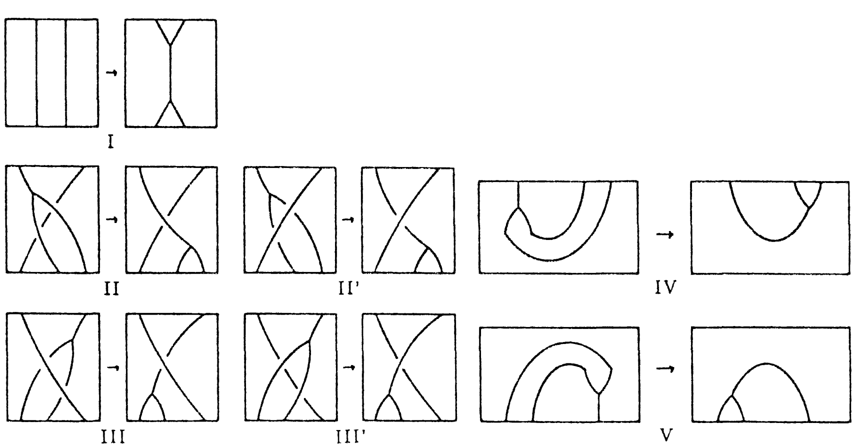


Fig. 28

In the case I, let the colours of the regions and the strings be as in Fig. 29. Then we show that the homomorphism $\sum_{p=1,\phi} F_{T_p}$ is the identity homomorphism. A vector space V_1 with basis $\{e_\sigma \otimes e_{a_2} \otimes e_\sigma\}_{a_2=1,\phi}$ is assigned to the bottom line of the diagram T_p and a vector space V_2 with basis $\{e_\sigma \otimes e_{a'_2} \otimes e_\sigma\}_{a'_2=1,\phi}$ is assigned to the top line. With respect to these basis, the matrix representa-

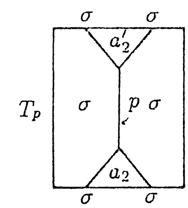


Fig. 29

tions for the homomorphisms $F_{T_1}: V_1 \to V_2$ and $F_{T_\phi}: V_1 \to V_2$ are

$$F_{T_1}: \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad F_{T_\phi}: \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

by the definition of the operators for diagrams $Y_{\sigma\sigma}$ and $Y^{\sigma\sigma}$.

Then the homomorphism $\sum_{p=1,\phi} F_{T_p}$ is the identity. Except for the above case, the invariance of the homomorphisms under move I is trivial.

In the case II, II′, III and III′, we need some more elementary calculations. For example, we consider the case pictured in Fig. 30.

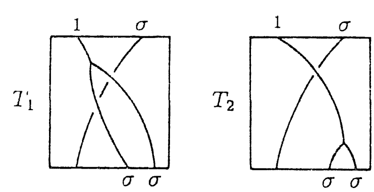


Fig. 30

Let V_1 be a vector space over $\boldsymbol{C}$ with basis $\{e_{i_1} \otimes e_\sigma \otimes e_{i_2} \otimes e_\sigma\}_{i_1, i_2 = 1, \phi}$ and V_2 a vector space over $\boldsymbol{C}$ with basis $\{e_1 \otimes e_1,\ e_\phi \otimes e_\phi\}$. The restriction of F_{T_1} and F_{T_2} to V_1 determines $\boldsymbol{C}$-linear homomorphisms $V_1 \to V_2$.

The matrices for their are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, one gets $F_{T_1}|_{V_1} = F_{T_2}|_{V_2}$. Similarly, comparing the matrix for F_{T_1} with that for F_{T_2} on the restriction of F_{T_1} and F_{T_2}, we can prove that $F_{T_1} = F_{T_2}$.

Let us consider the cases IV and V. The definition of the operators in (e) and (f) implies that for any fusing matrix $F\begin{bmatrix} a & d \\ b & c \end{bmatrix}$,

$$F\begin{bmatrix} a & d \\ b & c \end{bmatrix} = F\begin{bmatrix} d & c \\ a & b \end{bmatrix}.$$

Invariance of the homomorphisms in these cases follows from this. This completes the proof of the lemma. □

Now we are in position to prove the theorem.

Proof of Theorem 4.4. It suffices to verify that two Kirby moves on L do not change $\tau(M; L)$ (see [13]). The invariance under the first move, which is an elimination or insertion of an unknotted component with framing ± 1, can be derived from the definition of scalars C and d_i. The second move is called Kirby (+1)-move, under which two diagrams are related as shown in the two above pictures in Fig. 27. Let us prove invariance under this move. It is enough to show the equation

$$\sum_{i\in\{1,\sigma,\phi\}} C d_i F_{\Gamma_i} = F_{\Gamma_2}\,. \tag{4.6}$$

We show it by induction with respect to k. When $k=1$, (4.6) follows from Lemma 4.5. Suppose that it is true for $k=l$. We modify the diagram Γ_i illustrated in Fig. 26 to $\tilde{\Gamma}_{i,p}$ in Fig. 26. The diagram $\tilde{\Gamma}_{i,p}$ may be regarded as the composition of three diagrams Γ_1, Γ_i', and Γ_3 illustrated in Fig. 31. Similarly, we modify the diagram L_2 in Fig. 31 to the diagram $\tilde{\Gamma}_{2,p}$. The diagram $\tilde{\Gamma}_{2,p}$ may be thought of as the composition of three diagrams Γ_2, Γ_2', Γ_3 in Fig. 31.

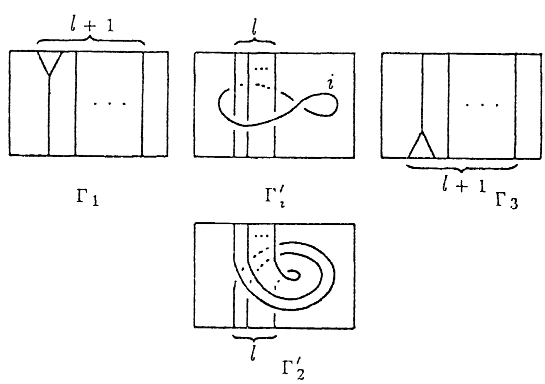


Fig. 31

Then from the assumption of the induction, it follows that

$$\sum_{i\in\{1,\sigma,\phi\}} d_i F_{\Gamma_i'} = F_{\Gamma_2'}\,.$$

Thus

$$\sum_{i\in\{1,\sigma,\phi\}} d_i F_{\tilde{\Gamma}_{i,p}} = F_{\tilde{\Gamma}_{2,p}}\,.$$

Lemma 4.6 implies (4.6). This completes the proof. □

We put $\tau(M; L)=\tau(M)$.

Example 4.7. Since S^3 is obtained from an unknotted circle with framing 1,

$$\tau(S^3)=C\sum_{i\in\{1,\sigma,\phi\}} d_i\delta_i v_i=1\,.$$

Since $S^2\times S^1$ is obtained from an unknotted circle with framing 0,

$$\tau(S^2\times S^1)=\sum_{i\in\{1,\sigma,\phi\}} d_i\delta_i=2\,.$$

Proposition 4.8. *The invariant $\tau(M)$ has the following properties.*

(1) $\tau(M_1\#M_2)=\tau(M_1)\tau(M_2)$

(2) $\tau(-M)=\overline{\tau(M)}$, *where '$-M$' is a 3-manifold M with reversed orientation and the bar is the complex conjugation.*

Proof. For (1), let us M_L be a 3-manifold obtained by Dehn surgery along a framed link L in S^3. Choosing framed links L_1 and L_2 with $M_{L_1}=M_1$, $M_{L_2}=M_2$, we obtain the equation $M_{L_1\cup L_2}=M_1\# M_2$, where $L_1\cup L_2$ denotes disjoint union (L_1 and L_2 are separated by a 2-sphere), We note that for fixed colouring λ_i of L_i ($i=1, 2$),

$$\tau_{L_1\cup L_2,\lambda_1\cup\lambda_2}=\tau_{L_1,\lambda_1}\cdot\tau_{L_2,\lambda_2}\,,$$

where $\lambda_1\cup\lambda_2$ denotes the colouring of $L_1\cup L_2$ induced from the colourings λ_1, λ_2 of L_1, L_2. So (1) follows from the definition τ.

For (2), one knows that $(-M_L)=M_{\bar{L}}$, while $\bar{L}$ is the mirror image of L. Since $\tau_{\bar{L},\lambda}=\overline{\tau_{L,\lambda}}$ from the definition of homomorphisms in (b) and $\sigma(\bar{L})=-\sigma(L)$, one derives (2). □

Remark 4.9. *For the lens spaces* $L(7, 1)$ *and* $L(7, 2)$,

$$\tau(L(7,1))=\exp\frac{3}{4}\pi\sqrt{-1},\qquad \tau(L(7,2))=\exp\left(-\frac{\pi\sqrt{-1}}{4}\right).$$

Thus τ is not a homotopy invariant. In general, for the lens space $L(m, 1)$,

$$\tau(L(m,1))=\begin{cases}\exp\dfrac{(m-1)\pi\sqrt{-1}}{8} & \text{if } m \text{ is odd,}\\[2mm] \exp\left(-\dfrac{\pi\sqrt{-1}}{8}\right)\left(1+\exp\dfrac{m\pi\sqrt{-1}}{8}\right) & \text{if } m \text{ is even.}\end{cases}$$

For the lens space $L(m, 2)$,

$$\tau(L(m,2))=\begin{cases}1 & \text{if } m=4l+1,\ l=1,2,\cdots\\ \exp\left(-\dfrac{\pi\sqrt{-1}}{4}\right) & \text{if } m=4l-1,\ l=1,2,\cdots\end{cases}$$

Let M be a closed connected oriented 3-manifold and T a coloured (0, 0)-tangle diagram in M. For a pair (M, T), put

$$\tau(M,T)=C^{\sigma(L)}\sum_{\lambda\in col(L)}\prod_{i=1}^{m} d_{\lambda(L_i)}\tau_{\Gamma(L)\cup T,\lambda}\,.$$

By the same discussion as in § 2, we may have a projectively linear representation

$$\rho: SL(2,\boldsymbol{Z})\longrightarrow GL(Z(T^2))/\langle C\rangle\,,$$

where corresponding matrices S, T are defined by the formulas

$$S=\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \quad T=\begin{pmatrix} 1 & & \\ & \exp\frac{\pi\sqrt{-1}}{8} & \\ & & -1 \end{pmatrix}.$$

We can also obtain 'Verlinde's formula' for this algebra.

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