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A Remark on the C^∞-Goursat Problem II

By

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§ 1. Introduction and Results

Let us consider the following operator with constant coefficients:

$$\mathscr{L}(\partial_t, \partial_x, \partial_y) = \sum_{i+j+|\alpha|\leqq m} a_{ij\alpha}\partial_t^i\partial_x^j\partial_y^\alpha, \quad a_{ij\alpha}: \text{constant}, \tag{1.1}$$

$$\partial_t = \partial/\partial_t, \qquad \partial_x = \partial/\partial_x, \qquad \partial_y = (\partial/\partial_{y_1}, \partial/\partial_{y_2}, \dots, \partial/\partial_{y_n}).$$

In this paper we assume that the hypersurface $t = 0$ is m_2-tuple characteristic, namely

1) $a_{ij\alpha} = 0$ for $i + j + |\alpha| = m$, $i > m - m_2 \equiv m_1$,
2) $\sum_{j+|\alpha|=m_2} a_{m_1,j,\alpha}\zeta^j\eta^\alpha \neq 0.$

We consider the following Goursat problem:

$$\text{(G)} \quad \begin{cases} \mathscr{L}u = 0, \\ \partial_t^i u(0, x, y) = \phi_i(x, y) \in \mathscr{E}_{(x,y)}, \qquad 0 \leqq i \leqq m_1 - 1, \\ \partial_x^j u(t, 0, y) = \psi_j(t, y) \in \mathscr{E}_{(t,y)}, \qquad 0 \leqq j \leqq m_2 - 1, \\ \qquad (x \in R^1, y \in R^n, t \in R^1_+ \text{ (or } t \in R^1_-)), \\ \text{where} \quad \partial_x^j\phi_i(0, y) = \partial_t^i\psi_j(0, y), \quad 0 \leqq i \leqq m_1 - 1, \quad 0 \leqq j \leqq m_2 - 1. \end{cases}$$

We say that the Goursat problem (G) is $\mathscr{E}$-wellposed for $t \geqq 0$ (or for $t \leqq 0$) if for any data $\{\phi_i\}$ $\{\psi_j\}$ there exists a unique solution $u(t, x, y) \in \mathscr{E}_{(t,x,y)}$ for $t \geqq 0$ (or $t \leqq 0$). If the Goursat problem is $\mathscr{E}$-wellposed for $t \geqq 0$ (or for $t \leqq 0$) then the linear mapping $\{\{\phi_i\}, \{\psi_j\}\} \to u(t, x, y)$ is continuous from $\prod \mathscr{E}_{(x,y)} \times \prod \mathscr{E}_{(t,y)}$ into $\mathscr{E}_{(t,x,y)}$ for $t \geqq 0$ (or for $t \leqq 0$). T. Nishitani [3] had considered the following operator:

$$\text{(N)} \qquad \sum_{\substack{i+j+|\alpha|\leqq m \\ i\leqq m-m_2}} a_{ij\alpha}\partial_t^i\partial_x^j\partial_y^\alpha, \qquad a_{m-m_2,m_2,0} \neq 0,$$

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and he had obtained a necessary and sufficient condition for $\mathscr{E}$-wellposedness. For this operator (N), we obtained Levi condition [2]. Let us call the operator (N) which was treated by Nishitani "N-type". In this paper we will show that if (G) is $\mathscr{E}$-wellposed then $\mathscr{L}$ is N-type.

Remark 1.1. "Operator $\mathscr{L}$ is N-type" means that

$$a_{m_1,m_2,0} \neq 0 \qquad \text{and} \qquad a_{i,j,\alpha} = 0 \qquad \text{for} \quad i > m_1 .$$

Theorem. *If the Goursat problem* (G) *is $\mathscr{E}$-wellposed for $t \geqq 0$ and for $t \leqq 0$ then $\mathscr{L}$ is* N-*type.*

Remark 1.2. In [1] we proved this theorem under some assumptions. The paper "A remark on the C^∞-Goursat problem I" does not exist, but we regard [1] as "A remark on the C^∞-Goursat problem I".

Proposition 1.3. *If* (G) *is $\mathscr{E}$-wellposed for $t \geqq 0$ then $a_{m_1,m_2,0} \neq 0$.*

The proof is given in [1].

§2. Simple Case (where $\mathscr{L}$ does not include ∂_y)

At first we treat the operator which does not include ∂_y. Taking account of Proposition 1.3, we consider the following operator:

$$L(\partial_t, \partial_x) = \partial_t^{m_1}\partial_x^{m_2} - \sum_{i+j\leqq m} a_{ij}\partial_t^i\partial_x^j , \tag{2.1}$$

$$m = m_1 + m_2 , \qquad a_{ij}: \text{constant} , \qquad a_{ij} = 0 \qquad \text{for} \quad i + j = m, \quad i \geqq m_1 ,$$

$$L(\partial_t, \partial_x)u = 0. \tag{2.2}$$

Let the following Goursat problem be (G′):

$$(\mathrm{G}') \qquad \begin{cases} L(\partial_t, \partial_x)u = 0 , \\ \partial_t^i u(0, x) = \phi_i(x) \in \mathscr{E}_x , \qquad 0 \leqq i \leqq m_1 - 1 , \\ \partial_x^j u(t, 0) = \psi_j(t) \in \mathscr{E}_t , \qquad 0 \leqq j \leqq m_2 - 1 , \\ \partial_x^j \phi_i(0) = \partial_t^i \psi_j(0) , \qquad 0 \leqq i \leqq m_1 - 1 , \qquad 0 \leqq j \leqq m_2 - 1 , \\ \qquad (t \geqq 0 \text{ or } t \leqq 0, \ x \in R^1) . \end{cases}$$

We set

$$\Omega = \{(i, j); a_{ij} \neq 0\} , \qquad ((m_1, m_2) \notin \Omega) . \tag{2.3}$$

Theorem 2.1. *If there exists (i, j) in Ω such that $i > m_1$ $(i + j < m)$ then the Goursat problem* (G′) *is not $\mathscr{E}$-wellposed for $t \geqq 0$ or for $t \leqq 0$.*

For to prove Theorem 2.1, we are going to show that the continuity from data to solution does not hold under the assumption of the theorem 2.1. More precisely we construct Goursat data $\{\phi_i(x; \xi), \psi_j(t; \xi)\}$ which have the following properties:

1) The growth order of Goursat data is at most polynomial of $|\xi|$,

2) We denote the solution of (G′) with previous Goursat data by $u(t, x; \xi)$. The growth order of $\partial_t^{m_1} u\ (0, x_\xi; \xi)$ is exponential of $|\xi|$, where x_ξ is bounded for large $|\xi|$.

Under the assumption of theorem 2.1, we consider Ω in R_+^2. Let ℓ be the straight line through (m_1, m_2) which has the following properties:

1) All elements (i, j) in Ω exist under ℓ or on ℓ,

2) There exists at least one element (i, j) in Ω on ℓ.

Let the slope of ℓ be $-p/q$. $(0 < q < p$, p and q are relatively prime). Here we put $m_1 p + m_2 q = J$ and

$$\Gamma(k) = \{(i, j); pi + qj = k, (i, j) \in \Omega\}, \qquad k = 0, 1, \dots, J. \tag{2.4}$$

Obviously, it holds that

$$\bigcup_{k=0}^{J} \Gamma(k) = \Omega. \tag{2.5}$$

Let

$$L_J(\tau, \zeta) = \tau^{m_1}\zeta^{m_2} - \sum_{(i,j) \in \Gamma(J)} a_{ij}\tau^i\zeta^j \tag{2.6}$$

and consider the roots of $L_J(1, \zeta) = 0$. We put

$$L_J(1, \zeta) = \zeta^\rho(\zeta - \alpha_1)^{n(1)}(\zeta - \alpha_2)^{n(2)} \dots (\zeta - \alpha_N)^{n(N)}, \tag{2.7}$$

where $\rho \geqq 0$, $\alpha_i \neq \alpha_j$ if $i \neq j$, $|\alpha_1| \geqq |\alpha_i| > 0$, α_i is $n(i)$-tuple root and $\rho + n(1) + n(2) + \cdots + n(N) = m_2$. Let (2.8) be a formal solution of (2.2):

$$u(t, x) = \sum_{r,s} u_{r,s} t^r x^s / r!s!. \tag{2.8}$$

We consider the following Goursat data:

$$\begin{cases} u_{r,s} = 0, & 0 \leqq s \leqq \rho - 1, \\ u_{r,s} = \alpha_1^s \xi^{pr+qs}, & \rho \leqq s \leqq m_2 - 1, \\ u_{r,s} = 0, & s \geqq m_2, \quad 0 \leqq r \leqq m_1 - 1, \end{cases} \tag{2.9}$$

namely

$$(2.9') \quad \begin{cases} \partial_x^j u(t,0) = 0\,, \qquad 0 \leqq j \leqq \rho - 1\,, \\ \partial_x^j u(t,0) = \sum_r (u_{r,j}/r!)t^r = (\alpha_1 \xi^q)^j \sum_r (\xi^{pr}/r!)t^r = (\alpha_1 \xi^q)^j \exp(\xi^p t)\,, \\ \quad \rho \leqq j \leqq m_2 - 1\,, \\ \partial_t^i u(0,x) = \xi^{pi} \sum_{s=\rho}^{m_2-1} (\alpha_1^s \xi^{qs}/s!)x^s\,, \qquad 0 \leqq i \leqq m_1 - 1\,. \end{cases}$$

Putting (2.8) in (2.2) and comparing the coefficients of $t^r x^s$, we have

$$(2.10) \qquad u_{r+m_1,s+m_2} = \sum_{(i,j)\in\Omega} a_{ij} u_{r+i,s+j}\,.$$

Here we introduce the following notion.

Definition 2.2. The order of u_{rs} is higher than the order of $u_{r's'}$ (or the order of $u_{r's'}$ is lower than that of u_{rs}) if and only if $pr + qs > pr' + qs'$ or else $pr + qs = pr' + qs'$ and $s' < s$.

By (2.9) and (2.10), we can determine u_{rs} successively and we have the following estimate.

Lemma 2.3. *$u_{r,s}$ is a polynomial of ξ with degree $pr + qs$ and has the following estimate*:

$$(2.11) \qquad |u_{r,s}| \leqq C^s |\xi|^{pr+qs} \qquad \text{for large } |\xi|\,,$$

where C is a constant independent of r, s and ξ.

Proof. We prove this by induction. Taking C as $|\alpha_1| < C$, Goursat data (2.9) satisfy (2.11). Suppose that $u_{r's'}$ satisfies (2.11) if its order is lower than that of $u_{r+m_1,s+m_2}$, then the following holds:

$$(2.12) \quad |u_{r+m_1,s+m_2}| \leqq A \sum_{(i,j)\in\Omega} |u_{r+i,s+j}| \leqq A \sum_{(ij)\in\Omega} C^{s+j} |\xi|^{p(r+i)+q(s+j)}\,, \qquad \text{where} \quad A = \max_{(i,j)\in\Omega} |a_{ij}|\,.$$

If $(i, j) \in \Gamma(J)$, (i, j) is on ℓ. So, we have

$$(2.13) \quad \Gamma(J) \subset \{(m_1 + q, m_2 - p), (m_1 + 2q, m_2 - 2p), \dots, (m_1 + h_0 q, m_2 - h_0 p)\}\,, \quad h_0 = [m_2/p]\,.$$

Thus, (2.12) becomes

$$|u_{r+m_1,s+m_2}| \leqq C^{s+m_2} |\xi|^{p(r+m_1)+q(s+m_2)} \times \left\{ A \sum_{(i,j)\in\Gamma(J)} C^{j-m_2} |\xi|^{-J+(pi+qj)} + \sum_{(i,j)\notin\Gamma(J)} AC^{j-m_2} |\xi|^{-J+(pi+qj)} \right\}$$

$$\leqq C^{s+m_2}|\xi|^{pr+qs+J}\{(Ah_0/C^p)+(AN_1C^{m_1}/|\xi|)\}\,,$$

where N_1 is the number elements of Ω.

Here we can suppose $C>1$. First we take C large enough to have $(Ah_0/C^p)<1/2$. Next we take $|\xi|$ large enough to have $(AN_1C^{m_1}/|\xi|)<1/2$. Then we have

$$|u_{r+m_1,s+m_2}|<C^{s+m_2}|\xi|^{p(r+m_1)+q(s+m_2)}\,. \tag{2.14}$$

q.e.d.

By Lemma 2.3 we have

Proposition 2.4. *The formal solution* (2.8) *of the Goursat problem* (2.2)–(2.9) *converges uniformly.*

Then (2.8) is a true solution of (2.2)–(2.9). By Lemma 2.3, we can put

$$u_{r,s}=\sum_{k=0}^{pr+qs}u_{r,s}^{(k)}\xi^{pr+qs-k}\,. \tag{2.15}$$

The following holds:

Lemma 2.5. *For $r\geqq m_1$ and $s\geqq\rho$, the leading coefficient of $u_{r,s}$ is α_1^s, namely*

$$u_{r,s}^{(0)}=\alpha_1^s\qquad \textit{for}\quad r\geqq m_1\,,\ s\geqq\rho\,. \tag{2.16}$$

Proof. We prove this by induction. Putting (2.15) in (2.10) and comparing the coefficient of $\xi^{pr+qs+J}$, we have

$$u_{r+m_1,s+m_2}^{(0)}=\sum_{(i,j)\in\Gamma(J)}a_{ij}u_{r+i,s+j}^{(0)}\,. \tag{2.17}$$

Goursat data (2.9) satisfy (2.16). Suppose that the every term of right-hand side of (2.17) satisfies (2.16). We have

$$\begin{aligned}u_{r+m_1,s+m_2}^{(0)}&=\sum_{(i,j)\in\Gamma(J)}a_{ij}\alpha_1^{s+j}\\&=\alpha_1^s\sum_{(i,j)\in\Gamma(J)}a_{ij}\alpha_1^j=\alpha_1^s\alpha_1^{m_2}=\alpha_1^{s+m_2}\,.\end{aligned} \tag{2.18}$$

Here we used the fact that α_1 is a root of $L_J(1,\zeta)=0$. q.e.d.

By (2.8) we have

$$\partial_t^{m_1}u(0,x)=\sum_{s\geqq0}(u_{m_1,s}/s!)x^s\,. \tag{2.19}$$

So we must estimate $u_{m_1,s}$. The lower order terms of $u_{m_1,s}$ on ξ have the following estimate.

Proposition 2.6. *For $\rho \leqq s < |\xi|^{\mu}$, $0 < \mu < 1/\rho'$, there exists constant C and $\sigma > 0$ such that*

$$|u_{m_1,s} - \alpha_1^s \xi^{pm_1+qs}| \leqq C|\alpha_1|^s|\xi|^{pm_1+qs-\sigma} . \tag{2.20}$$

Concerning ρ, refer to (2.7) *and we set* $\rho' = \max_{1 \leqq h \leqq N} n(h)$.

This proposition is the most important estimate to prove Theorem 2.1. The proof of proposition 2.6 is complicated. So we prove this later in § 3.

Now let us prove theorem 2.1. We set

$$x_\xi = 1/|\xi|^{\varepsilon}, \qquad \text{where} \quad 0 < q - \mu < \varepsilon < q . \tag{2.21}$$

The following holds:

$$\begin{aligned} &\left| \sum_{s \geqq 0} u_{m_1,s} x_\xi^s/s! - \xi^{m_1 p} \sum_{s \geqq 0} (\alpha_1 \xi^q)^s x_\xi^s/s! \right| \\ &\quad \leqq \sum_{s \leqq |\xi|^{\mu}} |u_{m_1,s} - \xi^{m_1 p}(\alpha_1 \xi^q)^s| x_\xi^s/s! + \sum_{s > |\xi|^{\mu}} |u_{m_1,s}| x_\xi^s/s! \\ &\quad + |\xi|^{m_1 p} \sum_{s > |\xi|^{\mu}} |\alpha_1 \xi^q|^s x_\xi^s/s! . \end{aligned} \tag{2.22}$$

Let the argument of α_1 be θ and let the argument of ξ be $-\theta/q$, that is

$$\alpha_1 = |\alpha_1| \exp(\theta i) \tag{2.23}$$

and

$$\xi = |\xi| \exp(-\theta i/q) . \tag{2.24}$$

Then we have

$$\alpha_1 \xi^q = |\alpha_1| \exp(\theta i)|\xi|^q \exp(-\theta i) = |\alpha_1||\xi|^q . \tag{2.25}$$

Using Proposition 2.6, (2.22) becomes (2.26):

$$\begin{aligned} &|\partial_t^{m_1} u(0, x_\xi) - \xi^{m_1 p} \exp(|\alpha_1||\xi|^{q-\varepsilon})| \\ &\quad \leqq C|\xi|^{m_1 p - \sigma} \exp(|\alpha_1||\xi|^{q-\varepsilon}) + R_1 + R_2 , \end{aligned} \tag{2.26}$$

where R_1 and R_2 are the second and third terms of the right-hand side of (2.22) respectively. By Lemma 2.3, we have

$$R_1 = \sum_{s > |\xi|^{\mu}} |u_{m_1,s}| x_\xi^s/s! \leqq \sum_{s > |\xi|^{\mu}} C^s |\xi|^{pm_1+qs} |\xi|^{-\varepsilon s}/s! . \tag{2.27}$$

Here we recall Stirling's formula:

$$s! = \sqrt{2\pi} s^{s+(1/2)} \exp(-s + (\theta'/12s), \qquad 0 \leqq \theta' \leqq 1 . \tag{2.28}$$

By (2.27), (2.28) and (2.21), we have the following for large $|\xi|$:

$$R_1 \leqq \sum_{s>|\xi|^\mu} C^s|\xi|^{m_1p+(q-\varepsilon)s}e^s/s^s \leqq \sum_{s>|\xi|^\mu} (eC)^s|\xi|^{m_1p+(q-\varepsilon-\mu)s} \tag{2.29}$$

$$< \text{constant}/|\xi| \qquad (\text{for large } |\xi|)\,.$$

By the same way, we have

$$R_2 \leqq \text{constant}/|\xi|\,. \tag{2.30}$$

Dividing (2.26) by $|\xi|^{m_1p}\exp(|\alpha_1||\xi|^{q-\varepsilon})$ we have

$$\left|\frac{\partial_t^{m_1}u(0, x_\xi)}{|\xi|^{m_1p}\exp(|\alpha_1||\xi|^{q-\varepsilon})} - \exp\left(-\frac{m_1p\theta}{q}i\right)\right| \leqq \frac{C'}{|\xi|^\sigma} \tag{2.31}$$

for large $|\xi|$, where $\sigma > 0$, C'; positive constant.

Now, we recall Goursat data (2.9′). Because of $\xi^p = |\xi|^p\exp(-pi\theta/q)$, if the real part of $\exp(-pi\theta/q) \geqq 0$ (or $\leqq 0$) we consider the Goursat problem for $t \leqq 0$ (or $t \geqq 0$), then Goursat data have at most polynomial order of $|\xi|$. So if we assume the $\mathscr{E}$-wellposedness then the solution $u(t, x)$ has at most polynomial order of $|\xi|$. When $|\xi| \to \infty$, (2.31) becomes $1 \leqq 0$ because of $q - \varepsilon > 0$. This is a contradiction. q.e.d.

§ 3. The Proof of Proposition 2.6

Recall (2.9), (2.10), (2.15) and (2.16):

$$\begin{cases} u_{r,s} = 0\,, & 0 \leqq s \leqq \rho - 1\,, \\ u_{r,s} = \alpha_1^s\xi^{pr+qs}\,, & \rho \leqq s \leqq m_2 - 1\,, \quad r \geqq 0\,, \\ u_{r,s} = 0\,, & s \geqq m_2\,, \quad 0 \leqq r \leqq m_2 - 1\,, \end{cases} \tag{2.9}$$

$$u_{r+m_1,s+m_2} = \sum_{(i,j)\in\Omega} a_{ij}u_{r+i,s+j}\,, \tag{2.10}$$

$$u_{r,s} = \sum_{k=0}^{pr+qs} u_{r,s}^{(k)}\xi^{pr+qs-k}\,, \tag{2.15}$$

$$u_{r,s}^{(0)} = \alpha_1^s \qquad \text{for} \quad r \geqq m_1\,, \quad s \geqq \rho\,. \tag{2.16}$$

Because of (2.4) and (2.5), we rewrite (2.10):

$$u_{r+m_1,s+m_2} = \sum_{d=0}^{J}\sum_{pi+qj=J-d} a_{ij}u_{r+i,s+j}\,. \tag{3.1}$$

Putting (2.15) in (3.1) we have

$$\sum_{k=0}^{pr+qs+J} u_{r+m_1,s+m_2}^{(k)}\xi^{pr+qs+J-k} = \sum_{d=0}^{J}\sum_{pi+qj=J-d} a_{ij}\sum_{k'\geqq 0} u_{r+i,s+j}^{(k')}\xi^{pr+qs+J-\mathrm{d}-k'}\,. \tag{3.2}$$

Comparing the coefficient of $\xi^{pr+qs+J-k}$, we have

$$u^{(k)}_{r+m_1,s+m_2} = \sum_{d=0}^{J} \sum_{pi+qj=J-d} a_{ij} u^{(k-d)}_{r+i,s+j}, \qquad 0 \leqq k \leqq pr+qs+J. \tag{3.3}$$

At first we notice that

Lemma 3.1. *When $r \geqq m_1(k+1)$, $u^{(k)}_{r,s}$ is independent of r.*

We prove this lemma by induction with respect to k and s. Moreover we have the following lemma. This is the key lemma to prove Proposition 2.6.

Lemma 3.2. *When $r \geqq m_1(k+1)$, $u^{(k)}_{r,s}$ has the following expression:*

$$u^{(k)}_{r,s} = \sum_{h=1}^{N} \sum_{\nu=0}^{n(h)k} z_h(\nu, k)(s-\rho)^\nu \alpha_h^s, \tag{3.4}$$

where $0^0 = 1$ *and* $(s-\rho)^\nu \equiv 0$ *for* $s-\rho < 0, \quad \nu \geqq 0.$

Here $z_h(\nu, k)$ is independent of r and has the following estimate:

$$|z_h(\nu, k)| \leqq M_1^k M_2^{n(h)k-\nu}/\nu!, \qquad (M_1, M_2;\text{ constant}). \tag{3.5}$$

Concerning the definition of $n(h)$, refer to (2.7).

The proof of Lemma 3.2 is fairly complicated. So we prove this later in §4. In the proof of Proposition 2.6, we use the following:

Corollary 3.3. *When $r \geqq m_1(k+1)$, $u^{(k)}_{r,s}$ has the following estimate:*

$$|u^{(k)}_{r,s}| \leqq C|\alpha_1|^s M^k s^{\rho' k}, \tag{3.6}$$

where $\rho' = \max_{1 \leqq h \leqq N} n(h)$ and C, M are constants.

Proof. By Lemma 3.2 we have

$$\begin{aligned} |u^{(k)}_{r,s}| &\leqq \sum_{h=1}^{N} \sum_{\nu=0}^{n(h)k} |z_h(\nu, k)|(s-\rho)^\nu |\alpha_h|^s \\ &\leqq \sum_{h=1}^{N} \sum_{\nu=0}^{\rho' k} (M_1^k M_2^{\rho' k-\nu}/\nu!)(s-\rho)^\nu |\alpha_1|^s \\ &\leqq N|\alpha_1|^s s^{\rho' k} M_1^k M_2^{\rho' k} \sum_{\nu=0}^{\rho' k} 1/(\nu! M_2^\nu) \\ &= |\alpha_1|^s s^{\rho' k}(M_1 M_2^{\rho'})^k N \exp(1/M_2) \\ &\leqq C|\alpha_1|^s M^k s^{\rho' k}. \end{aligned} \tag{3.7}$$

q.e.d.

In order to prove Proposition 2.6 we prepare some lemmas. By (2.10) we have

$$(3.8)\quad u_{r,s} = \sum_{(i,j)\in\Omega} a_{ij}u_{r+i-m_1,s+j-m_2}$$
$$= \sum_{i_1j_1} a_{i_1j_1} \sum_{i_2j_2} a_{i_2j_2}u_{r+i_1+i_2-2m_1,s+j_1+j_2-2m_2}\cdots$$
$$= \sum_{i_1j_1i_2j_2\ldots i_Kj_K} a_{i_1j_1}a_{i_2j_2}\cdots a_{i_Kj_K}u_{r+i_1+i_2\cdots+i_K-Km_1,s+j_1+j_2+\cdots+j_K-Km_2}\,.$$

Especially, for $r = m_1$, it becomes

$$(3.9)\qquad u_{m_1,n} = \sum_{i_1j_1\ldots i_Kj_K} a_{i_1j_1}\cdots a_{i_Kj_K}u_{m_1+i_1+\cdots+i_K-Km_1,n+j_1+\cdots+j_K-Km_2}\,.$$

In the right-hand side of (3.9), we replace $u_{r's'}$ by lower order ones successively. Finally each term of the right-hand side of (3.9) arrives at the Goursat data. By Lemma 2.3, $u_{m_1,n}$ is the polynomial of ξ with degree $pm_1 + qn$. Let us pay attention to the coefficient of $\xi^{pm_1+qn-\lambda}$. We recall the Goursat data (2.9). We are going to seek for Goursat data u_{rs} which satisfy (3.10):

$$(3.10)\qquad pr + qs = pm_1 + qn - \lambda\,, \qquad \rho \leqq s \leqq m_2 - 1\,.$$

Lemma 3.4. *The set of integers $\{qn - \lambda - qk; \rho \leqq k \leqq \rho + p - 1\}$ is equal to the set of integers $\{0, 1, 2, \ldots, p - 1\}$ modulo p.*

By (2.13) we have $\rho = m_2 - hp$, $1 \leqq h \leqq [m_2/p]$. So there exists one of Goursat data $u_{r,s}$ which satisfies (3.10). Let it be u_{r_0,s_0}, namely

$$(3.11)\qquad pr_0 + qs_0 = pm_1 + qn - \lambda\,, \qquad \rho \leqq s_0 \leqq m_2 - 1\,,$$

and

$$(3.12)\qquad \begin{cases} m_1 + i_1 + i_2 + \cdots + i_K - Km_1 = r_0 = (pm_1 + qn - \lambda - s_0q)/p\,, \\ n + j_1 + j_2 + \cdots + j_K - Km_2 = s_0\,. \end{cases}$$

We are going to estimate the coefficient of u_{r_0,s_0} in the right-hand side of (3.9). For this, we want to estimate K in (3.12). By (3.12) we have

$$(3.13)\qquad \begin{cases} pm_1 + p(i_1 + i_2 + \cdots + i_K) - pKm_1 = pm_1 + qn - \lambda - s_0q\,, \\ qn + q(j_1 + j_2 + \cdots + j_K) - qKm_2 = qs_0\,, \end{cases}$$

therefore the following holds:

$$(3.14)\quad p(i_1 + i_2 + \cdots + i_K) + q(j_1 + j_2 + \cdots + j_K) - K(pm_1 + qm_2) = -\lambda\,.$$

Considering $pm_1 + qm_2 = J$, (3.14) becomes the following:

$$(3.15)\qquad \sum_{k=1}^{K} (pi_k + qj_k - J) = -\lambda\,.$$

Let the number of $(i_k, j_k) \in \Gamma'$ $(=\Omega - \Gamma(J))$ in (3.15) be K'. The number of

$(i_k, j_k) \in \Gamma(J)$ becomes $K - K'$. According to the definition of $\Gamma(k)$ it holds that

$$\begin{cases} pi_k + qj_k - J = 0\,, & \text{for} \quad (i_k, j_k) \in \Gamma(J)\,, \\ pi_k + qj_k - J \leqq -1\,, & \text{for} \quad (i_k, j_k) \in \Gamma'\,. \end{cases} \tag{3.16}$$

By (3.15) and (3.16), we obtain

$$K' \leqq \lambda\,. \tag{3.17}$$

Recall the first equation in (3.12). The following holds:

$$\sum_{k=1}^{K} (i_k - m_1) = (qn - \lambda - s_0 q)/p\,. \tag{3.18}$$

According to the definition of $\Gamma(J)$ it holds that

$$i_k - m_1 \geqq q \geqq 1 \qquad \text{for} \quad (i_k, j_k) \in \Gamma(J)\,. \tag{3.19}$$

By (3.18), (3.19) and (3.17), it holds that

$$1(K - K') - m_1 K' \leqq (qn - \lambda - s_0 q)/p\,, \tag{3.20}$$

that is,

$$K \leqq (q/p)(n - s_0) + (m_1 + 1)K' - (\lambda/p)\,.$$

Thus we arrive at

$$K \leqq (q/p)n + (m_1 + 1)\lambda\,.$$

Finally we have

$$K < n + (m_1 + 1)\lambda\,. \tag{3.21}$$

Therefore the coefficient of u_{r_0, s_0} is estimated by $(AN_1)^{n+(m_1+1)\lambda}$. Here N_1 is the number of the elements of Ω and A is a constant satisfying $|a_{ij}| \leqq A$ for (i, j) in Ω. On the other hand, the number of $u_{r,s}$'s which satisfy (3.10) is at most m_2. Then we obtain the following lemma.

Lemma 3.5. *Let*

$$u_{m_1, n} = \sum_{\lambda=0}^{pm_1+qn} u_{m_1,n}^{(\lambda)} \xi^{pm_1+qn-\lambda}\,. \tag{3.22}$$

It holds that

$$|u_{m_1,n}^{(\lambda)}| \leqq C_1 (C_2)^{n+(m_1+1)\lambda}\,, \tag{3.23}$$

where C_1, C_2 are constants independent of n and λ.

This lemma is a very rough estimate. We use this lemma for large λ. For small λ, we need more delicate estimate. To obtain this, we use Corollary

3.3. In the right-hand side of (3.9), the Goursat data $u_{r,s}$ with $r > m_1(\lambda+1)+m_2$ must pass through $u_{r,s}$ with $m_1(\lambda+1) \leqq r \leqq m_1(\lambda+1)+m_2$. Let us notice $u_{r,s}$ with $m_1(\lambda+1) \leqq r \leqq m_1(\lambda+1)+m_2$. Let one of them be u_{r_0,s_0}:

$$m_1(\lambda+1) \leqq r_0 \leqq m_1(\lambda+1)+m_2\,, \tag{3.24}$$

$$pm_1+qn-\lambda \leqq pr_0+qs_0 \leqq pm_1+qn\,, \tag{3.25}$$

$$\begin{cases} m_1+i_1+i_2+\cdots+i_K-Km_1=r_0\,, \\ n+j_1+j_2+\cdots+j_K-Km_2=s_0\,. \end{cases} \tag{3.26}$$

Let us estimate K which satisfies (3.24), (3.25) and (3.26). By (3.26) we have

$$\sum_{k=1}^{K}(pi_k+qj_k-J)=pr_0+qs_0-(pm_1+qn)\,. \tag{3.27}$$

Let the number of $(i_k, j_k) \in \Gamma'$ in (3.27) be K''. Then the number of $(i_k, j_k) \in \Gamma(J)$ becomes $K-K''$. By (3.16), (3.25) and (3.27) we have

$$K'' \leqq \lambda\,. \tag{3.28}$$

The first equation of (3.26) is

$$\sum_{k=1}^{K}(i_k-m_1)=r_0-m_1\,. \tag{3.29}$$

By (3.19) and (3.29), it holds that

$$(K-K'')-m_1K'' \leqq r_0-m_1\,. \tag{3.30}$$

Therefore we have

$$\begin{aligned} K &\leqq (m_1+1)K''+r_0-m_1 \leqq (m_1+1)\lambda+m_1(\lambda+1)+m_2-m_1 \\ &= \lambda(2m_1+1)+m_2\,. \end{aligned} \tag{3.31}$$

Then the coefficient of u_{r_0,s_0} in the right-hand side of (3.9) is estimated by

$$(AN_1)^{\lambda(2m_1+1)+m_2}\,. \tag{3.32}$$

u_{r_0,s_0} is the polynomial of ξ with degree pr_0+qs_0. We want to estimate the coefficient of degree $pm_1+qn-\lambda$ of u_{r_0,s_0}. Putting

$$pr_0+qs_0-\lambda'=pm_1+qn-\lambda\,, \tag{3.33}$$

λ' satisfies

$$\lambda'=\lambda+pr_0+qs_0-(pm_1+qn)$$

and

$$0 \leqq \lambda' \leqq \lambda\,. \tag{3.34}$$

According to Corollary 3.3, we have

(3.35) $$|u^{(\lambda')}_{r_0,s_0}| \leqq C|\alpha_1|^{s_0} M^{\lambda'} s_0^{\rho'\lambda'} \leqq C|\alpha_1|^n M^\lambda n^{\rho'\lambda}.$$

By (3.10) and $r > m_1(\lambda + 1) + m_2$, it holds that

(3.36) $$\lambda < (qn - qs - pm_2)/(pm_1 + 1), \qquad \rho \leqq s \leqq m_2 - 1.$$

After all we have the following:

Lemma 3.6. *When*

(3.37) $$\lambda < (qn - qm_2 - pm_2 + q)/(pm_1 + 1)$$

it holds that

(3.38) $$\begin{aligned} |u^{(\lambda)}_{m_1,n}| &\leqq (AN_1)^{\lambda(2m_1+1)+m_2} C|\alpha_1|^n M^\lambda n^{\rho'\lambda} \lambda(m_2 + 1) \\ &\leqq C_1 C_2{}^\lambda |\alpha_1|^n n^{\rho'\lambda}, \qquad \textit{where} \quad C_1, C_2 \textit{ are constants}. \end{aligned}$$

Finally let us prove Proposition 2.6. Putting

(3.39) $$q/(pm_1 + 1) = \omega_1 \qquad \text{and} \qquad (qm_2 + pm_2 - q)/(pm_1 + 1) = \omega_2,$$

(3.37) becomes (3.40):

(3.40) $$\lambda < \omega_1 n - \omega_2.$$

Recall

(3.22) $$u_{m_1,n} = \sum u^{(\lambda)}_{m_1,n} \xi^{pm_1+qn-\lambda},$$

and let

(3.41) $$Q = \sum_{\lambda \geqq 1} u^{(\lambda)}_{m_1,n} \xi^{-\lambda}.$$

By Lemma 2.5, (2.20) is equivalent to (3.42).

(3.42) $$|Q| < C|\alpha_1|^n |\xi|^{-\sigma}, \qquad \sigma > 0, \qquad \text{for} \quad \rho \leqq n \leqq |\xi|^\mu.$$

Hereafter we are going to prove (3.42). We decompose Q as follows:

(3.43) $$\begin{aligned} Q &= \sum_{1 \leqq \lambda < \omega_1 n - \omega_2} u^{(\lambda)}_{m_1,n} \xi^{-\lambda} + \sum_{\lambda \geqq \omega_1 n - \omega_2} u^{(\lambda)}_{m_1,n} \xi^{-\lambda} \\ &= Q_1 + Q_2. \end{aligned}$$

First we consider Q_1. According to Lemma 3.6 and assumption of Prop. 2.6 we have

(3.44) $$\begin{aligned} |Q_1| &\leqq \sum_{1 \leqq \lambda < \omega_1 n - \omega_2} |u^{(\lambda)}_{m_1,n}| |\xi|^{-\lambda} \\ &\leqq C_1 \sum_{1 \leqq \lambda < \omega_1 n - \omega_2} C_2^\lambda |\alpha_1|^n n^{\lambda\rho'} |\xi|^{-\lambda} \end{aligned}$$

$$\leqq C_1|\alpha_1|^n \sum_{1\leqq\lambda} C_2^\lambda(|\xi|^\mu)^{\lambda\rho'}|\xi|^{-\lambda}$$

$$= C_1|\alpha_1|^n \sum_{1\leqq\lambda} (C_2|\xi|^{\mu\rho'-1})^\lambda$$

$$\leqq C_3|\alpha_1|^n|\xi|^{\mu\rho'-1} \qquad \text{for } |\xi| \text{ large},$$

where C_3 is constant and $\mu\rho' < 1$.

Next we consider Q_2. By Lemma 3.5 we have

$$(3.45) \qquad |Q_2| \leqq \sum_{\lambda\geqq\omega_1 n-\omega_2} |u_{m_1,n}^{(\lambda)}||\xi|^{-\lambda}$$

$$\leqq C_1 \sum_{\lambda\geqq\omega_1 n-\omega_2} (C_2)^{n+(m_1+1)\lambda}|\xi|^{-\lambda}$$

$$= C_1|\alpha_1|^n \sum_{\lambda\geqq\omega_1 n-\omega_2} (C_2/|\alpha_1|)^n C_2^{(m_1+1)\lambda}|\xi|^{-\lambda},$$

$$\lambda \geqq \omega_1 n - \omega_2, \quad \text{that is,} \quad n \leqq (\lambda/\omega_1) + (\omega_2/\omega_1),$$

and

$$(3.45') \quad |Q_2| \leqq C_1 \sum_{\lambda\geqq 1} (C_2')^{\lambda+\omega_2+(m_1+1)\lambda}|\xi|^{-\lambda}$$

$$\leqq \text{constant} \sum_{\lambda\geqq 1} (C_2'^{(m_1+2)}/|\xi|)^\lambda \leqq \text{constant}\,|\xi|^{-1} \quad \text{for large } |\xi|.$$

Thus we have proved (3.42).

§4. The Proof of Lemma 3.2

Putting (3.4) into (3.3), we have

$$(4.1) \qquad \sum_{h=1}^{N}\sum_{\nu=0}^{n(h)k} z_h(\nu, k)(s + m_2 - \rho)^\nu \alpha_h^{s+m_2}$$

$$= \sum_{d=0}^{J}\sum_{(i,j)\in\Gamma(J-d)} a_{ij}\sum_{h=1}^{N}\sum_{\nu=0}^{n(h)(k-d)} z_h(\nu, k-d)(s + j - \rho)^\nu \alpha_h^{s+j}.$$

Using the following equality;

$$(s + m_2 - \rho)^\nu = \{(s - \rho) + m_2\}^\nu = \sum_{n=0}^{\nu}\binom{\nu}{n}(s-\rho)^n m_2{}^{\nu-n},$$

we rewrite (4.1) and obtain (4.1');

$$(4.1') \quad \sum_{h=1}^{N}\sum_{\nu=0}^{n(h)k}\sum_{n=0}^{\nu} z_h(\nu, k)\binom{\nu}{n}(s-\rho)^n m_2{}^{\nu-n}\alpha_h^{s+m_2}$$

$$= \sum_{d=0}^{J}\sum_{(i,j)\in\Gamma(J-d)} a_{ij}\sum_{h=1}^{N}\sum_{\nu=0}^{n(h)(k-d)}\sum_{n=0}^{\nu} z_h(\nu, k-d)\binom{\nu}{n}(s-\rho)^n j^{\nu-n}\alpha_h^{s+j}.$$

We take the coefficients of $(s-\rho)^n\alpha_h^s$ of both sides of (4.1′) equal. We have

$$\sum_{\nu=n}^{kn(h)} z_h(\nu, k)\binom{\nu}{n} m_2^{\nu-n}\alpha_h^{m_2} - \sum_{(i,j)\in\Gamma(J)}\sum_{\nu=n}^{kn(h)} a_{ij}z_h(\nu,k)\binom{\nu}{n} j^{\nu-n}(\alpha_h)^j \tag{4.2}$$

$$= \sum_{d=1}^{J}\sum_{(i,j)\in\Gamma(J-d)} a_{ij}\sum_{\nu=n}^{(k-d)n(h)} z_h(\nu, k-d)\binom{\nu}{n} j^{\nu-n}\alpha_h^j\,,$$

$$0 \leqq n \leqq kn(h)\,,\quad 1 \leqq h \leqq N\,.$$

When $(k-1)n(h) < n \leqq kn(h)$, the right-hand side of (4.2) vanishes and the left-hand side is the following:

$$\sum_{\nu=n}^{kn(h)} z_h(\nu, k)\binom{\nu}{n}\left\{m_2{}^{\nu-n}\alpha_h^{m_2} - \sum_{(i,j)\in\Gamma(J)} a_{ij}j^{\nu-n}\alpha_h^j\right\}. \tag{4.3}$$

According to the equality:

$$y^t = \sum_{g=1}^{t} b_g y(y-1)(y-2)\ldots(y-g+1) \tag{4.4}$$

it holds that

$$m_2^t\alpha_h^{m_2} - \sum_{(i,j)\in\Gamma(J)} a_{ij}j^t\alpha_h^j = \sum_{g=1}^{t} b_g\left\{m_2(m_2-1)\ldots(m_2-g+1)\alpha_h^{m_2}\right. \tag{4.5}$$

$$\left. - \sum_{(i,j)\in\Gamma(J)} a_{ij}j(j-1)\ldots(j-g+1)\alpha_h^j\right\}$$

$$= \sum_{g=1}^{t} b_g\alpha_h^g(d/d\zeta)^g L_J(1, \alpha_h)\,.$$

As α_h is the $n(h)$-tuple root of $L_J(1,\zeta)=0$, the following holds:

$$(d/d\zeta)^g L_J(1,\alpha_h) = 0\,,\qquad 0 \leqq g \leqq n(h)-1\,. \tag{4.6}$$

By (4.5) and (4.6), we have (4.3) $= 0$. Therefore (4.2) holds for $(k-1)n(h) < n \leqq kn(h)$ and for any $\{z_h(\nu,k); (k-1)n(h) < \nu \leqq kn(h)\}$. Next we consider (4.2) for $n \leqq (k-1)n(h)$. By (4.6) we can rewrite (4.2) as follows:

$$z_h(n+n(h), k)\binom{n+n(h)}{k}\alpha_h^{n(h)}(d/d\zeta)^{n(h)}L_J(1,\alpha_h) \tag{4.7}$$

$$= -\sum_{\nu=n+n(h)+1}^{kn(h)} z_h(\nu,k)\binom{\nu}{n}\left\{m_2{}^{\nu-n}\alpha_h^{m_2} - \sum_{(i,j)\in\Gamma(J)} a_{ij}j^{\nu-n}\alpha_h^j\right\}$$

$$+ \sum_{d=1}^{J}\sum_{(i,j)\in\Gamma(J-d)} a_{ij}\sum_{\nu=n}^{(k-d)n(h)} z_h(\nu,k-d)\binom{\nu}{n} j^{\nu-n}\alpha_h^j\,,$$

$$0 \leqq n \leqq (k-1)n(h)\,.$$

As α_h is the $n(h)$-tuple root of $L_J(1, \zeta) = 0$, we have

$$(d/d\zeta)^{n(h)} L_J(1, \alpha_h) \neq 0 . \tag{4.8}$$

Therefore if we assume that the right-hand side of (4.7) is given then $z_h(\nu, k)$ $(n(h) \leqq \nu \leqq kn(h))$ is determined by (4.7). However $\{z_h(\nu, k);\ 0 \leqq \nu \leqq n(h) - 1\}$ are not determined by (4.7). These are determined by the following way. Recall (3.4):

$$u_{r,s}^{(k)} = \sum_{h=1}^{N} \sum_{\nu=0}^{n(h)k} z_h(\nu, k)(s-\rho)^\nu \alpha_h^s . \tag{3.4}$$

By (2.9) and (2.15), we have

$$u_{r,s}^{(k)} = 0 , \qquad (k \geqq 1,\ 0 \leqq s \leqq m_2 - 1) . \tag{4.9}$$

In the case $0 \leqq s \leqq \rho - 1$, it holds that $s - \rho < 0$. Then the right-hand side of (3.4) vanishes. We consider the case $\rho \leqq s \leqq m_2 - 1$, and set the right-hand side of (3.4) 0. We have

$$\sum_{h=1}^{N} \sum_{\nu=0}^{n(h)k} z_h(\nu, k)(s-\rho)^\nu \alpha_h^s = 0 , \qquad \rho \leqq s \leqq m_2 - 1 . \tag{4.10}$$

More precisely, (4.10) becomes

$$\tag{4.10$'$} \begin{cases} \sum_{h=1}^{N} z_h(0, k) = 0 , \quad (s = \rho) , \\ \sum_{h=1}^{N} \sum_{\nu=0}^{n(h)-1} z_h(\nu, k)\alpha_h = -\sum_{h=1}^{N} \sum_{\nu=n(h)}^{n(h)k} z_h(\nu, k)\alpha_h , \quad (s = \rho + 1) , \\ \sum_{h=1}^{N} \sum_{\nu=0}^{n(h)-1} z_h(\nu, k)2^\nu\alpha_h^2 = -\sum_{h=1}^{N} \sum_{\nu=n(h)}^{n(h)k} z_h(\nu, k)2^\nu\alpha_h^2 , \quad (s = \rho + 2) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \sum_{h=1}^{N} \sum_{\nu=0}^{n(h)-1} z_h(\nu, k)(m_2'-1)^\nu\alpha_h^{m_2'-1} = -\sum_{h=1}^{N} \sum_{\nu=n(h)}^{n(h)k} z_h(\nu, k)(m_2'-1)^\nu\alpha_h^{m_2'-1} , \\ \qquad (s = m_2 - 1 = m_2' + \rho - 1 \quad \text{and} \quad m_2' = m_2 - \rho) . \end{cases}$$

We consider that (4.10′) is a system of equations and it's unknowns are $\{z_h(\nu, k);$ $0 \leqq \nu \leqq n(h) - 1,\ 1 \leqq h \leqq N\}$. Let the coefficient matrix of (4.10′) be Δ. We have

$$|\det \Delta| = \left| \prod_{i<j} (\alpha_i - \alpha_j)^{n(i)n(j)} \prod_{i=1}^{N} \alpha_i^{n(i)(n(i)-1)} \right| \neq 0 . \tag{4.11}$$

Therefore if we assume that the right-hand side of (4.10′) is given, $\{z_h(\nu, k);$ $0 \leqq \nu \leqq n(h) - 1,\ 1 \leqq h \leqq N\}$ are determined uniquely.

Now we determine $\{z_h(\nu, k)\}$ in the following way. First, by (2.16) and (3.4) we have

$$z_1(0,0)=1\,,\qquad z_h(0,0)=0\qquad 1<h\leqq N\,. \tag{4.12}$$

Assuming that $\{z_h(\nu, k');\ k'<k\}$ are already determined, we determine $z_h(kn(h), k)$ by (4.7). Next, we determine $z_h(kn(h)-1, k)$ by (4.7), and so on. At last we determine $z_h(n(h), k)$ by (4.7). Finally we determine $\{z_h(\nu, k);\ 0\leqq \nu<n(h), 1\leqq h\leqq N\}$ by solving the system of equations (4.10′). We can prove (3.5) by induction. Thus we complete the proof of Lemma 3.2.

Remark 4.1. The coefficients $\{z_k(\nu, k);\ 0\leqq \nu\leqq kn(h),\ 1\leqq h\leqq N\}$ which satisfy (4.2) and (4.12) are determined uniquely. (4.2) is a sufficient condition for (4.1). However (4.2) is not a necessary condition for (4.1), therefore the expression in the right-hand side of (3.4) is not unique.

§5. General Case

At last we consider the general case where the operator $\mathscr{L}$ includes ∂_y;

$$\mathscr{L}(\partial_t, \partial_x, \partial_y)u(t,x,y)=0\,. \tag{5.1}$$

Putting

$$u(t,x,y)=\exp(i\eta y)v(t,x)\,,\qquad \eta\in\mathbb{R}^n\,, \tag{5.2}$$

(5.1) becomes (5.3):

$$\mathscr{L}(\partial_t, \partial_x, i\eta)v(t,x)=0\,. \tag{5.3}$$

If $\mathscr{L}(\partial_t, \partial_x, \partial_y)$ is not N-type, $\mathscr{L}(\partial_t, \partial_x, i\eta)$ is not N-type with respect to some η, too. So we can reduce the general case to the case of §2.

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References

[1] Hasegawa, Y., Some remarks on the C^∞-Goursat problem, *J. Math. Kyoto Univ.*, **28** (1988), 37–51.

[2] ———, On the Levi condition for Goursat problem, *J. Math. Kyoto Univ.*, **27** (1987), 1–25.

[3] Nishitani, T., On the $\mathscr{E}$-wellposedness for the Goursat problem with constant coefficients, *J. Math. Kyoto Univ.*, **20** (1980), 179–190.