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A Generalization of φ-conditional Expectation and Operator Valued Weight

By

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Abstract

Let M be a von Neumann algebra and N a von Neumann subalgebra of M. For any n.f.s. weights φ and ψ on M and N, respectively, we construct a normal map $E: M_+ \to \hat{N}_+$, which is the φ-conditional expectation if $\psi = \varphi|_N$, and is the operator valued weight if $\sigma_t^\psi = \sigma_t^\varphi|_N$ ($\forall t \in \mathbf{R}$).

Introduction

Let M be a von Neumann algebra with a normal faithful semifinite (n.f.s.) weight φ, and N be a von Neumann subalgebra of M with an n.f.s. weight ψ. The conditional expectations or the operator valued weights from M to N have been studied by several authors.

In [7], Takesaki showed that there exists a faitheful normal norm 1 projection (which is also called a conditional expectation) E which satisfies $\varphi = \varphi \circ E$ if and only if $\varphi|_N$ is semifinite and $\sigma_t^\varphi(N) = N$ ($\forall t \in \mathbf{R}$).

In [3], Haagerup showed that there exists an n.f.s. operator valued weight E which satisfies $\varphi = \psi \circ E$ if and only if $\sigma_t^\varphi|_N = \sigma_t^\psi$ ($\forall t \in \mathbf{R}$).

In another direction, when $\psi = \varphi|_N$, Accardi and Cecchini constructed in [1] the normal completely positive map E which satisfies

$$(E(x_1^* x_2) J_N \eta_\psi(y_1) | J_N \eta_\psi(y_2)) = (y_1^* y_2 J_M \eta_\varphi(x_1) | J_M \eta_\varphi(x_2)),$$
$$\forall x_1,\ x_2 \in \mathfrak{n}_\varphi, \qquad \forall y_1,\ y_2 \in \mathfrak{n}_\psi .$$

This map E is called the φ-conditional expectation and is a norm 1 projection if φ satisfies $\sigma_t^\varphi(N) = N$ $\forall t \in \mathbf{R}$.

In this paper, we generalize this Accardi and Cecchini's construction to the case that ψ is not necessarily equal to $\varphi|_N$. And we show that if φ and ψ satisfy $\sigma_t^\varphi|_N = \sigma_t^\psi$, $\forall t \in \mathbf{R}$, then the constructed map is the operator valued weight.

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Notations

Throughout this paper, M will denote a von Neumann algebra and N a von Neumann subalgebra of M. We also assume that $(M, \mathfrak{H}, J_M, \mathscr{P}_M)$ and $(N, \mathfrak{K}, J_N, \mathscr{P}_N)$ are standard forms and φ and ψ are n.f.s. weights on M and N, respectively.

Then we can define the linear map $x \in \mathfrak{n}_\varphi \mapsto \eta_\varphi(x) \in \mathfrak{H}$ canonically, and the linear map $x \in \mathfrak{m}_\varphi \mapsto \theta_\varphi(x) \in M_*$ as $\theta_\varphi(y^*z) = \omega_{J_M\eta_\varphi(y), J_M\eta_\varphi(z)}$ for $y, z \in \mathfrak{n}_\varphi$. The map $\theta_\varphi(x)$ has the following properties.

$$\text{(i)} \quad 0 \le x \Rightarrow 0 \le \theta_\varphi(x)\,.$$

$$\text{(ii)} \quad f \in M_*^+\,,\ f \le \varphi \Rightarrow f = \theta_\varphi(x) \qquad (\exists x \in \mathfrak{m}_\varphi \cap M_+)\,.$$

$$\text{(iii)} \quad \varphi(x) = \sup\{\langle x, \theta_\varphi(y)\rangle; y \in \mathfrak{m}_\varphi \cap M_+, \|y\| < 1\} \qquad (\forall x \in M_+)\,.$$

The analogous objects $\eta_\psi(y)$, $\theta_\psi(y)$ for N are defined with respect to ψ.

Let $\hat{N}_+$ be the extended positive part of N [3]. An *N-weight* on M is a map $T: M_+ \to \hat{N}_+$ which satisfies the following conditions:

$$\text{(i)} \quad T(\lambda x) = \lambda T(x) \qquad (\lambda \ge 0, x \in M_+)\,.$$

$$\text{(ii)} \quad T(x + y) = T(x) + T(y) \qquad (x, y \in M_+)\,.$$

Moreover, we say that T is an operator valued weight if

$$\text{(iii)} \quad T(a^*xa) = a^*T(x)a\,, \qquad (x \in M_+, a \in N).$$

We put

$$\mathfrak{n}_T = \{x \in M; T(x^*x) \in N_+\}\,,$$

$$\mathfrak{m}_T = \mathfrak{n}_T{}^*\mathfrak{n}_T = span\{x^*y; x, y \in \mathfrak{n}_T\}\,.$$

We say that T is *normal* if

$$x_i \nearrow x \Rightarrow T(x_i) \nearrow T(x) \qquad (x_i, x \in M_+)\,.$$

T is *faithful* if $T(x^*x) = 0$ implies $x = 0$, and T is *semifinite* if $\mathfrak{m}_T$ is σ-weakly dense in M.

§ 1. A Generalization of φ-conditional Expectation

For φ and ψ, we put

$$\mathfrak{N} = \mathfrak{N}_{\varphi,\psi}$$

$$= \{a \in M; \text{there exists } \lambda > 0 \text{ such that } \varphi(y^*a^*ay) \le \lambda\psi(y^*y) \text{ for any } y \in N\}\,.$$

$$\mathfrak{M} = \mathfrak{M}_{\varphi,\psi} = span\{b^*a \in M; a, b \in \mathfrak{N}_{\varphi,\psi}\}\,.$$

Since $\mathfrak{N}$ is a left ideal, there exists a projection $e = e_{\varphi,\psi} \in M$ such that

$$\overline{\mathfrak{N}}^{\sigma-w} = Me_{\varphi,\psi}.$$

Lemma 1.1. (i) $\mathfrak{N}N \subset \mathfrak{N}$, $N\mathfrak{M}N \subset \mathfrak{M}$.
(ii) $\mathfrak{N}\mathfrak{n}_\psi \subset \mathfrak{n}_\varphi$, $\mathfrak{n}_\psi^*\mathfrak{M}\mathfrak{n}_\psi \subset \mathfrak{m}_\varphi$.
(iii) $e_{\varphi,\psi} \in N' \cap M$.

Proof. (i) and (ii) are immediate.

(iii) For each unitary $u \in N$ and each $x \in \mathfrak{M} \cap M^+$, $u^*xu \in \mathfrak{M} \cap M^+$ from (i), hence we have $u^*e_{\varphi,\psi}u \leq e_{\varphi,\psi}$ and $ue_{\varphi,\psi}u^* \leq e_{\varphi,\psi}$. Therefore for each unitary $u \in N$, we have $u^*e_{\varphi,\psi}u = e_{\varphi,\psi}$, this implies $e_{\varphi,\psi} \in N' \cap M$ □

Example 1.2. (i) *If* $\varphi|_N \leq \lambda\psi$ *for some* $\lambda > 0$, *then* $e_{\varphi,\psi} = 1$.
(ii) *If* $\sigma_t^\varphi(y) = \sigma_t^\psi(y)(\forall y \in N, \forall t \in \mathbf{R})$, *then* $e_{\varphi,\psi} = 1$.

Proof. (ii) By [3], there exists a unique n.f.s. operator valued weight $F: M_+ \to \hat{N}_+$ such that $\varphi = \psi \circ F$. For any $a \in \mathfrak{n}_F$, we have

$$\varphi(y^*a^*ay) = \psi(F(y^*a^*ay)) = \psi(y^*F(a^*a)y) \leq \|F(a^*a)\| \psi(y^*y) \qquad (\forall y \in N).$$

Hence $\mathfrak{n}_F \subset \mathfrak{N}$, this implies $e_{\varphi,\psi} = 1$. □

The weight φ is said to be ψ-absolutely continuous in [4] if $M = N$, $\varphi + \psi$ is semifinite, and $\mathrm{Ker}(D\psi; D(\varphi+\psi))_{-i/2} = 0$. The next Proposition shows that if $M = N$ and M is a factor, then $e_{\varphi,\psi} = 1$ is equivalent to $\varphi \leq \lambda\psi$ for some $\lambda > 0$.

Proposition 1.3. *If* $M = N$ *and* $e_{\varphi,\psi} = 1$, *then the following statements are satisfied.*

(i) $\varphi + \psi$ *is an n.f.s. weight.*
(ii) φ *is ψ-absolutely continuous.*
(iii) *If* M *is a factor, then there exists* $\lambda > 0$ *such that* $\varphi \leq \lambda\psi$.

Proof. (i) From Lemma 1.1, $\mathfrak{N}\mathfrak{n}_\psi \subset \mathfrak{n}_\varphi \cap \mathfrak{n}_\psi = \mathfrak{n}_{\varphi+\psi}$.

(ii) Let $d = (D\psi; D(\varphi+\psi))_{-i/2}$ and assume $d\xi = 0$, $\xi \in \mathfrak{H}$. We choose a sequence $\{x_n\}_{n\in N} \subset \mathfrak{n}_{\varphi+\psi}$ such that $\lim_{n\to\infty}\eta_{\varphi+\psi}(x_n) = J\xi$ $(J = J_N = J_M)$. Then

$$\lim_{n\to\infty} \eta_\psi(x_n) = \lim_{n\to\infty} JdJ\eta_{\varphi+\psi}(x_n) = Jd\xi = 0.$$

Since, if $a \in \mathfrak{N}$, there exists $\lambda > 0$ such that $\|\eta_\varphi(ax_n)\|^2 \leq \lambda\|\eta_\psi(x_n)\|^2$, we have that $\lim_{n\to\infty}\eta_\varphi(ax_n) = 0$.

Consequently, for any $a \in \mathfrak{N}$,

$$\begin{aligned}\|aJ\xi\|^2 &= \lim_{n\to\infty} \|\eta_{\varphi+\psi}(ax_n)\|^2 \\ &= \lim_{n\to\infty} \|\eta_\varphi(ax_n)\|^2 + \lim_{n\to\infty} \|\eta_\psi(ax_n)\|^2 = 0.\end{aligned}$$

Since $e_{\varphi,\psi} = 1$, this implies $\xi = 0$.

(iii) By (ii), d has the inverse. Hence we have

$$\eta_{\varphi+\psi}(y) = Jd^{-1}J\eta_\psi(y) \qquad (\forall y \in \mathfrak{n}_{\varphi+\psi}).$$

Let $f \in \mathfrak{N}$ be a nonzero projection. Then there exists a $\lambda > 0$ such that

$$\|fJd^{-1}J\eta_\psi(y)\|^2 = \|\eta_{\varphi+\psi}(fy)\|^2 \leq (1+\lambda)\|\eta_\psi(y)\|^2 \qquad (\forall y \in \mathfrak{n}_{\varphi+\psi}).$$

It follows from above that $fJd^{-1}J$ is bounded. Let $(Jd^{-1}J)^* = vh$ be the polar decomposition, then h is affiliated to M', for $Jd^{-1}J$ is affiliated to M'. Thus we have $hf \supset fh$. Moreover, from

$$fh = fhv^*v = fJd^{-1}Jv$$

we have that fh is bounded, so that $fhf(\supset fh)$ is bounded. Hence if $h = \int \mu dE_\mu$ is the spectral decomposition, then $\overline{fhf} = \int \mu d(fE_\mu f)$ and there exists $\mu_0 > 0$ such that $f(1 - E_{\mu_0})f = 0$. On the other hand, since M is a factor, the induction $x \in M' \mapsto fxf \in fM'f$ is isomorphic. Hence, $1 - E_{\mu_0} = 0$. This means that h is bounded, so that $Jd^{-1}J$ is bounded.

Consequently, there exists $\lambda > 0$, such that

$$\begin{aligned} \varphi(y^*y) &\leq \|\eta_{\varphi+\psi}(y)\|^2 = \|Jd^{-1}J\eta_\psi(y)\|^2 \\ &\leq \lambda\|\eta_\psi(y)\|^2 = \lambda\psi(y^*y) \qquad (\forall y \in \mathfrak{n}_{\varphi+\psi}). \end{aligned}$$

Since $\mathfrak{N}\mathfrak{n}_\psi \subset \mathfrak{n}_{\varphi+\psi}$, it follows that $\varphi(y^*y) \leq \lambda\psi(y^*y)$ for any $y \in \mathfrak{n}_\psi$. □

For each $a \in \mathfrak{N}$, we have a unique bounded linear map $V_a\colon \mathfrak{K} \to \mathfrak{H}$ which satisfies

$$V_a\eta_\psi(y) = \eta_\varphi(ay) \qquad (\forall y \in \mathfrak{n}_\psi).$$

It is easy to see that

$$xV_ay = V_{xay} \qquad (\forall x \in M, \forall y \in N, \forall a \in \mathfrak{N}).$$

Lemma 1.4. *If $a, b \in \mathfrak{N}$, then*

$$a^*a \leq b^*b \Rightarrow \omega_{J_M V_a \xi} \leq \omega_{J_M V_b \xi} \text{ on } M \qquad (\forall \xi \in \mathfrak{K}).$$

Proof. If $y \in \mathfrak{n}_\psi$, then

$$\omega_{J_M V_a \eta_\psi(y)} = \theta_\varphi(y^*a^*ay) \leq \theta_\varphi(y^*b^*by) = \omega_{J_M V_b \eta_\psi(y)}.$$

Since $\eta_\psi(\mathfrak{n}_\psi)$ is dense in $\mathfrak{K}$,

$$\omega_{J_M V_a \xi} \leq \omega_{J_M V_b \xi}$$

hold for any $\xi \in \mathfrak{K}$. □

From the above Lemma, it is easy to see that if $a^*a = b^*b$ $(a, b \in \mathfrak{N})$, then $\omega_{J_M V_a \xi} = \omega_{J_M V_b \xi}$ on M for any $\xi \in \mathfrak{K}$. Moreover we see that

$$\{\omega_{J_M V_{a^{1/2}} \xi} \in M_*^+ : a \in \mathfrak{M} \cap M_+, \|a\| < 1\}$$

is upward directed for any $\xi \in \mathfrak{K}$.

For each $\xi \in \mathfrak{K}$, let φ_ξ be a normal weight on M such that

$$\varphi_\xi(x) = \sup\{\omega_{J_M V_{a^{1/2}} J_N \xi}(x) : a \in \mathfrak{M} \cap M_+, \|a\| < 1\} \qquad (\forall x \in M_+) .$$

Lemma 1.5. *If* $\xi, \eta \in \mathfrak{K}$, *then*

$$\omega_\xi = \omega_\eta \text{ on } N \Rightarrow \varphi_\xi = \varphi_\eta .$$

Proof. Note that there exists a partial isometry $u' \in N'$ such that $u'\xi = \eta$. Indeed, we can define u' by $u'y\xi = y\eta$ for $\forall y \in N$ and $u'\zeta = 0$ for $\forall \zeta \in [N\xi]^\perp$.

Let $u = J_N u' J_N \in N$. Then for any $a \in \mathfrak{M} \cap M_+$, $\|a\| < 1$, we have

$$\omega_{J_M V_{a^{1/2}} J_N \eta} = \omega_{J_M V_{a^{1/2}} u J_N \xi} = \omega_{J_M V_{a^{1/2}} u J_N \xi} = \omega_{J_M V_{(u^* a u)^{1/2}} J_N \xi} \leq \varphi_\xi .$$

Hence, $\varphi_\eta \leq \varphi_\xi$, so that $\varphi_\eta = \varphi_\xi$. □

For any $\omega \in N_*^+$, there exists a $\xi \in \mathfrak{K}$ such that $\omega = \omega_\xi$. We shall put

$$\varphi_\omega = \varphi_\xi .$$

By the above Lemma, this definition does not depend on the choice of ξ.

Lemma 1.6. *For any* $x \in M_+$, *there exists an element* $E(x) \in \hat{N}_+$ *such that*

$$\langle E(x), \omega \rangle = \langle x, \varphi_\omega \rangle \qquad (\forall \omega \in N_*^+) .$$

Proof. Since for any $x \in M_+$ and $a \in \mathfrak{M} \cap M_+$ the map

$$\xi \in \mathfrak{K} \mapsto \omega_{J_M V_{a^{1/2}} J_N \xi}(x) = \|x^{1/2} J_M V_{a^{1/2}} J_N \xi\|^2$$

is continuous, it follows that

$$\xi \in \mathfrak{K} \mapsto \varphi_\xi(x) = \sup\{\omega_{J_M V_{a^{1/2}} J_N \xi}(x); a \in \mathfrak{M} \cap M_+, \|a\| < 1\}$$

is a lower semicontinuous positive quadratic form. Hence there exists a positive operator h on $\mathfrak{K}$ such that $\|h^{1/2}\xi\|^2 = \varphi_\xi(x)$ $\forall \xi \in \mathfrak{K}$.

Since for any unitary $u' \in N'$, we have

$$\omega_{J_M V_{a^{1/2}} J_N u' \xi} = \omega_{J_M V_{a^{1/2}} u J_N \xi} = \omega_{J_M V_{(u^* a u)^{1/2}} J_N \xi}$$

where $u = J_N u' J_N$, it follows that $\varphi_{u'\xi} = \varphi_\xi$ ($\forall \xi \in \mathfrak{K}$). Therefore $h \equiv E(x)$ is affiliated to N. □

Theorem 1.7. *For given* φ *and* ψ, *there exists a normal* N*-weight* $E: M_+ \to \hat{N}_+$ *such that*

(i) $\langle E(x), \theta_\psi(y) \rangle = \langle \theta_\varphi(x), ye_{\varphi,\psi} \rangle \qquad (\forall x \in \mathfrak{m}_\varphi \cap M_+, \forall y \in \mathfrak{m}_\psi \cap N_+)$,

(ii) $\psi \circ E = \sup\{\varphi_{\theta_\psi(y)}; y \in \mathfrak{m}_\psi \cap N_+, \|y\| < 1\} \leq \varphi$.

Proof. (i) Let $E(x)$ be as in Lemma 1.6. Then it is easy to see that E is a normal N-weight on M. For any $x \in \mathfrak{m}_\varphi \cap M_+$ and $y \in \mathfrak{m}_\psi \cap N_+$, we have

$$\begin{aligned}\varphi_{\theta_\psi(y)}(x) &= \sup\{\omega_{J_M V_{a^{1/2}}\eta_\psi(y^{1/2})}(x);\ a \in \mathfrak{M} \cap M_+,\ \|a\| < 1\} \\ &= \sup\{\langle x, \theta_\varphi(y^{1/2}ay^{1/2})\rangle;\ a \in \mathfrak{M} \cap M_+,\ \|a\| < 1\} \qquad (1)\\ &= \sup\{\langle \theta_\varphi(x), y^{1/2}ay^{1/2}\rangle;\ a \in \mathfrak{M} \cap M_+,\ \|a\| < 1\} \\ &= \langle \theta_\varphi(x), y^{1/2}e_{\varphi,\psi}y^{1/2}\rangle = \langle \theta_\varphi(x), ye_{\varphi,\psi}\rangle . \qquad (2)\end{aligned}$$

(ii) Since

$$\psi = \sup\{\theta_\psi(y);\ y \in \mathfrak{m}_\psi \cap N_+,\ \|y\| < 1\},$$

it follows that

$$\begin{aligned}\psi \circ E &= \sup\{\theta_\psi(y) \circ E;\ y \in \mathfrak{m}_\psi \cap N_+,\ \|y\| < 1\} \\ &= \sup\{\varphi_{\theta_\psi(y)};\ y \in \mathfrak{m}_\psi \cap N_+,\ \|y\| < 1\} .\end{aligned}$$

From (1), we have $\varphi_{\theta_\psi(y)} \leq \varphi$ $(\forall y \in \mathfrak{m}_\psi \cap N_+, \|y\| < 1)$. Hence $\psi \circ E \leq \varphi$. □

In the rest of this paper, E will denote the map defined by Lemma 1.6.

Proposition 1.8. (i) *If $\psi = \varphi|_N$, then E has a unique linear extension, which is a φ-conditional expectation.*

(ii) *If $e_{\varphi,\psi} = 1$, then $\psi(E(x)) = \varphi(x)$ $(\forall x \in \mathfrak{m}_\varphi \cap M_+)$.*

Proof. (i) Since $1 \in \mathfrak{N}$, it follows that $\varphi_\omega \in M_*^+$ $(\forall \omega \in N_*^+)$. Hence $E(x) \in N_+$, $\forall x \in M_+$. Therefore, E has a unique linear extension from M to N. If we also denote this map by E, then E satisfies the following equation.

$$\langle E(x), \theta_\psi(y)\rangle = \langle \theta_\varphi(x), y\rangle \qquad (\forall x \in \mathfrak{m}_\varphi, \forall y \in \mathfrak{m}_\psi)$$

(ii) For any $x \in \mathfrak{m}_\varphi \cap M_+$,

$$\begin{aligned}\psi(E(x)) &= \sup\{\langle E(x), \theta_\psi(y)\rangle;\ y \in \mathfrak{m}_\psi \cap N_+,\ \|y\| < 1\} \\ &= \sup\{\langle \theta_\varphi(x), y\rangle;\ y \in \mathfrak{m}_\psi \cap N_+,\ \|y\| < 1\} \\ &= \langle \theta_\varphi(x), 1\rangle = \varphi(x) .\end{aligned}$$

□

§2. The Case Where $\sigma_t^\varphi|_N = \sigma_t^\psi$, $\forall t \in \mathbb{R}$

In this section, we assume that φ and ψ satisfy

$$\sigma_t^\varphi(y) = \sigma_t^\psi(y) \qquad (\forall y \in N, \forall t \in \mathbb{R}) .$$

Then by [3], there exists a unique n.f.s. operator valued weight $F: M_+ \to \hat{N}_+$ such that $\varphi = \psi \circ F$. Hence, from Example 1.2, we have $e_{\varphi,\psi} = 1$, in particular, $\mathfrak{N} = \mathfrak{n}_F$.

Lemma 2.1. *The following equalities hold.*

(i) $yJ_M V_a J_N \zeta = J_M V_a J_N y\zeta$ $(\forall y \in N, \forall \zeta \in \mathfrak{K}, \forall a \in \mathfrak{N} \cap \mathfrak{n}_\varphi)$.

(ii) $\varphi_{y\omega y^*} = \varphi_\omega(y^* \cdot y)$ $(\forall \omega \in N_*^+, \forall y \in N)$.

Proof. (i) Let M_a^φ and N_a^ψ be the sets of all entire analytic elements for φ and ψ, respectively. Then from the assumption, we have that $N_a^\psi \subset M_a^\varphi$.

By [2, Lemma 7], we have that

$$\eta_\varphi(ax) = J_M \sigma_{-i/2}^\varphi(x^*) J_M \eta_\varphi(a) \qquad (\forall a \in \mathfrak{n}_\varphi, \forall x \in M_a^\varphi) .$$

Hence, we get for $a \in \mathfrak{N} \cap \mathfrak{n}_\varphi$, $y \in N_a^\psi$ and $z \in N_a^\psi \cap \mathfrak{n}_\psi$

$$\begin{aligned} yJ_M V_a \eta_\psi(z) &= yJ_M \eta_\varphi(az) = y\sigma_{-i/2}^\varphi(z^*) J_M \eta_\varphi(a) \\ &= \sigma_{-i/2}^\varphi(\sigma_{i/2}^\varphi(y) z^*) J_M \eta_\varphi(a) \\ &= J_M \eta_\varphi(az\sigma_{i/2}^\varphi(y)^*) \\ &= J_M V_a \eta_\psi(z\sigma_{i/2}^\psi(y)^*) \\ &= J_M V_a J_N y J_N \eta_\psi(z) . \end{aligned}$$

Since $J_N \eta_\psi(N_a^\psi \cap \mathfrak{n}_\psi)$ is dense in $\mathfrak{K}$ and N_a^ψ is σ-weakly dense in N, we obtain (i).

(ii) Let $\{v_i\}_{i \in I} \subset \mathfrak{n}_\psi$ be a net such that

$$v_i \xrightarrow{s} 1 .$$

Since for any $a \in \mathfrak{N}$, we have $av_i \in \mathfrak{N} \cap \mathfrak{n}_\varphi$, using (i) we get

$$\begin{aligned} \omega_{yJ_M V_a J_N \zeta} &= \lim_{i \to \infty} \omega_{yJ_M V_{av_i} J_N \zeta} \\ &= \lim_{i \to \infty} \omega_{J_M V_{av_i} J_N y\zeta} \\ &= \omega_{J_M V_a J_N y\zeta} \quad (\forall \zeta \in \mathfrak{K}, \forall y \in N) . \end{aligned}$$

Thus, it follows that

$$\varphi_\zeta(y^* x y) = \varphi_{y\zeta}(x) \qquad \forall x \in M_+ . \qquad \square$$

Theorem 2.2. *E is equal to F.*

Proof. By the above Lemma, we have

$$\begin{aligned} \langle E(y^*xy), \omega \rangle &= \langle y^*xy, \varphi_\omega \rangle = \langle x, \varphi_{y\omega y^*} \rangle \\ &= \langle y^* E(x) y, \omega \rangle \qquad (\forall \omega \in N_*^+, \forall y \in N, \forall x \in M) . \end{aligned}$$

This means that E is an operator valued weight. Hence to prove the theorem, it is sufficient to show that $\psi \circ E = \varphi$. [3]

We can obtain $b_j \in M_a^\varphi$ $j \in J$ such that

$$b_j \in \mathfrak{M} \cap M_+ , \qquad \|b_j\| < 1 , \qquad \sigma_\alpha^\varphi(b_j) \xrightarrow{s^*} 1 \qquad (\forall \alpha \in \mathrm{C}) .$$

In fact, using some $a_j \in \mathfrak{M} \cap M_+$ $j \in J$ which satisfies

$$\|a_j\| < 1 \qquad (\forall j \in J) \qquad \text{and} \qquad a_j \xrightarrow{s^*} 1 ,$$

we may define b_j by

$$b_j = \sqrt{\frac{1}{\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sigma_t^{\varphi}(a_j) dt .$$

Similarly, we can obtain $y_k \in N_a^{\psi}$ $(k \in K)$ such that

$$y_k \in \mathfrak{n}_{\psi} \cap \mathfrak{n}_{\psi}^*, \qquad \|y_k\| < 1 \quad (\forall k \in K), \qquad \sigma_{\alpha}^{\psi}(y_k) \xrightarrow{s^*} 1 \quad (\forall \alpha \in \mathbb{C}).$$

Since b_j and y_k are analytic, we have that

$$\begin{aligned} J_M V_{xb_j} \eta_{\psi}(y_k^*) &= J_M \eta_{\varphi}(x b_j y_k^*) = \sigma_{-i/2}^{\varphi}(y_k b_j) J_M \eta_{\varphi}(x) \\ &= \sigma_{-i/2}^{\psi}(y_k) \sigma_{-i/2}^{\varphi}(b_j) J_M \eta_{\varphi}(x) \to J_M \eta_{\varphi}(x) \qquad (j, k \to \infty) \end{aligned}$$

for any $x \in \mathfrak{n}_{\varphi}$.

Therefore, we have that for $x \in \mathfrak{n}_{\varphi}$, $\|x\| < 1$

$$\begin{aligned} \theta_{\varphi}(x^* x) &= \lim_{j,k \to \infty} \omega_{J_M V_{xb_j} \eta_{\psi}(y_k^*)} \\ &\leq \sup_k \varphi_{J_N \eta_{\psi}(y_k^*)} = \sup_k \varphi_{\theta_{\psi}(y_k^* y_k)} \\ &= \sup_k \theta_{\psi}(y_k^* y_k) \circ E \\ &\leq \psi \circ E \qquad (\forall x \in \mathfrak{n}_{\varphi}, \|x\| < 1). \end{aligned}$$

This implies $\varphi \leq \psi \circ E$. Therefore, using Theorem 1.7, we have $\varphi = \psi \circ E$. □

Proposition 2.3. *We have that*

$$\mathfrak{M} \cap \mathfrak{m}_{\varphi} \cap M_+ = \{x \in \mathfrak{m}_{\varphi} \cap M_+ ; \text{there exists } \lambda > 0 \text{ such that } \theta_{\varphi}(x)|_N \leq \lambda \psi\} .$$

Proof. According to [6, proposition 2.17], it is true that for any $y \in N_a^{\psi} \cap \mathfrak{n}_{\psi} \cap \mathfrak{n}_{\psi}^*$

$$\psi(\sigma_{i/2}^{\psi}(y) \sigma_{-i/2}^{\psi}(y^*)) = \psi(y^* y) .$$

Thus, if $a \in \mathfrak{m}_{\varphi} \cap M_+$ and there exists $\lambda > 0$ such that $\theta_{\varphi}(a)|_N \leq \lambda \psi$, then for any $y \in N_a^{\psi} \cap \mathfrak{n}_{\psi} \cap \mathfrak{n}_{\psi}^*$, we have

$$\begin{aligned} \varphi(y^* a y) &= \|\eta_{\varphi}(a^{1/2} y)\|^2 = \|J_M \sigma_{-i/2}^{\psi}(y^*) J_M \eta_{\varphi}(a^{1/2})\|^2 \\ &= \langle \sigma_{-i/2}^{\psi}(y^*)^* \sigma_{-i/2}^{\psi}(y^*), \theta_{\varphi}(a) \rangle \\ &\leq \lambda \psi(y^* y) . \end{aligned}$$

Using the density of $\eta_{\psi}(N_a^{\psi} \cap \mathfrak{n}_{\psi} \cap \mathfrak{n}_{\psi}^*)$ in $\eta_{\psi}(\mathfrak{n}_{\psi})$, we have that $a \in \mathfrak{M} \cap M_+$.

Conversely, we assume $a \in \mathfrak{M} \cap \mathfrak{m}_{\varphi} \cap M_+$, then for some $\lambda > 0$

$$\begin{aligned} \langle y^* y, \theta_{\varphi}(a) \rangle &= \|J_M y J_M \eta_{\varphi}(a^{1/2})\|^2 = \|\eta_{\varphi}(a^{1/2} \sigma_{i/2}^{\psi}(y)^*)\|^2 \\ &\leq \lambda \psi(\sigma_{i/2}^{\psi}(y) \sigma_{i/2}^{\psi}(y)^*) = \lambda \psi(y^* y) \qquad (\forall y \in N_a^{\psi} \cap \mathfrak{n}_{\psi} \cap \mathfrak{n}_{\psi}^*). \end{aligned}$$

By the same argument as above, we have $\theta_{\varphi}(a)|_N \leq \lambda \psi$. □

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