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# Unitarily Invariant Norms under Which the Map $A \to |A|$ Is Lipschitz Continuous

By

Hideki KOSAKI*

**Abstract**

We will characterize the unitarily invariant norms (for compact operators) under which the map $A \to |A| = (A^*A)^{1/2}$ is Lipschitz-continuous. Although the map is not Lipschitz-continuous for the trace class norm, we will obtain a certain Lipschitz-type estimate by making use of the Macaev ideal.

## § 0. Introduction

In [9] E. B. Davies showed the following Lipschitz-type estimates in the Schatten $p$-norm $(1 < p < +\infty)$:

$$\begin{cases} \| |A|X - X|A| \|_p \leqq \text{Const.} \|AX - XA\|_p\,; & A = A^* \in C_p\,, \\ \| |A| - |B| \|_p \leqq \text{Const.} \|A - B\|_p\,; & A, B \in C_p\,. \end{cases}$$

Related results can be found in [1], [2], [3], [14] and [15]. For $p = 1$ and $+\infty$ (where $\|\cdot\|_\infty = \|\cdot\|$, the usual operator norm) the above Lipschitz-type estimates are known to fail. Instead some weaker estimates have been investigated by several authors ([7], [13], [16], [18]). See also [6] for some recent results.

An obvious next problem is to characterize unitarily invariant norms (of compact operators) under which the map $A \to |A| = (A^*A)^{1/2}$ is Lipschitz-continuous. In the present article we will obtain quite a complete solution to this problem based on very powerful analysis in [9] and Arazy's result, [4].

One of the difficulties of dealing with the map $A \to |A|$ is its non-linearity. A very clever trick in [9] is to reduce the desired Lipschitz-continuity to the boundedness of a certain linear operator (Schur-Hadamard multiplier, etc.). Therefore, interpolation (for linear operators) is at our disposal, and in § 2 all

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* Department of Mathematics, College of General Education, Kyushu University, Fukuoka 810, Japan.

the Lipschitz-type estimates in [9] are shown to remain valid for a symmetrically normed ideal (see §1 for its belief explanation) which is an interpolation space between some $C_{p_1}$ and $C_{p_2}$, $1 < p_1 < p_2 < +\infty$.

What is probably more interesting is that the converse is also true. For example, we can show that the Lipschitz-continuity of the map $A \to |A|$ implies the boundedness (relative to the relevant norm) of the triangle projection ([17]). Therefore, we can use Arazy's theorem, [4], stating that a symmetrically normed ideal possesses the above-mentioned interpolation property if and only if the triangle projection is bounded. This converse result will be proved in §3.

Although

$$\| |A| - |B| \|_p \leqq \text{Const.} \|A - B\|_p$$

is not valid for $p = 1, +\infty$, we will obtain Lipschitiz-type estimates involving these norms in §4. For example, if the above left side is replaced by the norm of the Macaev ideal (see [11]), the result remains valid for $p = 1$. The dual version can be also obtained by using the "predual" of the Macaev ideal. The Macaev ideal plays important roles in analysis on compact operators ([11], [12]). Its importance is also emphasized in the recent book [8], where relationship between this ideal and the Dixmier (non-normal) trace is discussed.

## §1. Symmetrically Normed Ideal ([11], [19])

In this section we collect basic facts on symmetrically normed ideals (of compact operators on a Hilbert space), and details on this subject matter can be found in [11], [19].

Let $f$ be the space of the sequences with finitely many non-zero terms. A norm $\Phi(\cdot)$ on $f$ (with normalization $\Phi(1, 0, 0, \dots) = 1$) is called a symmetric norm if $\Phi(\xi_1, \xi_2, \dots)$ is invariant under the permutations (of terms) and

$$\Phi(\xi_1, \xi_2, \dots) = \Phi(|\xi_1|, |\xi_2|, \dots).$$

Let $S_\Phi$ be the Banach space of sequences $a = \{a_n\}_{n=1,2,\dots}$ satisfying

$$\sup_m \Phi(a_1, a_2, \dots, a_m, 0, 0, \dots) \ (= \Phi(a), \text{ the extension of } \Phi) < +\infty,$$

and let $S_\Phi^{(0)}$ be the closure of $f$ relative to the norm $\Phi$.

Throughout let $H$ be a separable Hilbert space. For a compact operator $A$ on $H$ let $s_n(A)$ $(n = 1, 2, \dots)$ be the $n$-th singular number of $A$, that is, the $n$-th largest (with multiplicities counted) eigenvalue of $|A|$. We now introduce two Banach spaces $I(S_\Phi)$, $I(S_\Phi^{(0)})$ consisting of compact operators. A compact operator $A$ belongs to $I(S_\Phi)$ if the associated sequence $s(A) = \{s_n(A)\}_{n=1,2,\dots}$ lies in $S_\Phi$. The space $I(S_\Phi)$ is a Banach space under the norm

$$\|A\|_{I(S_\Phi)} = \Phi(s(A)).$$

The second Banach space $I(S_\Phi^{(0)})$ is defined as the closure (in $I(S_\Phi)$) of the space of the finite rank operators. The space $I(S_\Phi)$ may or may not be a separable Banach space while $I(S_\Phi^{(0)})$ is always separable. The both spaces are two sided ideals in $B(H)$, the bounded operators, and $\|\cdot\|_{I(S_\Phi)}$ is symmetric in the sense that

$$\|XAY\|_{I(S_\Phi)} \leqq \|X\| \, \|A\|_{I(S_\Phi)} \|Y\| ,$$

where $\|\cdot\|$ denotes the usual operator norm (throughout the article). In particular (and actually equivalently) we get the unitary invariance

$$\|UAV\|_{I(S_\Phi)} = \|A\|_{I(S_\Phi)} \qquad \text{for unitaries} \quad U, V.$$

Basic properties of these Banach spaces are:

1. $I(S_\Phi)$ is separable if and only if $I(S_\Phi) = I(S_\Phi^{(0)})$. (This can be checked by just looking at $\Phi$, i.e., mononormalizing in [11] or regular in [19].)

2. Any separable symmetrically normed ideal is of the form $(I(S_\Phi^{(0)}), \|\cdot\|_{I(S_\Phi)})$ for some symmetric norm $\Phi$.

3. For a given $\Phi$ (not equivalent to $\Phi_\infty$ defined later) we define the dual norm $\Phi'$ by

$$\Phi'(\xi) = \sup\left\{\left|\sum_{i=1}^{\infty} \xi_i\zeta_i\right| : \zeta = \{\zeta_i\} \in f \text{ and } \Phi(\zeta) \leqq 1\right\}.$$

Then the dual space $I(S_\Phi^{(0)})^*$ can be identified with $I(S_{\Phi'})$. Here the duality is given by the bilinear form

$$(A, B) \in I(S_\Phi^{(0)}) \times (I(S_{\Phi'}) \mapsto \mathrm{Tr}(AB) \in \mathbb{C} .$$

If we set $\Phi_p(\xi) = \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p}$, $1 \leqq p \leqq +\infty$ (with the usual convention for $p = +\infty$), we get $I(S_{\Phi_p}) = I(S_{\Phi_p}^{(0)}) = C_p$, the Schatten $C_p$-ideal, and $I(S_{\Phi_\infty}^{(0)}) = C_\infty$, the compact operators. In the rest of the article, we will deal with either $I(S_\Phi)$ or $I(S_\Phi^{(0)})$ which is strictly smaller than $C_\infty$. Whenever there is no possibility of confusion, the norm $\|\cdot\|_{I(S_\Phi)}$ will be denoted by $|||\cdot|||$.

For later use we list some properties of symmetrically normed ideals.

**Lemma 1.** *We have*

$$\left|\left|\left|\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right|\right|\right| = \left|\left|\left|\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right|\right|\right| = \left|\left|\left|\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}\right|\right|\right| = \left|\left|\left|\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}\right|\right|\right| = |||A||| .$$

*Proof.* The result follows from the obvious facts:

$$s_n\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = s_n\left(\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}\right) = s_n(A) ,$$

$$s_n\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) = s_n\left(\left|\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right|\right) = s_n\left(\begin{bmatrix} 0 & 0 \\ 0 & |A| \end{bmatrix}\right)$$

$$= s_n(|A|) = s_n(A)\,,$$

$$s_n\left(\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}\right) = s_n(|A^*|) = s_n(A)\,. \qquad \text{Q.E.D.}$$

**Lemma 2.** *We have*

$$\left|\left|\left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right|\right|\right| \leqq |||A||| + |||B||| + |||C||| + |||D||| \leqq 4\left|\left|\left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right|\right|\right|\,.$$

*Proof.* The first inequality follows from the triangle inequality and Lemma 1. To show the second, notice

$$\begin{aligned}\left|\left|\left|\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right|\right|\right| &= \left|\left|\left|\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right|\right|\right| \\ &\leqq \left\|\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\| \left|\left|\left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right|\right|\right| \left\|\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\| \\ &= \left|\left|\left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right|\right|\right|\,.\end{aligned}$$

Therefore, Lemma 1 shows

$$|||A||| \leqq \left|\left|\left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right|\right|\right|\,.$$

We similarly show that $|||B|||$, $|||C|||$, and $|||D|||$ are majorized by the same quantity. Q.E.D.

The next two results are Theorems 5.1 and 6.3 in Chap. III, [11], respectively.

**Lemma 3.** *Let $X$ be a bounded operator. If there is a sequence $\{X_n\}$ in $I\{S_\Phi)$ converging to $X$ in the weak operator topology and* $\sup_n |||X_n||| < +\infty$, *then $X$ belongs to $I(S_\Phi)$ and* $|||X||| \leqq \sup_n |||X_n|||$.

**Lemma 4.** *Assume $A \in I(S_\Phi^{(0)})$. If a sequence $\{X_n\}$ of self-adjoint operators converges to $X$ in the strong operator topology, then $X_nA \to XA$, $AX_n \to AX$, and $X_nAX_n \to XAX$ in the norm* $|||\cdot|||$.

## §2. Lipschitz-Type Estimates for Commutators

In this section we show that certain Lipschitz-type estimates are valid in a symmetrically normed ideal $(I(S_\Phi), |||\cdot|||)$ which is an interpolation space between $C_{p_1}$ and $C_{p_2}$, $1 < p_1 < p_2 < +\infty$. The reader can find details on the general interpolation theory in [5]. (Information on interpolation spaces between symmetrically normed ideals can be found in [4], [10].) In our set-up (since $C_{p_1} \subseteqq C_{p_2}$) the assumption means the following: We must have $C_{p_1} \subseteqq I(S_\Phi) \subseteqq C_{p_2}$ with continuous inclusion operators. Let $T$ be a linear mapping from $C_{p_2}$ into itself. Whenever $T(C_{p_1}) \subseteqq C_{p_1}$ and $T$ is bounded relative to $\|\cdot\|_{p_2}$ and $\|\cdot\|_{p_1}$, we must have $T(I(S_\Phi)) \subseteqq I(S_\Phi)$ and $T$ has to be bounded relative to $\||\cdot|\|$.

The central core for analysis in [9] was the next result based on the theory of Volterra operators ([12]).

**Lemma 5.** (*Corollary* 5 and *Corollary* 6 *in* [9]) *There is a constant* $\gamma_p$, $1 < p < \infty$, *satisfying the following*:

(i) *For any* $\lambda_i$, $\mu_i > 0$, $i = 1, 2, \ldots, n$ *and any* $n \times n$-*matrix* $A = [A_{ij}]$, *the* $n \times n$-*matrix* $B = [B_{ij}]$, $B_{ij} = (\lambda_i - \mu_i) \times (\lambda_i + \mu_j)^{-1} \times A_{ij}$, *satisfies* $\|B\|_p \leqq \gamma_p \|A\|_p$.

(ii) *For any* $\lambda_i$, $\mu_i \geqq 0$, $i = 1, 2, \ldots, n$, *and any* $n \times n$-*matrix* $C = [C_{ij}]$, *we have*

$$\|[(\lambda_i - \mu_j)C_{ij}]_{ij}\|_p \leqq \gamma_p \|[(\lambda_i + \mu_j)C_{ij}]_{ij}\|_p .$$

Obviously (i) and (ii) are equivalent. Let us emphasize that $\gamma_p$ is an absolute constant which does not depend on $n$, $A$, and the choice of $\lambda_i$'s and $\mu_i$'s. As was shown in [9], this lemma is based on the facts

$$(1)\qquad \begin{cases} \|P_{\pi/4}(A)\|_p \leqq \dfrac{1}{4}\gamma_p \|A\|_p \\ P_\theta(A) = U_\theta P_{\pi/4}(U_\theta^* A) \\ B = -A - \displaystyle\int_0^{\pi/2} P_\theta(A) g'(\theta) d\theta \end{cases}$$

(see p. 150, 151 for the definitions of $P_\theta$, $U_\theta$, $g(\theta)$, etc.). Our interpolation assumption (applied to the linear operator $P_{\pi/4}$) immediately implies $\||P_{\pi/4}(A)|\| \leqq$ Const. $\||A|\|$. Therefore, by repeating the arguments in [9], we conclude that Lemma 5 remains valid for $\||\cdot|\|$.

**Theorem 6.** *Assume that a symmetrically normed ideal* $(I(S_\Phi), \||\cdot|\|)$ *is an interpolation space between* $C_{p_1}$ *and* $C_{p_2}$, $1 < p_1 < p_2 < +\infty$. *For a bounded operator* $X$ *we have*

(i) $\|||A|X - X|B||\| \leqq$ *Const.* $\||AX - XB|\|$; $A = A^*$, $B = B^* \in I(S_\Phi)$,

(ii) $\|||A|X - X|A||\| \leqq$ *Const.* $\||AX - XA|\|$; $A \in I(S_\Phi)$, $X = X^*$,

(iii) $\||A_1X - XA_2|\| \leqq$ *Const.* $\||A_1X + XA_2|\|$; $A_1$, $A_2 \in I(S_\Phi)_+$.

*Proof.* (i) Since Lemma 5 is valid for $\||\cdot|\|$, the identical arguments as in the proof of Theorem 7, [9], together with Lemma 2 show

$$\|||a|x - x|a||\| \leqq \text{Const.} \||ax - xa|\| , \qquad a = a^* ,$$

for $n \times n$-matrices $a$, $x$. The operator $A \in I(S_\Phi)$ being compact, we can find an increasing sequence $\{p_n\}$ of projections with $\dim p_n H = n$, $p_n \to 1$ strongly and $[p_n, A] = 0$. Setting $a_n = p_n A p_n$ and $x_n = p_n X p_n$, we see $[a_n, x_n] = p_n[A, X]p_n$ and $[|a_n|, x_n] = p_n[|A|, x]p_n$. The above inequality for matrices then implies

$$\begin{aligned} |||p_n[|A|, X]p_n||| &\leqq \text{Const.}\, |||p_n[A, X]p_n||| \\ &\leqq \text{Const.}\, \|p_n\|\, |||[A, X]|||\, \|p_n\| \\ &\leqq \text{Const.}\, |||[A, X]|||\,. \end{aligned}$$

Since $p_n[|A|, X]p_n \to [|A|, X]$ strongly (hence weakly), Lemma 3 says

$$|||[|A|, X]||| \leqq \text{Const.}\, |||[A, X]|||\,,$$

which is exactly (i) with $A = B$. The general case can be obtained by applying this special case to

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} (= \tilde{A}^*)\,, \qquad \tilde{X} = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}.$$

In fact, we get

$$\left|\left|\left| \begin{bmatrix} 0 & |A|X - X|B| \\ |B|X^* - X^*|A| & 0 \end{bmatrix} \right|\right|\right| \leqq \text{Const.} \left|\left|\left| \begin{bmatrix} 0 & AX - XB \\ BX^* - X^*A & 0 \end{bmatrix} \right|\right|\right|.$$

Since $(|B|X^* - X^*|A|)^* = -(|A|X - X|B|)$, we have

$$\begin{aligned} |||\,|A|X - X|B|\,||| &= \frac{1}{2}\{|||\,|A|X - X|B|\,||| + |||\,|B|X^* - X^*|A|\,|||\} \\ &\leqq \text{the above left side} \qquad \text{(by Lemma 2)}\,. \end{aligned}$$

We similarly get

$$\begin{aligned} \text{the above right side} &\leqq |||AX - XB||| + |||BX^* - X^*A||| \qquad \text{(by Lemma 2)} \\ &= 2|||AX - XB|||\,. \end{aligned}$$

(ii) For $A$, $B \in I(S_\Phi)$, we set

$$a = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} (= a^*)\,, \qquad b = \begin{bmatrix} 0 & B^* \\ B & 0 \end{bmatrix} (= b^*)\,, \qquad x = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}.$$

Notice that

$$|a| = \begin{bmatrix} |A^*| & 0 \\ 0 & |A| \end{bmatrix} \quad \text{and} \quad |b| = \begin{bmatrix} |B| & 0 \\ 0 & |B^*| \end{bmatrix}.$$

Hence (i) applied to $a$, $b$, $x$ implies

$$\left|\left|\left|\begin{bmatrix} 0 & |A^*|X - X|B^*| \\ |A|X - X|B| & 0 \end{bmatrix}\right|\right|\right| \leqq \text{Const.} \left|\left|\left|\begin{bmatrix} AX - XB & 0 \\ 0 & A^*X - XB^* \end{bmatrix}\right|\right|\right| .$$

This estimate and Lemma 2 show

$$(2) \qquad |||\,|A|X - X|B|\,||| \leqq \text{Const.}\{|||AX - XB||| + |||A^*X - XB^*|||\} .$$

When $A = B$ and $X = X^*$, (since $(A^*X - XA^*)^* = -(AX - XA)$) (2) reduces to (ii).

(iii) This can be obtained by applying (i) (with $A = B$) to

$$a = \begin{bmatrix} A_1 & 0 \\ 0 & -A_2 \end{bmatrix} (= a^*) , \qquad x = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} . \qquad \text{Q.E.D.}$$

The theorem remains valid for a (separable) symmetrically normed ideal $I(S_\Phi^{(0)})$, the norm of $I(S_\Phi^{(0)})$ being just the restriction of $|||\cdot||| = \|\cdot\|_{I(S_\Phi)}$.

**Corollary 7.** *Let $(I(S_\Phi), |||\cdot|||)$ be as in Theorem* 6 *and $A$, $B \in B(H)$. If $A - B$ belongs to $I(S_\Phi)$, then so does $|A| - |B|$ and*

$$|||\,|A| - |B|\,||| \leqq \textit{Const.}\, |||A - B||| .$$

*Proof.* When $A$, $B \in I(S_\Phi)$, by setting $X = 1$ in (2) we get

$$\begin{aligned} |||\,|A| - |B|\,||| &\leqq \text{Const.}\{|||A - B||| + |||A^* - B^*|||\} \\ &= 2\ \text{Const.}\, |||A - B||| . \end{aligned}$$

To deal with the general case, we choose an increasing sequence $\{p_n\}_{n=1,2,\dots}$ of finite rank projections tending to 1 strongly. Since $p_nAp_n$, $p_nBp_n$ are finite rank operators ($\subseteqq I(S_\Phi)$), the above estimate implies

$$\begin{aligned} |||\,|p_nAp_n| - |p_nBp_n|\,||| &\leqq \text{Const.}\, |||p_n(A - B)p_n||| \\ &\leqq \text{Const.}\, |||A - B||| \qquad (< +\infty \text{ by the assumption}) . \end{aligned}$$

Since $|p_nAp_n| - |p_nBp_n| \to |A| - |B|$ strongly, Lemma 3 guarantees $|A| - |B| \in I(S_\Phi)$ as well as the desired inequality. Q.E.D.

This perturbation result fails for the trace class ideal $C_1$ and for $C_\infty$. However, different perturbation results will be obtained in §4.

## §3. The Converse of Theorem 6

Let $I$ be either $I(S_\Phi)$ or $I(S_\Phi^{(0)})$.

**Proposition 8.** *For a symmetrically normed ideal $(I, |||\cdot|||)$ the following seven conditions are equivalent*:

( i ) *There exists a constant* $K$ *such that for each* $n \in \mathbb{N}_+$, $n \times n$*-matrix* $C=[C_{ij}]$, *and* $\lambda_i,\ \mu_i \geqq 0$ $(i=1,2,\dots,n)$ *we have*

$$|||[(\lambda_i-\mu_j)C_{ij}]||| \leqq K|||[(\lambda_i+\mu_j)C_{ij}]||| .$$

( ii ) $|||\,|A|X - X|A|\,||| \leqq Const.|||AX-XA|||$ *for* $A=A^*\in I,\ X\in B(H)$.

(iii) $|||\,|A|X - X|B|\,||| \leqq Const.|||AX-XB|||$ *for* $A=A^*,\quad B=B^*\in I$, $X\in B(H)$.

(iv) $|||\,|A|X - X|B|\,||| \leqq Const.\{|||AX-XB||| + |||A^*X-XB^*|||\}$ *for* $A,\ B\in I,\ X\in B(H)$.

( v ) $|||\,|A|X - X|A|\,||| \leqq Const.|||AX-XA|||$ *for* $A\in I,\ X=X^*\in B(H)$.

(vi) $|||A_1X - XA_2||| \leqq Const.|||A_1X+XA_2|||$ *for* $A_i\in I_+,\ X\in B(H)$.

(vii) $|||\,|A|-|B|\,||| \leqq Const.|||A-B|||$ *for* $A,\ B\in I$.

*Here the six constants in* (ii) $\sim$ (vii) *do not depend on involved operators.*

*Proof.* In §2 we actually showed the implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v), (ii) ⇒ (vi), and (iv) ⇒ (vii). Since (vi) ⇒ (i) is obvious, it suffices to prove (v) ⇒ (ii) and (vii) ⇒ (ii).

(v) ⇒ (ii). For $A=A^*\in I$ and $X\in B(H)$ we set

$$a=\begin{bmatrix} A & 0 \\ 0 & A\end{bmatrix}, \qquad x=\begin{bmatrix} 0 & X \\ X^* & 0\end{bmatrix}\ (=x^*) .$$

Applying (v) to $a$, $x$, we get

$$\left|\left|\left|\begin{bmatrix} 0 & |A|X-X|A| \\ |A|X^*-X^*|A| & 0\end{bmatrix}\right|\right|\right| \leqq \text{Const.}\left|\left|\left|\begin{bmatrix} 0 & AX-XA \\ AX^*-X^*A & 0\end{bmatrix}\right|\right|\right| .$$

Since $(|A|X^* - X^*|A|)^* = -(|A|X - X|A|)$ and $(AX^*-X^*A)^* = -(A^*X-XA^*) = -(AX-XA)$, as before we obtain (ii) by using Lemma 2.

(vii) ⇒ (ii). The "semi-group theory trick" in the proof of Theorem 1, [1], shows (ii) with the additional assumption $X=X^*$. Then by using the same trick as in (v) ⇒ (ii) we can drop the self-adjointness of $X$. Q.E.D.

**Theorem 9.** *Assume that for a separable symmetrically normed ideal* $(I=I(S_\Phi^{(0)}),\ |||\cdot|||)$ *one (hence all) of the seven equivalent estimates in Proposition 8 is satisfied. Then* $I$ *is an interpolation space between* $C_{p_1}$ *and* $C_{p_2}$, $1<p_1<p_2<+\infty$.

To prove the theorem, we prepare the following lemma:

**Lemma 10.** *Under the same assumption as in the above theorem, there is a constant* $\tilde{K}$ *such that for each* $n\in\mathbb{N}_+$ *and* $n\times n$*-matrix* $C=[C_{ij}]$ we have

$$\left|\left|\left|\begin{bmatrix} C_{11} & \cdots & & C_{1n} \\ & C_{22} & & \vdots \\ & & \ddots & \\ 0 & & & C_{nn}\end{bmatrix}\right|\right|\right| \leqq \tilde{K}|||C||| .$$

*Proof.* For each $k \in \mathbb{N}_+$, we set

$$D_k(=D_k(C)) = \left[\frac{k^i - k^j}{k^i + k^j} C_{ij}\right]_{ij}.$$

Proposition 8, (i), implies $|||D_k||| \leqq K|||C|||$, where $K$ does not depend upon $k$ (and $n$). Letting $k \to \infty$, we have

$$\left|\left|\left|\begin{bmatrix} 0 & & -C_{ij} \\ & \ddots & \\ C_{ij} & & 0 \end{bmatrix}\right|\right|\right| \leqq K|||C|||.$$

Notice that

$$\begin{bmatrix} C_{11} & & C_{ij} \\ & \ddots & \\ 0 & & C_{nn} \end{bmatrix} = \frac{1}{2}\left\{C - \begin{bmatrix} 0 & & -C_{ij} \\ & \ddots & \\ C_{ij} & & 0 \end{bmatrix} + \begin{bmatrix} C_{11} & & 0 \\ & \ddots & \\ 0 & & C_{nn} \end{bmatrix}\right\}.$$

Since

$$\left|\left|\left|\begin{bmatrix} C_{11} & & 0 \\ & \ddots & \\ 0 & & C_{nn} \end{bmatrix}\right|\right|\right| \leqq |||C||| \qquad (\text{for any } |||\cdot|||)$$

is known, $\tilde{K} = 2^{-1}(1 + K + 1)$ does the job. Q.E.D.

*Proof of Theorem* 9. Let us identify $H$ with $l^2(\mathbb{N}_+)$. By using the canonical basis $\{e_i\}_{i=1,2,\dots}$, one can represent an operator as an (infinite) matrix. For an infinite matrix $C = [C_{ij}]$, we set

$$T(C) = \begin{bmatrix} C_{11} & & & C_{ij} \\ & C_{22} & & \\ & & \ddots & \\ 0 & & & \end{bmatrix}.$$

We also set

$$p_n = \begin{bmatrix} \overbrace{1 \; \ddots \; 1}^{n} & \\ & 0 \end{bmatrix} \qquad (n\text{-dimensional projection}).$$

For each $C \in I = I(S_\Phi^{(0)})$, Lemma 4 guarantees that $\{p_n C p_n\}$ is Cauchy in $|||\cdot|||$. Since the constant $\tilde{K}$ in Lemma 10 does not depend on $n$, $\{T(p_n C p_n)\}$ is also Cauchy in $I$ and there is an element $Y$ in $I$ such that $\lim_{n\to\infty} |||T(p_n C p_n) - Y||| = 0$. Take a vector $\xi$ in $p_m H$ ($m \in \mathbb{N}_+$). Since $\|\cdot\| \leqq |||\cdot|||$, we have

$$\|T(p_n C p_n)\xi - Y\xi\|_H \leqq \|T(p_n C p_n) - Y\| \|\xi\|_H \to 0$$

as $n \to \infty$. For $n \geqq m$, $T(p_n C p_n)\xi$ obviously does not depend upon $n$ so that we conclude

$$T(p_n C p_n)\xi = Y\xi \qquad \text{for} \qquad n \geqq m \qquad (\xi \in p_m H).$$

For each $i$, $j$, by choosing $n \geqq i, j$ we get

$$y_{ij} = (Ye_i, e_j) = (T(p_n C p_n)e_i, e_j) = \begin{cases} C_{ij} & \text{if } i \leqq j, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore, we conclude $Y = T(C)$ (and hence $T(C) \in I$) and $|||T(C)||| = \lim_{n\to\infty} |||T(p_n C p_n)|||$. Since

$$|||T(p_n C p_n)||| \leqq \tilde{K}|||p_n C p_n||| \leqq \tilde{K}|||C|||,$$

we conclude that $|||T(C)||| \leqq \tilde{K}|||C|||$, that is, $T: I \to I$ is bounded. Thanks to Arazy's characterization (Corollaries 3.4, 4.12, [4]) the ideal $I$ is an interpolation space between $C_{p_1}$ and $C_{p_2}$, $1 < p_1 < p_2 < +\infty$. Q.E.D.

Many other characterizations for $I(S_\Phi^{(0)})$ to be such an interpolation space are given in [4]. Also as remarked in p. 458, [4], these characterizations are valid for a (not necessarily separable) ideal $I(S_\Phi)$ by the simple duality argument. We remark that Theorem 9 also remains valid for $I = I(S_\Phi)$ by the duality. In fact, let us assume that $\|\cdot\|_{I(S_\Phi)}$ satisfies the inequality

$$\left\|\left[\frac{\lambda_i - \mu_j}{\lambda_i + \mu_j} C_{ij}\right]\right\|_{I(S_\Phi)} \leqq K\|[C_{ij}]\|_{I(S_\Phi)}.$$

Then the dual norm $\Phi'$ (see §1) satisfies

$$\left\|\left[\left(\frac{\lambda_i - \mu_j}{\lambda_i + \mu_j}\right) C_{ij}\right]\right\|_{I(S_{\Phi'})} = \sup_D \left|\operatorname{Tr}\left(\left[\frac{\lambda_i - \mu_j}{\lambda_i + \mu_j} C_{ij}\right][D_{ij}]\right)\right|,$$

where the sup is taken over all $n \times n$-matrices $D$ with $\|D\|_{I(S_\Phi)} \leqq 1$. It is elementary to see

$$\operatorname{Tr}\left(\left[\frac{\lambda_i - \mu_j}{\lambda_i + \mu_j} C_{ij}\right][D_{ij}]\right) = -\operatorname{Tr}\left([C_{ij}]\left[\frac{\mu_i - \lambda_j}{\mu_i + \lambda_j} D_{ij}\right]\right).$$

Hence we get

$$\left|\operatorname{Tr}\left(\left[\frac{\lambda_i - \mu_j}{\lambda_i + \mu_j} C_{ij}\right][D_{ij}]\right)\right| \leqq \|C\|_{I(S_{\Phi'})} \times \left\|\left[\frac{\mu_i - \lambda_j}{\mu_i + \lambda_j} D_{ij}\right]\right\|_{I(S_\Phi)} \leqq \|C\|_{I(S_{\Phi'})} \times K\|D\|_{I(S_\Phi)} \qquad \text{(by the assumption)},$$

and

$$\left\| \left[ \frac{\lambda_i - \mu_j}{\lambda_i + \mu_j} C_{ij} \right] \right\|_{I(S_{\Phi'})} \leqq K \|C\|_{I(S_{\Phi'})} . \tag{3}$$

Proposition 8, (i), thus remains valid for the dual norm $\|\cdot\|_{I(S_{\Phi'})}$, and Theorem 9 shows that the separable $I(S_{\Phi'}^{(0)})$ is an interpolation space between $C_{q_1}$ and $C_{q_2}$, $1 < q_2 < q_1 < +\infty$. But this means that $I(S_{\Phi'}^{(0)})^* = I(S_\Phi)$ is an interpolation space between $C_{p_1}$ and $C_{p_2}$ with $1 < p_1 < p_2 < +\infty$, $p_i^{-1} + q_i^{-1} = 1$. Combining the above with other characterizations given in [4], we have proved the following main result in the article:

**Theorem 11.** *Let $I$ be either $I(S_\Phi^{(0)})$ or $I(S_\Phi)$. The following conditions are equivalent*:

(a) *One of the seven estimates in Proposition* 8 *is valid* (*for example*, $\||A| - |B|\| \leqq Const. \||A - B|\|$ *for* $A, B \in I$).

(b) *$I$ is an interpolation space between $C_{p_1}$ and $C_{p_2}$, $1 < p_1 < p_2 < +\infty$.*

(c) *The triangle projection $T$ is bounded relative to $\||\cdot|\|$.*

(d) *The Macaev theorem remains valid for $I$, that is, whenever a Volterra operator $A$ satisfies $Im\, A \in I$, we must have $Re\, A \in I$ and*

$$\||Re\, A|\| \leqq Const. \||Im\, A|\|.$$

(e) *The Boyd indices* (*see* [4] *for details*) *of $\Phi$ are non-trivial.*

The last condition is very useful because one can check Lipschitz-continuity of the map $A \to |A|$ by just looking at the norm $\Phi$ on the sequence space $f$. Define the discrete dilation operators $D_m$, $D_{1/m}$ $(m = 1, 2, \dots)$ on $f$ by

$$\begin{cases} D_m(\xi) = (\underbrace{\xi_1, \dots, \xi_1}_{m \text{ times}}, \underbrace{\xi_2, \dots, \xi_2}_{m \text{ times}}, \dots) , \\ D_{1/m}(\xi) = \left( \sum_{i=1}^{m} \frac{\xi_i}{m}, \sum_{i=m+1}^{2m} \frac{\xi_i}{m}, \dots \right) . \end{cases}$$

Then compute the norms $\|D_m\|$ and $\|D_{1/m}\|$ (relative to $\Phi(\cdot)$). The Boyd indices $(p_\Phi, q_\Phi)$ are defined by

$$\begin{cases} p_\Phi = \sup_m \dfrac{\log m}{\log \|D_m\|} & \left( = \lim_{m \to \infty} \dfrac{\log m}{\log \|D_m\|} \right) , \\ q_\Phi = \inf_m \dfrac{\log(1/m)}{\log \|D_{1/m}\|} & \left( = \lim_{m \to \infty} \dfrac{\log(1/m)}{\log \|D_{1/m}\|} \right) . \end{cases}$$

It is easy to see $1 \leqq p_\Phi \leqq q_\Phi \leqq +\infty$. (For $\Phi_p$ corresponding to the Schatten ideal $C_p$, we easily see $p_{\Phi_p} = q_{\Phi_p} = p$.) Non-triviality in the last condition (e) means $1 \lneqq p_\Phi \leqq q_\Phi \lneqq +\infty$.

Let $C_{pq}$ $(1 \leqq p \leqq +\infty, 1 \leqq q \leqq +\infty)$ be the non-commutative analogue of the Lorentz space (see [4] or [19]) consisting of all compact operators such that

$$\|A\|_{pq} = \left(\sum_{i=1}^{\infty} i^{(q/p)-1} s_i(A)^q\right)^{1/q}$$

$$(= \sup_i (i^{1/p} s_i(A)) \text{ if } q = +\infty)$$

is finite. Note that $\|\cdot\|_{pq}$ is a norm only if $q \leqq p$, but when $p > 1$ there is a norm on $C_{pq}$ equivalent to $\|\cdot\|_{pq}$. It is well-known that $C_{pq}$ $(1 < p < \infty, 1 \leqq q \leqq +\infty)$ is an interpolation space (the $K$-method can be used, [5]) between $C_{p_1}$ and $C_{p_2}$ $(1 < p_1 < p < p_2 < +\infty)$. Therefore, the map $A \to |A|$ is Lipschitz-continuous relative to $\|\cdot\|_{pq}$ $(1 < p < +\infty, 1 \leqq q \leqq +\infty)$.

The above characterization roughly says that the map $A \to |A|$ is Lipschitz-continuous when the "geometry" of a symmetrically normed ideal in question is "good". However, the example presented after Corollary 4.6, [4], shows: there exists a non-uniformly convex symmetrically normed ideal in which the map $A \to |A|$ is Lipschitz-continuous.

## § 4. Estimates in the Operator and Trace Class Norms

As was mentioned in § 0, the map $A \to |A|$ is not Lipschitz-continuous for $\|\cdot\|$ and $\|\cdot\|_1$. Instead the following estimates are known ([13], [16]):

$$\begin{cases} \||A| - |B|\| \leqq \dfrac{2}{\pi}\|A - B\|\left\{2 + \log\dfrac{\|A\| + \|B\|}{\|A - B\|}\right\}; & A, B \in B(H), \\ \||A| - |B|\|_1 \leqq \sqrt{2}\,\|A + B\|_1^{1/2}\|A - B\|_1^{1/2}; & A, B \in C_1. \end{cases}$$

In this section we obtain different (and probably more natural) estimates for $|A| - |B|$ by making use of the ideals introduced by V. I. Macaev.

For a sequence $\xi = \{\xi_i\}_{i=1,2,\ldots}$, let $\{\xi_i^*\}_{i=1,2,\ldots}$ be the non-increasing rearrangement of $\{|\xi_1|, |\xi_2|, \ldots\}$. We introduce the (dual) symmetric norms $\Phi_\Omega$, $\Phi_\omega$ (on $f$) defined by

$$\begin{cases} \Phi_\Omega(\xi) = \sup_n \left(\sum_{i=1}^{n} \xi_i^* \Big/ \sum_{i=1}^{n} (2i-1)^{-1}\right), \\ \Phi_\omega(\xi) = \sum_{i=1}^{\infty} (2i-1)^{-1}\xi_i^*. \end{cases}$$

The corresponding symmetrically normed ideals $I(S_{\Phi_\omega}) (= I(S_{\Phi_\omega}^{(0)}))$ and $I(S_{\Phi_\Omega}) \supsetneqq I(S_{\Phi_\Omega}^{(0)}) (= \{A \in C_\infty : \lim_{n\to\infty} \left(\sum_{i=1}^{n} s_i(A) \Big/ \sum_{i=1}^{n} (2i-1)^{-1}\right) = 0\})$ satisfy

$$\begin{cases} I(S_{\Phi_\Omega}^{(0)})^* = I(S_{\Phi_\omega}), \\ I(S_{\Phi_\omega})^* = I(S_{\phi_\Omega}). \end{cases}$$

These ideals were introduced by V. I. Macaev and play important roles in analysis of compact operators (see [11], [12] for details and typical applications). Notice that $I(S_{\Phi_\Omega})$ (resp. $I(S_{\Phi_\omega})$) is "slightly" larger (resp. smaller) than $C_1$ (resp. $C_\infty$):

$$\begin{cases} I(S_{\Phi_\Omega}) \subsetneqq C_p, & p > 1\,, \\ C_p \subsetneqq I(S_{\Phi_\omega})\,, & p < +\infty\,. \end{cases}$$

In what follows, the norms of $I(S_{\Phi_\Omega})$ and $I(S_{\Phi_\omega})$ will be denoted by $\|\cdot\|_\Omega$ and $\|\cdot\|_\omega$ respectively. Recall that the proof of (1) (in §2) was based on Theorem 6.3 in Chap. III, [12]. If one starts from Theorem 2.2 in Chap. III, [12], instead, one obtains

$$\|P_{\pi/4}(A)\|_\Omega \leqq \|A\|_1\,.$$

Therefore, by repeating the arguments in p. 150, [9], we get

$$\left\|\left[\frac{\lambda_i - \mu_j}{\lambda_i + \mu_j} C_{ij}\right]\right\|_\Omega \leqq \text{Const.}\,\|C\|_1\,. \tag{4}$$

(Here the obvious fact $\|\cdot\|_\Omega \leqq \|\cdot\|_1$ is used.) Hence, the same arguments as in §2 show (among other commutator estimates) the next perturbation result.

**Theorem 12.** *If $A, B \in B(H)$ satisfy $A - B \in C_1$, then $|A| - |B|$ belongs to the ideal $I(S_{\Phi_\Omega})$ and*

$$\||A| - |B|\|_\Omega \leqq \textit{Const.}\,\|A - B\|_1\,.$$

By the obvious modification of the proof of (3), from (4) we get

$$\left\|\left[\frac{\lambda_i - \mu_j}{\lambda_i + \mu_j} C_{ij}\right]\right\| \leqq \text{Const.}\,\|C\|_\omega\,.$$

We then would like to show a dual version of the previous theorem. However, notice that Lemma 3 is not valid for $C_\infty$ = the compact operators. Starting from the assumption $A = A^* \in I(S_{\Phi_\omega})$, $X \in B(H)$, we get

$$\|p_n[|A|, X]p_n\| \leqq \text{Const.}\,\|[A, X]\|_\omega$$

as in the proof of Theorem 6. Since $Y \in B(H) \to \|Y\| = \sup\{\|Y\xi\|_H : \xi \in H, \|\xi\|_H \leqq 1\}$ is lower semi-continuous relative to the strong operator topology, (without using Lemma 3) we conclude

$$\begin{aligned} \|[|A|, X]\| &\leqq \liminf_{n \to \infty} \|p_n[|A|, X]p_n\| \\ &\leqq \text{Const.}\,\|[A, X]\|_\omega\,. \end{aligned}$$

Hence, (2) in the proof of Theorem 6 is still valid and we get

$$\||A| - |B|\| \leqq \text{Const.}\,\|A - B\|_\omega\,; \qquad A, B \in I(S_{\Phi_\omega})\,.$$

**Theorem 13.** *If* $A, B \in B(H)$ *satisfy* $A - B \in I(S_{\Phi_\omega})$, *then* $|A| - |B|$ *is a compact operator and*

$$\| |A| - |B| \| \leqq Const. \|A - B\|_\omega .$$

*Proof.* The arguments in the second half of the proof of Corollary 7 (but Lemma 3 is replaced by the above-mentioned lower semi-continuity of $Y \to \|Y\|$) show the desired inequality. The compactness of $|A| - |B|$ follows from the following standard argument: Let $B(H)/C_\infty$ be the Calkin algebra and $\pi$: $B(H) \to B(H)/C_\infty$ be the natural projection. We have $\pi(A) = \pi(B)$ because $A - B$ is compact. Since $\pi$ is a $C^*$-algebra homomorphism, we conclude $\pi(|A|) = |\pi(A)| = |\pi(B)| = \pi(|B|)$, i.e., $|A| - |B| \in C_\infty$. (Q.E.D.)

When $A$ is an $n \times n$-matrix, $s_i(A) = 0$ for $i \geqq n + 1$. Consequently we get

$$\begin{cases} \sum_{i=1}^{n} s_i(A) \Big/ \sum_{i=1}^{n} (2i-1)^{-1} \left( = \|A\|_1 \Big/ \sum_{i=1}^{n} (2i-1)^{-1} \right) \leqq \|A\|_\Omega , \\ \|A\|_\omega = \sum_{i=1}^{n} (2i-1)^{-1} s_i(A) \leqq \|A\| \sum_{i=1}^{n} (2i-1)^{-1} . \end{cases}$$

For the second estimate the obvious fact $s_1(A) = \|A\| \geqq s_2(A) \geqq s_3(A) \geqq \cdots$ was used. We thus get the next result for (finite) matrices.

**Corollary 14.** (*Theorem* 14, [9]) *There exists a constant* $K$ *such that for any* $n \times n$*-matrices* $A$, $B$ $(n \geqq 2)$ *we have*

$$\begin{cases} \| |A| - |B| \| \leqq (\log n) K \|A - B\| , \\ \| |A| - |B| \|_1 \leqq (\log n) K \|A - B\|_1 . \end{cases}$$

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