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Implementation of Comparative Probability by States

By

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Abstract

Let $\mathscr{H}$ be an infinite dimensional Hilbert space and $\mathscr{P}(\mathscr{H})$ the set of all (orthogonal) projections on $\mathscr{H}$. A comparative probability on $\mathscr{P}(\mathscr{H})$ is a linear preorder $\preceq$ on $\mathscr{P}(\mathscr{H})$ such that $\mathbf{O} \preceq P \preceq \mathbf{1}$, $\mathbf{1} \npreceq \mathbf{O}$ and such that if $P \perp R$, $Q \perp R$, then $P \preceq Q \Leftrightarrow P + R \preceq Q + R$ for all P, Q, R in $\mathscr{P}(\mathscr{H})$. In an earlier paper [1], it was shown that weak continuity of $\preceq$ was a sufficient and necessary condition for $\preceq$ to be implemented by a normal state on $\mathscr{B}(\mathscr{H})$, the bounded linear operators on $\mathscr{H}$. In this sequel to [1] we prove that uniform continuity is sufficient and necessary for implementation of $\preceq$ by a state.

§ 1. Introduction

We will generally use the same notation as that of [1], to which this paper is a sequel. Let $\mathscr{H}$ be a Hilbert space and E a (closed) subspace of $\mathscr{H}$. $\mathscr{P}(E)$ denotes the set of all (orthogonal) projections on E and P_E denotes the corresponding projection, with P_ϕ denoting the projection onto the one dimensional subspace spanned by ϕ. We drop the E and ϕ if no reference to the subspaces is required. $\mathscr{P}_1(E)$ is the subset of all one dimensional projections on E. Lower case Roman subscripts as in P_j or P_{ϕ_k} will generally be used for indexing sequences and nets. $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the natural numbers, the reals and the complex numbers respectively. $P_{\mathscr{H}}$ is denoted by $\mathbf{1}_{\mathscr{H}}$ or just $\mathbf{1}$ if no confusion arises and the zero vector is denoted by $\mathbf{O}$. The orthogonal complement of P (i.e. $\mathbf{1} - P$) is denoted by $P^\perp$. If P, $Q \in \mathscr{P}(\mathscr{H})$ and $P \leq Q^\perp$ then we write $P \perp Q$. Finally, we use $P_j \xrightarrow{u} P$ and $P_j \xrightarrow{w} P$ to mean that the net (or sequence) P_j converges to P in the uniform and weak operator topologies respectively.

Definition 1.1. *Let $\mathscr{H}$ be any Hilbert space. A preorder relation $\preceq$ on $\mathscr{P}(\mathscr{H})$ is called a comparative probability* (CP) *iff the following axioms are*

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satisfied by all P, Q, $R \in \mathscr{P}(\mathscr{H})$:

A1 $P \preceq Q$ *or* $Q \preceq P$,

A2 $P \preceq Q$ *and* $Q \preceq R \Rightarrow P \preceq R$,

A3 $\mathbb{O} \preceq P \preceq \mathbb{1}$, $\mathbb{1} \not\preceq \mathbb{O}$.

A4 *If* $P \perp R$, $Q \perp R$, *then* $P \preceq Q \Leftrightarrow P + R \preceq Q + R$. □

We note that axiom A4 is equivalent to the following: If $P \perp R$, $Q \perp R$, then $P \prec Q \Leftrightarrow P + R \prec Q + R$. Recall that a Gleason measure is a σ-additive measure on $\mathscr{P}(\mathscr{H})$, that is a σ-orthoadditive mapping μ: $\mathscr{P}(\mathscr{H}) \to [0, 1]$ satisfying $\mu(\mathbb{1}) = 1$. If $\dim \mathscr{H} \neq 2$ and $\mathscr{H}$ remains separable, then Gleason's theorem [3] says that μ may be extended to a normal state on $\mathscr{B}(\mathscr{H})$. For $\mathscr{H}$ not separable, one trivially verifies that if the σ-additivity is replaced by complete additivity, then μ can still be extended to a normal state on $\mathscr{B}(\mathscr{H})$.

More recent work (see [2] for a comprehensive review, including references of the original papers) has generalized Gleason's theorem to include cases where μ is just an additive measure (i.e. finitely orthoadditive map) acting on the projections of arbitrary Von Neumann algebras. Essentially no new "exceptions" appear beyond the case $\dim \mathscr{H} = 2$ which appears in Gleason's theorem. The generalization may be stated as follows: ([2] Theorem 12.1)

Theorem 1.2. *Let* $\mathscr{A}$ *be a Von Neumann algebra without a direct summand of type* I_2 *and let* μ *be an additive measure on the projections of* $\mathscr{A}$. *Then* μ *can be extended to a state* $\tilde{\mu}$ *on* $\mathscr{A}$. *Further, if* μ *is* σ*-additive, then* $\tilde{\mu}$ *is normal if and only if* μ *has a support.* □

As in [1] we wish to establish sufficient and necessary conditions for a CP $\preceq$ on $\mathscr{P}(\mathscr{H})$, where $\mathscr{H}$ is any infinite dimensional Hilbert space, to be implemented by a state ω on $\mathscr{B}(\mathscr{H})$ according to the prescription: $P \preceq Q \Leftrightarrow \omega(P) \leq \omega(Q)$. Where it exists, the implementing state is unique if $\mathscr{H}$ is infinite dimensional but this is generally not true for $\mathscr{H}$ finite dimensional [6], [5]. The proof offered here is also valid for the problem of [1]. Not every CP can be implemented in this way as the following counter example[1] shows: Let $\mathscr{H}$ be any Hilbert space of dimension at least three and let P_ϕ and P_ψ be mutually orthogonal (one dimensional) projections of $\mathscr{P}(\mathscr{H})$. Define states ω_ϕ and ω_ψ on $\mathscr{B}(\mathscr{H})$ by $\omega_\phi(P_\phi) = 1$ and $\omega_\psi(P_\psi) = 1$. Let $\trianglelefteq$ be defined by $P \trianglelefteq Q$ if $\omega_\phi(P) < \omega_\phi(Q)$ or if $\omega_\phi(P) = \omega_\phi(Q)$ and $\omega_\psi(P) \leq \omega_\psi(Q)$. One verifies that $\trianglelefteq$ is indeed a CP and also that no state can implement it. A missing crucial ingredient in this CP is (uniform) continuity. This continuity, unlike the case of additive measures (= states), is not automatic for CP's, and may be defined as follows:

[1] Communicated to the author by A. Paszkiewicz

Definition 1.3. *Let* $\mathcal{T}$ *be a locally convex topology on* $\mathcal{B}(\mathcal{H})$ *and* $\preceq$ *a* CP *on* $\mathcal{P}(\mathcal{H})$. *We say* $\preceq$ *is* $\mathcal{T}$ *continuous if whenever a net* P_j *converges to* P *in the* $\mathcal{T}$ *topology and* $Q \preceq P_j \preceq R$ $\forall j$, *then* $Q \preceq P \preceq R$. □

Recall that the $\preceq$ (interval) topology on $\mathcal{P}(\mathcal{H})$ is generated by a neighbourhood base of $\preceq$ intervals of $\mathcal{P}(\mathcal{H})$. For any uniformly continuous CP $\preceq$ on $\mathcal{P}(\mathcal{H})$, addition is separately $\preceq$ continuous (Proposition 2.10) in the sense that if P_j is a net in $\mathcal{P}(\mathcal{H})$ which $\preceq$ converges to $P \in \mathcal{P}(\mathcal{H})$ (we denote this convergence by $P_j \overset{\preceq}{\rightarrow} P$) and there exists $Q \in \mathcal{P}(\mathcal{H})$ such that $P \perp Q$ and $P_j \perp Q$ $\forall j$, then $Q + P_j \overset{\preceq}{\rightarrow} Q + P$. Joint $\preceq$ continuity of addition is harder to establish. We give a formal definition of joint continuity of addition on $\mathcal{P}(\mathcal{H})$:

Definition 1.4. *Addition on* $\mathcal{P}(\mathcal{H})$ *is said to be jointly* $\preceq$ *continuous if whenever* P, $Q \in \mathcal{P}(\mathcal{H})$ *and the nets* P_j *and* Q_j *in* $\mathcal{P}(\mathcal{H})$ *satisfy* $P_j \overset{\preceq}{\rightarrow} P$, $Q_j \overset{\preceq}{\rightarrow} Q$, $P \perp Q$ *and* $P_j \perp Q_j$ $\forall j$, *then* $P_j + Q_j \overset{\preceq}{\rightarrow} P + Q$. □

§ 2. Uniformly Continuous CP's

From now on, $\mathcal{H}$ denotes a complex infinite dimensional Hilbert space which is not necessarily separable, and $\preceq$ denotes a uniformly continuous CP on $\mathcal{P}(\mathcal{H})$.

Definition 2.1. *Let* E *be a subspace of* $\mathcal{H}$. *We define* $\mathcal{D}(E)$ *to be the set of all projections* $P_F \in \mathcal{P}(E)$ *such that the rank of* P_F *and the rank of* $(P_E - P_F)$ *have the same cardinality.* □

Lemma 2.2. *Let* G *be any infinite dimensional subspace of* $\mathcal{H}$. *Then* $\mathcal{D}(G)$ *is uniformly connected.*

Proof. Let $P_E = \sum_{j \in \Lambda} P_{\phi_j}$ and $P_F = \sum_{j \in \Lambda} P_{\psi_j}$ be both in $\mathcal{D}(G)$ (where the summands in each case are mutually orthogonal) and suppose that $P_E \perp P_F$. Define the function $f: t \in [0, 1] \mapsto \sum_{j \in \Lambda} P_{t\phi_j + s\psi_j} \in \mathcal{D}(G)$, where $s = \sqrt{1 - t^2}$. One easily shows that f is *uniformly* continuous. Now suppose $P_E \not\perp P_F$. There exist projections $P_{E'}$ and $P_{F'}$, both in $\mathcal{D}(G)$, such that $P_{E'} \leq P_E$ and $P_{F'} \leq P_F$ and such that $P_{E'} \perp P_{F'}$; this can be established, for example, by a simple application of Zorn's lemma. Using the result of the case $P_E \perp P_F$, we can construct a uniformly continuous path along the route $P_E \rightarrow P_E^{\perp} \rightarrow P_{E'} \rightarrow P_{F'} \rightarrow P_F^{\perp} \rightarrow P_F$. □

We remark that the uniform connectedness of $\mathcal{D}(\mathcal{H})$ implies that nets may be replaced by sequences in handling convergence to any projection $Q \in \mathcal{P}(\mathcal{H})$ such that $Q \cong P$ for some $P \in \mathcal{D}(\mathcal{H})$.

Lemma 2.3. *Let* P_E, P_F *and* P_K, *all in* $\mathcal{D}(\mathcal{H})$, *be mutually orthogonal with* $\mathbf{0} \cong P_E \prec P_F$. *Put* $P_G = P_K + P_F + P_E$. *Suppose that* $\preceq$ *is implemented by a state when restricted to* $\mathcal{P}(E + F)$, *then there exists a sequence* P_{G_j} *in* $\mathcal{P}(G)$

satisfying the following:

i. $P_{G_j} \prec P_K + P_F \ \forall j$

ii. $P_{G_j} \xrightarrow{u} P_K + P_F$

iii. *If* P_L, $P_M \in \mathscr{P}(F + K)$ *are mutually orthogonal, then there exist sequences* S_j *and* T_j *in* $\mathscr{P}(G)$ *such that* $S_j \perp T_j$ *and* $S_j + T_j \leq P_{G_j} \ \forall j$, *and such that* $S_j \xrightarrow{u} P_L$, $T_j \xrightarrow{u} P_M$.

Proof. Let $P_E = \sum_{j \in \Lambda} P_{\phi_j}$ and $P_F = \sum_{j \in \Lambda} P_{\psi_j}$. For each $t \in [0, 1]$, define the function $f_t: K + E \to G$ by $f_t(\phi_j) = s\phi_j + t\psi_j \ \forall j \in \Lambda$ and $f_t(\xi) = \xi \ \forall \xi \in K$, where $s = \sqrt{1 - t^2}$. Then the restriction of f_t to any subspace J of $K + E$ is a unitary operator onto $f_t(J)$ and the map $t \mapsto P_{f_t(J)}$ is uniformly and hence $\preceq$ continuous. Thus there exists a sequence of the form $P_{f_{t_j}(E)}$ such that $P_{f_{t_j}(E)} \xrightarrow{u} P_F$ and $P_{f_{t_j}(E)} \preceq P_F \ \forall j$. But for any r such that $0 < r < 1$, P_E is in the linear span of the set $\{P_{f_t(E)}: r \leq t \leq 1\} \subset \mathscr{P}(E + F)$. Since $\preceq$ is implemented by a state on $\mathscr{P}(E + F)$, we cannot have $P_{f_t(E)} \cong P_F \ \forall t \geq r$. Thus the t_j may be chosen so as to give the strict inequality $P_{f_{t_j}(E)} \prec P_F \ \forall j$ so that the sequence $P_{f_{t_j}(K+E)}$ satisfies items (i) and (ii).

To show that this sequence also satisfies item (iii) we put $\tilde{L} = f_1^{-1}(L)$ and $\tilde{M} = f_1^{-1}(M)$. Then the sequences $P_{f_{t_j}(\tilde{L})}$ and $P_{f_{t_j}(\tilde{M})}$ will satisfy the requirements of item (iii). □

Proposition 2.4. *The following are both true*:

i. *Let* $P_G \in \mathscr{D}(\mathscr{H})$ *and let* $P \in \mathscr{P}(\mathscr{H})$ *be such that* $\mathbb{O} \prec P$. *Then there exists* $Q \in \mathscr{D}(G)$ *such that* $Q \preceq P$.

ii. *Let* $\mathscr{A} \subset \mathscr{D}(\mathscr{H})$ *be a set of mutually orthogonal projections. Suppose that there exists* $R \in \mathscr{P}(\mathscr{H})$ *such that* $\mathbb{O} \prec R$ *and such that* $R' \in \mathscr{A} \Rightarrow R \preceq R'$, *then* $\mathscr{A}$ *is a finite set.*

Proof. i. First we claim that there exists $Q \in \mathscr{D}(\mathscr{H})$ such that $Q \preceq P$. Suppose that this is false. Then there exists $P_E \in \mathscr{P}(\mathscr{H})$ such that $P_E \leq P_{E'}$ for some $P_{E'} \in \mathscr{D}(\mathscr{H})$ and such that $\mathbb{O} \prec P_E \preceq R \ \forall R \in \mathscr{D}(\mathscr{H})$. Now we can clearly construct two sequences P_j and Q_j, both in $\mathscr{D}(E^{\perp})$, such that $P_j \perp P_k$, $Q_j \perp Q_k$ $\forall j, k \in \mathbb{N}$ with $j \neq k$, and such that $P_j \perp Q_k$, $P_j \cong Q_j \ \forall j, k \in \mathbb{N}$. Let $P_F = \sum_{j=1}^{\infty} P_j$ and define S_n to be $P_E + \sum_{j=2}^{n} P_j$. Clearly, $S_n \in \mathscr{D}(\mathscr{H}) \ \forall n$ and since $\mathbb{O} \prec P_E$, S_n is a strictly $\preceq$ increasing sequence. As $\mathscr{D}(F)$, and hence also $P_E + \mathscr{D}(F)$, are uniformly connected, there exists $S \in \mathscr{D}(F)$ such that $S_n \xrightarrow{\preceq} P_E + S$. Thus for some $n_0 \in \mathbb{N}$, $S \prec S_{n_0}$, so that by A4, $S + P_E \preceq S + Q_{n_0+1} \prec S_{n_0} + Q_{n_0+1} \cong S_{n_0+1}$. The contradiction verifies the claim.

Now suppose that the assertion in (i) is false. Then there exists $R \in \mathscr{P}(\mathscr{H})$ such that $\mathbb{O} \prec R \preceq Q \ \forall Q \in \mathscr{D}(G)$. The claim above implies that there exists

$P_L \in \mathscr{D}(\mathscr{H})$ such that $P_L \prec R$. As pointed out in Lemma 2.2, there exist $P_E \in \mathscr{D}(L)$ and $P_F \in \mathscr{D}(G)$ such that $P_E \perp P_F$. Let $P_K \in \mathscr{D}(F)$ be such that $P_K \prec P_F - P_K$ and let $P_{E_1}, P_{E_2} \in \mathscr{D}(E)$ be such that $P_{E_1} \preceq P_{E_2}$. Then A4 gives $P_K + P_{E_1} \prec P_F + P_{E_2} - P_K$. Since $\mathscr{D}(K + E_1)$ is $\preceq$ connected and $P_{E_1} \in \mathscr{D}(K + E_1)$, there exists $S \in \mathscr{D}(K + E_1)$ such that $S \in \preceq\inf \mathscr{D}(K)$. As $P_{E_2} \perp (P_K + P_{E_1})$, $\exists T \in \mathscr{D}(\{K + E_1\}^{\perp})$ such that $\mathbf{O} \prec T \prec R$. Let the sequence S_j in $\mathscr{D}(K)$ be $\preceq$ convergent to S. For each $j \in \mathbf{N}$, let $S_j' \leq S_j$ be such that $S_j' \cong S_j - S_j'$. Then $T + S \preceq T + (S_j - S_j') \prec S_j' + (S_j - S_j') \to S$, a contradiction.

ii. Suppose that the statement is not true, so that $\mathscr{A}$ is an infinite set. By taking a subset if necessary, we may assume that $\mathscr{A} = \{Q_j : j \in \mathbf{N}\}$, that is $\mathscr{A}$ is countable. We may also assume that $\sum_{j \in \mathbf{N}} Q_j \leq Q - \sum_{j \in \mathbf{N}} Q_j$, for some $Q \in \mathscr{D}(\mathscr{H})$ such that $\sum_{j \in \mathbf{N}} Q_j < Q$. Define P_n to be $\sum_{j=n}^{\infty} Q_j$. Then P_n is a strictly $\preceq$ decreasing sequence. Now $\forall n \in \mathbf{N}$, $R \prec P_n$ and $P_n \in \mathscr{D}(\mathscr{H})$; thus, by (i) and by the uniform connectedness of $\mathscr{D}(\mathscr{H})$, there exists $P_0 \in \mathscr{D}(\mathscr{H})$ such that $P_0 \in \preceq \inf_{n \in \mathbf{N}} P_n$. Put $Q - \sum_{j \in \mathbf{N}} Q_j = P_F$ and $\mathbf{1} - \sum_{j \in \mathbf{N}} Q_j = P_K$. Then $P_0 \preceq P_F$ and $P_F \in \mathscr{D}(K)$. Hence, by (i) and by the connectedness of $\mathscr{D}(K)$, we may assume that $P_0 \in \mathscr{D}(K)$. Let $R' \in \mathscr{D}(K)$ be such that $R' \perp P_0$ and $\mathbf{O} \prec R' \preceq R$. Then, for some large enough n, $P_n \prec P_0 + R'$ and for such n, $P_n = P_{n+1} + Q_n \prec P_0 + R' \preceq P_0 + Q_n \Rightarrow P_{n+1} \prec P_0$, by A4. The contradiction gives the required result. □

Corollary 2.5. *The following are both true*:

i. *If P_j is a sequence of mutually orthogonal projections of $\mathscr{P}(\mathscr{H})$, then $P_j \overset{\preceq}{\to} \mathbf{O}$.*

ii. *$\mathscr{P}(E)$ is $\preceq$ connected if $P_E \in \mathscr{D}(\mathscr{H})$.*

Proof. **i.** Clear.

ii. In view of Proposition 2.4 (i), it is sufficient to show that P_E is a $\preceq$ limit point of $\mathscr{D}(E)$. If $P_E \cong \mathbf{O}$ or $P_E \cong \mathbf{1}$ then this is trivial, so assume that $\mathbf{O} \prec P_E \prec \mathbf{1}$ and suppose that the result is false. Then there exists $Q \in \mathscr{P}(E)$ such that $\mathbf{O} \prec Q$ and such that $P \preceq P_E - Q\ \forall P \in \mathscr{D}(E)$. Now there exist $S \in \mathscr{D}(E)$ and $T \in \mathscr{D}(E^{\perp})$ such that $S \cong T \prec Q$. A4 gives $P_E - Q + T \prec P_E = P_E - S + S \cong P_E - S + T \Rightarrow P \preceq P_E - Q \prec P_E - S\ \forall P \in \mathscr{D}(E)$, a contradiction, since $P_E - S \in \mathscr{D}(E)$. □

Lemma 2.6. *Let $P_E \in \mathscr{D}(\mathscr{H})$ be such that $P_E \preceq P_E^{\perp}$ and let P_1, P_2, Q_1 and Q_2, all in $\mathscr{P}(E)$, be such that $P_1 \perp Q_1$, $P_2 \perp Q_2$, $P_1 \preceq P_2$ and $Q_1 \preceq Q_2$. Then $P_1 + Q_1 \preceq P_2 + Q_2$. Hence for any R_1, $R_2 \in \mathscr{P}(E)$ and $Q \in \mathscr{P}(\mathscr{H})$ such that $R_1 \leq Q$ and $R_2 \leq Q$ we have $R_1 \preceq R_2 \Leftrightarrow Q - R_2 \preceq Q - R_1$.*

Proof. Since $P_E^{\perp} \in \mathscr{D}(\mathscr{H})$ and $P_E \preceq P_E^{\perp}$, we can find, using the $\preceq$ connectedness of $\mathscr{P}(E^{\perp})$ and the continuity of $\preceq$ on $\mathscr{P}(\mathscr{H})$ (and hence on $\mathscr{P}(E^{\perp})$),

$R \leq P_E^{\perp}$ such that $P_1 \preceq R \preceq P_2$. This gives, by axiom A4, $P_1 + Q_1 \preceq R + Q_1 \preceq R + Q_2 \preceq P_2 + Q_2$, as required. We note that this result implies the following: if $P \leq P_E$ and $Q \leq P_E$, then $P \preceq Q \Leftrightarrow P_E - Q \preceq P_E - P$. To prove the second part we let $R = R_1 \vee R_2$ (the minimal projection in $\mathscr{P}(\mathscr{H})$ containing R_1 and R_2 as sub-projections). As $R \preceq R^{\perp}$, the first result gives $R_1 \preceq R_2 \Leftrightarrow R - R_2 \preceq R - R_1$. Axiom A4 gives $R - R_2 \preceq R - R_1 \Leftrightarrow Q - R_2 \preceq Q - R_1$, giving the required result. □

Proposition 2.7. *$\mathscr{P}(\mathscr{H})$ is $\preceq$ connected.*

Proof. It will suffice to show that $\mathbb{I} \in \preceq\sup \mathscr{D}(\mathscr{H})$. Now suppose that $\mathbb{I} \notin \preceq\sup \mathscr{D}(\mathscr{H})$. Then $\exists P_G \in \mathscr{P}(\mathscr{H})$ such that $\mathbb{O} \prec P_G \prec \mathbb{I}$ and such that $Q \preceq P_G^{\perp}\ \forall Q \in \mathscr{D}(\mathscr{H})$. We examine two cases:

Case 1 $\mathscr{D}(G^{\perp})$ is $\preceq$ dense in $\mathscr{P}(G^{\perp})$.

Clearly $P_G \notin \mathscr{D}(\mathscr{H})$, being of too small a rank. Take $Q \in \mathscr{D}(G^{\perp})$, then $P_G \preceq Q^{\perp}$. As $\mathscr{D}(G^{\perp})$ is $\preceq$ dense in $\mathscr{P}(G^{\perp})$ and $\preceq$ connected, $\exists P \in \mathscr{D}(G^{\perp})$ such that $P \cong Q^{\perp}$. Let R_1, $R_2 \in \mathscr{D}(G^{\perp})$ be such that $R_1 \perp R_2$, $(R_1 + R_2) \perp P$, $R_1 \preceq R_2$ and $R_1 \prec P_G$. Axiom A4 gives $R_1 + P_G \preceq R_1 + P \preceq P + R_2$. Since $R_1 + P_G \perp P + R_2$, we have $R_1 + P_G \preceq (R_1 + P_G)^{\perp}$ so that, by Lemma 2.6, $\mathbb{I} - P_G \prec \mathbb{I} - R_1$, a contradiction.

Case 2 $\mathscr{D}(G^{\perp})$ is not $\preceq$ dense in $\mathscr{P}(G^{\perp})$.

This implies that $\mathbb{O} \prec P_G$ by Corollary 2.5 (ii) and that there exists $P_K \in \mathscr{P}(G^{\perp})$ such that $\mathbb{O} \prec P_K \prec P_G^{\perp}$, $P_K \preceq P_G$ and such that $Q \preceq P_G^{\perp} - P_K\ \forall Q \in \mathscr{D}(G^{\perp})$. Put $P_J = P_K + P_G$. Then $\exists S, T \in \mathscr{D}(J^{\perp})$ such that $S \perp T$, $S \prec P_K$, and $S \preceq T$. Axiom A4 gives $S + P_K \preceq S + P_G \preceq T + P_G \Rightarrow S + P_K \preceq (S + P_K)^{\perp}$ so that, by Lemma 2.6, $P_G^{\perp} - P_K \prec P_G^{\perp} - S \in \mathscr{D}(G^{\perp})$, a contradiction. This completes the proof. □

We remark that, by Lemma 2.2 and Proposition 2.7, if G is any infinite dimensional subspace of $\mathscr{H}$, then $\mathscr{D}(G)$, and hence also $\mathscr{P}(G)$, are $\preceq$ connected in $\mathscr{P}(\mathscr{H})$. The following results, which we list without proof, are easy consequences of the foregoing results (cf. [1], Theorem 2.3).

Theorem 2.8. *The following statements are all true*:

i. *$\mathscr{P}(\mathscr{H})$ is $\preceq$ compact and hence every nonempty subset of $\mathscr{P}(\mathscr{H})$ has an* inf *and a* sup *with respect to $\preceq$.*

ii. *Let $\mathscr{A} \subset \mathscr{P}(\mathscr{H})$ be any set of mutually orthogonal projections such that $\forall P \in \mathscr{A}$, $\mathbb{O} \prec P$. Then $\mathscr{A}$ is at most a countably infinite set.* □

Joint $\preceq$ continuity on $\mathscr{P}(\mathscr{H})$ is essentially a strengthening of axiom A4.

Proposition 2.9. *The following statements are equivalent.*

i. *Addition is jointly $\preceq$ continuous on $\mathscr{P}(\mathscr{H})$.*

ii. *Let* P_1, P_2, Q_1 *and* Q_2, *all in* $\mathscr{P}(\mathscr{H})$, *be such that* $P_1 \perp Q_1$, $P_2 \perp Q_2$, $P_1 \preceq P_2$ *and* $Q_1 \preceq Q_2$, *then* $P_1 + Q_1 \preceq P_2 + Q_2$

Proof. i. $\Rightarrow$ ii. First we remark that the joint $\preceq$ continuity of addition immediately gives $P_1 + Q_1 \cong P_2 + Q_2$ if $P_1 \cong P_2$ and $Q_1 \cong Q_2$. We shall assume that $P_2 + Q_2 \prec \mathbb{I}$ and that $\mathbb{O} \prec P_1$, $\mathbb{O} \prec Q_1$, $\mathbb{O} \prec P_2$, $\mathbb{O} \prec Q_2$, otherwise the result is trivially true. Assume, with no loss of generality, that $R \leq Q_2^\perp$ for some $R \in \mathscr{D}(\mathscr{H})$. Since $\mathbb{O} \prec P_2 \prec Q_2^\perp$, there exist by Proposition 2.7, P_1', $P_2' \in \mathscr{D}(\mathscr{H})$ such that $P_1' \leq P_2' \leq Q_2^\perp$, $P_1' \cong P_1$ and $P_2' \cong P_2$. Hence there similarly exist Q_1', $Q_2' \in \mathscr{D}(\mathscr{H})$ such that $Q_1' \leq Q_2' \leq P_2'^\perp$, $Q_1' \cong Q_1$ and $Q_2' \cong Q_2$. Thus $P_1 + Q_1 \cong P_1' + Q_1' \preceq P_2' + Q_2' \cong P_2 + Q_2$, as required.

ii. $\Rightarrow$ i.

Let P, Q and the nets P_j, $Q_j : j \in \mathscr{J}$, all in $\mathscr{P}(\mathscr{H})$, be such that $P_j \overset{\preceq}{\to} P$, $Q_j \overset{\preceq}{\to} Q$, $P \perp Q$ and $P_j \perp Q_j \, \forall j$. First we consider the case where both P_j and Q_j are monotone $\preceq$ increasing and assume that $\mathbb{O} \prec P$ and $\mathbb{O} \prec Q$, lest the result be trivial. Hence we may also assume that P, $Q \in \mathscr{D}(\mathscr{H})$. Now item (ii) implies that $P_j + Q_j$ is monotone $\preceq$ increasing and hence $\preceq$ convergent to $R \in \preceq\sup_j (P_j + Q_j)$. Clearly $R \preceq P + Q$. Suppose that $R \prec P + Q$. Then, by hypothesis, there exist $\tilde{P} < P$ and $\tilde{Q} < Q$ such that $R \prec \tilde{P} + \tilde{Q} \prec P + Q$. This implies that for all j large enough, we have $P_j + Q_j \prec \tilde{P} + \tilde{Q}$, $\tilde{P} \prec P_j \prec P$ and $\tilde{Q} \prec Q_j \prec Q$. The contradiction gives $R \cong P + Q$. For the case where both P_j and Q_j are monotone $\preceq$ decreasing a similar strategy gives the same result.

Now we consider the case where P_j is monotone $\preceq$ increasing and Q_j is monotone $\preceq$ decreasing and consider only the case $\mathbb{O} \prec P$, otherwise the result is trivially true. Hence we may assume here that P is of infinite rank. Now there exists, by Theorem 2.8, a subnet $P_{j_k} + Q_{j_k}$ of $P_j + Q_j$ which is $\preceq$ convergent to R, say. We claim that $R \cong P + Q$. To show this we suppose first that $R \prec P + Q$. As P is of infinite rank, there exists $\tilde{P} < P$ such that $\tilde{P}^\perp$ is of infinite rank and such that $R \prec \tilde{P} + Q \prec P + Q$. This in turn implies that there exists $\tilde{Q} < \tilde{P}^\perp$ such that $R \prec \tilde{P} + Q \prec \tilde{P} + \tilde{Q} \prec P + Q$. Hence there exists $j_0 \in \mathscr{J}$ such that $\forall j_k > j_0$, $P_{j_k} + Q_{j_k} \prec \tilde{P} + Q$ and $\tilde{P} \prec P_{j_k} \prec P$, $Q \prec Q_{j_k} \prec \tilde{Q}$. The last two inequalities give, by hypothesis, $\tilde{P} + Q \prec P_{j_k} + Q_{j_k} \, \forall j_k > j_0$, a contradiction. A similar argument establishes that we cannot have $P + Q \prec R$, and the claim is verified. Finally, we note that for the general case, an easy argument involving subnets of $P_j + Q_j$ completes the proof. □

Proposition 2.10. *Addition is separately* $\preceq$ *continuous on* $\mathscr{P}(\mathscr{H})$.

Proof. Let P, P_G and the net $P_j : j \in \mathscr{J}$, all in $\mathscr{P}(\mathscr{H})$, be such that $P_j \overset{\preceq}{\to} P$, $P \perp P_G$ and $P_j \perp P_G \, \forall j$. If $P_G^\perp$ is of finite rank, then every subnet of $P_G + P_j$ has itself a subnet which uniformly converges to a projection of the form

$P_G + P'$ where $P' \perp P_G$ and, by the uniform continuity of $\preceq$, $P' \cong P$, hence the result. For $P_G^{\perp}$ of infinite rank, we assume for the moment that P_j is monotone $\preceq$ increasing, and hence $\preceq$ convergent to R, say. Clearly $R \preceq P + P_G$. If $R \prec P + P_G$, then, since $\mathscr{D}(G^{\perp})$ and hence $P_G + \mathscr{D}(G^{\perp})$ are $\preceq$ connected, there exists $P' \in \mathscr{D}(G^{\perp})$ such that, for all j, $P_G + P_j \preceq P_G + P' \prec P_G + P$, contradicting $P_j \overset{\preceq}{\to} P$. Hence $P_G + P_j \overset{\preceq}{\to} P_G + P$. A similar argument gives the same result if we assume P_j to be monotone $\preceq$ decreasing. For P_j arbitrary, we note that every subnet of P_j has a subnet that $\preceq$ converges to $P_G + P$, and the result follows. □

Corollary 2.11. *$\mathscr{P}(\mathscr{H})$ is $\preceq$ first countable.*

Proof. Let $P_E \in \mathscr{D}(\mathscr{H})$ be such that $\mathbb{O} \prec P_E \preceq P_E^{\perp}$ and let the sequence Q_j in $\mathscr{D}(E)$ be such that $\mathbb{O} \prec Q_j\ \forall j$ and $Q_j \overset{\preceq}{\to} \mathbb{O}$. Then Lemma 2.6 and Corollary 2.5 (ii) imply that $P_E - Q_j \overset{\preceq}{\to} P_E$ and Proposition 2.10 gives $P_E^{\perp} + P_E - Q_j \overset{\preceq}{\to} \mathbb{1}$. Put $P_E^{\perp} + P_E - Q_j = P_j$ then the two sets of $\preceq$ intervals $[\mathbb{O}, Q_j) : j \in \mathbb{N}$ and $(P_j, \mathbb{1}] : j \in \mathbb{N}$ form countable $\preceq$ neighbourhood bases for $\mathbb{O}$ and $\mathbb{1}$ respectively. The remark at the end of Lemma 2.2 completes the proof. □

§ 3. Implementability of Continuous CP's

In this section, we prove a number of results which will culminate in the construction of an additive measure on $\mathscr{P}(\mathscr{H})$. This measure will, by Theorem 1.2, lead to a unique state which implements $\preceq$.

Lemma 3.1. *For $n \in \mathbb{N}$, let the sets $\{P_j : 1 \leq j \leq n\}$ and $\{Q_j : 1 \leq j \leq n\}$, each of which consists of mutually orthogonal projections in $\mathscr{P}(\mathscr{H})$, be such that $P_j \perp Q_k\ \forall j, k$ and such that for each j, $P_j \preceq Q_j$. Then $\sum_{j=1}^{n} P_j \preceq \sum_{j=1}^{n} Q_j$.*

Proof. Let $m \in \mathbb{N}$ be such that $m < n$ and suppose that $\sum_{j=1}^{m} P_j \preceq \sum_{j=1}^{m} Q_j$. Axiom **A4** gives $\sum_{j=1}^{m+1} P_j = P_{m+1} + \sum_{j=1}^{m} P_j \preceq P_{m+1} + \sum_{j=1}^{m} Q_j \preceq Q_{m+1} + \sum_{j=1}^{m} Q_j = \sum_{j=1}^{m+1} Q_j$. Since $P_1 \preceq Q_1$ the result follows by induction. □

Lemma 3.2. *Given any $n \in \mathbb{N}$, there exists a set $\{P_{E_j} : j \in K(n)\}$, where $K(n) = \{1, 2, 3, \ldots, 2^n\}$, of mutually orthogonal projections such that $P_{E_j} \in \mathscr{D}(\mathscr{H})\ \forall j$, $P_{E_j} \cong P_{E_k}\ \forall j, k$ and such that $\sum_{j \in K(n)} P_{E_j} = \mathbb{1}$.*

Proof. Suppose the result to be true for some $n_0 \in \mathbb{N}$. For each $j \in K(n_0)$, $\preceq$ is obviously a uniformly continuous CP when restricted to $\mathscr{P}(E_j)$ and because $\mathscr{D}(E_j)$ is $\preceq$ connected and $\preceq$ dense in $\mathscr{P}(E_j)$, there exists $P_{\tilde{E}_j} \in \mathscr{D}(E_j)$ $(\subset \mathscr{D}(\mathscr{H}))$ such that $P_{E_j} - P_{\tilde{E}_j} \cong P_{\tilde{E}_j}$. This, together with Lemma 3.1, is enough to show that the result is also true for both $n = 1$ and $n = n_0 + 1$ and hence, by induction, for all $n \in \mathbf{N}$. $\square$

Definition 3.3. *A set $\{P_{E_j} : j \in K(n)\}$ satisfying the assertion of Proposition 3.2 is said to be an equipartition of the identity of order 2^n. Let $\mathscr{L} = \{P_{E_j} : j \in K(n)\}$ be an equipartition of the identity of order 2^n, we define $\mathscr{S}(\mathscr{L}) \subset \mathscr{D}(\mathscr{H})$ to be the set $\left\{\sum_{j \in K} P_{E_j} : K \subset K(n),\ K \neq \varnothing,\ K \neq K(n)\right\}$. It is clear that there exists a sequence of equipartitions $\mathscr{L}_n$ of the identity such that $\mathscr{L}_n$ is of order 2^n and such that the $\mathscr{L}_n$ are "nested" in the sense that $\mathscr{S}(\mathscr{L}_n) \subset \mathscr{S}(\mathscr{L}_{n+1})\ \forall n \in \mathbf{N}$. When there is such nesting, we define $\mathscr{S}_\infty$ to be $\bigcup_{n \in \mathbf{N}} \mathscr{S}(\mathscr{L}_n)$.* $\square$

From now on we work with a fixed set of nested equipartitions $\mathscr{L}_n$ as set out in Definition 3.3.

Proposition 3.4. *$\mathscr{S}_\infty$ is $\preceq$ dense in $\mathscr{P}(\mathscr{H})$.*

Proof. Assuming the axiom of choice, let $P_j \in \mathscr{L}_j\ \forall j \in \mathbf{N}$. Clearly $P_j \overset{\preceq}{\to} \mathbf{O}$ and $P_j^\perp (\in \mathscr{L}_j) \overset{\preceq}{\to} \mathbf{I}$. Now let $P \in \mathscr{P}(\mathscr{H})$ be such that $\mathbf{O} \prec P \prec \mathbf{I}$. Then, because $\mathbf{O} \prec P^\perp$, there exists, for some $n \in \mathbf{N}$, $R \in \mathscr{L}_n$ such that $R \prec P$. We may assume, without loss of generality, that $R \perp P$. We wish to show that P is a $\preceq$ limit point of $\mathscr{S}_\infty$. Consider the set $\Gamma = \{Q \in \mathscr{S}_\infty : Q \preceq P\}$. Let $Q_0 \in \preceq\sup \Gamma$. As $Q_0 \preceq P \leq R^\perp$, we may also assume that $Q_0 \perp R$. Because of the nesting of the $\mathscr{L}_j$ and because $Q_0 \leq R^\perp$, we have $Q_0 \in \preceq\sup \tilde{\Gamma}$ where $\tilde{\Gamma} = \{Q \in \mathscr{S}_\infty : Q \preceq P, Q \leq R^\perp\}$. We now show that $Q_0 \cong P$. Suppose this is false, so that $Q_0 \prec P$. Then there exists $R_0 \in \mathscr{L}_m$ for some m such that $R_0 \leq R$ and such that $Q_0 + R_0 \prec P$. Now let the sequence Q_j in $\tilde{\Gamma}$ $\preceq$ converge to Q_0. Then there clearly exists n such that $Q_0 \prec Q_n + R_0$. As $Q_j \preceq Q_0\ \forall j$, **A4** gives $Q_0 \prec Q_n + R_0 \preceq Q_0 + R_0 \prec P$. Since $Q_n + R_0 \in \mathscr{S}_\infty$, this contradicts $Q_0 \in \preceq\sup \Gamma$; hence the result. $\square$

Definition 3.5. *Define the function $\mu: \mathscr{S}_\infty \to [0, 1]$ as follows:*

In the notation of Definition 3.3, if $\sum_{j \in K} P_{E_j} \in \mathscr{S}(\mathscr{L}_n)$, then $\mu\left(\sum_{j \in K} P_{E_j}\right) = 2^{-n}\sharp(K)$, where $\sharp(K)$ is the cardinality of the set K. It is trivially easy to verify that μ is a well defined function because of the nested structure of $\mathscr{S}_\infty$. $\square$

Proposition 3.6. *The function μ in Definition 3.5 is (finitely) additive and $\preceq$ continuous.*

Proof. The additivity is obvious from the definition. To show the continuity, we first note that if $P_j \in \mathscr{S}_\infty\ \forall j$, then $P_j \overset{\preceq}{\to} \mathbf{O} \Leftrightarrow \mu(P_j) \to 0$. Now let P_j be a $\preceq$ increasing sequence in $\mathscr{S}_\infty$ which $\preceq$ converges to $P \in \mathscr{S}_\infty$. Because of the nesting in $\mathscr{S}_\infty$, we may assume without loss of generality that $P_j \leq P\ \forall j$.

Thus $P - P_j \overset{\preceq}{\to} 0$ and hence $\mu(P) = \lim \mu(P_j)$, by the above remark and by additivity. We similarly reach the same conclusion if P_j is a $\preceq$ decreasing sequence. Now let $\mu(P_j)$ be any sequence in $\mathscr{S}_\infty$ which $\preceq$ converges to $P \in \mathscr{S}_\infty$. If $\mu(P_j)$ does not converge to $\mu(P)$, then P_j has either a $\preceq$ decreasing or a $\preceq$ increasing subsequence P_{j_k} such that $\mu(P_{j_k}) \not\to \mu(P)$, in contradiction with the results established above. This leads to the required result. □

The continuity proved in Proposition 3.6 and the $\preceq$ density of $\mathscr{S}_\infty$ in $\mathscr{P}(\mathscr{H})$ allows us to extend μ to a $\preceq$ continuous function on all of $\mathscr{P}(\mathscr{H})$. Accordingly we regard the domain of μ to be all of $\mathscr{P}(\mathscr{H})$ from now on. Clearly μ satisfies $P \preceq Q \Leftrightarrow \mu(P) \leq \mu(Q)\ \forall P, Q \in \mathscr{P}(\mathscr{H})$. It is clear that joint $\preceq$ continuity of addition would immediately imply additivity of μ on all of $\mathscr{P}(\mathscr{H})$ which, by Theorem 1.2, would lead to the implementation of $\preceq$ by a state. However, in view of Lemma 2.6 and Proposition 2.9, we have the following obvious result:

Lemma 3.7. *Let $P_E \in \mathscr{D}(\mathscr{H})$. If $P_E \preceq P_E^\perp$ then μ is additive on $\mathscr{P}(E)$.* □

Lemma 3.8. *Let S_1, $S_2 \in \mathscr{S}_\infty$ be mutually orthogonal. If $P_1 \leq S_1$ and $P_2 \leq S_2$ then $\mu(P_1 + P_2) = \mu(P_1) + \mu(P_2)$.*

Proof. We first look at the case $P_1 \in \mathscr{S}_\infty$. Now there clearly exists a sequence Q_j in $\mathscr{S}_\infty$ such that $Q_j \leq S_2$ and such that $Q_j \overset{\preceq}{\to} P_2$. Separate $\preceq$ continuity of addition, $\preceq$ continuity of μ and additivity of μ on $\mathscr{S}_\infty$ lead to $\mu(P_1 + P_2) = \lim_{j\to\infty} \mu(P_1 + Q_j) = \lim_{j\to\infty} \{\mu(P_1) + \mu(Q_j)\} = \mu(P_1) + \mu(P_2)$ as required. If $P_1 \notin \mathscr{S}_\infty$, then we again have $\mu(P_1 + P_2) = \lim_{j\to\infty} \mu(P_1 + Q_j) = \lim_{j\to\infty} \{\mu(P_1) + \mu(Q_j)\}$ $= \mu(P_1) + \mu(P_2)$, where we have used the additivity proved in the case $P_1 \in \mathscr{S}_\infty$. This completes the proof. □

Lemma 3.9. *Suppose we have another sequence of nested equipartitions $\tilde{\mathscr{L}}_n$ of the identity such that $\tilde{\mathscr{L}}_n$ is of order 2^n. Let $\tilde{\mathscr{S}}_\infty = \bigcup_{n\in\mathbb{N}} \mathscr{S}(\tilde{\mathscr{L}}_n)$, in the notation of Definition 3.3. Suppose that for some $n \in \mathbb{N}$ there exist $P_E \in \mathscr{L}_n$ and $P_{\tilde{E}} \in \tilde{\mathscr{L}}_n$ such that $P_E \leq P_{\tilde{E}}$, then the following are true:*

i. $P_E = P_{\tilde{E}}$.

ii. *If $\tilde{\mu}$ is the function constructed from $\tilde{\mathscr{S}}_\infty$, as set out in Definition 3.5, then $\tilde{\mu} = \mu$.*

Proof. i. Assume the hypothesis and suppose the result is not true so that $P_E < P_{\tilde{E}}$. Let $\mathscr{L}_n = \{P_j : j \in K(n)\}$, where $P_1 = P_E$ and $\tilde{\mathscr{L}}_n = \{\tilde{P}_j : j \in K(n)\}$, where $\tilde{P}_1 = P_{\tilde{E}}$. Then for each $j \in K(n)$, there exists $Q_j \in \mathscr{D}(\mathscr{H})$ such that $Q_j \cong P_j$ and $Q_j < \tilde{P}_j$, with $Q_1 = P_1$. Define $\mathscr{A}$ to be $\{Q_j : j \in K(n)\}$. We claim that $\mu\left(\sum_{k=1}^{2^n} Q_k\right) = \sum_{k=1}^{2^n} \mu(Q_k)$. Suppose inductively that $\mu\left(\sum_{Q\in\mathscr{B}} Q\right) = \sum_{Q\in\mathscr{B}} \mu(Q)$ for any subset $\mathscr{B}$ of $\mathscr{A}$ of cardinality N, where $1 < N < 2^n$. Now let $\{Q_{j_k} : 1 \leq k \leq N+1\}$ be any subset of $\mathscr{A}$ of cardinality $N+1$ such that $j_k \leq j_{k+1}$ for all k. The inductive hypothesis implies that $\sum_{k=2}^{N+1} \tilde{Q}_{j_k} \cong \sum_{k=2}^{N+1} P_k$ so that application of axiom A4 and additivity of μ on $\mathscr{S}_\infty$ yields $\mu\left(\sum_{k=1}^{N+1} Q_{j_k}\right) = \mu\left(Q_{j_1} + \sum_{k=2}^{N+1} Q_{j_k}\right) = \mu\left(P_1 + \sum_{k=2}^{N+1} Q_{j_k}\right) = \mu\left(P_1 + \sum_{k=2}^{N+1} P_k\right) = \sum_{k=1}^{N+1} \mu(P_k) = \sum_{k=1}^{N+1} \mu(Q_{j_k})$. We remark that this argument also demonstrates that μ is additive on any subset of $\mathscr{A}$ of cardinality 2, and so verifying the claim by induction. Since $Q_k \cong P_k \; \forall k \in K(n)$, this additivity gives $\mu\left(\sum_{k=1}^{2^n} Q_k\right) = 1$, contradicting $Q_k \prec \tilde{P}_k \; \forall k \in K(n)$. This completes proof.

ii. The proof of item (i) shows that μ is additive on $\tilde{\mathscr{L}}_n$. We now show that μ is also additive on $\tilde{\mathscr{L}}_m$ for any $m \in \mathbb{N}$. If $m > n$ then $\tilde{\mathscr{L}}_m$ induces an equipartition $\tilde{\mathscr{B}}$ of P_E and $\mathscr{L}_m$ induces equipartitions $\mathscr{B}$ of P_E and $\mathscr{C}$ of $P_E^\perp$. As $P_E \preceq P_E^\perp$, Lemma 2.6 implies that $\tilde{\mathscr{B}} \cup \mathscr{C}$ is also an equipartition of the identity of order 2^m. Application of item (i.) shows that μ is additive on $\tilde{\mathscr{B}} \cup \mathscr{C}$ and another application shows additivity on $\tilde{\mathscr{L}}_m$ and hence on all of $\tilde{\mathscr{S}}_\infty$. This immediately gives $\mu = \tilde{\mu}$. □

Proposition 3.10. *Let $P_E \in \mathscr{S}_\infty$, then μ is additive on $\mathscr{P}(E)$.*

Proof. Let $P, Q \in \mathscr{P}(E)$ be mutually orthogonal. We can clearly construct a set of nested equipartitions $\tilde{\mathscr{L}}_n$ of the identity such that there exist $\tilde{S} \in \tilde{\mathscr{S}}_\infty = \bigcup_{n \in \mathbb{N}} \mathscr{S}(\tilde{\mathscr{L}}_n)$ and $Q' \in \mathscr{P}(\mathscr{H})$ satisfying $P \leq \tilde{S}$ and $Q \cong Q' \leq \tilde{S}^\perp$. By Lemma 3.9, we can also easily ensure that $\tilde{\mathscr{S}}_\infty$ is such that $\tilde{\mu} = \mu$. Lemma 3.8 gives $\tilde{\mu}(P+Q) = \tilde{\mu}(P+Q') = \tilde{\mu}(P) + \tilde{\mu}(Q') = \tilde{\mu}(P) + \tilde{\mu}(Q)$ as required. □

Corollary 3.11. *Let $P_E \in \mathscr{D}(\mathscr{H})$ be such that $P_F \leq P_E$ for some $P_F \in \mathscr{S}_\infty$, then μ is additive on $\mathscr{P}(E)$.*

Proof. If $P_E \prec \mathbb{1}$ then by Lemma 3.9, the problem reduces to that of Proposition 3.10. So we assume that $P_E \cong \mathbb{1}$. By enlarging P_E where necessary, we may also assume that $P_E - P_F \in \mathscr{D}(\mathscr{H})$ and that $\mathbb{O} \prec P_E - P_F$. By Lemma 2.3, there exists a sequence P_{G_j} in $\mathscr{D}(\mathscr{H})$ (where, in the notation of Lemma 2.3, P_{G_j} is of the form $P_{f_{t_j}(E^\perp + F)}$, and where $P_{f_1(E^\perp)} = P_E - P_F$, $P_{f_t(F)} = P_F \forall t$) such that, for all j, $P_{G_j} \prec P_E$, $P_F \leq P_{G_j}$ and such that $P_{G_j} \xrightarrow{u} P_E$. Now let $P, Q \in \mathscr{P}(E)$ be mutually orthogonal. We choose that the sequence P_{G_j} also satisfies item (iii) of Lemma 2.3 so that there exist sequences P_j and Q_j such that, for each j, $P_j \perp Q_j$, $P_j + Q_j \leq G_j$ and such that $P_j \xrightarrow{u} P$, $Q_j \xrightarrow{u} Q$. Since $P_j + Q_j \xrightarrow{u} P + Q$, and since, by Proposition 3.10, μ is additive on each $\mathscr{P}(G_j)$, the result follows at once. □

Corollary 3.12. *Let* $\tilde{\mathscr{L}}_n$ *be another sequence of nested equipartitions of the identity as set out in Lemma* 3.9 *and let* $\tilde{\mu}$ *be the function on* $\mathscr{P}(\mathscr{H})$ *constructed from the* $\tilde{\mathscr{L}}_n$. *Then* $\tilde{\mu} = \mu$.

Proof. Let $P_{\tilde{E}} \in \tilde{\mathscr{S}}_\infty$ and let $P_E \in \mathscr{S}_\infty$. Let $P_G \leq P_E$ and $P_{\tilde{G}} \leq P_{\tilde{E}}$ be mutually orthogonal (see Lemma 2.2). The following two cases are sufficient for the proof.

Case 1 $\mathbb{O} \prec P_G$ and $\mathbb{O} \prec P_{\tilde{G}}$.

We assume without loss of generality that $P_G + P_{\tilde{G}} \prec \mathbb{I} - P_G - P_{\tilde{G}}$ and hence, by Lemma 3.7, we may also assume that $P_G \in \mathscr{L}_n$ and that $P_{\tilde{G}} \in \tilde{\mathscr{L}}_n$ for some $n \in \mathbb{N}$. Another sequence $\mathscr{L}'_k$ of nested equipartitions of the identity, with the associated function μ', can be constructed such that by an appropriate choice of the equipartition $\mathscr{L}'_n$, Lemma 3.9 gives $\mu = \mu' = \tilde{\mu}$ as required.

Case 2 $\mathbb{O} \cong P_{\tilde{G}}$.

By Corollary 3.11, μ is additive on $\mathscr{P}(G^\perp)$ and $\tilde{\mu}$ on $\mathscr{P}(\tilde{G}^\perp)$. Let $\mathscr{L}'_n$ be a sequence of nested equipartitions of P_F, where $P_F = \mathbb{I} - P_G - P_{\tilde{G}}$, such that $\mathscr{L}'_n = \{P_{n1}, P_{n2}, \dots, P_{n2^n}\}$. Then $\{\mathscr{L}''_n : n \in \mathbb{N}\}$, where $\mathscr{L}''_n = \{P_{n1} + P_{\tilde{G}}, P_{n2}, P_{n3}, \dots, P_{n2^n}\}$ is a sequence of nested equipartitions of $P_G^\perp$. Since $\tilde{\mu}$ is additive on $\mathscr{P}(F)$, $\tilde{\mu} = \mu$ on $\mathscr{P}(G^\perp)$. As $P_E \in \mathscr{S}_\infty$ is arbitrary, we have $\tilde{\mu} = \mu$ on $\mathscr{S}_\infty$, and hence on $\mathscr{P}(\mathscr{H})$. □

Proposition 3.13. μ *is an additive measure on* $\mathscr{P}(\mathscr{H})$. *If* $\preceq$ *is weakly continuous, then* μ *is completely additive.*

Proof. Let $P_F, P_G \in \mathscr{P}(\mathscr{H})$ be mutually orthogonal. Clearly, one can construct a sequence of nested equipartitions of the identity such that if μ' is the associated function, then $\mu'(P + Q) = \mu'(P) + \mu'(Q)$. Corollary 3.12 gives the required additivity.

Now let $\preceq$ be weakly continuous and let $\mathscr{A} = \{P_j : j \in \Lambda\}$ be a set of mutually orthogonal projections of $\mathscr{P}(\mathscr{H})$. By Theorem 2.8, the set $\mathscr{B}$ of projections $P \in \mathscr{A}$ satisfying $\mu(P) \neq 0$ is, at most, countably infinite. Additivity and weak continuity of μ ensure that $\sum_{P \in \mathscr{B}} \mu(P) = \mu\left(\sum_{P \in \mathscr{B}} P\right)$. Weak continuity also implies that $\mu\left(\sum_{P \in \mathscr{A} \setminus \mathscr{B}} P\right) = 0$ (see [1], Proposition 2.4 (ii)); this leads to the required complete additivity. □

Theorem 1.2 and Proposition 3.13 lead to the final result:

Theorem 3.14. *Let* $\mathscr{H}$ *be an infinite dimensional* (*not necessarily separable*) *Hilbert space and let* $\preceq$ *be a* CP *on* $\mathscr{P}(\mathscr{H})$. *Then* $\preceq$ *can be implemented by a* (*unique*) *state* $\mu_\preceq$ *on* $\mathscr{B}(\mathscr{H})$ *if and only if* $\preceq$ *is uniformly continuous. If* $\preceq$ *is weakly continuous, then* $\mu_\preceq$ *is normal.* □

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