

Existence and Smoothing Effect of Solutions for the Zakharov Equations

By

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§1. Introduction and Theorems

In the present paper we consider the unique local solvability and the smoothing effect for the Zakharov equations:

$$(1.1) \quad i \frac{\partial E}{\partial t} + \Delta E = nE, \quad t > 0, \quad x \in \mathbf{R}^N,$$

$$(1.2) \quad \frac{\partial^2 n}{\partial t^2} - \Delta n = \Delta |E|^2, \quad t > 0, \quad x \in \mathbf{R}^N,$$

$$(1.3) \quad E(0, x) = E_0(x), \quad n(0, x) = n_0(x), \quad \frac{\partial}{\partial t} n(0, x) = n_1(0, x),$$

where E is a function from $\mathbf{R}_t^+ \times \mathbf{R}_x^N$ to \mathbf{C}^N , n is a function from $\mathbf{R}_t^+ \times \mathbf{R}_x^N$ to \mathbf{R} and $1 \leq N \leq 3$. (1.1)-(1.3) describe the long wave Langmuir turbulence in a plasma (see [20]). $E(t, x)$ denotes the slowly varying envelope of the highly oscillatory electric field and $n(t, x)$ denotes the deviation of the ion density from its equilibrium. When (1.2) depends on the ion sound speed c , that is, (1.2) is replaced by

$$(1.4) \quad \frac{1}{c^2} \frac{\partial^2 n}{\partial t^2} - \Delta n = \Delta |E|^2,$$

it is thought that (1.1) and (1.4) converge to

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$$(1.5) \quad i\frac{\partial E}{\partial t} + \Delta E = nE, \quad n = -|E|^2$$

as $c \rightarrow \infty$ (see [2], [17] and [20]). (1.5) is just the nonlinear Schrödinger equation and it is conjectured that the solutions of (1.1)-(1.2) and the solution of (1.5) have some common properties (see, e.g., [12, §1. Introduction]). The Zakharov equations (1.1)-(1.3) have not yet been studied well, while the nonlinear Schrödinger equation (1.5) has extensively been studied (see, e.g., [3]-[9], [11], [15] and [18]).

In [17] C. Sulem and P.L. Sulem proved by using the Galerkin method that if $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus (H^{m-2} \cap \dot{H}^{-1})$, $m \geq 3$ and $1 \leq N \leq 3$, then (1.1)-(1.3) have the unique local solutions $(E, n) \in L^\infty(0, T; H^m) \oplus L^\infty(0, T; H^{m-1})$ for some $T > 0$. Here H^m denotes the standard Sobolev space $H^m(\mathbb{R}^N)$. \dot{H}^m denotes the homogeneous Sobolev space consisting of all tempered distributions u with $|\xi|^m \hat{u} \in L^2 \equiv L^2(\mathbb{R}^N)$, where \hat{u} is the Fourier transform of u . In [12] Schochet and Weinstein showed a similar result for (1.1), (1.4) and (1.3) by the different method, but the existence time T of the local solutions does not depend on the parameter c in [12] (see also [2]). In both [12] and [17], the assumption $n_1 \in \dot{H}^{-1}$ is needed for the construction of the local solutions. This assumption is rather strong, because $\mathcal{S} \not\subset \dot{H}^{-1}$ for $N=1, 2$. For example, $e^{-|x|^2}$ is not in \dot{H}^{-1} for $N=1, 2$. Furthermore, the uniqueness of the solutions $(E(t), n(t))$ for (1.1)-(1.3) is proved only in the class $H^m \oplus H^{m-1}$, $m \geq 3$, in the previous results. In this paper we first show the unique local existence, result in $H^2 \oplus H^1$ for (1.1)-(1.3). $H^2 \oplus H^1$ seems more natural than the class in the previous results, because the solutions in $H^2 \oplus H^1$ are the so-called strong solutions.

We next investigate the smoothing effect of the solutions for (1.1)-(1.3). It is well known that the nonlinear Schrödinger equation (1.5) has the drastic smoothing effect (see, e.g., [4], [7]-[10] and [15]). In [7] and [8] it is proved that if $E(0) = E_0 \in H^1$, $|x|^k E_0 \in L^2$, $k \geq 1$ and $1 \leq N \leq 3$, then the solution $E(t)$ of (1.5) is in $H_{loc}^k \equiv H_{loc}^k(\mathbb{R}^N)$ for $t > 0$ as long as $E(t)$ exists. In [4] and [15] the smoothing effect of different type for (1.5) is proved, that is, if $E(0) = E_0 \in H^k$, $k \geq 1$ and $1 \leq N \leq 3$, then the solution $E(t)$ of (1.5) satisfies.

$$\int_0^T \|\varphi(1-\Delta)^{k/2+1/4} E(t)\|_{L^2}^2 dt \leq C$$

for $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $0 < T < T_{max}$, where T_{max} is the maximal existence time of $E(t)$. On the other hand, there seems to be no result concerning the smoothing property of (1.1)-(1.3). The Zakharov system consists of the Schrödinger equa-

tion (1.1) and the wave equation (1.2), and we cannot expect the smoothing effect of the wave equation part. Accordingly, we cannot expect the drastic smoothing effect for (1.1)-(1.3) like the single nonlinear Schrödinger equation. Nevertheless, we can prove that the solution $E(t)$ of the Schrödinger part for (1.1)-(1.3) has some smoothing properties.

Before we state the main results in this paper, we define several function spaces. Let $W^{m,p}$ denote the Sobolev space

$$W^{m,p} = \{f \in \mathcal{S}' ; \|f\|_{W^{m,p}} \equiv \|(1-\Delta)^{m/2} f\|_{L^p} < \infty\}$$

for $m \in \mathbf{R}$ and $1 < p < \infty$. We put $H^m \equiv W^{m,2}$. Let $H^{m,s}$ denote the weighted Sobolev space

$$H^{m,s} = \{f \in \mathcal{S}' ; \|f\|_{H^{m,s}} \equiv \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2} f\|_{L^2} < \infty\}$$

for $m, s \in \mathbf{R}$. For a Banach space X and $T > 0$, we define $W^{m,p}(0, T; X)$ by

$$W^{m,p}(0, T; X) \equiv \{f(t) \in L^p(0, T; X) ; [\sum_{j=0}^m \int_0^T \|\frac{d^j}{dt^j} f(t)\|_X^p dt]^{1/p} < \infty\},$$

if $1 \leq p < \infty$ and

$$W^{m,\infty}(0, T; X) \equiv \{f(t) \in L^\infty(0, T; X) ; \sum_{j=0}^m \sup_{0 \leq t \leq T} \|\frac{d^j}{dt^j} f(t)\|_X < \infty\}$$

if $p = \infty$.

The main results in this paper are the following.

Theorem 1.1. *Assume that $1 \leq N \leq 3$.*

(1) *Let $(E_0, n_0, n_1) \in H^2 \oplus H^1 \oplus L^2$. Then for some $T > 0$ there exist the unique strong solutions $(E(t), n(t))$ of (1.1)-(1.3) such that*

$$(1.6) \quad E \in \bigcap_{j=0}^1 C^j([0, T]; H^{2-2j}),$$

$$(1.7) \quad E \in \bigcap_{j=0}^1 W^{j,8/N}(0, T; W^{2-2j,4}),$$

$$(1.8) \quad n \in \bigcap_{j=0}^2 C^j([0, T]; H^{1-j}),$$

where T depends only on $\|E_0\|_{H^2}, \|n_0\|_{H^1}, \|n_1\|_{L^2}$ and N .

(2) *Let m be an even integer with $m \geq 4$. If $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$, then the solutions $(E(t), n(t))$ of (1.1)-(1.3) given by Part (1) satisfy*

$$(1.9) \quad E \in \bigcap_{j=0}^{m/2} C^j([0, T]; H^{m-2j}),$$

$$(1.10) \quad E \in \bigcap_{j=0}^{m/2} W^{j,8/N}(0, T; W^{m-2j,4}),$$

$$(1.11) \quad n \in \bigcap_{j=0}^3 C^j([0, T]; H^{m-1-j}),$$

and if $m \geq 6$,

$$(1.12) \quad n \in \bigcap_{j=4}^{m/2+1} C^j([0, T]; H^{m+2-2j}).$$

(3) Let m be an odd integer with $m \geq 3$. If $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$, then the solutions $(E(t), n(t))$ of (1.1)-(1.3) given by Part (1) satisfy

$$(1.13) \quad E \in \bigcap_{j=0}^{(m-1)/2} C^j([0, T]; H^{m-2j}),$$

$$(1.14) \quad E \in \bigcap_{j=0}^{(m-1)/2} W^{j,8/N}(0, T; W^{m-2j,4}),$$

$$(1.15) \quad n \in \bigcap_{j=0}^3 C^j([0, T]; H^{m-1-j}),$$

and if $m \geq 7$,

$$(1.16) \quad n \in \bigcap_{j=4}^{(m+1)/2} C^j([0, T]; H^{m+2-2j}).$$

Remark 1.1 (1) The solutions $(E(t), n(t))$ of (1.1)-(1.3) in Theorem 1.1 (1) satisfy (1.1) in the L^2 sense, while they satisfy (1.2) in the distribution sense. Therefore, the solutions in the class of Theorem 1.1(1) are called the strong solutions (for the weak solutions, see [12, Theorem 4] and [17, Theorem 1]).

(2) Theorem 1.1 (1) shows that the solutions of (1.1)-(1.3) are unique in the class of the strong solutions.

(3) In Theorem 1.1 we do not need the condition $n_1 \in \dot{H}^{-1}$, which was always assumed in the previous papers [12] and [17].

(4) In Theorems 1.1(2) and (3) the existence time T of the more regular solutions than the strong solutions is the same as that of the strong solutions. In the previous papers [12] and [17], T depends on the higher order Sobolev norms of the initial data, when the solutions are regular. Theorems 1.1(2) and (3) imply that if $(E_0, n_0, n_1) \in \cap_{k=1}^{\infty} H^k$, then the solutions $(E, n) \in C^\infty([0, T] \times \mathbb{R}^N)$.

(5) (1.7) (1.10) and (1.14) show that $E(t)$ has a smoothing property in a

certain sense like the solution of the single nonlinear Schrödinger equation (see [6], [11], [16] and [19]).

(6) (1.12) and (1.16) imply that $\frac{\partial^j n}{\partial t^j}$, $j \geq 3$, lose the regularity of Sobolev order 2 with respect to the spatial variables, each time we differentiate them in t . This may seem strange, since $n(t)$ is a solution of the wave equation. But (1.2) contains the solution $E(t)$ of the Schrödinger equation as the external force, which is why (1.12) and (1.16) occur.

Theorem 1.2. *Let m be an integer with $m \geq 2$. Assume that $1 \leq N \leq 3$ and $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$. Let $(E(t), n(t))$ and $T_{max} > 0$ be the solutions of (1.1)-(1.3) given by Theorem 1.1 and their maximal existence time, respectively.*

(1) *Let $\varphi \in C_0^\infty(\mathbf{R}^N)$. Then $E(t)$ satisfies*

$$(1.17) \quad \varphi E \in L^2(0, T; H^{m+1/2})$$

for any T with $0 < T < T_{max}$.

(2) *In addition, let $m \geq 4$. Put $k=1$ if $m \geq 4$ and $k=1$ or 2 if $m \geq 6$. If $E_0 \in H^{m,k}$, then*

$$(1.18) \quad E(t) \in H_{loc}^{m+k}, \quad 0 < t < T_{max}.$$

Remark 1.2. Theorem 1.2 shows the smoothing properties of the Zakharov equations (1.1)-(1.3). Part (1) is completely the same as in the case of the single nonlinear Schrödinger equation (see [4] and [15]). On the other hand, Part (2) is not so good as in the case of the single nonlinear Schrödinger equation (see [7]-[9]). This is because the Zakharov system contains the wave equation and it has the form such that the derivative loss occurs.

The difficulty of solving (1.1)-(1.3) is that when we use the standard iteration scheme, we meet with the loss of derivative, which comes from the second derivatives of $|E(t)|^2$ in (1.2). In the case of the single nonlinear Schrödinger equation the L^p-L^q estimate and the Strichartz estimate play an important role (see [5], [6], [11] and [19]). However, in the previous papers [12] and [17] they are not used, because the loss of derivative prevents us from using them. In our proof of Theorem 1.1 we first transform (1.1)-(1.2) into the system which does not have the derivative loss. For that purpose, we apply the technique developed by Shibata and Y. Tsutsumi [13], which was used to solve the fully nonlinear wave equation. After that we apply the L^p-L^q estimate and the Strichartz estimate to the resulting system, following Kato [11].

When we investigate the smoothing effect of (1.1)-(1.3), the derivative loss of (1.1)-(1.3) causes difficulty again. In addition, we can not expect the solution $n(t)$ of the wave equation part to have a smoothing property. However, we can derive the smoothing effect for the solution $E(t)$ of the Schrödinger equation part by using the smoothing effect peculiar to the Schrödinger equation (see [4], [7]-[10], [14] and [15]) and the difference between the Schrödinger equation and the wave equation. Especially in our proof of Theorem 1.2(2) the difference of the derivative in t between the Schrödinger equation and the wave equation plays an important role.

Our plan in this paper is as follows. In Section 2 we prepare several lemmas needed for the proofs of Theorems 1.1 and 1.2. In Section 3 we give the proof of Theorem 1.1 and state some results concerning the existence of global solutions for (1.1)-(1.3). In Section 4 we give the proof of Theorem 1.2.

Finally we conclude this section by giving several notations. Let (\cdot, \cdot) denote the scalar product in L^2 . We abbreviate $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x_k}$, $1 \leq k \leq N$, to ∂_t and ∂_k , $1 \leq k \leq N$, respectively. Let $\alpha(N) = \infty$ if $N = 1, 2$ and $\alpha(N) = \frac{2N}{N-2}$ if $N \geq 3$. By $U(t)$ we denote the evolution operator of the free Schrödinger equation. We put $\omega = \sqrt{-\Delta}$, $J_k = x_k + 2it\partial_k$, $1 \leq k \leq N$, and $M(t) = e^{ix^2/4it}$. Let $\rho \in C_0^\infty(\mathbf{R}^N)$ such that $\rho \geq 0$ and $\|\rho\|_{L^1} = 1$. We put $\rho_\varepsilon(x) = \varepsilon^{-N}\rho(x/\varepsilon)$ for $\varepsilon > 0$. Let $*$ denote the convolution with respect to the spatial variables. For $z \in \mathbf{C}$ we denote by \bar{z} the complex conjugate of z . In the course of calculations below various positive constants are simply denoted by C .

§2. Lemmas

In this section we summarize several lemmas needed for the proofs of Theorems 1.1 and 1.2.

We first state two lemmas concerning the space-time estimates of the evolution operator of the free Schrödinger equation.

Lemma 2.1. (i) *Let p and q be two positive constants such that $2 \leq p \leq \infty$ and $1/p + 1/q = 1$. Then,*

$$(2.1) \quad \|U(t)v\|_{L^p} \leq (4\pi|t|)^{-N/2+N/p} \|v\|_{L^q}, \quad t \neq 0.$$

(ii) *Let q and r be two positive constants such that $2 \leq q < \alpha(N)$ and $(N/2 - N/q)r = 2$. Then, there exists a positive constant K_1 depending only N and q*

such that

$$(2.2) \quad \|U(\cdot)v\|_{L^r(\mathbf{R}; L^q)} \leq K_1 \|v\|_{L^2}, \quad v \in L^2.$$

(2.1) in Lemma 2.1(i) is the well known $L^p - L^q$ estimate and (2.2) in Lemma 2.1(ii) is the Strichartz estimate. For the proof of Lemma 2.1, see [5, Lemma 1.2], [6, Proposition 4.4] and [16, Corollary 1 in §3].

Lemma 2.2. *Let q, r, q' and r' be four positive constants such that $1 \leq q', r' \leq 2, 1/q + 1/q' = 1, 1/r + 1/r' = 1, 2 \leq q < \alpha(N)$ and $(N/2 - N/q)r = 2$. Let I be any interval in \mathbf{R} . There exists a positive constant K_2 depending only on N and q' such that*

$$(2.3) \quad \left\| \int_0^t U(t-s)f(s)ds \right\|_{L^\infty(I; L^2)} \leq K_2 \|f\|_{L^{r'}(I; L^{q'})}, \quad f \in L^{r'}(I; L^{q'}).$$

Remark 2.1. (1) The constant K_2 in Lemma 2.2 does not depend on the interval I . (2.3) still holds with $I = \mathbf{R}$.

(2) In fact, a slightly stronger result holds than (2.3). That is,

$$(2.4) \quad \int_0^t U(t-s)f(s)ds \in C(\bar{I}; L^2)$$

under the same assumptions as in Lemma 2.2, where \bar{I} is the closure of I . This follows directly from the approximation of $f(t)$ by a sequence of smooth functions.

Lemma 2.2 is the version of Lemma 2.1 for the inhomogeneous linear Schrödinger equation. For the proof of Lemma 2.2, see [19, Lemmas 2.1 and 2.2].

We next state the local smoothing effect of the evolution operator for the free Schrödinger equation (see [4], [14] and [15]).

Lemma 2.3. *Let $T > 0$ and $\varphi \in C_0^\infty(\mathbf{R}^N)$.*

(i) *There exists a positive constant K_3 depending only on N, T and φ such that*

$$(2.5) \quad \|\varphi(1-\mathcal{D})^{1/4}U(\cdot)v\|_{L^2(0,T; L^2)} \leq K_3 \|v\|_{L^2}, \quad v \in L^2.$$

(ii) *There exists a positive constant K_4 depending only on N, T and φ such that*

$$(2.6) \quad \|\varphi(1-\mathcal{D})^{1/4} \int_0^t U(t-s)f(s)ds\|_{L^2(0,T; L^2)} \leq K_4 \|f\|_{L^1(0,T; L^2)}, \quad f \in L^1(0, T; L^2).$$

Proof. For the proof of Part (i), see [4, Theorem 2.1] and [14, (2) at page 701]. We briefly describe the proof of Part (ii).

We use a duality argument to prove (ii). Let \mathcal{Q} be a bounded open set which includes the support of φ . Let $\Phi(t, x) \in C_0^\infty((0, T) \times \mathcal{Q})$. We denote $(1 - \Delta)^{l/4}$ by B . Then, by using Schwarz's inequality and (2.5) we have

$$\begin{aligned} & \left| \int_0^T (B \int_0^t U(t-s)f(s)ds, \Phi(t))dt \right| \\ &= \left| \int_0^T \int_0^t (BU(t-s)f(s), \Phi(t))ds dt \right| \\ &\leq \int_0^T \int_0^t \|BU(t-s)f(s)\|_{L^2(\mathcal{Q})} \|\Phi(t)\|_{L^2(\mathcal{Q})} ds dt \\ &\leq \int_0^T \int_0^T \|BU(t-s)f(s)\|_{L^2(\mathcal{Q})} \|\Phi(t)\|_{L^2(\mathcal{Q})} dt ds \\ &\leq \int_0^T \|BU(\cdot -s)f(s)\|_{L^2(0,T; L^2(\mathcal{Q}))} \|\Phi\|_{L^2(0,T; L^2(\mathcal{Q}))} ds \\ &\leq C \int_0^T \|U(-s)f(s)\|_{L^2} ds \|\Phi\|_{L^2(0,T; L^2(\mathcal{Q}))} \\ &\leq C \|f\|_{L^1(0,T; L^2)} \|\Phi\|_{L^2(0,T; L^2(\mathcal{Q}))}. \end{aligned}$$

This completes the proof of (ii), since $C_0^\infty((0, T) \times \mathcal{Q})$ is dense in $L^2(0, T; L^2(\mathcal{Q}))$. \blacksquare

We finally state the lemma concerning the properties of the commutator $J_k(t)$, $1 \leq k \leq N$. This will be useful, when we consider the smoothing effect of (1.1)-(1.3) in the weighted Sobolev space.

Lemma 2.4. $J_k(t)$, $1 \leq k \leq N$, commute with $i\partial_t + \Delta$ and we have

$$\begin{aligned} J_k(t) &= M(t)(2it\partial_k)M(-t), \\ J_k(t)U(t-s) &= U(t-s)J_k(s) \end{aligned}$$

for $t, s \in \mathbf{R}$ and $1 \leq k \leq N$.

Lemma 2.4 follows from a direct calculation (see, e.g., [8] and [10]).

§3. Proof of Theorem 1.1

In this section we describe the proof of Theorem 1.1.

When we use the standard iteration scheme to solve (1.1)-(1.3), the loss of derivative occurs, as stated in §1. In fact, if $1 \leq N \leq 3$ and $E \in L^\infty(0, T; H^m)$ for some $m \geq 2$ and $T > 0$, we solve (1.2) to have $n \in L^\infty(0, T; H^{m-1})$. However, we have only $E \in L^\infty(0, T; H^{m-1})$ by (1.1) when $n \in L^\infty(0, T; H^{m-1})$.

Thus, we first consider the following system:

$$(3.1) \quad i\partial_t F + \Delta F - nF - \partial_t n(E_0 + \int_0^t F ds) = 0,$$

$$(3.2) \quad \partial_t^2 n - \Delta n - \Delta |E|^2 = 0,$$

$$(3.3) \quad (-\Delta + 1)E = iF - (n-1)(E_0 + \int_0^t F ds),$$

$$(3.4) \quad F(0) = i(\Delta E_0 - n_0 E_0), \quad n(0) = n_0, \quad \partial_t n(0) = n_1.$$

If we formally differentiate (1.1) in t and put $F = \partial_t E$, we obtain (3.1). (1.1) is also rewritten as (3.3) in terms of F . The loss of derivative does not occur for (3.1)-(3.4). This technique was used to solve the fully nonlinear wave equation in [13].

Proposition 3.1. *Assume that $1 \leq N \leq 3$. If $(E_0, n_0, n_1) \in H^2 \oplus H^1 \oplus L^2$, then for some $T > 0$ there exist the solutions $(F(t), n(t))$ of (3.1)-(3.4) such that*

$$(3.5) \quad F \in [\bigcap_{j=0}^1 C^j([0, T]; H^{-2j})] \cap L^{8/N}(0, T; L^4),$$

$$(3.6) \quad n \in \bigcap_{j=0}^2 C^j([0, T]; H^{1-j}),$$

$$(3.7) \quad E \in C([0, T]; H^2),$$

where T depends only on $N, \|E_0\|_{H^2}, \|n_0\|_{H^1}$ and $\|n_1\|_{L^2}$. Furthermore, $E(t) \in C^1([0, T]; L^2), \frac{\partial}{\partial t} E(t) = F(t), E(0) = E_0$ and $(E(t), n(t))$ are the solutions of (1.1)-(1.3) satisfying (1.6)-(1.8).

Proof. We put

$$a = \max \{ \|E_0\|_{L^2}, \|E_0\|_{L^4}, \|\Delta E_0 - n_0 E_0\|_{L^2}, \|n_0\|_{H^1} + \|n_1\|_{L^2} + \|\omega |E_0|^2\|_{L^2} \}.$$

We note by the Sobolev imbedding theorem that a depends only on $N, \|E_0\|_{H^2}, \|n_0\|_{H^1}$ and $\|n_1\|_{L^2}$. (2.2) gives us

$$(3.8) \quad \|U(\cdot) i(\Delta E_0 - n_0 E_0)\|_{L^{8/N}(\mathbb{R}; L^4)} \leq \delta a$$

for some $\delta > 0$. Since $U(t)$ is a unitary group in L^2 , we have

$$(3.9) \quad \|U(\cdot) i(\Delta E_0 - n_0 E_0)\|_{L^\infty(\mathbb{R}; L^2)} \leq a.$$

Let T be a small positive constant to be determined later. We put $I = (0, T)$. We define the Banach space X and its norm $\|\cdot\|$ as follows:

$$X = [L^\infty(I; L^2) \cap L^{8/N}(I; L^4)] \oplus [L^\infty(I; H^1) \cap W^{1,\infty}(I; L^2)]$$

and

$$\| (F, n) \| = \| F \|_{L^\infty(I; L^2)} + \| F \|_{L^{8/N}(I; L^4)} + \| n \|_{L^\infty(I; H^1)} + \left\| \frac{d}{dt} n \right\|_{L^\infty(I; L^2)}.$$

We put

$$Y = \{ (F, n) \in X; \| F \|_{L^\infty(I; L^2)} \leq 2a, \| F \|_{L^{8/N}(I; L^4)} \leq 2\delta a, \\ \| n \|_{L^\infty(I; H^1)} \leq 2a, \left\| \frac{d}{dt} n \right\|_{L^\infty(I; L^2)} \leq 2a \}.$$

We note that Y is a closed subset in X . For $(F, n) \in Y$ we define the nonlinear mapping $N[F, n](t)$ as follows:

$$(3.10) \quad N[F, n](t) = (N_1[F, n](t), N_2[F, n](t)), \\ N_1[F, n](t) = U(t) i(\Delta E_0 - n_0 E_0) \\ + \int_0^t U(t-s) \{ n(s) F(s) + \partial_s n(s) (E_0 + \int_0^s F \, d\tau) \} \, ds,$$

$$(3.11) \quad N_2[F, n](t) = \cos \omega t n_0 + \omega^{-1} \sin \omega t n_1 \\ + \int_0^t \omega^{-1} \sin \omega(t-s) \Delta |E(s)|^2 \, ds,$$

where

$$(3.12) \quad E(t) = (-\Delta + 1)^{-1} \{ iF - (n-1)(E_0 + \int_0^t F(s) \, ds) \}.$$

Since the fixed points (F, n) of $N[F, n]$ are the solutions of (3.1)-(3.4), we show that for sufficiently small $T > 0$ $N[F, n]$ is a contraction mapping from Y into Y .

We first show that for sufficiently small $T > 0$ $N[F, n]$ is a mapping from Y into Y . Let $(F, n) \in Y$. We take the L^4 norm of (3.10) and use Lemma 2.1(i) with $p=4$ and Hölder's inequality with $3/4=1/2+1/4$ to obtain

$$(3.13) \quad \| N_1[F, n](t) \|_{L^4} \leq \| U(t) i(\Delta E_0 - n_0 E_0) \|_{L^4} \\ + C \int_0^t |t-s|^{-N/4} \{ \| n \|_{L^2} \| F \|_{L^4} \\ + \left\| \frac{d}{dt} n \right\|_{L^2} (\| E_0 \|_{L^4} + \int_0^s \| F \|_{L^4} \, d\tau) \} \, ds.$$

We take the $L^{8/N}(I)$ norm of (3.13) and use the Hardy-Littlewood-Sobolev inequality and (3.8) to obtain

$$\begin{aligned}
 (3.14) \quad & \|N_1[F, n]\|_{L^{8/N}(I; L^4)} \leq \delta a \\
 & + CT^{1-N/4} \|n\|_{L^\infty(I; L^2)} \|F\|_{L^{8/N}(I; L^4)} \\
 & + CT^{1-N/8} \left\| \frac{d}{dt} n \right\|_{L^\infty(I; L^2)} \|E_0\|_{L^4} \\
 & + CT^{2-N/8} \left\| \frac{d}{dt} n \right\|_{L^\infty(I; L^2)} \|F\|_{L^{8/N}(I; L^4)} \\
 & \leq \delta a + C(T^{1-4/N} a + T^{1-N/8} \delta^{-1} a + T^{2-N/4} \delta a) .
 \end{aligned}$$

If we choose $T > 0$ so small that

$$C(T^{1-N/4} a + T^{1-N/8} \delta^{-1} a + T^{2-N/4} \delta a) \leq 1 ,$$

then (3.14) gives us

$$(3.15) \quad \|N_1[F, n]\|_{L^{8/N}(I; L^4)} \leq 2\delta a .$$

We next take the $L^\infty(I; L^2)$ norm of (3.10) and use Lemma 2.2(i) with $q' = 4/3$ and $r' = 8/(8-N)$ and (3.9) to obtain

$$\begin{aligned}
 (3.16) \quad & \|N_1[F, n]\|_{L^\infty(I; L^2)} \leq a \\
 & + CT^{1-N/4} \|n\|_{L^\infty(I; L^2)} \|F\|_{L^{8/N}(I; L^4)} \\
 & + CT^{1-N/8} \left\| \frac{d}{dt} n \right\|_{L^\infty(I; L^2)} \|E_0\|_{L^4} \\
 & + CT^{2-N/4} \left\| \frac{d}{dt} n \right\|_{L^\infty(I; L^2)} \|F\|_{L^{8/N}(I; L^4)} \\
 & \leq a + C(T^{1-N/4} \delta a + T^{1-N/8} a + T^{2-N/4} \delta a) a .
 \end{aligned}$$

If we choose $T > 0$ so small that

$$C(T^{1-N/4} \delta a + T^{1-N/8} a + T^{2-N/4} \delta a) \leq 1 ,$$

then (3.16) gives us

$$(3.17) \quad \|N_1[F, n]\|_{L^\infty(I; L^2)} \leq 2a .$$

By (3.12) we evaluate E to obtain

$$\begin{aligned}
 (3.18) \quad & \|E\|_{L^\infty(I; H^2)} \leq C[\|F\|_{L^\infty(I; L^2)} \\
 & + \|n\|_{L^\infty(I; L^4)} (\|E_0\|_{L^4} + \int_0^T \|F\|_{L^4} ds) \\
 & + \|E_0\|_{L^2} + \int_0^T \|F\|_{L^2} ds] \\
 & \leq C(a + a^2 + T^{1-N/8} \delta a^2 + T^{1-N/8} \delta a) , \quad (F, n) \in Y .
 \end{aligned}$$

Here we have used the Sobolev imbedding theorem at the second inequality.

Accordingly, we have by (3.11) and (3.18)

$$\begin{aligned}
 (3.19) \quad & \|N_2[F, n](t)\|_{H^1} \leq (1+t)a \\
 & + \int_0^t (1+s) \| \Delta |E(s)|^2 \|_{L^2} ds \\
 & \leq (1+T)a \\
 & + CT(1+T)(a+a^2+T^{1-N/8}\delta a^2+T^{1-N/8}\delta a)^2 \\
 & = a + [T+CT(1+T)(1+a+T^{1-N/8}\delta a+T^{1-N/8}\delta)^2]a, \quad t \in I.
 \end{aligned}$$

If we choose $T > 0$ so small that

$$T + CT(1+T)(1+a+T^{1-N/8}\delta a+T^{1-N/8}\delta)a \leq 1,$$

then (3.19) yields

$$(3.20) \quad \|N_2[F, n]\|_{L^\infty(I; H^1)} \leq 2a.$$

On the other hand, we have

$$\begin{aligned}
 (3.21) \quad & \frac{d}{dt} N_2[F, n](t) = -\omega \sin \omega t n_0 \\
 & + \cos \omega t n_1 + \omega^{-1} \sin \omega t \Delta |E_0|^2 \\
 & + \int_0^t \cos \omega(t-s) \Delta |E(s)|^2 ds.
 \end{aligned}$$

Therefore, we take the L^2 norm of (3.21) and use the Sobolev imbedding theorem and (3.18) to obtain

$$\begin{aligned}
 (3.22) \quad & \left\| \frac{d}{dt} N_2[F, n](t) \right\|_{L^2} \\
 & \leq a + C \int_0^t \| \Delta |E(s)|^2 \|_{L^2} ds \\
 & \leq a + CT \|E\|_{L^\infty(I; H^2)}^2 \\
 & \leq a + CT(1+a+T^{1-N/8}\delta a+T^{1-N/8}\delta)^2 a^2, \quad t \in I.
 \end{aligned}$$

If we choose $T > 0$ so small that

$$CT(1+a+T^{1-N/8}\delta a+T^{1-N/8}\delta)^2 a \leq 1,$$

then (3.22) yields

$$(3.23) \quad \left\| \frac{d}{dt} N_2[F, n](t) \right\|_{L^2} \leq 2a.$$

Therefore, (3.15), (3.17), (3.20) and (3.23) show that for sufficiently small $T > 0$ $N[F, n]$ is a mapping from Y into Y . In the same way as above we obtain

$$(3.24) \quad |||N[F, n] - N[F', n']||| \leq \frac{1}{2} |||(F, n) - (F', n')|||, \\ (F, n), (F', n') \in Y$$

for sufficiently small $T > 0$, which implies that $N[F, n]$ is a contraction mapping from Y into Y . Accordingly, there exist the unique fixed points (F, n) of $N[F, n]$ and $E(t)$ is determined by (F, n) in terms of (3.12). These (F, n, E) satisfy (3.1)-(3.4) in the integral form and

$$(3.25) \quad F \in L^\infty(I; L^2) \cap L^{8/N}(0, T; L^4),$$

$$(3.26) \quad n \in \bigcap_{j=0}^1 W^{j, \infty}(I; H^{1-j}),$$

$$(3.27) \quad E \in L^\infty(I; H^2).$$

(3.25)-(3.27) and the standard argument show that (F, n, E) are the solutions of (3.1)-(3.4) satisfying (3.5)-(3.7) (see, e.g., Remark 2.1(2)).

We next prove that $E \in C^1([0, T]; L^2)$ and $\frac{d}{dt}E(t) = F(t)$. We differentiate (3.3) in t to obtain

$$(3.28) \quad (-\Delta + 1) \frac{d}{dt}E = i \frac{d}{dt}F - (n-1)F \\ - \frac{d}{dt}n(E_0 + \int_0^t F ds) \text{ in } H^{-2}.$$

On the other hand, (3.1) gives us

$$(3.29) \quad (-\Delta + 1)F = i \frac{d}{dt}F - (n-1)F \\ - \frac{d}{dt}n(E_0 + \int_0^t F ds) \text{ in } H^{-2}.$$

Therefore, $\frac{d}{dt}E(t) = F(t)$ in H^{-2} . Furthermore, by (3.28) we have

$$(3.30) \quad \frac{d}{dt}E(t) = (-\Delta + 1)^{-1} [i \frac{d}{dt}F - (n-1)F \\ + \frac{d}{dt}n(E_0 + \int_0^t F ds)].$$

The right hand side of (3.30) is in $C([0, T]; L^2)$. Accordingly, $E(t) \in C^1([0, T]; L^2)$. By (3.3) and (3.4) we have

$$\begin{aligned}
 (3.31) \quad (-\mathcal{A}+1)E(0) &= iF_0 - (n_0-1)E_0 \\
 &= i \cdot i(\mathcal{A}E_0 - n_0E_0) - (n_0-1)E_0 \\
 &= (-\mathcal{A}+1)E_0,
 \end{aligned}$$

which shows $E(0) = E_0$.

It remains only to show that $E \in L^{8/N}(0, T; W^{2,4})$, which follows directly from (1.1), (3.5) and the regularity theorem of the elliptic equation.

Therefore, $(E(t), n(t))$ are the unique solutions of (1.1)-(1.3) satisfying (1.6)-(1.8). ■

In the same way as in the proof of Proposition 3.1 we have the following proposition.

Proposition 3.2. *Assume that $1 \leq N \leq 3$.*

- (1) *Let m be an even integer with $m \geq 4$. If $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$, then for some $\tilde{T} > 0$ there exist the unique solutions $(E(t), n(t))$ of (1.1)-(1.3) satisfying (1.9)-(1.12) with T replaced by \tilde{T} , where \tilde{T} depends only on $N, \|E_0\|_{H^m}, \|n_0\|_{H^{m-1}}$ and $\|n_1\|_{H^{m-2}}$.*
- (2) *Let m be an odd integer with $m \geq 3$. If $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$, then for some $\tilde{T} > 0$ there exist the unique solutions $(E(t), n(t))$ of (1.1)-(1.3) satisfying (1.13)-(1.16) with T replaced by \tilde{T} , where \tilde{T} depends only on $N, \|E_0\|_{H^m}, \|n_0\|_{H^{m-1}}$ and $\|n_1\|_{H^{m-2}}$.*

Remark 3.1. We note that in Proposition 3.2 \tilde{T} depends on the higher order Sobolev norms of (E_0, n_0, n_1) than the norms of $H^2 \oplus H^1 \oplus L^2$ and that \tilde{T} is less than T given by Proposition 3.1.

We omit the proof of Proposition 3.2, since it is similar to the proof of Proposition 3.1.

We can complete the proof of Theorem 3.1(2)-(3), if we prove the following proposition.

Proposition 3.3. *Assume that $1 \leq N \leq 3$. Let m be an integer with $m \geq 2$ and let $(E_0, n_0, n_1) \in H^{m+1} \oplus H^m \oplus H^{m-1}$. If Theorem 1.1 holds for m , then Theorem 1.1 also holds for $m+1$.*

Proof. Let $(E(t), n(t))$ be the solutions of (1.1)-(1.3) satisfying Theorem 1.1 for m . Since $(E_0, n_0, n_1) \in H^{m+1} \oplus H^m \oplus H^{m-1}$, by Proposition 3.2 and the

uniqueness of solutions we conclude that $(E(t), n(t))$ satisfy Theorem 1.1 for $m+1$ with T replaced by \tilde{T} .

We prove Proposition 3.3 only in the case where m is even, since the proof for odd m is the same. Let \tilde{T}_{max} be the maximal existence time of the solutions $(E(t), n(t))$ in $H^{m+1} \oplus H^m$. That is, for any T' with $0 < T' < \tilde{T}_{max}$ $(E(t), n(t))$ satisfy (1.13)-(1.16) with T and m replaced by T' and $m+1$. If $\tilde{T}_{max} > T$, the proof is completed. We suppose that $\tilde{T}_{max} \leq T$ and derive a contradiction. We divide the proof into three cases.

Case 1. Let m be an even integer with $m \geq 6$. Since $(E(t), n(t))$ satisfy (1.13)-(1.16) with T and m replaced by T' and $m+1$ for any T' with $0 < T' < \tilde{T}_{max}$, we differentiate (1.1) $m/2 - 1$ times in t and once in x_k , $1 \leq k \leq N$ to obtain

$$(3.32) \quad i \partial_t (\partial_k \partial_t^{m/2-1} E) + \Delta (\partial_k \partial_t^{m/2-1} E) \\ = \sum_{j=0}^{m/2-1} \binom{m/2-1}{j} [(\partial_k \partial_t^{m/2-1-j} n) (\partial_t^j E) + (\partial_t^{m/2-1-j} n) (\partial_k \partial_t^j E)] \\ \text{in } L^2, \quad 1 \leq k \leq N$$

for $0 \leq t < \tilde{T}_{max}$. By the assumption that Theorem 1.1 holds for m , we have

$$(3.33) \quad \partial_t^{m/2-1-j} n \in L^\infty(0, T; H^4), \quad 0 \leq j \leq m/2 - 2,$$

$$(3.34) \quad n \in L^\infty(0, T; H^5),$$

$$(3.35) \quad \partial_t^j E \in L^\infty(0, T; H^4), \quad 0 \leq j \leq m/2 - 2,$$

$$(3.36) \quad \partial_t^{m/2-1} E \in L^\infty(0, T; L^2).$$

Therefore, by (3.32)-(3.36) and the Sobolev imbedding theorem we obtain

$$(3.37) \quad \|\partial_k \partial_t^{m/2-1} E(t)\|_{H^2} \leq \|\partial_k \partial_t^{m/2-1} E(0)\|_{H^2} \\ + C \int_0^t \sum_{j=0}^{m/2-2} [\|(\partial_k \partial_t^{m/2-1-j} n) (\partial_t^j E)\|_{H^2} \\ + \|(\partial_t^{m/2-1-j} n) (\partial_k \partial_t^j E)\|_{H^2}] d\tau \\ + C \int_0^t \|(\partial_k n) (\partial_t^{m/2-1} E)\|_{H^2} d\tau \\ + C \int_0^t \|n \partial_k \partial_t^{m/2-1} E\|_{H^2} d\tau \\ \leq \|\partial_k \partial_t^{m/2-1} E(0)\|_{H^2} \\ + C \int_0^T \sum_{j=0}^{m/2-1} [\|\partial_t^{m/2-1-j} n\|_{H^4} \|\partial_t^j E\|_{H^3} \\ + \|\partial_t^{m/2-1-j} n\|_{H^3} \|\partial_t^j E\|_{H^4}] d\tau$$

$$\begin{aligned}
 &+ C \int_0^t \|n\|_{H^5} \|\partial_t^{m/2-1} E\|_{H^2} d\tau \\
 &+ C \int_0^t \|n\|_{H^4} \|\partial_t^{m/2-1} E\|_{H^2} d\tau \\
 &\leq C + C \int_0^t \|\partial_k \partial_t^{m/2-1} E\|_{H^2} d\tau
 \end{aligned}$$

for $t \in [0, \tilde{T}_{max})$ and $1 \leq k \leq N$. (3.37) and Gronwall's inequality yield

$$(3.38) \quad \partial_k \partial_t^{m/2-1} E \in L^\infty(0, \tilde{T}_{max}; H^2), \quad 1 \leq k \leq N.$$

By differentiating (1.1) $m/2 - 2$ times in t , we have

$$\begin{aligned}
 (3.39) \quad &i \partial_t (\partial_t^{m/2-2} E) + \Delta (\partial_t^{m/2-2} E) \\
 &= \sum_{j=0}^{m/2-2} \binom{m/2-2}{j} (\partial_t^{m/2-2-j} n) (\partial_t^j E).
 \end{aligned}$$

Since Theorem 1.1 holds for m , we easily see by the Sobolev imbedding theorem that the right hand side of (3.39) belongs to $L^\infty(0, \tilde{T}_{max}; H^3)$. Therefore, by (3.39), (3.38) and the regularity theorem of the elliptic equation we obtain

$$(3.40) \quad \partial_t^{m/2-2} E \in L^\infty(0, \tilde{T}_{max}; H^5).$$

Repeating this procedure, we conclude that

$$(3.41) \quad E \in L^\infty(0, \tilde{T}_{max}; H^{m+1}).$$

(3.41) and (3.2) imply that

$$(3.42) \quad n \in L^\infty(0, \tilde{T}_{max}; H^m),$$

$$(3.43) \quad \partial_t n \in L^\infty(0, \tilde{T}_{max}; H^{m-1}).$$

(3.41)-(3.43) assure that by Proposition 3.2 we can extend the solutions $(E(t), n(t))$ as the solutions in $H^{m+1} \oplus H^m$ beyond $t = \tilde{T}_{max}$. This contradicts the definition of \tilde{T}_{max} .

Case 2. Let $m=4$. Since $(E(t), n(t))$ satisfy (1.13)-(1.16) with T and m replaced by T' and $m+1$ for any T' with $0 < T' < \tilde{T}_{max}$, we have by (1.1)

$$(3.44) \quad E \in C^3([0, \tilde{T}_{max}); H^{-1}).$$

We differentiate (1.1) twice in t and once in x_k , $1 \leq k \leq N$ to obtain

$$\begin{aligned}
 (3.45) \quad & i\partial_t(\partial_k\partial_i^2E)+4(\partial_k\partial_i^2E) \\
 & = \sum_{j=0}^2 \binom{2}{j} [(\partial_k\partial_i^{2-j}n)(\partial_i^jE)+(\partial_i^{2-j}n)(\partial_k\partial_i^jE)] \\
 & \text{in } H^{-2}, \quad 0 \leq t < \tilde{T}_{max}, \quad 1 \leq k \leq N.
 \end{aligned}$$

We rewrite (3.45) as the integral form:

$$\begin{aligned}
 (3.46) \quad & \partial_k\partial_i^2E(t) = U(t)\partial_k\partial_i^2E(0) \\
 & -i \int_0^t U(t-s) \left[\sum_{j=0}^2 \binom{2}{j} \{(\partial_k\partial_s^{2-j}n)(\partial_s^jE)+(\partial_s^{2-j}n)(\partial_k\partial_s^jE)\} \right] ds, \\
 & 0 \leq t < \tilde{T}_{max}, \quad 1 \leq k \leq N.
 \end{aligned}$$

We take the L^4 norm of (3.46) and use Lemma 2.1(i) to obtain

$$\begin{aligned}
 (3.47) \quad & \|\partial_k\partial_i^2E(t)\|_{L^4} \leq \|U(t)\partial_k\partial_i^2E(0)\|_{L^4} \\
 & + C \sum_{j=0}^2 \int_0^t |t-s|^{-N/4} [\|(\partial_k\partial_s^{2-j}n)(\partial_s^jE)\|_{L^{4/3}} \\
 & \quad + \|(\partial_s^{2-j}n)(\partial_k\partial_s^jE)\|_{L^{4/3}}] ds \\
 & \leq \|U(t)\partial_k\partial_i^2E(0)\|_{L^4} \\
 & + C \sum_{j=0}^1 \int_0^t |t-s|^{-N/4} [\|\partial_k\partial_s^{2-j}n\|_{L^2} \|\partial_s^jE\|_{L^4} \\
 & \quad + \|\partial_s^{2-j}n\|_{L^2} \|\partial_k\partial_s^jE\|_{L^4}] ds \\
 & + C \int_0^t |t-s|^{-N/4} \|\partial_k n\|_{L^4} \|\partial_s^2E\|_{L^2} ds \\
 & + C \int_0^t |t-s|^{-N/4} \|n\|_{L^4} \|\partial_k\partial_s^jE\|_{L^4} ds, \\
 & 0 \leq t < \tilde{T}_{max}, \quad 1 \leq k \leq N.
 \end{aligned}$$

By the definition of \tilde{T}_{max} we note that $\partial_k\partial_i^2E \in L^{8/N}(0, T'; L^4)$ for any T' with $0 < T' < \tilde{T}_{max}$. By the assumption that Theorem 1.1 holds for $m=4$, we have

$$(3.48) \quad \partial_i^{2-j}n \in L^\infty(0, T; H^{3-j}), \quad 0 \leq j \leq 2,$$

$$(3.49) \quad \partial_i^jE \in L^\infty(0, T; H^{4-2j}), \quad 0 \leq j \leq 2.$$

Therefore, by (3.47)-(3.49) and the Sobolev imbedding theorem we obtain

$$\begin{aligned}
 (3.50) \quad & \|\partial_k\partial_i^2E(t)\|_{L^4} \leq \|U(t)\partial_k\partial_i^2E(0)\|_{L^4} \\
 & + C + C \int_0^t |t-s|^{-N/4} \|\partial_k\partial_s^jE(s)\|_{L^4} ds, \\
 & 0 \leq t < \tilde{T}_{max}, \quad 1 \leq k \leq N.
 \end{aligned}$$

By Lemma 2.1(ii) with $q=4$ and $r=8/N$ we have

$$(3.51) \quad \|U(\cdot)\partial_k\partial_i^2E(0)\|_{L^4} \in L^{8/N}(\mathbf{R}).$$

Accordingly, (3.50), (3.51) and the theory of the Volterra type integral equation show that

$$(3.52) \quad \partial_k\partial_i^2E \in L^{8/N}(0, \tilde{T}_{max}; L^4), \quad 1 \leq k \leq N.$$

We next take the $L^\infty(0, \tilde{T}_{max}; L^2)$ norm of (3.46) and use Lemma 2.2 with $q'=4/3$ and $r'=8/(8-N)$ to obtain by (3.48) and (3.49)

$$(3.53) \quad \|\partial_k\partial_i^2E\|_{L^\infty(0, \tilde{T}_{max}; L^2)} \leq C + C\|\partial_k\partial_i^2E\|_{L^{8/N}(0, \tilde{T}_{max}; L^4)}, \quad 1 \leq k \leq N$$

in the same way as (3.47). (3.53) and (3.52) show that

$$(3.54) \quad \partial_k\partial_i^2E \in L^\infty(0, \tilde{T}_{max}; L^2), \quad 1 \leq k \leq N.$$

We differentiate (1.1) in t to obtain

$$(3.55) \quad i\partial_i^2E + 4(\partial_iE) = (\partial_i n)E + n\partial_tE \text{ in } L^2, \quad 0 \leq t < \tilde{T}_{max}.$$

(3.48), (3.49) and the Sobolev imbedding theorem imply that the right hand side of (3.55) belongs to $L^\infty(0, T; H^1)$. Accordingly, (3.54), (3.55) and the regularity theorem of the elliptic equation give us

$$(3.56) \quad \partial_iE \in L^\infty(0, \tilde{T}_{max}; H^3).$$

Repeating this procedure, we have

$$(3.57) \quad E \in L^\infty(0, \tilde{T}_{max}; H^5).$$

(3.57) and (1.2) imply that

$$(3.58) \quad n \in L^\infty(0, \tilde{T}_{max}; H^4),$$

$$(3.59) \quad \partial_i n \in L^\infty(0, \tilde{T}_{max}; H^3).$$

(3.57)-(3.59) assure that by Proposition 3.2 we can extend the solutions $(E(t), n(t))$ as the solutions in $H^5 \oplus H^4$ beyond $t = \tilde{T}_{max}$. This contradicts the definition of \tilde{T}_{max} .

Case 3. Let $m=2$. Since $(E(t), n(t))$ satisfy (1.13)-(1.15) with T and m replaced by T' and $m+1$ for any T' with $0 < T' < \tilde{T}_{max}$, we have by (1.1)

$$(3.60) \quad E \in C^2([0, \tilde{T}_{max}); H^{-1}).$$

We differentiate (1.1) in t and x_k , $1 \leq k \leq N$ to obtain

$$(3.61) \quad i\partial_t(\partial_k\partial_t E) + \Delta(\partial_k\partial_t E) = (\partial_k\partial_t n)E + (\partial_t n)(\partial_k E) \\ + (\partial_k n)(\partial_t E) + n(\partial_k\partial_t E) \text{ in } H^{-2}$$

for $0 \leq t < \tilde{T}_{max}$ and $1 \leq k \leq N$. We rewrite (3.61) as the integral form:

$$(3.62) \quad \partial_k\partial_t E(t) = U(t)\partial_k\partial_t E(0) \\ - i \int_0^t U(t-s)[(\partial_k\partial_s n)E + (\partial_s n)(\partial_k E) \\ + (\partial_k n)(\partial_s E) + n(\partial_k\partial_s E)] ds, \\ 0 \leq t < \tilde{T}_{max}, \quad 1 \leq k \leq N.$$

We take the L^4 norm of (3.62) and use Lemma 2.1(i) to obtain

$$(3.63) \quad \|\partial_k\partial_t E(t)\|_{L^4} \leq \|U(t)\partial_k\partial_t E(0)\|_{L^4} \\ + C \int_0^t |t-s|^{-N/4} [\|(\partial_k\partial_s n)E\|_{L^{4/3}} \\ + \|(\partial_s n)(\partial_k E)\|_{L^{4/3}} + \|(\partial_k n)(\partial_s E)\|_{L^{4/3}} \\ + \|n(\partial_k\partial_s E)\|_{L^{4/3}}] ds \\ \leq \|U(t)\partial_k\partial_t E(0)\|_{L^4} \\ + C \int_0^t |t-s|^{-N/4} [\|\partial_k\partial_s n\|_{L^2}\|E\|_{L^4} \\ + \|\partial_s n\|_{L^2}\|\partial_k E\|_{L^4} + \|\partial_k n\|_{L^2}\|\partial_s E\|_{L^4} \\ + \|n\|_{L^2}\|\partial_k\partial_s E\|_{L^4}] ds, \\ 0 \leq t < \tilde{T}_{max}, \quad 1 \leq k \leq N.$$

By the definition of \tilde{T}_{max} we note that $\partial_k\partial_t E \in L^{8/N}(0, T'; L^4)$ for any T' with $0 < T' < \tilde{T}_{max}$. By the assumption that Theorem 1.1 holds for m , we have

$$(3.64) \quad \partial_t^j n \in L^\infty(0, T; H^{1-j}), \quad j=0, 1,$$

$$(3.65) \quad \partial_t^j E \in L^\infty(0, T; H^{2-2j}), \quad j=0, 1,$$

$$(3.66) \quad \partial_t^j E \in L^{8/N}(0, T; W^{2-2j,4}), \quad j=0, 1.$$

(3.63)-(3.66) and the Sobolev imbedding theorem give us

$$(3.67) \quad \|\partial_k\partial_t E(t)\|_{L^4} \leq \|U(t)\partial_k\partial_t E(0)\|_{L^4} \\ + C + C \int_0^t |t-s|^{-N/4} \|\partial_s E\|_{L^4} ds \\ + C \int_0^t |t-s|^{-N/4} (\|\partial_k\partial_s n\|_{L^2} + \|\partial_k\partial_s E\|_{L^4}) ds, \\ 0 \leq t < \tilde{T}_{max}, \quad 1 \leq k \leq N.$$

On the other hand, by (1.2) we have

$$(3.68) \quad \begin{aligned} \partial_t^2 n - \Delta n &= \Delta |E|^2 \\ &= \sum_{j=1}^N 2(|\nabla E_j|^2 + \operatorname{Re} \Delta E_j \bar{E}_j). \end{aligned}$$

Since we have by (1.1)

$$\Delta E_j = -i \partial_t E_j + n E_j, \quad 1 \leq j \leq N,$$

we obtain by (3.68)

$$(3.69) \quad \partial_t^2 n - \Delta n = 2 \sum_{j=1}^N |\nabla E_j|^2 + 2 \sum_{j=1}^N \operatorname{Im} \partial_t E_j \bar{E}_j + n |E|^2.$$

(3.69) yields

$$(3.70) \quad \begin{aligned} &\|\partial_k \nabla n(t)\|_{L^2} + \|\partial_k \partial_t n(t)\|_{L^2} \\ &\leq C(\|\partial_k \nabla n_0\|_{L^2} + \|\partial_k n_1\|_{L^2}) \\ &\quad + C \int_0^t \sum_{j=1}^N \|\partial_k |\nabla E_j|^2\|_{L^2} ds \\ &\quad + C \int_0^t \sum_{j=1}^N \|\partial_k (\bar{E}_j \partial_s E_j)\|_{L^2} ds \\ &\quad + C \int_0^t \|\partial_k (n |E|^2)\|_{L^2} ds \\ &\leq C(\|\partial_k \nabla n_0\|_{L^2} + \|\partial_k n_1\|_{L^2}) \\ &\quad + C \int_0^t \sum_{j=1}^N \|\nabla E_j \cdot \nabla \partial_k \bar{E}_j\|_{L^2} ds \\ &\quad + C \int_0^t \sum_{j=1}^N (\|\partial_k \bar{E}_j\|_{L^2} \|\partial_s E_j\|_{L^2} + \|\bar{E}_j \partial_k \partial_s E_j\|_{L^2}) ds \\ &\quad + C \int_0^t (\|\partial_k n\|_{L^2} \|E\|_{L^2} + \|n E \partial_k \bar{E}\|_{L^2}) ds, \\ &0 \leq t < \tilde{T}_{max}, \quad 1 \leq k \leq N. \end{aligned}$$

By the definition of \tilde{T}_{max} we note that $\partial_k \nabla n, \partial_k \partial_t n \in L^\infty(0, T'; L^2)$ for any T' with $0 < T' < \tilde{T}_{max}$. By (3.64)-(3.66) and the Sobolev imbedding theorem we obtain

$$(3.71) \quad \begin{aligned} &\|\partial_k \nabla n(t)\|_{L^2} + \|\partial_k \partial_t n(t)\|_{L^2} \\ &\leq C + C \int_0^t \sum_{j=1}^N \|\partial_k E_j\|_{L^4} \|\partial_k \nabla E_j\|_{L^4} ds \end{aligned}$$

$$\begin{aligned}
 &+ C \int_0^t (\|\partial_k E\|_{L^4} \|\partial_s E\|_{L^4} + \|E\|_{L^4} \|\partial_k \partial_s E\|_{L^4}) ds \\
 &+ C \int_0^t (\|\partial_k n\|_{L^2} \|E\|_{L^\infty}^2 + \|n\|_{L^4} \|E\|_{L^\infty} \|\partial_k E\|_{L^4}) ds \\
 &\leq C + C \int_0^t \|\partial_k \partial_s E\|_{L^4} ds, \\
 &0 \leq t < \tilde{T}_{max}, \quad 1 \leq k \leq N.
 \end{aligned}$$

Inserting (3.71) into (3.67), we have

$$\begin{aligned}
 (3.72) \quad \|\partial_k \partial_t E(t)\|_{L^4} &\leq \|U(t) \partial_k \partial_t E(0)\|_{L^4} \\
 &+ C + C \int_0^t |t-s|^{-N/4} \|\partial_s E\|_{L^4} ds \\
 &+ C \int_0^t |t-s|^{-N/4} (C + C \int_0^s \|\partial_k \partial_\tau E\|_{L^4} d\tau) ds \\
 &+ C \int_0^t |t-s|^{-N/4} \|\partial_k \partial_s E\|_{L^4} ds \\
 &\leq \|U(t) \partial_k \partial_t E(0)\|_{L^4} \\
 &+ C + C \int_0^t |t-s|^{-N/4} \|\partial_s E\|_{L^4} ds \\
 &+ C \int_0^t (|t-s|^{(4-N)/4} + |t-s|^{-N/4}) \|\partial_k \partial_s E\|_{L^4} ds, \\
 &0 \leq t < \tilde{T}_{max}, \quad 1 \leq k \leq N.
 \end{aligned}$$

Here at the second inequality of (3.72) we have used the following identity:

$$\begin{aligned}
 &\int_0^t |t-s|^{-N/4} \int_0^s \|\partial_k \partial_\tau E\|_{L^4} d\tau ds \\
 &= (1-N/4)^{-1} \int_0^t |t-s|^{(4-N)/4} \|\partial_k \partial_s E\|_{L^4} ds.
 \end{aligned}$$

Lemma 2.1(ii) with $q=4$ implies that the first term at the right hand side of (3.72) belongs to $L^{8/N}(\mathbf{R})$, and (3.66) and the Hardy-Littlewood-Sobolev inequality imply that the second term at the right hand side of (3.72) belongs to $L^\infty(0, \tilde{T}_{max})$ for $N=1, 2$ and to $L^3(0, \tilde{T}_{max})$ for $N=3$. Therefore, (3.72) and the theory of the Volterra type integral equation yield

$$(3.73) \quad \partial_k \partial_t E \in L^{8/N}(0, \tilde{T}_{max}; L^4), \quad 1 \leq k \leq N.$$

We next take the $L^\infty(0, \tilde{T}_{max}; L^2)$ norm of (3.62) and use Lemma 2.2 with $q'=4/3$ and $r'=8/(8-N)$ to obtain by (3.64)-(3.66) and (3.71)

$$(3.74) \quad \begin{aligned} & \|\partial_k \partial_t E\|_{L^\infty(0, \tilde{T}_{max}; L^2)} \\ & \leq C + C \|\partial_k \partial_t E\|_{L^{8/N}(0, \tilde{T}_{max}; L^4)}, \quad 1 \leq k \leq N \end{aligned}$$

in the same way as (3.72). (3.74) and (3.73) show that

$$(3.75) \quad \partial_k \partial_t E \in L^\infty(0, \tilde{T}_{max}; L^2), \quad 1 \leq k \leq N.$$

(3.64)-(3.66) and the Sobolev imbedding theorem imply that the right hand side of (1.1) belongs to $L^\infty(0, T; H^1)$. Accordingly, (3.75), (3.65), (1.1) and the regularity theorem of the elliptic equation give us

$$(3.76) \quad E \in L^\infty(0, \tilde{T}_{max}; H^3).$$

(3.76) and (1.2) imply that

$$(3.77) \quad n \in L^\infty(0, \tilde{T}_{max}; H^2),$$

$$(3.78) \quad \partial_t n \in L^\infty(0, \tilde{T}_{max}; H^1).$$

(3.76)-(3.78) assure that by Proposition 3.2 we can extend the solutions $(E(t), n(t))$ as the solutions in $H^3 \oplus H^2$ beyond $t = \tilde{T}_{max}$. This contradicts the definition of \tilde{T}_{max} .

Thus, the proof for even m is complete. In the same way as above we can prove Proposition 3.3 for odd m . ■

By combining Propositions 3.1, 3.3 and the induction argument we obtain Theorem 1.1(2)-(3). Thus, the proof of Theorem 1.1 is completed.

We conclude this section by giving the following theorem concerning the existence of global solutions for (1.1)-(1.3).

Theorem 3.4. (1) Assume $N=1$. Let m be an integer with $m \geq 2$. If $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$ and $n_1 \in \dot{H}^{-1}$, then the existence time T of the solutions in Theorem 1.1 can be chosen as $T = +\infty$. Furthermore, if $E_0, n_0, n_1 \in \cap_{m=1}^\infty H^m$ and $n_1 \in \dot{H}^{-1}$, then the solutions $E(t, x)$ and $n(t, x)$ are in $C^\infty([0, \infty) \times \mathbf{R})$.
 (2) Assume $N=2$. Let m be an integer with $m \geq 2$. There exists $\delta > 0$ such that if $(E_0, n_0, n_1) \in H^m \oplus H^{m-1} \oplus H^{m-2}$, $n_1 \in \dot{H}^{-1}$ and $\|E_0\|_{L^2} < \delta$, then the existence time T of the solutions in Theorem 1.1 can be chosen as $T = +\infty$. In addition, if E_0, n_0 , and n_1 are in $\cap_{m=1}^\infty H^m$, then the solutions $E(t, x)$ and $n(t, x)$ are in $C^\infty([0, \infty) \times \mathbf{R}^2)$.

The a priori estimates needed for the proof of existence of global solutions are already established by C. Sulem and P.L. Sulem [17, Proof of Theorme 2]

and by H. Added and S. Added [1, Proof of Theorem] (see also [12]). The proof of the a priori estimates requires the assumption $n_1 \in \dot{H}^{-1}$, because the energy identity of (1.1)-(1.3) contains the \dot{H}^{-1} norm of n_1 . Those a priori estimates and Theorem 1.1 show Theorem 3.4.

§4. Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2. We first describe the proof of Part (1) of Theorem 1.2.

Proof of Theorem 1.2(1). Let $(E(t), n(t))$ be the solutions of (1.1)-(1.3) in the class of Theorem 1.1.

We first assume that m is even. Then we can easily see by (1.1) that $E(t) \in C^{m/2+1}([0, T_{max}); H^{-2})$. We differentiate (1.1) $m/2$ times in t to obtain

$$(4.1) \quad i\partial_t(\partial_t^{m/2}E) + A(\partial_t^{m/2}E) = \sum_{j=0}^{m/2} \binom{m/2}{j} (\partial_t^{m/2-j}n)(\partial_t^jE)$$

in H^{-2} , $0 \leq t < T_{max}$.

We rewrite (4.1) as the integral form:

$$(4.2) \quad \partial_t^{m/2}E(t) = U(t)\partial_t^{m/2}E(0) - i \int_0^t U(t-s) \sum_{j=0}^{m/2} \binom{m/2}{j} (\partial_s^{m/2-j}n)(\partial_s^jE) ds,$$

$0 \leq t < T_{max}$.

Since $(E(t), n(t))$ are in the class of Theorem 1.1, we have

$$(4.3) \quad E \in \bigcap_{j=0}^{m/2} C^j([0, T_{max}); H^{m-2j}),$$

$$(4.4) \quad E \in \bigcap_{j=0}^{m/2} W^{j, 8/N}(0, T; W^{m-2j, 4}), \quad 0 < T < T_{max},$$

$$(4.5) \quad n \in \bigcap_{j=0}^3 C^j([0, T_{max}); H^{m-1-j}),$$

and if $m \geq 6$.

$$(4.6) \quad n \in \bigcap_{j=4}^{m/2+1} C^j([0, T_{max}); H^{m+2-2j}).$$

On the other hand, by the Sobolev imbedding theorem we have

$$\begin{aligned}
 (4.7) \quad \left\| \sum_{j=0}^{m/2} \binom{m/2}{j} (\partial_s^{m/2-j} n) (\partial_s^j E) \right\|_{L^2} &\leq C \sum_{j=0}^{m/2-1} \|\partial_s^{m/2-j} n\|_{L^2} \|\partial_s^j E\|_{L^\infty} \\
 &\quad + \|n\|_{L^4} \|\partial_s^{m/2} E\|_{L^4} \\
 &\leq C \sum_{j=0}^{m/2-1} \|\partial_s^{m/2-j} n\|_{L^2} \|\partial_s^j E\|_{H^2} \\
 &\quad + C \|n\|_{H^1} \|\partial_s^{m/2} E\|_{L^4}.
 \end{aligned}$$

(4.3)-(4.7) give us

$$(4.8) \quad \sum_{j=0}^{m/2} \binom{m/2}{j} (\partial_s^{m/2-j} n) (\partial_s^j E) \in L^{8/N}(0, T; L^2), \quad 0 < T < T_{max}.$$

Noting $\partial_t^{m/2} E(0) \in L^2$, we apply Lemma 2.3 to (4.2) and use (4.8) to obtain

$$\begin{aligned}
 (4.9) \quad \partial_t^{m/2} E &\in L^2(0, T; H^{1/2}(|x| < R)), \\
 &0 < T < T_{max}, \quad R > 0.
 \end{aligned}$$

We differentiate (1.1) $m/2-1$ times in t to obtain

$$\begin{aligned}
 (4.10) \quad i \partial_t (\partial_t^{m/2-1} E) + \Delta (\partial_t^{m/2-1} E) \\
 = \sum_{j=0}^{m/2-1} \binom{m/2-1}{j} (\partial_t^{m/2-1-j} n) (\partial_t^j E), \\
 0 \leq t < T_{max}.
 \end{aligned}$$

(4.3), (4.5), (4.6) and the Sobolev imbedding theorem imply that the right hand side of (4.10) is in $L^\infty(0, T; H^1)$ for $0 < T < T_{max}$. Therefore, (4.9), (4.10) and the local regularity theorem of the elliptic equation yield

$$(4.11) \quad \partial_t^{m/2-1} E \in L^2(0, T; H^{5/2}(|x| < R)), \quad 0 < T < T_{max}, \quad R > 0.$$

If $m/2-1 < 0$, we repeat the same argument as above until we have

$$(4.12) \quad E \in L^2(0, T; H^{m+1/2}(|x| < R)), \quad 0 < T < T_{max}, \quad R > 0.$$

Thus, the proof for even m is completed. We similarly prove Theorem 1.2(1) for m odd. \blacksquare

We next state the proof of Part (2) of Theorem 1.2.

Proof of Theorem 1.2(2). We prove Theorem 1.2(2) only in the case where m is an even integer with $m \geq 6$ and $k=2$, since the proof for the other cases is the same.

Let $(E(t), n(t))$ be the solutions of (1.1)-(1.3) in the class of Theorem 1.2(2) for any T with $0 < T < T_{max}$. Let T be an arbitrarily fixed constant with

$0 < T < T_{max}$. We put $n_\epsilon(t) = \rho_\epsilon * n(t)$ and put

$$A(n) = \sum_{j=0}^3 \|\partial_t^j n\|_{L^\infty(0, T; H^{m-1-j})} + \sum_{j=4}^{m/2+1} \|\partial_t^j n\|_{L^\infty(0, T; H^{m+2-2j})}.$$

We note that $n_\epsilon(t) \in C^{m/2+1}([0, T]; H^s)$ for $s > 0$ and $A(n_\epsilon) \leq A(n)$. Let $E_{0\epsilon} \in \mathcal{S}$ such that $E_{0\epsilon} \rightarrow E_0$ in $H^{m,2}$ as $\epsilon \rightarrow 0$ and $\|E_{0\epsilon}\|_{H^{m,2}} \leq 2\|E_0\|_{H^{m,2}}$. We consider the following linear Schrödinger equation:

$$(4.13) \quad i\partial_t E_\epsilon + \Delta E_\epsilon = n_\epsilon E_\epsilon, \quad 0 \leq t \leq T, \quad x \in \mathbf{R}^N,$$

$$(4.14) \quad E_\epsilon(0, x) = E_{0\epsilon}(x).$$

By the theory of evolution equation we have the unique solution $E_\epsilon(t)$ of (4.13) and (4.14) such that $E_\epsilon \in C^{m/2+1}([0, T]; H^s)$ for $s > 0$ and

$$(4.15) \quad \|\partial_t^j E_\epsilon\|_{L^\infty(0, T; H^{m-2j})} \leq C_1, \quad 0 \leq j \leq m/2,$$

where C_1 depends only on $T, A(n), \|E_0\|_{H^m}$ and N but not on ϵ . We put $g_l(x) = (1 + |x/l|^2)^{-2}$ for a positive integer l . A simple calculation gives us

$$(4.16) \quad |\mathcal{F}\{(1 + |x|^2)g_l^2(x)\}| \leq C_2(1 + |x|^2)^{1/2}g_l^2(x),$$

$$(4.17) \quad |\mathcal{F}\{(1 + |x|^2)^2g_l^2(x)\}| \leq C_3(1 + |x|^2)^{3/2}g_l^2(x),$$

where C_2 and C_3 do not depend on l .

We consider the scalar product in L^2 between (4.13) and $(1 + |x|^2)g_l^2(x)E_\epsilon(t)$ and take the imaginary part of the resulting equation to obtain by (4.16)

$$(4.18) \quad \begin{aligned} \|(1 + |x|^2)^{1/2}g_l E_\epsilon(t)\|_{L^2}^2 &\leq \|(1 + |x|^2)^{1/2}g_l E_{0\epsilon}\|_{L^2}^2 \\ &\quad + C \int_0^t \sum_{j=1}^N \|g_l \mathcal{F} E_{\epsilon j}\|_{L^2} \|(1 + |x|^2)^{1/2}g_l E_\epsilon\|_{L^2} ds \\ &\leq \|E_0\|_{H^{m,2}}^2 \\ &\quad + C \int_0^t \sum_{j=1}^N \|\mathcal{F} E_{\epsilon j}\|_{L^2} \|(1 + |x|^2)^{1/2}g_l E_\epsilon\|_{L^2} ds, \\ &\quad 0 \leq t \leq T. \end{aligned}$$

(4.15), (4.18) and Gronwall's inequality yield

$$(4.19) \quad \|(1 + |x|^2)^{1/2}g_l E_\epsilon\|_{L^\infty(0, T; L^2)} \leq C_4,$$

where C_4 depends only on $T, \|E_0\|_{H^{m,2}}, A(n)$ and N but not on l and ϵ . Letting $l \rightarrow \infty$ in (4.19), by Fatou's lemma we obtain

$$(4.20) \quad \|(1 + |x|^2)^{1/2} E_{\mathbf{e}}\|_{L^\infty(0, T; L^2)} \leq C_4.$$

We next differentiate (4.13) in x_k , $1 \leq k \leq N$ and take the imaginary part of the scalar product in L^2 between the resulting equation and $(1 + |x|^2)g_i^2(x)\partial_k E_{\mathbf{e}}(t)$ to obtain by (4.18)

$$(4.21) \quad \begin{aligned} & \|(1 + |x|^2)^{1/2} g_i \partial_k E_{\mathbf{e}}(t)\|_{L^2}^2 \leq \|(1 + |x|^2)^{1/2} g_i \partial_k E_{0\mathbf{e}}\|_{L^2}^2 \\ & + C \int_0^t \left[\sum_{j=1}^N \|g_i \partial_k \nabla E_{\mathbf{e}j}\|_{L^2} \|(1 + |x|^2)^{1/2} g_i \partial_k E_{\mathbf{e}}\|_{L^2} \right. \\ & \quad \left. + \|\partial_k n_{\mathbf{e}}\|_{L^\infty} \|(1 + |x|^2)^{1/2} g_i E_{\mathbf{e}}\|_{L^2} \|(1 + |x|^2)^{1/2} g_i \partial_k E_{\mathbf{e}}\|_{L^2} \right] ds \\ & \leq C \|E_0\|_{H^{m,2}}^2 \\ & + C \int_0^t \left[\sum_{j=0}^N \|\partial_k \nabla E_{\mathbf{e}}\|_{L^2} \right. \\ & \quad \left. + \|\partial_k n_{\mathbf{e}}\|_{L^\infty} \|(1 + |x|^2)^{1/2} g_i E_{\mathbf{e}}\|_{L^2} \|(1 + |x|^2)^{1/2} g_i \partial_k E_{\mathbf{e}}\|_{L^2} \right] ds, \\ & 0 \leq t \leq T, \quad 1 \leq k \leq N. \end{aligned}$$

(4.15), (4.21), the Sobolev imbedding theorem and Gronwall's inequality yield

$$(4.22) \quad \|(1 + |x|^2)^{1/2} g_i \partial_k E_{\mathbf{e}}\|_{L^\infty(0, T; L^2)} \leq C_5, \quad 1 \leq k \leq N,$$

where C_5 depends only on T , $\|E_0\|_{H^{m,2}}$, $A(n)$ and N but not on l and ε . Letting $l \rightarrow \infty$ in (4.22), by Fatou's lemma we obtain

$$(4.23) \quad \|(1 + |x|^2)^{1/2} \partial_k E_{\mathbf{e}}\|_{L^\infty(0, T; L^2)} \leq C_5, \quad 1 \leq k \leq N.$$

We take the imaginary part of the scalar product in L^2 between (4.13) and $(1 + |x|^2)^2 g_i^2 E_{\mathbf{e}}(t)$ to obtain by (4.17)

$$(4.24) \quad \begin{aligned} & \|(1 + |x|^2) E_{\mathbf{e}}(t)\|_{L^2}^2 \leq \|(1 + |x|^2) g_i E_{0\mathbf{e}}\|_{L^2}^2 \\ & + C \int_0^t \sum_{j=1}^N \|(1 + |x|^2)^{1/2} g_i \nabla E_{\mathbf{e}j}\|_{L^2} \|(1 + |x|^2) g_i E_{\mathbf{e}}\|_{L^2} ds \\ & \leq C \|E_0\|_{H^{m,2}}^2 \\ & + C \int_0^t \sum_{j=1}^N \|(1 + |x|^2)^{1/2} \nabla E_{\mathbf{e}j}\|_{L^2} \|(1 + |x|^2) g_i E_{\mathbf{e}}\|_{L^2} ds, \\ & 0 \leq t \leq T. \end{aligned}$$

(4.23), (4.24) and Gronwall's inequality yield

$$(4.25) \quad \|(1 + |x|^2) g_i E_{\mathbf{e}}\|_{L^\infty(0, T; L^2)} \leq C_6,$$

where C_6 depends only on T , $\|E_0\|_{H^{m,2}}$, $A(n)$ and N but not on l and ε . Letting $l \rightarrow \infty$ in (4.25), by Fatou's lemma we have

$$(4.26) \quad \|(1 + |x|^2)E_\varepsilon\|_{L^\infty(0,T;L^2)} \leq C_6.$$

We next show that if

$$(4.27) \quad \|(1 + |x|^2)\partial_t^p E_\varepsilon\|_{L^\infty(0,T;L^2)} \leq C(q), \quad 0 \leq p \leq q,$$

$$(4.28) \quad \|(1 + |x|^2)^{1/2}\partial_k^p E_\varepsilon\|_{L^\infty(0,T;L^2)} \leq C(q), \quad 1 \leq k \leq N, \quad 0 \leq p \leq q$$

for some integer q with $0 \leq q \leq m/2 - 2$, then (4.27) and (4.28) also hold with q and $C(q)$ replaced by $q+1$ and $C(q+1)$, where $C(q)$ depends only on $T, \|E_0\|_{H^{m,2}}, A(n), N$ and q but not on ε . We differentiate (4.13) $q+1$ times in t and take the imaginary part of the scalar product in L^2 between the resulting equation and $(1 + |x|^2)g_t^2 \partial_t^{q+1} E_\varepsilon(t)$ to obtain by (4.16)

$$(4.29) \quad \begin{aligned} & \|(1 + |x|^2)^{1/2} g_t \partial_t^{q+1} E_\varepsilon(t)\|_{L^2}^2 \leq \|(1 + |x|^2)^{1/2} g_t \partial_t^{q+1} E_\varepsilon(0)\|_{L^2}^2 \\ & \quad + C \int_0^t \sum_{j=1}^N \|g_t \nabla \partial_s^{q+1} E_{\varepsilon,j}\|_{L^2} \|(1 + |x|^2)^{1/2} g_t \partial_s^{q+1} E_\varepsilon\|_{L^2} ds \\ & \quad + C \int_0^t \sum_{j=0}^q \|\partial_s^{q+1-j} n_\varepsilon\|_{L^\infty} \|(1 + |x|^2)^{1/2} g_t \partial_s^j E_\varepsilon\|_{L^2} \\ & \quad \quad \times \|(1 + |x|^2)^{1/2} g_t \partial_s^{q+1} E_\varepsilon\|_{L^2} ds \\ & \leq C + C \int_0^t \left[\sum_{j=1}^N \|\nabla \partial_s^{q+1} E_{\varepsilon,j}\|_{L^2} \right. \\ & \quad \left. + \sum_{j=0}^q \|\partial_s^{q+1-j} n_\varepsilon\|_{L^\infty} \|(1 + |x|^2)^{1/2} g_t \partial_s^j E_\varepsilon\|_{L^2} \right] \\ & \quad \quad \times \|(1 + |x|^2)^{1/2} g_t \partial_s^{q+1} E_\varepsilon\|_{L^2} ds. \end{aligned}$$

(4.15), (4.29), the Sobolev imbedding theorem and Gronwall's inequality yield

$$(4.30) \quad \|(1 + |x|^2)^{1/2} g_t \partial_t^{q+1} E_\varepsilon\|_{L^\infty(0,T;L^2)} \leq C_7,$$

where C_7 depends only on $T, \|E_0\|_{H^{m,2}}, A(n)$ and N but not on l and ε . Letting $l \rightarrow \infty$ in (4.30), by Fatou's lemma we obtain

$$(4.31) \quad \|(1 + |x|^2)^{1/2} \partial_t^{q+1} E_\varepsilon\|_{L^\infty(0,T;L^2)} \leq C_7.$$

We next differentiate (4.13) $q+1$ times in t and once in $x_k, 1 \leq k \leq N$, and take the imaginary part of the scalar product in L^2 between the resulting equation and $(1 + |x|^2)g_k^2 \partial_k \partial_t^{q+1} E_\varepsilon(t)$ to obtain by (4.16)

$$(4.32) \quad \begin{aligned} & \|(1 + |x|^2)^{1/2} g_t \partial_k \partial_t^{q+1} E_\varepsilon(t)\|_{L^2}^2 \leq \|(1 + |x|^2)^{1/2} g_t \partial_k \partial_t^{q+1} E_\varepsilon(0)\|_{L^2}^2 \\ & \quad + C \int_0^t \sum_{j=1}^N \|g_t \partial_k \nabla \partial_s^{q+1} E_{\varepsilon,j}\|_{L^2} \|(1 + |x|^2)^{1/2} g_t \partial_k \partial_s^{q+1} E_\varepsilon\|_{L^2} ds \end{aligned}$$

$$\begin{aligned}
 &+ C \int_0^t \left[\sum_{j=0}^{q+1} \|\partial_k \partial_s^{q+1-j} n_\varepsilon\|_{L^\infty} \|(1+|x|^2)^{1/2} g_t \partial_s^j E_\varepsilon\|_{L^2} \right. \\
 &\quad \left. + \sum_{j=0}^q \|\partial_s^{q+1-j} n_\varepsilon\|_{L^\infty} \|(1+|x|^2)^{1/2} g_t \partial_k \partial_s^j E_\varepsilon\|_{L^2} \right] \\
 &\quad \times \|(1+|x|^2)^{1/2} g_t \partial_k \partial_s^{q+1} E_\varepsilon\|_{L^2} ds \\
 &\leq C + C \int_0^t \left[\sum_{j=1}^N \|\partial_k \nabla \partial_s^{q+1} E_{\varepsilon_j}\|_{L^2} \right. \\
 &\quad \left. + \sum_{j=0}^{q+1} \|\partial_k \partial_s^{q+1-j} n_\varepsilon\|_{L^\infty} \|(1+|x|^2)^{1/2} \partial_s^j E_\varepsilon\|_{L^2} \right. \\
 &\quad \left. + \sum_{j=0}^q \|\partial_s^{q+1-j} n_\varepsilon\|_{L^\infty} \|(1+|x|^2)^{1/2} \partial_k \partial_s^j E_\varepsilon\|_{L^2} \right] \\
 &\quad \times \|(1+|x|^2)^{1/2} g_t \partial_k \partial_s^{q+1} E_\varepsilon\|_{L^2} ds, \\
 &\quad 0 \leq t \leq T, \quad 1 \leq k \leq N.
 \end{aligned}$$

(4.15), (4.27), (4.28), (4.31), (4.32), the Sobolev imbedding theorem and Gronwall’s inequality yield

$$(4.33) \quad \|(1+|x|^2)^{1/2} g_t \partial_k \partial_t^{q+1} E_\varepsilon\|_{L^\infty(0,T;L^2)} \leq C_8, \quad 1 \leq k \leq N,$$

where C_8 depends only on $T, \|E_0\|_{H^{m,2}}, A(n)$ and N but not on l and ε . Letting $l \rightarrow \infty$ in (4.33), by Fatou’s lemma we obtain

$$(4.34) \quad \|(1+|x|^2)^{1/2} \partial_t^{q+1} E_\varepsilon\|_{L^\infty(0,T;L^2)} \leq C_8, \quad 1 \leq k \leq N.$$

We differentiate (4.13) $q+1$ times in t again and take the imaginary part of the scalar product in L^2 between the resulting equation and $(1+|x|^2)^2 g_t E_\varepsilon(t)$ to obtain by (4.17)

$$\begin{aligned}
 (4.35) \quad &\|(1+|x|^2) g_t \partial_t^{q+1} E_\varepsilon(t)\|_{L^2}^2 \leq \|(1+|x|^2) g_t \partial_t^{q+1} E_\varepsilon(0)\|_{L^2}^2 \\
 &+ C \int_0^t \sum_{j=1}^N \|(1+|x|^2)^{1/2} g_t \nabla \partial_s^{q+1} E_{\varepsilon_j}\|_{L^2}^2 \|(1+|x|^2) g_t \partial_s^{q+1} E_\varepsilon\|_{L^2} ds \\
 &+ C \int_0^t \sum_{j=0}^{q+1} \|\partial_s^{q+1-j} n_\varepsilon\|_{L^\infty} \|(1+|x|^2) g_t \partial_s^j E_\varepsilon\|_{L^2} \\
 &\quad \times \|(1+|x|^2) g_t \partial_s^{q+1} E_\varepsilon\|_{L^2} ds \\
 &\leq C + C \int_0^t \left[\sum_{j=1}^N \|(1+|x|^2)^{1/2} \nabla \partial_s^{q+1} E_{\varepsilon_j}\|_{L^2} \right. \\
 &\quad \left. + \sum_{j=0}^q \|\partial_s^{q+1-j} n_\varepsilon\|_{L^\infty} \|(1+|x|^2) \partial_s^j E_\varepsilon\|_{L^2} \right] \\
 &\quad \times \|(1+|x|^2) g_t \partial_s^{q+1} E_\varepsilon\|_{L^2} ds, \\
 &\quad 0 \leq t \leq T.
 \end{aligned}$$

(4.15), (4.27), (4.28), (4.34), the Sobolev imbedding theorem and Gronwall's inequality yield

$$(4.36) \quad \|(1 + |x|^2)g_l \partial_t^{q+1} E_\varepsilon\|_{L^\infty(0, T; L^2)} \leq C_9,$$

where C_9 depends only on $T, \|E_0\|_{H^{m,2}}, A(n)$ and N but not on l and ε . Letting $l \rightarrow \infty$ in (4.36) by Fatou's lemma we obtain

$$(4.37) \quad \|(1 + |x|^2) \partial_t^{q+1} E_\varepsilon\|_{L^\infty(0, T; L^2)} \leq C_9.$$

Hence, (4.23), (4.26) and the induction argument imply that

$$(4.38) \quad \|(1 + |x|^2) \partial_t^j E_\varepsilon\|_{L^\infty(0, T; L^2)} \leq C_{10}, \quad 0 \leq j \leq m/2 - j,$$

$$(4.39) \quad \|(1 + |x|^2)^{1/2} \partial_k \partial_t^j E_\varepsilon\|_{L^\infty(0, T; L^2)} \leq C_{11}, \quad 0 \leq j \leq m/2 - 1, \quad 1 \leq k \leq N,$$

where C_{10} and C_{11} depend only on $T, \|E_0\|_{H^{m,2}}, A(n)$ and N but not on ε .

Noting that $E_{0\varepsilon} \in S$ and $n_\varepsilon \in C^{m/2+1}([0, T]; H^s)$ for $s > 0$, we can similarly show that $(1 + |x|^2) \partial_t^{m/2} E \in L^\infty(0, T; L^2)$ and $(1 + |x|^2)^{1/2} \partial_k \partial_t^{m/2} E \in L^\infty(0, T; L^2), 1 \leq k \leq N$. We differentiate (4.13) $m/2$ times in t to obtain

$$(4.40) \quad \begin{aligned} i \partial_t (\partial_t^{m/2} E_\varepsilon) + A(\partial_t^{m/2} E_\varepsilon) \\ = \sum_{j=0}^{m/2} \binom{m/2}{j} (\partial_t^{m/2-j} n_\varepsilon) (\partial_t^j E_\varepsilon), \quad 0 \leq t \leq T. \end{aligned}$$

We rewrite (4.40) as the integral form:

$$(4.41) \quad \begin{aligned} \partial_t^{m/2} E_\varepsilon(t) &= U(t) \partial_t^{m/2} E_\varepsilon(0) \\ &\quad - i \int_0^t U(t-s) \sum_{j=0}^{m/2} \binom{m/2}{j} (\partial_s^{m/2-j} n_\varepsilon) (\partial_s^j E_\varepsilon) ds, \\ &\quad 0 \leq t \leq T. \end{aligned}$$

By (4.41) and Lemma 2.4 we obtain

$$(4.42) \quad \begin{aligned} J_k^2(t) \partial_t^{m/2} E_\varepsilon(t) &= U(t) x_k^2 \partial_t^{m/2} E_\varepsilon(0) \\ &\quad - i \int_0^t U(t-s) \sum_{j=0}^{m/2} \binom{m/2}{j} J_k^2(s) \{(\partial_s^{m/2-j} n_\varepsilon) (\partial_s^j E_\varepsilon)\} ds, \\ &\quad 0 \leq t \leq T, \quad 1 \leq k \leq N. \end{aligned}$$

We take the L^2 norm of (4.42) to obtain

$$(4.43) \quad \begin{aligned} \|J_k^2 \partial_t^{m/2} E_\varepsilon(t)\|_{L^2} &\leq \|x_k^2 \partial_t^{m/2} E_\varepsilon(0)\|_{L^2} \\ &\quad + C \int_0^t \sum_{j=0}^{m/2} \|M(s) (2is \partial_k)^2 \{M(-s) (\partial_s^{m/2-j} n_\varepsilon) (\partial_s^j E_\varepsilon)\}\|_{L^2} ds \end{aligned}$$

$$\begin{aligned}
 &\leq C + C \int_0^t s^2 \sum_{j=0}^{m/2} [\|(\partial_k^2 \partial_s^{m/2-j} n_e)(\partial_s^j E_e)\|_{L^2} \\
 &\quad + \|(\partial_k \partial_s^{m/2-j} n_e) \partial_k (M(-s) \partial_s^j E_e)\|_{L^2} \\
 &\quad + \|(\partial_s^{m/2-j} n_e) \partial_k^2 (M(-s) \partial_s^j E_e)\|_{L^2}] ds \\
 &\leq C + C \int_0^t s^2 \sum_{j=0}^{m/2-1} [\|\partial_k^2 \partial_s^{m/2-j} n_e\|_{L^2} \|\partial_s^j E_e\|_{L^\infty} \\
 &\quad + \|\partial_k \partial_s^{m/2-j} n_e\|_{L^4} \|\partial_k (M(-s) \partial_s^j E_e)\|_{L^4} \\
 &\quad + \|\partial_s^{m/2-j} n_e\|_{L^\infty} \|\partial_k^2 (M(-s) \partial_s^j E_e)\|_{L^2}] ds \\
 &\quad + C \int_0^t s^2 \|\partial_k^2 n_e\|_{L^\infty} \|\partial_s^{m/2} E_e\|_{L^2} ds \\
 &\quad + C \int_0^t s^2 \|\partial_k n_e\|_{L^\infty} \|\partial_k (M(-s) \partial_s^{m/2} E_e)\|_{L^2} ds \\
 &\quad + C \int_0^t s^2 \|n_e\|_{L^\infty} \|\partial_k^2 (M(-s) \partial_s^{m/2} E_e)\|_{L^2} ds, \\
 &\quad 1 \leq k \leq N.
 \end{aligned}$$

On the other hand, by the Sobolev imbedding theorem and the interpolation inequality we have

$$\begin{aligned}
 (4.44) \quad &\|\partial_k (M(-s) \partial_s^j E_e)\|_{L^4} \\
 &\leq C \left[\sum_{i=1}^N \|\partial_i^2 (M(-s) \partial_s^j E_e)\|_{L^2} + \|\partial_k (M(-s) \partial_s^j E_e)\|_{L^2} \right],
 \end{aligned}$$

$$\begin{aligned}
 (4.45) \quad &\|\partial_k (M(-s) \partial_s^j E_e)\|_{L^2} \\
 &\leq C \left[\sum_{i=1}^N \|\partial_i^2 (M(-s) \partial_s^j E_e)\|_{L^2} + \|\partial_s^j E_e\|_{L^2} \right].
 \end{aligned}$$

(4.43)-(4.45), (4.15), (4.38), (4.39) and the Sobolev imbedding theorem give us

$$\begin{aligned}
 (4.46) \quad &\sum_{k=1}^N \|J_k^2 \partial_k^{m/2} E_e(t)\|_{L^2} \leq C \\
 &\quad + C \int_0^t s^2 \sum_{j=0}^{m/2-1} [\|\partial_s^{m/2-j} n_e\|_{H^2} \|\partial_s^j E_e\|_{H^2} \\
 &\quad + \|\partial_s^{m/2-j} n_e\|_{H^2} \left\{ \sum_{k=1}^N \|\partial_k^2 (M(-s) \partial_s^j E_e)\|_{L^2} \right. \\
 &\quad \left. + \sum_{k=1}^N \|\partial_k (M(-s) \partial_s^j E_e)\|_{L^2} \right\}] ds \\
 &\quad + C \int_0^t s^2 \|n_e\|_{H^4} \|\partial_s^{m/2} E_e\|_{L^2} ds
 \end{aligned}$$

$$\begin{aligned}
 &+ C \int_0^t s^2 \|n_\varepsilon\|_{H^3} \sum_{k=1}^N \|\partial_k(M(-s)\partial_s^{m/2}E_\varepsilon)\|_{L^2} ds \\
 &+ C \int_0^t s^2 \|n_\varepsilon\|_{H^2} \sum_{k=1}^N \|\partial_k^2(M(-s)\partial_s^{m/2}E_\varepsilon)\|_{L^2} ds \\
 \leq &C_{12} + C_{13} \int_0^t (2s)^2 \sum_{k=1}^N \|\partial_k^2(M(-s)\partial_s^{m/2}E_\varepsilon)\|_{L^2} ds \\
 = &C_{12} + C_{13} \int_0^t \sum_{k=1}^N \|J_k^2 \partial_s^{m/2}E_\varepsilon\|_{L^2} ds, \\
 &0 \leq t \leq T,
 \end{aligned}$$

where C_{12} and C_{13} depend only on $T, \|E_0\|_{H^{m,2}}, A(n)$ and N but not on ε . (4.46) and Gronwall's inequality yield

$$\begin{aligned}
 (4.47) \quad &4t^2 \sum_{k=1}^N \|\partial_k^2(M(-t)\partial_t^{m/2}E_\varepsilon(t))\|_{L^2} \\
 &= \sum_{k=1}^N \|J_k^2 \partial_t^{m/2}E_\varepsilon(t)\|_{L^2} \leq C_{14}, \quad 0 < t \leq T,
 \end{aligned}$$

where C_{14} depends only on $T, \|E_0\|_{H^{m,2}}, A(n)$ and N but not on ε .

We can easily see that

$$(4.48) \quad E_\varepsilon(t) \rightarrow E(t) \text{ in } \bigcap_{j=0}^{m/2} C^j([0, T]; H^{m-2j})$$

as $\varepsilon \rightarrow 0$. Therefore, (4.47) and (4.48) imply that

$$(4.49) \quad 4t^2 \sum_{k=1}^N \|\partial_k^2(M(-t)\partial_t^{m/2}E(t))\|_{L^2} \leq C_{14}, \quad 0 < t \leq T.$$

(4.49) and the definition of $M(t)$ show that

$$\begin{aligned}
 (4.50) \quad &\partial_t^{m/2}E \in L^\infty(\tau, T; H^2(|x| < R)), \\
 &0 < \tau < T, \quad R > 0.
 \end{aligned}$$

We differentiate (1.1) $m/2 - 1$ times in t to obtain

$$\begin{aligned}
 (4.51) \quad &i\partial_t^{m/2}E + A(\partial_t^{m/2-1}E) \\
 &= \sum_{j=0}^{m/2-1} \binom{m/2-1}{j} (\partial_t^{m/2-j}n)(\partial_t^jE), \quad 0 \leq t \leq T.
 \end{aligned}$$

Since $(E(t), n(t))$ is in the class of Theorem 1.1(2), the Sobolev imbedding theorem implies that the right hand side of (4.51) belongs to $L^\infty(0, T; H^2)$. Therefore, (4.50), (4.51) and the local regularity theorem of the elliptic equation give us

$$(4.52) \quad \partial_t^{m/2-1} E \in L^\infty(\tau, T; H^4(|x| < R)), \\ 0 < \tau < T, \quad R > 0.$$

Repeating the above argument, we conclude that

$$(4.53) \quad E \in L^\infty(\tau, T; H^{m+2}(|x| < R)), \\ 0 < \tau < T, \quad R > 0.$$

Thus, since T is an arbitrary constant with $0 < T < T_{max}$, the proof is completed in the case where m is an even integer with $m \geq 6$ and $k=2$.

In the same way as above we can prove Theorem 1.2(2) for the other cases. \square

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