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On the Uniqueness for the Cauchy Problem for Elliptic Equations with Triple Characteristics

By

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Contents

§ 1. Introduction

In this paper we shall prove an uniqueness theorem for the Cauchy problem for certain elliptic differential operator $P(x, D)$ in a neighbourhood of the origin in $\boldsymbol{R}^n$ of order $m \geq 1$ with C^∞-coefficients and the principal symbol $P_m(x, \xi)$ of the form

$$P_m(x, \xi) = Q_1(x, \xi)^2 Q_2(x, \xi) \tag{1.1}$$

where Q_i $(i=1, 2)$ is a homogeneous polynomial in ξ of degree m_i with C^∞-coefficients such that

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if $m_i \geq 1$, for every $\xi' \in \mathbb{R}^{n-1}\backslash(0)$ the zeros τ of $Q_i(0, (\tau, \xi'))$ are non-real and simple.

Our theorem is an extension of the Watanabe's theorem [6] in case of C^∞-coefficients. Our main result is the following.

Theorem 1.1. *Let $P(x, D)$ be a differential operator in an open neighborhood Ω of the origin in $\mathbb{R}^n$ of order $m \geq 1$ with C^∞-coefficients and the symbol $P(x, \xi)$. Let $P(x, \xi) = P_m(x, \xi) + \cdots + P_0(x, \xi)$ with homogeneous polynomials $P_j(x, \xi)$ in ξ of degree j. We assume the followings.*

(i) *The principal symbol P_m of P takes the form* (1.1) *with Q_i as above.*

(ii) *If $P_m = \partial_{\xi_1} P_m = \partial_{\xi_1}^2 P_m = 0$ at $(0, (\tau_0, \eta_0))$ with non-real τ_0 and $\eta_0 \in \mathbb{R}^{n-1}\backslash(0)$, there exists an open conic set Γ in $\Omega \times (\mathbb{C}^n\backslash(0))$ containing $(0, (\tau_0, \eta_0))$ satisfying the following condition.*

$$\begin{aligned}&(|\zeta|\,|(\partial_\xi P_m)(x, \zeta)| + |(\partial_x P_m)(x, \zeta)|)\,|P_{m-1}(x, \zeta)| \\ &\quad \leq C|P_m(x, \zeta)|^{2/3}(|P_m(x, \zeta)|^{1/3}|\zeta|^{m-1} + |(P_m + P_{m-1})(x, \zeta)|\,|\zeta|^{m/3} \\ &\qquad + (1+|\zeta|)^{(4m/3)-(3/2)}) \\ &\quad \text{for } (x, \zeta) \in \Gamma \text{ with } (\partial_{\xi_1} P_m)(x, \zeta) = 0\,.\end{aligned}$$

Under the assumptions (i), (ii) *there exists an open neighbourhood $\Omega' \subset \Omega$ of the origin in $\mathbb{R}^n$ such that every $u \in C^\infty(\Omega)$ satisfying $P(x, D)u = 0$ in Ω and $u|_{x_1 \leq 0} = 0$ vanishes in Ω'.*

Now, we give simple examples of differential operators $P(x, D)$ satisfying the assumptions of Theorem 1.1.

Example 1.1. *Let $p(x, \xi) = (\xi_1 - i\xi_2)^2(\xi_1 - i\xi_2 - a(x)\xi_2)$ and $q(x, \xi) = b(x)\xi_2^2$ where a, b are C^∞-functions in an open neighbourhood of the origin in $\mathbb{R}^2$. We assume that $a(0) = 0$ and $|b(x)|\,|da(x)| \leq C|a(x)|$ near the origin. Then $P(x, D) = p(x, D) + q(x, D)$ satisfies the assumptions in Theorem* 1.1.

Indeed we have $\partial_{\xi_1} p = 3(\xi_1 - i\xi_2)(\xi_1 - i\xi_2 - \frac{2}{3}a(x)\xi_2)$. When $\xi_1 - i\xi_2 - \frac{2}{3}a(x)\xi_2 = 0$, $(x, \xi) \in \mathbb{R}^2 \times \mathbb{C}^2$ and when $|x'|$ is small, we have that

$$\begin{aligned}&(|\partial_x p| + |\partial_\xi p|\,|\xi|)\,|q| \leq C(|da|\,|b| + |a|)\,|a|^2|\xi_2|^5\,, \\ &|p|\,|\xi|^2 \geq \delta|a|^3\,|\xi_2|^5 \quad \text{for some} \quad \delta > 0\,.\end{aligned}$$

Thus the inequality in the assumption (ii) in the theorem holds when $\xi_1 = i\xi_2 + \frac{2}{3}a(x)\xi_2$, $(x, \xi) \in \mathbb{R}^2 \times \mathbb{C}^2$ and $|x'|$ is small. This means that the assumption in the theorem holds for $P(x, D)$.

Example 1.2. *Let* $p(x,\xi)=(\xi_1^2+\cdots+\xi_n^2)^2(\xi_1^2+\cdots+\xi_n^2+a(x',\xi'))$ *where* $a(x',\xi')=|x'|^{2k}(\xi_2^2+\cdots+\xi_n^2)+x_2^{2k_2}\xi_2^2+\cdots+x_n^{2k_n}\xi_n^2$ *with* $k, k_j\in \boldsymbol{N}, k_j>k>0$. *Here we use a notation that* $x'=(x_2,\cdots,x_n)$. *Let* $q(x,\xi)=c_1(x,\xi')\xi_1+c_0(x,\xi')$ *where* c_1, c_0 *are respectively homogeneous polynomials in* ξ' *of degree* 4, 5 *with* C^∞*-coefficients in an open neighbourhood of the origin in* $\boldsymbol{R}^n$. *We assume that* $|c_1(x,\zeta)||\zeta|+|c_0(x,\zeta)|\le C|x'||\zeta|^5$ *for small* $|x|$ *and* $\zeta\in \boldsymbol{C}^{n-1}$. *Then* $P(x, D)=p(x,D)+q(x,D)$ *satisfies the assumptions in Theorem* 1.1.

Indeed we have $\partial_{\xi_1}p=6(\xi_1^2+\cdots+\xi_n^2)\xi_1(\xi_1^2+\cdots+\xi_n^2+\frac{2}{3}a)$. Assume that $\xi_1^2+\cdots+\xi_n^2+\frac{2}{3}a=0$, $(x,\xi)\in \boldsymbol{R}^n\times \boldsymbol{C}^n$ with $|Im\,\xi'|<\frac{1}{2}|Re\,\xi'|$ and $|x'|$ is small. Then $|x'||\partial_{x_j}a|\le C|a|$ and $|\partial_{\xi_j}a||\xi|\le C|a|$, because $|a|\ge \delta|\xi'|^2|x'|^{2k}$ for some $\delta>0$. Since $|\partial_x p|+|\partial_\xi p||\xi|\le C(|\partial_{x'}a|+|\partial_{\xi'}a||\xi|)|a|^2$ and $|p||\xi|^5=\frac{4}{27}|a|^2|\xi|^5$, the inequality in the assumption (ii) in the theorem holds. This means that the assumption in the theorem holds for $P(x, D)$.

The main part of proof of Theorem 1.1 is to derive Carleman estimate for some third order elliptic operators in the following proposition.

Proposition 1.1. *Let* $P=p(x,D)+q(x,D)$ *be a pseudo-differential operator on* $\boldsymbol{R}^n$ *with* $p(x,\xi)$, $q(x,\xi)$ *of the form*

$$p(x,\xi)=(\xi_1-\lambda(x,\xi'))^2(\xi_1-\lambda(x,\xi')+c(x,\xi')),\ \lambda, c\in S^1_{1,0}(\boldsymbol{R}^n\times \boldsymbol{R}^{n-1});$$

$$q(x,\xi)=\sum_{j=0}^{2}a_j(x,\xi')\xi_1^j,\quad a_j(x,\xi')\in S^{2-j}_{1,0}(\boldsymbol{R}^n\times \boldsymbol{R}^{n-1})$$

with $C|Im\lambda|\ge\langle\xi'\rangle$, $|Im\lambda(x,\xi')|\ge 2|c(x,\xi')|$. *Assume that*

$$(|\partial_x p|+|\partial_{\xi'}p||\xi'|)|q|\le C|p|^{2/3}(|p|^{1/3}|\xi'|^2+|p+q||\xi'|+|\xi'|^{5/2}+1)$$

for all $(x,\xi)\in \boldsymbol{R}^n\times(\boldsymbol{C}\times \boldsymbol{R}^{n-1})$ *with* $\partial_{\xi_1}p(x,\xi)=0$. *Then there exist constants* $\tau_0>0$ *and* $C_0>0$ *such that if* $\tau T^2>\tau_0$ *and* $T^{-1}>\tau_0$,

$$T^{-1/2}\sum_{1\le|\alpha|+|\beta|\le 2}\|E_{(|\alpha|-|\beta|)/2}P^{(\alpha)}_{(\beta)}u\|^{(\tau)}_T+|||u|||^{(\tau)}_{T,6}\le C_0\|Pu\|^{(\tau)}_T,\ u\in \mathcal{S}_{T/2}(\boldsymbol{R}^n)\,.$$

Here, by definition

$$\mathcal{S}_T(\boldsymbol{R}^n)=\{u\in \mathcal{S}(\boldsymbol{R}^n);\ supp\,u\subset[0,T]\times \boldsymbol{R}^{n-1}\}$$

where $\mathcal{S}(\boldsymbol{R}^n)$ *denotes the space of all rapidly decreasing* C^∞*-functions on* $\boldsymbol{R}^n$;

$$||u||_T^{(\tau)} = ||e^{\tau(x_1-T)^2/2} u||_{L^2(\boldsymbol{R}^n)} \quad \mathit{for} \quad u \in \mathcal{S}(\boldsymbol{R}^n);$$
$$|||u|||_{T,s}^{(\tau)} = \sum_{\substack{i/2+j \leq s/2 \\ i,j \in \boldsymbol{Z}_+}} \tau^{3/2-i/2-j}\, T^{-i/2-j}\, ||E_{i/2}\, D_1^j\, u||_T^{(\tau)} \quad \mathit{for} \quad u \in \mathcal{S}(\boldsymbol{R}^n),$$
$$s = 0, 1, \cdots, 6, T>0,$$

where $\boldsymbol{Z}_+$ denotes the set of all non-negative integers and

$$D_k = \frac{1}{i}\,\partial_{x_k}, \quad E_s = \langle D' \rangle^s;$$
$$P_{(\beta)}^{(\alpha)} = [\partial_\xi^\alpha\, \partial_x^\beta (p+q)]\,(x, D).$$

The assumption (ii) in Theorem 1.1 is a translation of that of the above proposition. The assumption in the proposition ensures a factorization of P in the proposition into first order operators being differential operators in x_1 and pseudodifferential operators of Beals-Fefferman's class in x'.

When $c(x, \xi') \equiv 0$ in the above proposition, our assumption on P makes no condition on $q(x, \xi)$. Carleman estimates for elliptic pseudo-differential operators with smooth characteristics of arbitrary high multiplicity were studied by Watanabe-Zuily [7]. But our result is stronger than theirs in our case.

This paper is organized as follows. We devote ourselves to prove Proposition 1.1 from §2 to §7. Theorem 1.1 is proved as a corollary of the proposition in the next two sections. In §2 we carry out local factorization of the operator P in Proposition 1.1 modulo negligible terms. In §3 we derive local Carleman estimates for factorized operators. In §4 we prove Proposition 1.1 by patching local Carleman estimates which follow from the results in §2 and §3. Several facts on pseudo-differential operators used to prove Proposition 1.1 are collected in §5. In §6 we prove Carleman estimates for first order factors which are essential in the argument in §4. In §7 we prove lemmas in §3 on symbolic calculus. In §8 we prove the invariance of the assumptions in Theorem 1.1 under changes of variables such as $y_1 = x_1 - \varphi(x')$, $y_j = x_j (j \geq 2)$ where $\varphi \in C^\infty$ with $\varphi(0)=0$, $d\varphi(0)=0$. In §9 we prove Theorem 1.1 using the result in §8, Proposition 1.1, and theorems of Calderón [2], Mizohata [5], and Hörmander [4].

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§2. Factorization

Let λ, $c\in S^1_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$ satisfying

$$C\,|Im\,\lambda(x,\xi')|\geq\langle\xi'\rangle\,, \tag{2.1}$$

$$2\,|c(x,\xi')|\leq|Im\,\lambda(x,\xi')|\,. \tag{2.2}$$

Let

$$p(x,\xi)=(\xi_1-\lambda(x,\xi'))^2\,(\xi_1-\lambda(x,\xi')+c(x,\xi'))\,, \tag{2.3}$$

$$q(x,\xi)=\sum_{j=0}^{2}a_j(x,\xi')\,(\xi_1-\lambda(x,\xi'))^j \quad\text{with}\quad a_j\in S^{2-j}_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})\,, \tag{2.4}$$

and set

$$P=p+q\,. \tag{2.5}$$

We have

$$\partial_{\xi_1}p(x,\xi)=3\,(\xi_1-\lambda_1(x,\xi'))\,(\xi_1-\lambda_2(x,\xi')) \tag{2.6}$$

with

$$\lambda_1=\lambda,\ \lambda_2=\lambda-\frac{2}{3}\,c\,. \tag{2.7}$$

Then,

$$p(x,\xi)=(\xi_1-\lambda_2(x,\xi'))^2\,(\xi_1-\lambda_2(x,\xi')-c(x,\xi'))+\frac{4}{27}\,c(x,\xi')^3\,. \tag{2.8}$$

Since $\lambda_1-\lambda_2=\dfrac{2}{3}\,c$, q can be expressed as

$$q(x,\xi)=\sum_{j=0}^{2}b_{lj}(x,\xi')\,(\xi_1-\lambda_l(x,\xi'))^j\,,\quad l=1,2 \tag{2.9}$$

with $b_{1j}=a_j$, $b_{2j}\in S^{2-j}_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$ satisfying

$$b_{2j}-b_{1j}=cd_j\quad\text{for some}\quad d_j\in S^{1-j}_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})\,.$$

Setting

$$g_1=b_{10}\,,\quad g_2=\frac{4}{27}\,c^3+b_{20}\,,\quad c_l=(-1)^l\,c \tag{2.10}$$

for $l=1,2$, and

$$p_l(x,\xi)=(\xi_1-\lambda_l(x,\xi'))^2\,(\xi_1-\lambda_l(x,\xi')-c_l(x,\xi'))+g_l(x,\xi') \tag{2.11}$$

we have

$$P(x,\xi)=p_l(x,\xi)+\sum_{j=1}^{2}b_{lj}(x,\xi')\,(\xi_1-\lambda_l(x,\xi'))^j\,. \tag{2.12}$$

Now we deduce the estimates of derivatives of g_l from the assumption of Proposition 1.1.

Lemma 2.1. *Assume that*

$$(|\partial_x p|+|\partial_{\xi'} p|\,|\xi'|)|q|\leq C|p|^{2/3}(|p|^{1/3}|\xi'|^2+|P|\,|\xi'|+|\xi'|^{5/2}+1) \tag{2.13}$$
when $\partial_{\xi_1} p(x,\xi)=0$, $(x,\xi)\in \boldsymbol{R}^n\times(\boldsymbol{C}_{\xi_1}\times \boldsymbol{R}^{n-1}_{\xi'})$.

Choose $\chi\in C^\infty(\boldsymbol{R})$ *satisfying* $\chi(t)=0$ *when* $t\leq\frac{1}{2}$ *and* $\chi(t)=1$ *when* $t\geq 1$. *Define* $\Phi_l\in C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$ *by*

$$\Phi_l(x,\xi')^{-1}=\frac{1}{\langle g_l(x,\xi')\rangle^{1/3}}\chi(\langle g_l(x,\xi')\rangle\langle\xi'\rangle^{-3/2})$$
$$+\frac{1}{\langle\xi'\rangle^{1/2}}(1-\chi)(2^{-1}\langle g_l(x,\xi')\rangle\langle\xi'\rangle^{-3/2}) \tag{2.14}$$

where $\langle z\rangle=(1+|z|)^{1/2}$ *for* $z\in\boldsymbol{C}$. *We also define* $\varphi_l\in C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$ *by*

$$\varphi_l(x,\xi')=\langle\xi'\rangle^{-1}\Phi_l(x,\xi')\,. \tag{2.15}$$

Then, we have

$$|\partial_x^\alpha\partial_{\xi'}^\beta g_l(x,\xi')|\leq C_{\alpha\beta}\,\Phi_l^{3-|\beta|}(x,\xi')\,\varphi_l^{-|\alpha|}(x,\xi')\quad \textit{for any } \alpha,\beta\,. \tag{2.16}$$

Proof. From (2.6), $\partial_{\xi_1}p=0$ means that $\xi_1=\lambda_1(x,\xi')$ or $\xi_1=\lambda_2(x,\xi')$. If $\xi_1=\lambda_2(x,\xi')$,

$$\partial_{x_j}p(x,\xi)=\frac{4}{9}(\partial_{x_j}c)(x,\xi')\,c(x,\xi')^2\,,$$
$$\partial_{\xi_j}p(x,\xi)=\frac{4}{9}(\partial_{\xi_j}c)(x,\xi')\,c(x,\xi')^2\,,$$
$$p(x,\xi)=\frac{4}{27}c(x,\xi)^3\,,\quad q(x,\xi)=b_{20}(x,\xi)\,,$$
$$P(x,\xi)=g_2(x,\xi')\,.$$

So from (2.13),

$$\Big(\sum_{j=1}^n|\partial_{x_j}c|+\sum_{j=2}^n|\partial_{\xi_j}c|\,|\xi'|\Big)|b_{20}|\leq C(|c|\,|\xi'|^2+|g_2|\,|\xi'|+|\xi'|^{5/2}+1)\,. \tag{2.17}$$

Now we shall show (2.16) for $l=1,2$. First we have

$$\partial_x^\alpha\partial_{\xi'}^\beta g_2(x,\xi')=\frac{4}{27}\sum_{\substack{\sum_{i=1}^3\alpha^{(i)}=\alpha\\ \sum_{i=1}^3\beta^{(i)}=\beta}}\frac{\alpha!\,\beta!}{\prod_{i=1}^3(\alpha^{(i)}!\,\beta^{(i)}!)}\prod_{i=1}^3\partial_x^{\alpha^{(i)}}\partial_{\xi'}^{\beta^{(i)}}c(x,\xi')$$
$$+\partial_x^\alpha\partial_{\xi'}^\beta b_{20}(x,\xi')\,. \tag{2.18}$$

We note that

$$C^{-1}\langle\xi'\rangle^{1/2}\leq\Phi_l\leq C\langle\xi'\rangle, \quad C^{-1}\langle\xi'\rangle^{-1/2}\leq\varphi_l\leq C \quad \text{for some } C. \tag{2.19}$$

To estimate the second term on the right hand side of (2.18) we show that for each l

$$\begin{aligned}&|\partial_x^\alpha \partial_{\xi'}^\beta a(x,\xi')|\leq C_{\alpha\beta}\,\Phi_l^{3-|\beta|}(x,\xi')\,\varphi_l^{-|\alpha|}(x,\xi')\\ &\quad \text{if} \quad a\in S^2_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1}) \quad \text{and} \quad |\alpha|+|\beta|>0.\end{aligned} \tag{2.20}$$

Indeed, from (2.19)

$$\begin{aligned}\langle\xi'\rangle^2 &= (\Phi_l^{-3}(x,\xi')\langle\xi'\rangle^2)\,\Phi_l^3(x,\xi')\\ &\leq C(\Phi_l^{-1}(x,\xi')\langle\xi'\rangle)\,\Phi_l^3(x,\xi'),\end{aligned} \tag{2.21}$$

and from (2.15)

$$(\Phi_l^{-1}(x,\xi')\langle\xi'\rangle)^{|\alpha|+|\beta|}\langle\xi'\rangle^{-|\beta|} = \Phi_l^{-|\beta|}(x,\xi')\,\varphi_l^{-|\alpha|}(x,\xi'). \tag{2.22}$$

Thus, if $|\alpha|+|\beta|\geq 1$, using (2.19), (2.21) we get

$$\begin{aligned}|\partial_x^\alpha \partial_{\xi'}^\beta a(x,\xi')| &\leq C_{\alpha\beta}\langle\xi'\rangle^{2-|\beta|}\\ &\leq C'_{\alpha\beta}(\Phi_l^{-1}(x,\xi')\langle\xi'\rangle)\,\Phi_l^3(x,\xi')\langle\xi'\rangle^{-|\beta|}\\ &\leq C''_{\alpha\beta}(\Phi_l^{-1}(x,\xi')\langle\xi'\rangle)^{|\alpha|+|\beta|}\,\Phi_l^3(x,\xi')\langle\xi'\rangle^{-|\beta|}.\end{aligned}$$

This inequality and (2.22) mean (2.20). From (2.18) and (2.20), in order to show that (2.15) holds when $|\alpha|+|\beta|>0$, it suffices to show that

$$\begin{aligned}&\Big|\prod_{i=1}^3 \partial_x^{\alpha(i)}\partial_{\xi'}^{\beta(i)} c(x,\xi')\Big|\leq C_{\alpha\beta}\,\Phi_2^{3-|\beta|}(x,\xi')\,\varphi_2^{-|\alpha|}(x,\xi')\\ &\quad \text{when} \quad |\alpha|+|\beta|>0,\ \sum_{i=1}^3 \alpha^{(i)}=\alpha,\ \sum_{i=1}^3 \beta^{(i)}=\beta.\end{aligned} \tag{2.23}$$

Set

$$\begin{aligned}A_1 &= \{(x,\xi')\in\boldsymbol{R}^n\times\boldsymbol{R}^{n-1};\ \Phi_l(x,\xi')\leq|c(x,\xi')|\},\\ A_2 &= \{(x,\xi')\in\boldsymbol{R}^n\times\boldsymbol{R}^{n-1};\ \Phi_l(x,\xi')\geq|c(x,\xi')|\}.\end{aligned}$$

Case 1. Assume $(x,\xi')\in A_1$. We devide our argument into two subcases:

$$|\alpha^{(i)}|+|\beta^{(i)}|\leq 1 \qquad \text{for any} \quad i, \tag{2.24}$$

$$|\alpha^{(i)}|+|\beta^{(i)}|\geq 2 \qquad \text{for some} \quad i. \tag{2.25}$$

First we assume (2.24). Put $J=\{i;\ |\alpha^{(i)}|+|\beta^{(i)}|\neq 0\}$ and choose $i_0\in J$. Then,

$$\begin{aligned}\prod_{i=1}^3 \partial_x^{\alpha(i)}\partial_{\xi'}^{\beta(i)} c(x,\xi') &= \frac{27}{4}g_2(x,\xi')\prod_{i\in J}\frac{\partial_x^{\alpha(i)}\partial_{\xi'}^{\beta(i)}c(x,\xi')}{c(x,\xi')}\\ &\quad -\frac{27}{4}b_{20}(x,\xi')\prod_{i\in J}\frac{\partial_x^{\alpha(i)}\partial_{\xi'}^{\beta(i)}c(x,\xi')}{c(x,\xi')} = I+II.\end{aligned} \tag{2.26}$$

Since $\#(J)=|\alpha|+|\beta|$, and since $(x, \xi')\in A_1$,

$$|I|\leq C(\Phi_2(x, \xi')^{-1}\langle\xi'\rangle)^{|\alpha|+|\beta|}\langle\xi'\rangle^{-|\beta|}\,\Phi_2^3(x, \xi')\,. \tag{2.27}$$

On the other hand, from (2.17) and the same reason as above,

$$\begin{aligned}|II|&\leq C_1\langle\xi'\rangle^{-|\beta^{(i_0)}|}(|c(x, \xi')|\,|\xi'|^2+|\xi'|\,|g_2(x, \xi')|+|\xi'|^{5/2}+1)|c(x, \xi')|^{-1}\\&\quad\times\prod_{i\in J\setminus\{i_0\}}\frac{|\partial_x^{\alpha(i)}\partial_{\xi'}^{\beta(i)}c(x, \xi')|}{|c(x, \xi')|}\\&\leq C_2\Big\{\Big(\frac{\langle\xi'\rangle}{|c(x, \xi')|}\Big)^{\#(J)-1}\langle\xi'\rangle^{2-|\beta|}\\&\quad+\Big(\frac{\langle\xi'\rangle}{|c(x, \xi')|}\Big)^{\#(J)}\langle\xi'\rangle^{-|\beta|}(|g_2(x, \xi')|+\langle\xi'\rangle^{3/2})\Big\}\\&\leq C_3\{(\Phi_2^{-1}(x, \xi')\langle\xi'\rangle)^{|\alpha|+|\beta|-1}\langle\xi'\rangle^{2-|\beta|}\\&\quad+(\Phi_2^{-1}(x, \xi')\langle\xi'\rangle)^{|\alpha|+|\beta|}\langle\xi'\rangle^{-|\beta|}\,\Phi_2^3(x, \xi')\}\,.\end{aligned}$$

Applying (2.21) to the first term in the last expression we get

$$|II|\leq C_4(\Phi_2^{-1}(x, \xi')\langle\xi'\rangle)^{|\alpha|+|\beta|}\langle\xi'\rangle^{-|\beta|}\,\Phi_2^3(x, \xi')\,. \tag{2.28}$$

(2.27), (2.28), and (2.22) mean (2.23). Next we assume (2.25). We have with the set J as above,

$$\begin{aligned}\prod_{i=1}^{3}|\partial_x^{\alpha(i)}\partial_{\xi'}^{\beta(i)}c(x, \xi')|&\leq C_1\Big(\frac{\langle\xi'\rangle}{|c(x, \xi')|}\Big)^{\#(J)}\langle\xi'\rangle^{-|\beta|}|c(x, \xi')|^3\\&\leq C_2\Big(\frac{\langle\xi'\rangle}{|c(x, \xi')|}\Big)^{\#(J)}\langle\xi'\rangle^{-|\beta|}(|g_2(x, \xi')|+|b_{20}(x, \xi')|)\\&\leq C_3(\Phi_2^{-1}(x, \xi')\langle\xi'\rangle)^{|\alpha|+|\beta|-1}\langle\xi'\rangle^{-|\beta|}(\Phi_2^3(x, \xi')+\langle\xi'\rangle^2)\,,\end{aligned}$$

because $\#(J)\leq|\alpha|+|\beta|-1$. From (2.19), (2.21), and (2.22) we obtain (2.23).

Case 2. Assume that $(x, \xi)\in A_2$, and define the set J as in Case 1. Noting that $\#(J)\leq\min\{3, |\alpha|+|\beta|\}$ we have

$$\begin{aligned}\prod_{i=1}^{3}|\partial_x^{\alpha(i)}\partial_{\xi'}^{\beta(i)}c(x, \xi')|&\leq C\langle\xi'\rangle^{\#(J)-|\beta|}|c(x, \xi')|^{3-\#(J)}\\&\leq C\langle\xi'\rangle^{\#(J)-|\beta|}\,\Phi_2(x, \xi')^{3-\#(J)}\\&\leq C'\Big(\frac{\langle\xi'\rangle}{\Phi_2(x, \xi')}\Big)^{|\alpha|+|\beta|}\langle\xi'\rangle^{-|\beta|}\,\Phi_2(x, \xi')\,.\end{aligned}$$

Thus (2.22) means (2.23). Q.E.D.

Corollary 2.1. *Φ_l and φ_l satisfy the estimates*

$$|\partial_x^\alpha\partial_{\xi'}^\beta\,\Phi_l(x, \xi')|\leq C_{\alpha\beta}\,\Phi_l^{1-|\beta|}\,\varphi_l^{-|\alpha|}(x, \xi')\,, \tag{2.29}$$

$$|\partial_x^\alpha \partial_{\xi'}^\beta \varphi_l(x, \xi')| \leq C_{\alpha\beta}\, \Phi_l^{-|\beta|}(x, \xi')\, \varphi_l^{1-|\alpha|}(x, \xi') . \tag{2.30}$$

One can deduce these from (2.16) by using the following lemma which is frequently used in the proof of Lemma 2.3.

Lemma 2.2. *Let $U\subset \boldsymbol{R}^n$ and $V\subset \boldsymbol{R}^m$ be open sets, let $F\in C^\infty(U)$, and let f: $V\to U$ be C^∞-mapping with $f=(f_1, \cdots, f_n)$. Suppose that there exist positive functions $Z(y)$, $N_j(y)$ $(j=1, \cdots, n)$ on U, and $M_\alpha(x)$ $(\alpha\in \boldsymbol{Z}_+^m)$ on V with $M_{\alpha+\beta}=M_\alpha M_\beta$ satisfying*

$$|F|_\alpha := \sup_{y\in U} |\partial^\alpha F(y)|\ Z(y)^{-1} N(y)^\alpha < +\infty ,$$

$$|f_j|_\alpha := \sup_{x\in V} |\partial^\alpha f_j(x)|\, M_\alpha(x)^{-1} N_j(f(x))^{-1} < +\infty .$$

Set

$$|F|_L = \max_{|\alpha|\leq L} |F|_\alpha , \quad |f|_\alpha = \max_{i=1,\cdots,n} \max_{\beta\leq\alpha} |f_i|_\beta .$$

Then

$$|\partial^\alpha(F\circ f)(x)| \leq |F|_{|\alpha|}(|f|_\alpha+1)^{|\alpha|} n^{|\alpha|} 2^{(|\alpha|(|\alpha|-1))/2} Z(f(x))\, M_\alpha(x) \quad \textit{for any} \quad \alpha .$$

Proof of this lemma is straightforward.

Now we define a symbol class for a pair of positive C^∞ functions Φ and φ on $\boldsymbol{R}^n \times \boldsymbol{R}^{n-1}$ satisfying that

(i) there exist $C>0$ and $c>0$ such that

$$c(1+|\xi'|)^{1/2} \leq \Phi(x, \xi') \leq C(1+|\xi'|) ,$$
$$c(1+|\xi'|)^{-1/2} \leq \varphi(x, \xi') \leq C ; \tag{2.31}$$

(ii) for any $\alpha\in \boldsymbol{Z}_+^n$ and $\beta\in \boldsymbol{Z}_+^{n-1}$ there exists $C_{\alpha\beta}>0$ such that

$$|\partial_x^\alpha \partial_{\xi'}^\beta \Phi(x, \xi')| \leq C_{\alpha\beta}\, \Phi(x, \xi')^{1-|\beta|}\, \varphi(x, \xi')^{-|\alpha|} ,$$
$$|\partial_x^\alpha \partial_{\xi'}^\beta \varphi(x, \xi')| \leq C_{\alpha\beta}\, \Phi(x, \xi')^{-|\beta|}\, \varphi(x, \xi')^{1-|\alpha|} ; \tag{2.32}$$

(iii) there exists $C'>0$ such that

$$C'^{-1}(1+|\xi'|) \leq \frac{\Phi(x, \xi')}{\varphi(x, \xi')} \leq C'(1+|\xi'|) . \tag{2.33}$$

For M, $m\in \boldsymbol{R}$ we say that a function $a\in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$ belongs to the set $S_{\Phi,\varphi}^{M,m}$ if a satisfies the estimates that for any α, β there exists $C_{\alpha\beta}>0$ such that

$$|\partial_x^\alpha \partial_{\xi'}^\beta a(x, \xi')| \leq C_{\alpha\beta}\, \Phi^{M-|\beta|}(x, \xi')\, \varphi^{m-|\alpha|}(x, \xi') . \tag{2.34}$$

Φ_l and φ_l in Lemma 2.1 satisfy (2.31)~(2.33) for each l. Now we shall prove the main lemma in this section. This gives a local factorization of P in Proposition 1.1 into first order factors for which Carleman estimates are deduced in Proposition 3.2.

Lemma 2.3. *One can find two families of a finite number of C^∞ functions on $\boldsymbol{R}^n \times \boldsymbol{R}^{n-1}$, $\{\psi_{jk}\}_{k\in I}(j=0, 1)$ with $\psi_{jk}\neq 0$ for any j, k having the following properties.*

(1) *$\{\psi_{1k}\}_{k\in I}$ is a finite partitions of unity of $\boldsymbol{R}^n\times\boldsymbol{R}^{n-1}$.*

(2) *$\psi_{0k}=1$ on a neighbourhood of supp ψ_{1k}.*

(3) *For each $k\in I$ there exists $l\in\{1, 2\}$ such that $\psi_{jk}\in S^{0,0}_{\Phi_l,\varphi_l}$ and one of the following* (I), (II), (III) *holds.*

(I) (i) $\sup_{(x,\xi')\in supp\psi_{0k}}|c(x, \xi')|\Phi_l^{-1}(x, \xi')<+\infty$.
$p_l(x, \xi)=\prod_{j=1}^3(\xi_1-\lambda_l(x, \xi')-A_j(x, \xi'))$ on a neighbourhood of supp ψ_{0k} as polynomials in ξ_1 where $A_j\in S^{1,0}_{\Phi_l,\varphi_l}$ depending on k with

(ii) $\inf_{|\xi'|\geq R}|Im(\lambda_l(x, \xi')+A_j(x, \xi'))|\langle\xi'\rangle^{-1}>0$ *for some* $R>0$,

(iii) $\inf_{(x,\xi')\in supp\psi_{0k}}|A_j(x, \xi')-A_{j'}(x, \xi')|\Phi_l^{-1}(x, \xi')>0$ *if* $j\neq j'$.

(II) $p_l(x, \xi)=(\xi_1-\lambda_l(x, \xi')-c_l(x, \xi')-A_1(x, \xi'))\prod_{j=2}^3(\xi_1-\lambda_l(x, \xi')-A_j(x, \xi'))$ *on a neighbourhood of supp ψ_{0k} as polynomials in ξ_1 where $A_j\in S^{1,0}_{\Phi_l,\varphi_l}$ depending on k with*

(i) $\inf_{|\xi'|\geq R}|Im(\lambda_l(x, \xi')+A_j(x, \xi'))|\langle\xi'\rangle^{-1}>0$ *for some* $R>0$ *if* $j=2, 3$,

(ii) $\inf_{|\xi'|\geq R}|Im(\lambda_l(x, \xi')+c_l(x, \xi')+A_1(x, \xi'))|\langle\xi'\rangle^{-1}>0$ *for some* $R>0$,

(iii) $\inf_{(x,\xi')\in supp\psi_{0k}}|c(x, \xi')|\Phi_l^{-1}(x, \xi')>0$,

(iv) $\inf_{(x,\xi')\in supp\psi_{0k}}|c_l(x, \xi')+A_1(x, \xi')-A_j(x, \xi')||c(x, \xi')|^{-1}>0$ *if* $j=2, 3$,

(v) $A_2(x, \xi')\neq A_3(x, \xi')$ *for* $(x, \xi')\in supp\ \psi_{0k}$,

there exists an open set U containing supp ψ_{0k} such that

(vi) $\sup_{(x,\xi')\in U}|\partial_x^\alpha\partial_{\xi'}^\beta A_j(x, \xi')||A_2(x, \xi')-A_3(x, \xi')|^{-1}\Phi_l^{-|\beta|}(x, \xi')\varphi_l^{-|\alpha|}(x, \xi')$ $<+\infty$ *for* $j=2,3$ *and* $\alpha\in\boldsymbol{Z}_+^{n-1}$, $\beta\in\boldsymbol{Z}_+^{n-1}$.

(III) $\Phi_l(x, \xi')\leq C\langle\xi'\rangle^{1/2}$ *on supp ψ_{0k} for some* $C>0$.

Proof. Step 1. In this step we shall deduce the algebraic equations with parameters to decompose p_l. Set

$$D_l = \{(x, \xi')\in\boldsymbol{R}^n\times\boldsymbol{R}^{n-1};\ \langle g_{l'}(x, \xi')\rangle>\frac{1}{100}\langle g_l(x, \xi')\rangle\}\quad(l\neq l'),$$

$$\mathcal{D}_{l2}(\varepsilon) = \{(x, \xi)\in\boldsymbol{R}^n\times\boldsymbol{R}^{n-1};\ \varepsilon\langle c(x, \xi')\rangle^3>\langle g_l(x, \xi')\rangle\},$$

$$\mathcal{D}_{l2}(\varepsilon) = \{(x, \xi')\in\boldsymbol{R}^n\times\boldsymbol{R}^{n-1};\ \langle g_l(x, \xi')\rangle>\frac{\varepsilon}{10}\langle c(x, \xi')\rangle^3\},$$

$$\Gamma_l(N) = \{(x, \xi')\in\boldsymbol{R}^n\times\boldsymbol{R}^{n-1};\ \langle g_l(x, \xi')\rangle>N\langle\xi'\rangle^{3/2}\},$$

$$D_{lj}(\varepsilon, N) = \Gamma_l(N)\cap\mathcal{D}_{lj}(\varepsilon)\cap D_l,$$

$$\Gamma_{0l}(N) = \{(x, \xi')\in \boldsymbol{R}^n\times\boldsymbol{R}^{n-1};\ \langle g_l(x, \xi')\rangle<2N\langle\xi'\rangle^{3/2}\}\ ,$$

where $0<\varepsilon<1$, $N>1$ which are determined in Step 3.

We assume $N\geq 500$. Then

$$|c(x, \xi')|>1 \quad \text{on} \quad D_{l1}(\varepsilon, N)\ ,$$

$$\min\{|g_j(x, \xi')|\ ; j=1, 2\}>1 \quad \text{on} \quad \Gamma_l(N)\cap D_l\ .$$

We have

$$p_l(x, \xi) = c_l(x, \xi')^3 f_1(Z_l(x, \xi), w_l(x, \xi')) \quad \text{for} \quad (x, \xi')\in D_{l1}(\varepsilon, N)\ , \tag{2.35}$$

$$p_l(x, \xi) = |g_l(x, \xi')|\, f_2(\tilde{Z}_l(x, \xi), \tilde{w}_l(x, \xi')) \quad \text{for} \quad (x, \xi')\in D_{l2}(\varepsilon, N)\ , \tag{2.36}$$

where

$$f_1(z, w) = z^3(z-1)+w_2 w_1^3\ , \tag{2.37}$$

$$f_2(z, w) = z^2(z-w_2)+w_1\ , \tag{2.38}$$

for $z\in \boldsymbol{C}$, $w=(w_1, w_2)\in \boldsymbol{C}^2$, and by definition,

$$Z_l(x, \xi) = c_l(x, \xi')^{-1}(\xi_1-\lambda_l(x, \xi'))\ ,$$
$$w_l(x, \xi') \equiv (w_{l1}(x, \xi'), w_{l2}(x, \xi')) = \left(\frac{|g_l(x, \xi')|^{1/3}}{c_l(x, \xi')}, \frac{g_l(x, \xi')}{|g_l(x, \xi')|}\right)$$

for $(x, \xi')\in D_{l1}(\xi, N)$,

$$\tilde{Z}_l(x, \xi) = |g_l(x, \xi')|^{-1/3}(\xi_1-\lambda_l(x, \xi'))$$
$$\tilde{w}_l(x, \xi') \equiv (\tilde{w}_{l1}(x, \xi'), \tilde{w}_{l2}(x, \xi')) = \left(\frac{g_l(x, \xi')}{|g_l(x, \xi')|}, \frac{c_l(x, \xi')}{|g_l(x, \xi')|^{1/3}}\right)$$

for $(x, \xi')\in D_{l2}(\varepsilon, N)$. We have

$$|w_{l2}(x, \xi')| = 1\ , \quad |w_{l1}(x, \xi')|\leq 2\varepsilon^{1/3}\ , \tag{2.39}$$

$$|\tilde{w}_{l1}(x, \xi')| = 1\ , \quad |\tilde{w}_{l2}(x, \xi')|\leq 20\varepsilon^{-1/3}\ . \tag{2.40}$$

If we denote by $D(w)$ the descriminant of polynomial $f_2(z, w)$ of Z, we have

$$D(w) = (27w_1-4w_2^3)\, w_1$$

and

$$D(\tilde{w}_l(x, \xi')) = 27|g_l(x, \xi')|^{-2}\left(g_l(x, \xi')+(-1)^{l+1}\frac{4}{27}c(x, \xi')^3\right) g_l(x, \xi')\ .$$

Using the equality

$$g_l+(-1)^{l+1}\frac{4}{27}c^3 = g_{l'}+(-1)^l c\left(-\frac{2}{3}a_1+\frac{4}{9}ca_2\right) \quad \text{if} \quad l\neq l'$$

we see that there exists a constant C_0 such that

$$|D(\tilde{w}_l(x, \xi'))| \geq \frac{1}{20} \quad \text{if} \quad N^2 \varepsilon \geq C_0 \quad \text{and} \quad N \geq 500 . \tag{2.41}$$

Step 2. In this step we factorize $f_i(z, w)$ as a polynomial in z locally. We first consider $f_1(z, w)$. From the implicit function theorem there exists $\delta_0>0$ such that for any pair of positive numbers δ, R with $\delta R^3<\delta_0$ there exist holomorphic functions μ_1, A, B in $w\in B^1_\delta(0)\times B^1_R(0)$ satisfying

$$f_1(z, w) = (z-\mu_1(w))\,(z^2+2A(w)\,z+B(w)) \quad \text{for} \quad w\in B^1_\delta(0)\times B^1_R(0)$$

with $\mu_1(0, w_2)=1$, $\mu_1(w)\neq 0$ everywhere, $A(0, w_2)=B(0, w_2)=0$. Here, $B^m_r(0)$ denotes the open ball with the center at the origin in C^m and the radius r. A simple calculation shows that there exists a holomorphic function $D(w)$ on $B^1_\delta(0)\times B^1_R(0)$ satisfying

$$A(w)^2-B(w) = w_1^3\, D(w),\ D(0, w_2) = -w_2 .$$

We take δ, R as $\delta=(\delta_0/4)^{1/3}$, $R=3$.

Then one can choose a positive number δ_1 with $\delta_1<(\delta_0/4)^{1/3}$ and an open covering $\{U_{1j}\}_{j=1}^{k_1}$ of $(\overline{B^1_{\delta_1/2}(0)}\setminus\{0\})\times\overline{B^1_{1/2}(0)^c\cap B^1_2(0)}$ in $C\setminus\{0\}\times C$ such that

$$U_{1j} = (B^1_{\delta_1}(0)\cap U_1)\times B_{1j}\,,\quad j = 1, \cdots, k_1/2 \quad (k_1 \text{ is even})$$
$$U_{1j} = (B^1_{\delta_1}(0)\cap U_2)\times B_{1j}\,,\quad j = k_1/2+1, \cdots, k_1$$

where U_1, U_2 are two connected open sets in C with angles $<2\pi$ such that $U_1\cup U_2=C\setminus\{0\}$ and B_{1j} are open sets in $B^1_3(0)\setminus\{0\}$ with $\cup_{j=1}^{k_1} B_{1j}\supseteq\overline{B^1_{1/2}(0)^c\cap B^1_2(0)}$, and such that there exist holomorphic functions $\mu_{1jk}(w)$ in U_{1j}, $k=1,2$ satisfying

$$f_1(z, w) = (z-\mu_1(w))\prod_{k=1}^{2}(z-\mu_{1jk}(w)),\ w\in U_{1j}\,, \tag{2.42}$$

$$C_1|\mu_{1j1}(w)-\mu_{1j2}(w)| \geq |w|^{2/3}\,, \tag{2.43}$$

$$C_1|\mu_1(w)-\mu_{1jk}(w)| \geq 1\,, \tag{2.44}$$

$$|\partial^\alpha_w\, \mu_{1jk}(w)| \leq |w_1|^{(3/2)-\alpha_1} . \tag{2.45}$$

We also note that

$$|\partial^\alpha_w(1-\mu_1(w))|\,|w_1|^{\alpha_1}\leq C'_\alpha|w_1|^3 \quad \text{for} \quad w\in B^1_{\delta_1}(0)\times B^1_{3/2}(0) . \tag{2.46}$$

Next we take up $f_2(z, w)$. We set for $R'>0$, $\delta'>0$ which are to be determined in the next step

$$K(R', \delta') = \{w\in C^2;\ |D(w)|\geq\delta',\ |w|\leq R'\} .$$

One can find open balls U_{2j}, $j=1, \cdots, k_2$ in $\boldsymbol{C}^2$ and holomorphic functions μ_{2jk} on a neighbourhood of U_{2j}, $k=1, 2, 3, j=1, \cdots, k_2$ such that

$$|D(w)| \geq \frac{\delta'}{2} \quad \text{on} \quad U_{2j} \quad \text{and} \quad K(R', \delta') \subseteq \bigcup_{j=1}^{k_2} U_{2j}\,, \tag{2.46}$$

$$f_2(z, w) = \prod_{k=1}^{3} (z-\mu_{2jk}(w)) \quad \text{for} \quad w \in U_{2j}\,. \tag{2.47}$$

Step 3. In this step we shall define a family of non-negative functions in $\bigcup_{l=1}^{2} S^{0,0}_{\Phi_l,\varphi_l}$ where sum is greater than or equal to 1 such that on the support of each one, one of (I), (II), (III), in Lemma 2.3 will holds. We take ε, N as $\varepsilon = \min\{\frac{1}{2}, \left(\frac{\delta_1}{8}\right)^3\}$, $N=\max\{500, (C_0\,\varepsilon^{-1})^{1/2}\}$, and we take R', δ' in the Step 2 as $R'=(1+\left(\frac{20}{\varepsilon}\right)^{2/3})^{1/2}$, $\delta'=\frac{1}{20}$. We denote $\mathcal{D}_{lj}$, $\Gamma_l(N)$, $D_{lj}(\varepsilon, N)$ by $\mathcal{D}_{lj}$, Γ_l, D_{lj}. Choose $\chi_i \in C^\infty(\boldsymbol{R})$, $i=0, 1, 2$ so that $0 \leq \chi \leq 1$, supp $\chi_i \subseteq (1, \infty)$, $\chi_i=1$ on $[2, \infty)$, $\chi_i=1$ on a neighbourhood of supp χ_{i+1} for $i=0, 1$. Define C^∞ functions $\Psi_i^{(l,s)}(x, \xi')$ on $\boldsymbol{R}^n \times \boldsymbol{R}^{n-1}$ for $i=0, 1, 2$ and $l=1, 2$ by

$$\Psi_i^{(l,1)} = \chi_i(\varepsilon\langle c\rangle^3/\langle g_l\rangle)\,\chi_i(100\,\langle g_{l'}\rangle/\langle g_l\rangle)\,\chi_i(N^{-1}\langle g_l\rangle\,\langle\xi'\rangle^{-3/2})\,,$$
$$\Psi_i^{(l,2)} = \chi_i(\frac{10}{\varepsilon}\langle g_l\rangle/\langle c\rangle^3)\,\chi_i(100\,\langle g_{l'}\rangle/\langle g_l\rangle)\,\chi_i(N^{-1}\,\langle g_l\rangle\,\langle\xi'\rangle^{-3/2})\,,$$

where $l \neq l'$.

Then we have that *supp* $\Psi_i^{(l,s)} \subseteq D_{ls}$ and that

$$\sum_{s=1}^{2} \Psi^{(l,s)} \geq 1 \quad \text{on} \quad \Gamma_l(2N) \cap \tilde{D}_l \tag{2.48}$$

where $\tilde{D}_l = \{(x, \xi') \in \boldsymbol{R}^n \times \boldsymbol{R}^{n-1};\ \langle g_{l'}(x, \xi')\rangle > \frac{1}{50}\langle g_l(x, \xi')\rangle\}$. Now we define $\tilde{\Psi}_i^{(l,s,j)}$, $\Psi_i^{(l,0)}$, $\Lambda_k^{(l,s,j)} \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$ ($i=0, 1, 2$; $l, s=1, 2$; $k=1, 2, 3$; $j=1, \cdots, k_s$) with notations k_1, k_2 in Step 2 where $\tilde{\Psi}_i^{(l,s,j)}$, $\Psi_i^{(l,0)}$ are functions stated in the beggining of this step such that (II) (resp. (I)) in Lemma 2.3 holds on *supp* $\tilde{\Psi}_i^{(l,1,j)}$ (resp. *supp* $\Psi_i^{(l,2,j)}$) and (III) holds on *supp* $\Psi_i^{(l,0)}$, and where $\Lambda_k^{(l,1,j)}$ (resp. $\Lambda_k^{(l,2,j)}$) corresponds to Λ_k in the case (II) (resp. (I)).

To do so we choose $\varphi_{i1j} \in C^\infty(\boldsymbol{C}\setminus\{0\} \times \boldsymbol{C})$, $i=0, 1, 2, j=1, \cdots, k_1$ and $\varphi_{i2j} \in C_0^\infty(U_{2j})$, $i=0, 1, 2, j=1, \cdots, k_2$ so that

$$\textit{supp}\ \varphi_{isj} \subseteq U_{sj};\ \varphi_{isj} = 1 \quad \text{on a neighbourhood of} \quad \textit{supp}\ \varphi_{i+1sj} \quad \text{for} \quad i = 0, 1;$$
$$\sum_{j=1}^{k_1} \varphi_{21j} = 1 \quad \text{on a neighbourhood of} \quad \overline{(B^1_{\delta_1/2}(0)\setminus\{0\}) \times B^1_{1/2}(0)^c \setminus B^1_2(0)}$$

$$\text{in} \quad \boldsymbol{C}\setminus\{0\}\times\boldsymbol{C};$$
$$\sum_{j=1}^{k_2}\varphi_{22j}=1 \quad \text{on a neighbourhood of} \quad K(R',\delta')\,;$$
$$|\partial_w^\alpha\,\varphi_{i1j}(w)|\le C'_\alpha|w_1|^{-\alpha_1};\,0\le\varphi_{isj}\le 1\,. \tag{2.49}$$

$\tilde{\Psi}_i^{(l,s,j)}$, $\Lambda_k^{(l,s,j)}$ are defined as follows.

$$\tilde{\Psi}_i^{(l,s,j)}\equiv 0\,,\quad \Lambda_k^{(l,s,j)}\equiv 0 \quad \text{when} \quad D_{ls}=\emptyset\,.$$

When $D_{l1}\neq\emptyset$,

$$\tilde{\Psi}_i^{(l,s,j)}(x,\xi')=\begin{cases}\varphi_{i1j}(w_l(x,\xi'))\,\Psi_i^{(l,1)}(x,\xi') & (x,\xi')\in D_{l1}\,,\\ 0 & \text{otherwise}\end{cases}$$

$$\Lambda_1^{(l,1,j)}(x,\xi')=\begin{cases}c_l(x,\xi')\,(\mu_1(w_l(x,\xi'))-1)\,\tilde{\Psi}_0^{(l,1,j)}(x,\xi') & (x,\xi')\in D_{l1}\,,\\ 0 & \text{otherwise}\end{cases}$$

$$\Lambda_k^{(l,1,j)}(x,\xi')=\begin{cases}c_l(x,\xi')\,\mu_{1jk}(w_l(x,\xi'))\,\tilde{\Psi}_0^{(l,1,j)}(x,\xi') & (x,\xi')\in w_l^{-1}(U_{1j})\\ 0 & \text{otherwise}\end{cases}$$

for $k=2, 3$.

When $D_{l2}\neq\emptyset$,

$$\tilde{\Psi}_i^{(l,2,j)}(x,\xi')=\begin{cases}\varphi_{i2j}(\tilde{w}_l(x,\xi'))\,\Psi_i^{(l,2)}(x,\xi') & (x,\xi')\in D_{l2}\,,\\ 0 & \text{otherwise}\end{cases}$$

$$\Lambda_k^{(l,2,j)}(x,\xi')=\begin{cases}|g_l(x,\xi')|^{1/3}\,\mu_{2jk}(\tilde{w}_l(x,\xi'))\,\tilde{\Psi}_0^{(l,2,j)}(x,\xi') & (x,\xi')\in\tilde{w}_l^{-1}(U_{2j})\,.\\ 0 & \text{otherwise}\end{cases}$$

$\Psi_i^{(l,0)}$ is defined by

$$\Psi_i^{(l,0)}=(1-\chi_{2-i})\,(5^{-1}\langle g_l\rangle\langle\xi'\rangle^{-3/2}N^{-1})\,.$$

Since $\sum_{j=1}^{k_s}\tilde{\Psi}_i^{(l,s,j)}\ge\Psi_i^{(l,s)}$ from the definition of φ_{isj}, and since $\bigcap_{l=1}^2\Gamma_l(2N)\subseteq\bigcup_{l=1}^2(\Gamma_l(2N)\cap\tilde{D}_l)$, we have that

$$\sum_{l=1}^2\sum_{s=1}^2\sum_{j=1}^{k_s}\tilde{\Psi}_i^{(l,s,j)}\ge 1 \quad \text{on} \quad \bigcap_{l=1}^2\Gamma_l(2N)$$

in view of (2.48). Thus, since $\bigcup_{l=1}^2\Gamma_{0l}(\frac{5}{2}N)\cup\bigcap_{l=1}^2\Gamma_l(2N)=\boldsymbol{R}^n\times\boldsymbol{R}^{n-1}$, we have that

$$\sum_{l=1}^2\Big(\sum_{s=1}^2\sum_{j=1}^{k_s}\tilde{\Psi}_i^{(l,s,j)}+\Psi_i^{(l,0)}\Big)\ge 1\,. \tag{2.50}$$

Since $\tilde{\Psi}_{i-1}^{(l,s,j)}=1$ on a neighbourhood of $supp\,\tilde{\Psi}_i^{(l,s,j)}$ for $i=1, 2$, from (2.35) and (2.36) we have that the factorization in (i) (resp. (ii)) in Lemma 2.3 holds for $(x,\xi')\in supp\,\tilde{\Psi}_1^{(1,2,j)}$ (resp. $supp\,\tilde{\Psi}_1^{(l,1,j)}$) with Λ_k replaced by $\Lambda_k^{(l,2,j)}$ (resp. $\Lambda_k^{(l,1,j)}$).

Step 4. In this step we deduce the estimates of derivatives of functions defined in Step 3. To do so, we have to deduce the estimates of derivatives of functions $\langle g_l\rangle\langle\xi'\rangle^{-3/2}|_{\Gamma_l}$, $\langle c\rangle^3/\langle g_l\rangle|_{D_{l1}}$, $\langle g_{l'}\rangle/\langle g_l\rangle|_{\Gamma_l\cap D_l}$, $\langle g_l\rangle/\langle c\rangle^3|_{D_{l2}\cap\mathcal{D}_{l1}}$, w_l, $\tilde{w}_l$.

Definition 2.1. *For an open set U in $\boldsymbol{R}^n\times\boldsymbol{R}^{n-1}$ and a positive function $Z(x,\xi')$ on U we set*

$$S_l(U,Z)=\{a\in C^\infty(U);\ |a^{(\beta)}_{(\alpha)}(x,\xi')|\le C_{\alpha\beta}\,M^{(l)}_{\alpha,\beta}(x,\xi')\,Z(x,\xi)\quad \textit{for any}\quad \alpha,\beta\}$$

where $a^{(\beta)}_{(\alpha)}=\partial^\alpha_x\,\partial^\beta_{\xi'}\,a$ *and* $M^{(l)}_{\alpha,\beta}=\varphi_l^{-|\alpha|}\,\Phi_l^{-|\beta|}$.

Let us consider g_l on Γ_l. Since $\Phi_l^{-1}\ge\langle g_l\rangle^{-1/3}\ge\sqrt{2}^{-1/3}|g_l|^{-1/3}$ on Γ_l, Lemma 2.1 implies that

$$g_l|_{\Gamma_l}\in S_l(\Gamma_l,\ |g_l|\,|_{\Gamma_l})\quad\text{when}\quad \Gamma_l\neq\emptyset\,. \tag{2.51}$$

When $\Gamma_l\neq\emptyset$, taking in Lemma 2.2 $U=\Gamma_l$, $V=\boldsymbol{R}^2\setminus\{0\}$, $f=(Re[g_l|_{\Gamma_l}], Im[g_l|_{\Gamma_l}])$, $F(y)=Z(y)=|y|^s(s\in\boldsymbol{R})$, $N_i(y)=|y|$, $M_{(\alpha,\beta)}=M^{(l)}_{\alpha,\beta}|_{\Gamma_l}$ one obtain that

$$|g_l|^s|_{\Gamma_l}\in S_l(\Gamma_l,\ |g_l|^s|_{\Gamma_l})\,. \tag{2.52}$$

When $\Gamma_l\neq\emptyset$, one also obtain taking in Lemma 2.2 $U=\Gamma_l$, $V=(0,+\infty)$, $f=|g_l|\,|_{\Gamma_l}$, $F(y)=\langle y\rangle^{s/2}$, $N_1(y)=\langle y\rangle$, $M_{(\alpha,\beta)}=M^{(l)}_{\alpha,\beta}|_{\Gamma_l}$ that

$$\langle g_l\rangle^s|_{\Gamma_l}\in S_l(\Gamma_l,\langle g_l\rangle^s|_{\Gamma_l})\,. \tag{2.53}$$

Next we consider c on D_{l1}. Noting the inequalities

$$\begin{aligned}\langle\xi'\rangle&\le C|c|\varphi_l^{-1}\quad\text{on}\quad D_{l1}\,,\\ 1&\le C|c|\Phi_l^{-1}\quad\text{on}\quad D_{l1}\,,\end{aligned} \tag{2.54}$$

we see that $\langle\xi'\rangle^{1-|\beta|}\le C_{\alpha\beta}|c|M^{(l)}_{\alpha,\beta}$ on D_{l1} when $|\alpha|+|\beta|\neq0$. Thus we obtain

$$c|_{D_{l1}}\in S_l(D_{l1},\ |c|\,|_{D_{l1}})\quad\text{when}\quad D_{l1}\neq\emptyset\,. \tag{2.55}$$

From this and Lemma 2.2, we see that

$$\langle c\rangle^s|_{D_{l1}}\in S_l(D_{l1},\langle c\rangle^s|_{D_{l1}})\quad\text{when}\quad D_{l1}\neq\emptyset\,, \tag{2.56}$$

$$(c|_{D_{l1}})^{-k}\in S_l(D_{l1},(|c|\,|_{D_{l1}})^{-k})\quad\text{for}\quad k\in\boldsymbol{Z}\quad\text{when}\quad D_{l1}\neq\emptyset\,. \tag{2.57}$$

We need a lemma which follows from Leibniz rule.

Lemma 2.4. *Let $Z_i(x,\xi')$ $(i=1,2)$ be a positive function on an open set U in $\boldsymbol{R}^n\times\boldsymbol{R}^{n-1}$, and let $a_i\in S_l(U,Z_i)$ $(i=1,2)$. Then $a_1\,a_2\in S_l(U,Z_1\,Z_2)$.*

From (2.51), (2.52), (2.57) the above lemma implies that when $D_{l1}\neq\emptyset$,

$$w_{l1}\in S_l(D_{l1},\ |w_{l1}|) \quad\text{and}\quad w_{l2}\in S_l(D_{l1}, 1)\,. \tag{2.58}$$

Since on Γ_l the estimates (2.54) with $|c|$ replaced by $|g_l|^{1/3}$ and D_{l1} by Γ_l hold, we see that

$$c|_{D_{l2}}\in S_l(D_{l2},\ |g_l|^{1/3}|_{D_{l2}}) \quad\text{when}\quad D_{l2}\neq\emptyset\,. \tag{2.59}$$

From this, (2.51), and (2.52) Lemma 2.4 implies that

$$\tilde{w}_{lj}\in S_l(D_{l2}, 1)\,. \tag{2.60}$$

Now from (2.53) and (2.56) Lemma 2.4 implies that

$$\langle g_l\rangle\langle\xi'\rangle^{-3/2}|_{\Gamma_l}\in S_l(\Gamma_l, \langle g_l\rangle\langle\xi'\rangle^{-3/2}|_{\Gamma_l}) \quad\text{when}\quad \Gamma_l\neq\emptyset\,, \tag{2.61}$$

$$\langle c\rangle^3/\langle g_l\rangle|_{D_{l2}}\in S_l(D_{l2}, 1) \quad\text{when}\quad D_{l2}\neq\emptyset\,. \tag{2.62}$$

From (2.59), $c|_{D_{l2}\cap D_{l1}}\in S_l(D_{l2}\cap D_{l1}, \langle c\rangle|_{D_{l2}\cap D_{l1}})$ when $D_{l2}\cap\mathcal{D}_{l1}\neq\emptyset$ from which one obtain $\langle c\rangle^s|_{D_{l2}\cap D_{l1}}\in S_l(D_{l2}\cap D_{l1}, \langle c\rangle^s|_{D_{l2}\cap D_{l1}})$ by Lemma 2.2 when $D_{l2}\cap D_{l1}\neq\emptyset$. Thus from this and (2.53) Lemma 2.4 implies that

$$\langle c\rangle^3/\langle g_l\rangle|_{D_{l2}\cap D_{l1}} S_l(D_{l2}\cap D_{l1}, 1)\,. \tag{2.63}$$

Since (2.53) with Γ_l replaced by $\Gamma_l(N/100)$ also holds because of the fact that $(N/100)>1$, since $\Gamma_l\cap D_l\subseteq\Gamma_{l'}(N/100)$ if $l\neq l'$, and since $M^{(l')}_{\alpha,\beta}\leq 100^{(|\alpha|+|\beta|)/3}M^{(l)}_{\alpha,\beta}$ on $\Gamma_l\cap D_l$, we see that $\langle g_{l'}\rangle|_{\Gamma_l\cap D_l}\in S_l(\Gamma_l\cap D_l, \langle g_{l'}\rangle|_{\Gamma_l\cap D_l})$ if $l\neq l'$. From this and (2.53) Lemma 2.4 implies that

$$\langle g_{l'}\rangle/\langle g_l\rangle|_{\Gamma_l\cap D_l}\in S_l(\Gamma_l\cap D_l, \langle g_{l'}\rangle/\langle g_l\rangle|_{\Gamma_l\cap D_l}) \quad\text{when}\quad \Gamma_l\cap D_l\neq\emptyset\,. \tag{2.64}$$

Now we can show that $\Psi^{(l,j)}_i\in S^{0,0}_{\Psi_l,\varphi_l}$. When $\Gamma_l\neq\emptyset$, noting (2.61) and taking in Lemma 2.2 $U=\Gamma_l$, $V=(0,+\infty)$, $f=\langle g_l\rangle\langle\xi'\rangle^{-3/2}$, $F(y)=\chi_i(N^{-1}y)$, $N_1(y)=y$, $Z(y)=1$, $M_{(\alpha,\beta)}=M^{(l)}_{\alpha,\beta}$ one obtain that

$$\chi_i(N^{-1}\langle g_l\rangle\langle\xi'\rangle^{-3/2})|_{\Gamma_l}\in S_l(\Gamma_l, 1)\,. \tag{2.65}$$

Similar argument as above shows that

$$\chi_i(\varepsilon\langle c\rangle^3/\langle g_l\rangle)|_{D_{l1}}\in S_l(D_{l1}, 1) \quad\text{when}\quad D_{l1}\neq\emptyset\,, \tag{2.66}$$

$$\chi_i(100\,\langle g_{l'}\rangle/\langle g_l\rangle)|_{\Gamma_l\cap D_l}\in S_l(\Gamma_l\cap D_l, 1) \quad\text{when}\quad \Gamma_l\cap D_l\neq\emptyset\,, \tag{2.67}$$

$$\chi_i\Big(\frac{10}{\varepsilon}\frac{\langle g_l\rangle}{\langle c\rangle^3}\Big)|_{D_{l2}\cap D_{l1}}\in S_l(D_{l2}\cap D_{l1}, 1) \quad\text{when}\quad D_{l2}\cap D_{l1}\neq\emptyset\,. \tag{2.68}$$

Since the support of any first order derivatives of $\chi_i(10/\varepsilon\,\langle g_l\rangle/\langle c\rangle^3)$ contained in $D_{l1}\cap D_{l2}$, boundedness of χ_i and (2.68) imply that

$$\chi_i(\frac{10}{\varepsilon}\langle g_l\rangle/\langle c\rangle^3)|_{D_{l2}}\in S_l(D_{l2}, 1) \quad \text{when} \quad D_{l2}\neq\emptyset . \tag{2.69}$$

From (2.66), (2.67), and (2.69) Lemma 2.4 implies that

$$\Psi_i^{(l,j)}|_{D_{lj}}\in S_l(D_{lj}, 1) \quad \text{when} \quad D_{lj}\neq\emptyset . \tag{2.70}$$

Since $supp\ \Psi_i^{(l,j)}\subseteq D_{lj}$, one obtains that

$$\Psi_i^{(l,j)}\in S^{0,0}_{\Phi_l,\varphi_l} \quad \text{for} \quad j\neq 0 . \tag{2.71}$$

We also get that $\Psi_i^{(l,0)}\in S^{(l,0)}_{\Phi_l,\varphi_l}$, because (2.65) holds with N replaced by $N'>1$ and $supp\ \Psi_i^{(l,0)}\subseteq\Gamma_l$.

Now we can derive the estimates of derivatives of $\tilde{\Psi}_i^{(l,s,j)}$ and $\Lambda^{(l,s,j)}$. Noting the estimates (2.49), (2.58) and using Lemma 2.2 we see that $\varphi_{i1j}(w_l(\cdot))\in S_l(D_{l1}, 1)$ when $D_{l1}\neq\emptyset$. Using this, (2.71), and Lemma 2.4 we obtain that

$$\tilde{\Psi}_i^{(l,1,j)}\in S^{0,0}_{\Phi_l,\varphi_l} . \tag{2.72}$$

Using similar argument for $\tilde{\Psi}_i^{(l,2,j)}$ we get that $\tilde{\Psi}^{(l,2,j)}_{\Phi_l,\varphi_l}\in S^{0,0}_{\Phi_l,\varphi_l}$.

Using (2.46) and Lemma 2.2 we see that $\mu_1(w_l(\cdot))-1\in S_l(D_{l1}, \frac{|g_l|\,|_{D_{l1}}}{(|c|\,|_{D_{l1}})^3})$ when $D_{l1}\neq\emptyset$. This implies that $\Lambda_1^{(l,1,j)}|_{D_{l1}}\in S_l(D_{l1}, \frac{|g_l|\,|_{D_{l1}}}{(|c|\,|_{D_{l1}})^2})$ when $D_{l1}\neq\emptyset$. Noting $supp\ \Lambda_1^{(l,1,j)}\subseteq D_{l1}$ one obtains that $\Lambda_1^{(l,1,j)}\in S^{(1,0)}_{\Phi_l,\varphi_l}$. Similarly, noting $supp\ \Lambda_k^{(l,1,j)}\subseteq w_l^{-1}(U_{1j})$ when $D_{l1}\neq\emptyset$ and using (2.55), we see that

$$\Lambda_k^{(l,1,j)}|_{D_{l1}}\in S_l(D_{l1}, |c|\,|_{D_{l1}}|w_{l1}|^{3/2}) \quad \text{for} \quad k=2, 3 \quad \text{when} \quad D_{l1}\neq\emptyset . \tag{2.73}$$

From $|c|\,|w_{l1}|^{3/2}=\frac{|g_l|^{1/2}}{|c|^{1/2}}$ on D_{l1}, this implies that $\Lambda_k^{(l,1,j)}\in S^{1,0}_{\Phi_l,\varphi_l}$ for $k=2, 3$. Similar argument also shows that $\Lambda_k^{(l,2,j)}\in S^{1,0}_{\Phi_l,\varphi_l}$. Finally we shall derive other facts on $\tilde{\Psi}_i^{(l,1,j)}$ and $\Lambda_k^{(l,1,j)}$, $\tilde{\Psi}_i^{(l,2,j)}$ and $\Lambda_k^{(l,2,j)}$, $\Psi_i^{(l,0)}$ respectively corresponding to (II), (I), and (III) in Lemma 2.3. First we consider (II).

Since $\Phi_l=\langle g_l\rangle^{1/3}$ on Γ_l, the definition of D_{l1} implies (iii) with ψ_{0k} replaced by $\tilde{\Psi}_0^{(l,1,j)}$ when $\tilde{\Psi}_0^{(l,1,j)}\not\equiv 0$. (iv) and (v) with ψ_{0k}, Λ_j replaced by $\tilde{\Psi}_0^{(l,1,j)}$, $\Lambda_j^{(l,1,j')}$ follows from (2.43) and (2.44) when $\Psi_0^{(l,1,j')}\not\equiv 0$. (iv) with the same convention as before follows from (2.73) and (2.43). (i) follows from the fact that $p=p_l+b_{l0}$, (2.1), (2.2), and that for $k=2, 3$

$$(\lambda_l+\Lambda_k^{(l,1,j)})(x, \xi')=[\tilde{\Psi}_0^{(l,1,j)}(\lambda_l+\tilde{\Lambda}_k^{(l,1,j)})](x, \xi')+[(1-\Psi_0^{(l,1,j)})\lambda_l](x, \xi')$$
$$\text{on} \quad supp\ \Psi_0^{(l,1,j)}$$

where $\tilde{\Lambda}_k^{(l,1,j)}(x, \xi')=\tilde{\mu}_{1jk}(w_l(x, \xi'))\,c_l(x, \xi')$. (ii) follows from similar reason.

Nest, with the convention that ψ_{0k} and Λ_j replaced by $\tilde{\Psi}_i^{(l,2,j')}$ and $\Lambda_j^{(l,2,j')}$, (II) follows similarly. Finally (III) with ψ_{0k} replaced by $\tilde{\Psi}_i^{(l,0)}$ follows from the definition of Φ_l.

Step 5. Now we shall define ψ_{ik}. Set

$$\tilde{\Psi} = \sum_{l=1}^{2} \sum_{s=1}^{2} \sum_{j=1}^{k_s} \tilde{\Psi}_2^{(l,s,j)} + \sum_{l=1}^{2} \Psi_2^{(l,0)} .$$

Then $\tilde{\Psi} \geq 1$. We set

$$\tilde{\tilde{\Psi}}^{(l,s,j)} = \tilde{\Psi}_2^{(l,s,j)}/\tilde{\Psi},\ \tilde{\Psi}^{(l,0)} = \Psi_2^{(l,0)}/\tilde{\Psi} .$$

Since $\tilde{\Psi}^{-1} \in S^0_{1/2,1/2}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$, and since $\Phi_l \leq C\langle \xi' \rangle^{1/2}$, $\tilde{\Psi}^{(l,0)} \in S^{0,0}_{\Phi_l,\varphi_l}$. Using the fact that $1/100 \leq \frac{\langle g_2 \rangle}{\langle g_1 \rangle} \leq 100$ on $D_1 \cap D_2$ and that $\Phi_{l'} \leq C\langle \xi' \rangle^{1/2}$ on $\Gamma_{0l}(N) \cap D_{l'j}$ when $l \neq l'$, one can easily see $\tilde{\tilde{\Psi}}^{(l,s,j)} \in S^{0,0}_{\Phi_l,\varphi_l}$. Now we set

$$\tilde{\psi}_{1k} = \tilde{\Psi}^{(k,0)} (k = 1, 2)$$

$$\tilde{\psi}_{1k} = \begin{cases} \tilde{\tilde{\Psi}}^{(l,1,k-(2+(l-1)k_1))} & (3+(l-1)\,k_1+(l-1)\,k_2 \leq k \leq 2+lk_1+(l-1)\,k_2) \\ \tilde{\tilde{\Psi}}^{(l,2,k-(2+lk_1+(l-1)k_2))} & (3+lk_1+(l-1)\,k_2 \leq k \leq 2+l(k_1+k_2)) \end{cases}$$

and we define $\tilde{\psi}_{0k}$ by the same definition as above with $\tilde{\Psi}^{(k,0)}$ and $\tilde{\tilde{\Psi}}^{(\cdot,\cdot,\cdot)}$ replaced by $\Psi_1^{(k,0)}$ and $\tilde{\Psi}^{(\cdot,\cdot,\cdot)}$ respectively. Then ψ_{ik} is defined as follows:

$$\psi_{ik} = \tilde{\psi}_{ij(k)} , \quad k = 1, \cdots, k_0$$

where $\{j(1), \cdots, j(k_0)\} = \{j; \tilde{\psi}_{1j} \not\equiv 0\}$. This $\{\psi_{ik}\}$ has required properties from Step 4. The proof of Lemma 2.3 is complete.

§3. Local Carleman Estimates

Let $\Phi, \varphi \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$ be a pair of weight functions stated after Lemma 2.2. Let $\psi_i (i=0, 1) \in S^{0,0}_{\Phi,\varphi}$ satisfying that

$$\psi_1 \not\equiv 0 ,$$

$$\psi_0 = 1 \quad \text{on a neighbourhood of} \quad supp\, \psi_1 .$$

Let P be a pseudodifferential operator on $\boldsymbol{R}^n$ with the symbol $p(x, \xi)$ given by

$$p(x, \xi) = (\xi_1 - \lambda(x, \xi'))^2 (\xi_1 - \lambda(x, \xi') - c(x, \xi')) + g(x, \xi')$$

with

$$\lambda, c \in S^1_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1}) , \quad g \in S^3_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1}) \cap S^{3,0}_{\Phi,\varphi} .$$

We assume that

$$C\,|Im\,\lambda(x,\xi')|\geq\langle\xi'\rangle\,,\quad C\,|Im[(\lambda+c)(x,\xi')]|\geq\langle\xi'\rangle$$

for some positive constant C, and that one of the following (I), (II), (III) holds.
(I) There exist $A_i(i=1,2,3)\in S^{1,0}_{\Phi,\varphi}$ such that

$$p(x,\xi)=\prod_{i=1}^{3}(\xi_1-\lambda(x,\xi')-A_i(x,\xi'))\quad\text{when}\quad(x,\xi')\in supp\,\psi_0\,,$$

$$\sup_{(x,\xi')\in supp\psi_0}|c(x,\xi')|\,\Phi^{-1}(x,\xi')<+\infty\,,$$

$$\inf_{(x,\xi')\in supp\psi_0}|A_i(x,\xi')-A_j(x,\xi')|\,\Phi^{-1}(x,\xi')>0\quad\text{for any distinct}\quad i,j\,,$$

$$\min_{1\leq i\leq 3}\inf_{|\xi'|\geqq R}|Im[\lambda(x,\xi')+A_i(x,\xi')]|\langle\xi'\rangle^{-1}>0\quad\text{for some}\quad R>0\,.$$

(II) There exist $A_i(i=1,2,3)\in S^{1,0}_{\Phi,\varphi}$ such that

$$p(x,\xi)=(\xi_1-\lambda(x,\xi')-c(x,\xi')-A_1(x,\xi'))\prod_{i=2}^{3}(\xi_1-\lambda(x,\xi')-A_i(x,\xi'))$$

when $(x,\xi')\in supp\,\psi_0$,

$$\inf_{(x,\xi')\in supp\psi_0}|c(x,\xi')|\,\Phi^{-1}(x,\xi')>0\,,$$

$$\inf_{(x,\xi')\in supp\psi_0}|c(x,\xi')+A_1(x,\xi')-A_i(x,\xi')|\,|c(x,\xi')|^{-1}>0\quad\text{for}\quad i=2,3\,,$$

$$A_i(x,\xi')\,(i=2,3)\quad\text{are distinct when}\quad(x,\xi')\in supp\,\psi_0\,,$$

$$|\partial_x^\alpha\partial_{\xi'}^\beta A_i(x,\xi')|\leq C_{\alpha,\beta}|A_2(x,\xi')-A_3(x,\xi')|\,\Phi^{-|\beta|}(x,\xi')\,\varphi^{-|\alpha|}(x,\xi')$$

on a neighbourhood of $supp\,\psi_0$ for $i=2,3$,

$$\inf_{|\xi'|\geq R}|Im[(\lambda+c+A_1)(x,\xi')]|\langle\xi'\rangle^{-1}>0\quad\text{and}$$

$$\inf_{|\xi'|\geq R}|Im[(\lambda+A_i)(x,\xi')]|\langle\xi'\rangle^{-1}>0\quad\text{for}\quad i=2,3$$

with some $R>0$.
(III) $\sup_{(x,\xi')\in supp\psi_0}\Phi(x,\xi')\,(1+|\xi'|)^{-1/2}<+\infty$.
The main result of this section is the following proposition. This gives the estimates for P in Proposition 1.1 on supports of functions ψ_{jk} in Lemma 2.3. We set $\Psi_i=\psi_i(x,D')$.

Proposition 3.1. (1) *Assume that (I) or (II) in the above holds. Then there exists positive constants τ_0, T_0, C_0 such that*

$$\begin{aligned}T^{-1/2}A_1(\Psi_1 u)+A_2(\Psi_1 u)+A_3(\Psi_1 u)\\ \leq C_0(\|Pu\|_T^{(\tau)}+A_1(u)+T^{1/2}A_3(u)+T^{-1}R(u))\end{aligned}\tag{3.1}$$

for $u\in\mathcal{S}_T(\boldsymbol{R}^n)$ *when* $\tau T^2>\tau_0$ *and* $T<T_0$.
Here

$$A_1(u)=\sum_{1\leq|\alpha|+|\beta|\leq 2}\|E_{(|\alpha|-|\beta|)/2}P^{(\alpha)}_{(\beta)}u\|^{(\tau)}_T,$$

$$A_2(u)=|||u|||^{(\tau)}_{T,6}+T^{-1/2}\sum_{i\neq j}\|E_{1/2}(L_i\circ L_j)(x,D)u\|^{(\tau)}_T$$
$$+T^{-1}\sum_{i=1}^{3}\|E_1L_i(x,D)u\|^{(\tau)}_T,$$

$$A_3(u)=T^{-1}\sum_{i=1}^{2}\|E_1L_{0i}(x,D)u\|^{(\tau)}_T+T^{-1}\|E_1c(x,D')u\|^{(\tau)}_T,$$

$$R(u)=\sum_{i=0}^{2}\|E_{-1/2+i}D_1^{2-i}u\|^{(\tau)}_T,$$

$$L_{0i}(x,\xi')=\begin{cases}\xi_1-\lambda(x,\xi') & (i=1)\\ \xi_1-\lambda(x,\xi')-c(x,\xi') & (i=2)\end{cases}$$

$$L_i(x,\xi)=\xi_1-\lambda(x,\xi')-A_i(x,\xi')\quad \textit{except for that}$$
$$L_1(x,\xi)=\xi_1-\lambda(x,\xi')-c(x,\xi')-A_1(x,\xi')\quad \textit{in case (II)}.$$

(2) *Assume that (III) in the above holds. Then there exist positive constants* τ_0, T_0, C_0 *such that*

$$T^{-1/2}A_1(\Psi_1u)+B(\Psi_1u)\leq C_0(\|Pu\|^{(\tau)}_T+A_1(u)+T^{-1/2}R(u))\tag{3.2}$$

for $u\in\mathcal{S}_T(\boldsymbol{R}^n)$ *when* $\tau T^2>\tau_0$ *and* $T<T_0$.
Here

$$B(u)=|||u|||^{(\tau)}_{T,6}+\sum_{i\neq j}\tau^{-1}|||(L_{0i}\circ L_{0j})(x,D)u|||^{(\tau)}_{T,2}+\sum_{i=1}^{2}\tau^{-1/2}|||L_{0i}(x,D)u|||^{(\tau)}_{T,4}$$
$$+\tau^{-1/2}|||c(x,D')u|||^{(\tau)}_{T,4}+\tau^{-1}|||L_{01}(x,D)^2u|||^{(\tau)}_{T,2},$$

and the other notations are the same as in (1).

We shall prove this proposition in this section admitting one proposition and several lemmas which will be proved in later sections. We first prepare a proposition of Carleman estimate for first order factors in the factorizations of p and $p-g$ having the basic role for proof of Proposition 3.1.

Proposition 3.2. (1) *Let* $L(x,\xi)=\xi_1-a(x,\xi')-b(x,\xi')$ *with*

$$a\in S^1_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1}),\quad b\in S^{1,0}_{\Phi,\varphi},$$
$$\inf_{|\xi'|\geq R}|Im[(a+b)(x,\xi')]|\langle\xi'\rangle^{-1}>0\quad \textit{for some}\quad R>0.\tag{3.3}$$

Then there exist positive constants τ_0, T_0, C_0 *such that*

$$\tau^{-1} |||u|||_{T,2}^{(\tau)} \leq C_0 ||L(x, D) u||_T^{(\tau)}, \quad u \in \mathcal{S}_T(\boldsymbol{R}^n)$$

when $\tau T^2 > \tau_0$ *and* $T < T_0$.

(2) Let $L_i(x, \xi)$ $(i=1, 2, 3)$ be given by $L_i(x, \xi) = \xi_1 - a_i(x, \xi') - b_i(x, \xi')$ with $a_i \in S^1_{1,0}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$, $b_i \in S^{1;0}_{\Phi,\varphi}$ satisfying (3.3) with a, b replaced by a_i, b_i respectively. Then there exist positive constants τ_0, T_0, C_0 such that

$$\tau^{-1/2} |||u|||_{T,4}^{(\tau)} \leq C_0 ||(L_i \circ L_j)(x, D) u||_T^{(\tau)}, \quad u \in \mathcal{S}_T(\boldsymbol{R}^n) \quad \text{for any} \quad i, j,$$

$$|||u|||_{T,6}^{(\tau)} \leq C_0 \sum_{\sigma \in S_3} ||(L_{\sigma(1)} \circ L_{\sigma(2)} \circ L_{\sigma(3)})(x, D) u||_T^{(\tau)}, \quad u \in \mathcal{S}_T(\boldsymbol{R}^n)$$

when $\tau T^2 > \tau_0$ and $T < T_0$. Here S_3 is the symmetric group of degree 3.

Next we prepare some lemmas which need for the proof of both of (1) and (2) in Proposition 3.1.

Lemma 3.1. *Let* $a \in S^m_{1/2,1/2}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$. *Then there exist positive constant* C *such that for any* τ *and* T,

$$||a(x, D') u||_T^{(\tau)} \leq C ||E_m u||_T^{(\tau)}, \quad u \in \mathcal{S}(\boldsymbol{R}^n).$$

Next two lemmas give estimates for commutators.

Lemma 3.2. *Let* $L_i(x, \xi)$ $(i=1, 2, 3)$ *be the same as in Proposition* 3.2-(2). *Then there exist positive constants* τ_0, T_0, C_0 *such that*

$$C_0^{-1} ||(L_1 \circ L_2 \circ L_3)(x, D) u||_T^{(\tau)} \leq ||(L_1 L_2 L_3)(x, D) u||_T^{(\tau)} \leq C_0 ||(L_1 \circ L_2 \circ L_3)(x, D) u||_T^{(\tau)}$$

for $u \in \mathcal{S}_T(\boldsymbol{R}^n)$ *when* $\tau T^2 > \tau_0$ *and* $T < T_0$.

Lemma 3.3. *Let* $\chi \in S^{0;0}_{\Phi,\varphi}$. *Then we have that*

$$p \circ \chi - \chi \circ p = \sum_{1 \leq |\alpha| + |\beta| \leq 2} a_{\alpha\beta} \circ p^{(\alpha)}_{(\beta)} + \sum_{j=0}^{2} b_j \xi_1^{2-j}$$

with some $a_{\alpha\beta} \in S^{-|\alpha|, -|\beta|}_{\Phi,\varphi}$ *and* $b_j \in S^{-(1/2)+j}_{1/2,1/2}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$.

Lemma 3.4. *Let* $\chi \in S^{0;0}_{\Phi,\varphi}$. *Then we have that for* $b \in S^{-(1/2)+j}_{1/2,1/2}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$ *with* $j = 0, 1, 2$

$$(b\xi_1^{2-j}) \circ \chi = \sum_{k=j}^{2} a_k \xi_1^{2-k} \quad \textit{with some} \quad a_k \in S^{-(1/2)+((j+k)/2)}_{1/2,1/2}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1}).$$

Next two lemmas are ones for handling neglizible terms.

Lemma 3.5. *Let* $\chi \in S^{-3,-3}_{\Phi,\varphi}$. *Then there exist positive constants* τ_0, T_0, C_0

such that when $\tau T^2 > \tau_0$ *and* $T < T_0$.

$$B(\chi(x, D')u) + \|g(x, D')\chi(x, D')u\|_T^{(\tau)} \leq C_0(\|Pu\|_T^{(\tau)} + A_1(u) + R(u))$$

for $u \in \mathcal{S}_T(\mathbf{R}^n)$.

Lemma 3.6. *Let* $\chi \in S_{\Phi,\varphi}^{-3,-3}$. *Then there exist positive constants* τ_0, T_0, C_0 *such that when* $\tau T^2 > \tau_0$ *and* $T < T_0$,

$$T^{-1/2} A_1(\chi(x, D')u) + T^{-1/2} \sum_{1 \leq |\alpha| + |\beta| \leq 2} \|E_{(|\alpha| - |\beta|)/2}\, g_{(\beta)}^{(\alpha)}(x, D')\chi(x, D')u\|_T^{(\tau)}$$
$$\leq C_0(\|Pu\|_T^{(\tau)} + A_1(u) + T^{-1/2} R(u))$$

for $u \in \mathcal{S}_T(\mathbf{R}^n)$.

Now we start to prove Proposition 3.1.

Proof of (1). First we estimate $A_2(\Psi_1 u)$. We break up into $\Psi_0 \Psi_1$ and $(1 - \Psi_0)\Psi_1$. Then

$$A_2(\Psi_1 u) \leq A_2(\Psi_0 \Psi_1 u) + A_2((1 - \Psi_0)\Psi_1 u). \tag{3.4}$$

We take up the first term on the right hand side first. From Proposition 3.2 and Lemma 3.2 there exist positive constants τ_1, T_1, C_1 such that when $\tau T^2 > \tau_1$ and $T < T_1$,

$$A_2(u) \leq C_1 \|(L_1 L_2 L_3)(x, D)u\|_T^{(\tau)}, \quad u \in \mathcal{S}_T(\mathbf{R}^n). \tag{3.5}$$

We need a lemma to estimate $(L_1 L_2 L_3)(x, D)\Psi_0 - \Psi_0 P$.

Lemma 3.7. *Assume that (I) or (II) holds. Then if* $\chi \in S_{\Phi,\varphi}^{0,0}$ *with supp* $\chi \subseteq$ *supp* ψ_0,

$$(L_1 L_2 L_3) \circ \chi - \chi \circ p = \sum_{i \neq j} a_{ij} \circ L_i \circ L_j + \sum_{i=1}^{3} a_i \circ L_i$$
$$+ a_0 + \sum_{1 \leq |\alpha| + |\beta| \leq 2} a_{\alpha\beta} \circ p_{(\beta)}^{(\alpha)} + \sum_{i=0}^{2} b_i \xi_1^{2-i}$$

with $a_{ij} \in S_{\Phi,\varphi}^{0,-1}$, $a_i \in S_{\Phi,\varphi}^{0,-2}$ *for* $i \neq 0$, $a_0 \in S_{\Phi,\varphi}^{1,-2}$, $a_{\alpha\beta} \in S_{\Phi,\varphi}^{-|\beta|,-|\alpha|}$, $b_i \in S_{1/2,1/2}^{-(1/2)+i}(\mathbf{R}^n \times \mathbf{R}^{n-1})$.

We note that

$$S_{\Phi,\varphi}^{M,m} \subseteq \begin{cases} S_{1/2,1/2}^{(M-m)/2}(\mathbf{R}^n \times \mathbf{R}^{n-1}) & (-m \geq M \geq 0) \\ S_{1/2,1/2}^{-(M+m)/2}(\mathbf{R}^n \times \mathbf{R}^{n-1}) & (m \leq 0,\, M \leq 0). \end{cases} \tag{3.6}$$

Substituting $\Psi_0 u$ to u in (3.5), using Lemma 3.7 with $\chi=\psi_0$, and noting (3.6) we obtain the following inequality with the notations in Lemma 3.7: there exists a positive constant C_2 such that when $\tau T^2>\tau_1$ and $T<T_1$, for $u\in\mathcal{S}_T(\boldsymbol{R}^n)$

$$\begin{aligned} A_2(\Psi_0 u)&\leq C_1(\|\Psi_0 Pu\|_T^{(\tau)}+\sum_{i\neq j}\|a_{ij}(x,D')(L_i\circ L_j)(x,D)u\|_T^{(\tau)} \\ &\quad+\sum_{i=1}^{3}\|a_i(x,D')L_i(x,D)u\|_T^{(\tau)}+\|a_0(x,D')u\|_T^{(\tau)} \\ &\quad+\sum_{1\leq|\alpha|+|\beta|\leq 2}\|a_{\alpha\beta}(x,D')P_{(\beta)}^{(\alpha)}u\|_T^{(\tau)}+\sum_{i=0}^{2}\|b_i(x,D')D_1^{2-i}u\|_T^{(\tau)}) \\ &\leq C_2(\|Pu\|_T^{(\tau)}+\sum_{i\neq j}\|E_{1/2}(L_i\circ L_j)(x,D)u\|_T^{(\tau)} \\ &\quad+\sum_{i=1}^{3}\|E_1 L_i(x,D)u\|_T^{(\tau)}+\|E_{3/2}u\|_T^{(\tau)} \\ &\quad+\sum_{1\leq|\alpha|+|\beta|\leq 2}\|E_{(|\alpha|-|\beta|)/2}P_{(\beta)}^{(\alpha)}u\|_T^{(\tau)}+R(u)) \\ &\leq C_2(\|Pu\|_T^{(\tau)}+(T^{1/2}+T)A_2(u)+A_1(u)+2R(u)). \end{aligned} \tag{3.7}$$

From Lemma 3.4 there exists a positive constant C_3 such that for any τ, T

$$R(\Psi_1 u)\leq C_3 R(u), \quad u\in\mathcal{S}(\boldsymbol{R}^n). \tag{3.8}$$

From Lemma 3.3 there exists a positive constant C_4 such that for any τ, T

$$\|P\Psi_1 u\|_T^{(\tau)}\leq C_4(\|Pu\|_T^{(\tau)}+A_1(u)+R(u)), \quad u\in\mathcal{S}_T(\boldsymbol{R}^n). \tag{3.9}$$

Substituting $\Psi_1 u$ into u in (3.7) and using (3.8) and (3.9) we obtain that when $\tau T^2>\tau_1$ and $T<T_1$, for $u\in\mathcal{S}_T(\boldsymbol{R}^n)$

$$\begin{aligned} A_2(\Psi_0\Psi_1 u)&\leq C_2 C_4(\|Pu\|_T^{(\tau)}+A_1(u))+C_2(C_4+2C_3)R(u) \\ &\quad+C_2(T^{1/2}+T)A_2(\Psi_1 u)+C_2 A_1(\Psi_1 u). \end{aligned} \tag{3.10}$$

Next we handle the second term on the right hand side of (3.4).

Lemma 3.8. *Let* $\chi\in S_{\Phi,\varphi}^{-3,-3}$.

(1) *Assume that* (*I*) *holds.*

(i) *If* $i\neq j$,

$$\langle\xi'\rangle^{1/2}\circ L_i\circ L_j\circ\chi=\langle\xi'\rangle^{1/2}\circ L_{01}\circ L_{01}\circ\chi+a_{ij}\circ L_{01}+a'_{ij}$$

with $a_{ij}\in S_{\Phi,\varphi}^{0,-2}$, $a'_{ij}\in S_{\Phi,\varphi}^{0,-3}$.

(ii) $\langle\xi'\rangle\circ L_i\circ\chi=\langle\xi'\rangle\circ L_{01}\circ\chi+a_i$

with $a_i\in S_{\Phi,\varphi}^{0,-3}$.

(2) *Assume that* (*II*) *holds.*

(i) *If* $i\neq j$ *and* $i\neq 1, j\neq 1$, (*i*) *in* (1) *holds. If* $i\neq j$ *and one of* i *and* j *is equal* *to* 1,

$$\langle\xi'\rangle^{1/2}\circ L_i\circ L_j\circ\chi=\langle\xi'\rangle^{1/2}\circ L_{01}\circ L_{02}\circ\chi+\sum_{k=1}^{2}a_{ijk}\circ L_{0k}+a'_{ij}$$

with $a_{ijk}\in S^{0,-2}_{\Phi,\varphi}$, $a'_{ij}\in S^{0,-3}_{\Phi,\varphi}$.

(ii) *If* $i\neq 1$, (ii) *in* (1) *holds and we have*

$$\langle\xi'\rangle\circ L_1\circ\chi=\langle\xi'\rangle\circ L_{02}\circ\chi+a_1$$

with $a_1\in S^{0,-3}_{\Phi,\varphi}$.

Lemma 3.8 easily implies the following.

Corollary 3.1. *Assume (I) or (II) holds. Let* $\chi\in S^{-3,-3}_{\Phi,\varphi}$. *Then there exists a positive constant* C_0 *such that for any* τ, T, *and* $u\in\mathcal{S}(\mathbf{R}^n)$.

$$\sum_{i\neq j}||E_{1/2}(L_i\circ L_j)(x,D)\,\chi(x,D')\,u||^{(\tau)}_T$$
$$\leq C_0\Big(\sum_{k=1}^{2}||E_{1/2}(L_{01}\circ L_{0k})(x,D)\,\chi(x,D')\,u||^{(\tau)}_T+\sum_{k=1}^{2}||E_1\,L_{0k}(x,D)\,u||^{(\tau)}_T$$
$$+||E_{3/2}\,u||^{(\tau)}_T\Big),$$
$$\sum_{i=1}^{2}||E_1\,L_i(x,D)\,\chi(x,D')\,u||^{(\tau)}_T$$
$$\leq C_0\Big(\sum_{k=1}^{2}||E_1\,L_{0k}(x,D)\,\chi(x,D')\,u||^{(\tau)}_T+||E_{3/2}\,u||^{(\tau)}_T\Big).$$

From the fact that $\psi_0=1$ on a neighbourhood of *supp* ψ_1 we have

$$(I-\Psi_0)\,\Psi_1\in OpS^{-N,-N}_{\Phi,\varphi}\quad\text{for any}\quad N>0\,.$$

Noting this and using Corollary 3.1 we see that there exists a positive constant C_5 such that for any τ, T, and $u\in\mathcal{S}(\mathbf{R}^n)$

$$T^{-1/2}\sum_{i\neq j}||E_{1/2}(L_i\circ L_j)(x,D)(I-\Psi_0)\,\Psi_1\,u||^{(\tau)}_T+T^{-1}\sum_{k=1}^{3}||E_1\,L_k(x,D)(I-\Psi_0)\,\Psi_1\,u||^{(\tau)}_T$$
$$\leq C_5\Big(T^{-1/2}\sum_{k=1}^{2}||E_{1/2}(L_{01}\circ L_{0k})(x,D)\,(I-\Psi_0)\,\Psi_1\,u||^{(\tau)}_T+T^{1/2}A_3(u)+T^{-1/2}R(u)$$
$$+T^{-1}\sum_{k=1}^{2}||E_1\,L_{0k}(x,D)\,(I-\Psi_0)\,\Psi_1\,u||^{(\tau)}_T+T^{-1}R(u)\Big).$$

Using this we obtain that for any τ, T, and $u\in\mathcal{S}(\mathbf{R}^n)$

$$A_2((I-\Psi_0)\,\Psi_1\,u)\leq\max(C_5,1)\,B((I-\Psi_0)\,\Psi_1\,u)$$
$$+C_5\,T^{1/2}A_3(u)+C_5(T^{1/2}+1)\,T^{-1}R(u)\,.$$

From Lemma 3.5 there exist positive constants $\tau_2>\tau_1$ and $T_2<T_1$ and C_6 such

that

$$B((I-\Psi_0)\,\Psi_1\,u)\leq C_6(||Pu||_T^{(\tau)}+A_1(u)+R(u))\,,\quad u\in\mathcal{S}_T(\boldsymbol{R}^n) \tag{3.11}$$

when $\tau T^2>\tau_2$ and $T<T_2$.
Substituting this inequality to the above one we get that when $\tau T^2>\tau_2$ and $T<T_2$, for $u\in\mathcal{S}_T(\boldsymbol{R}^n)$

$$\begin{aligned}A_2((I-\Psi_0)\,\Psi_1\,u)\leq&\max(C_5,1)\,C_6(||Pu||_T^{(\tau)}+A_1(u))+C_5\,T^{1/2}\,A_3(u)\\&+\{\max(C_5,1)\,C_6\,T+C_5(T^{1/2}+1)\}\,T^{-1}\,R(u)\,.\end{aligned}\tag{3.12}$$

From (3.4), (3.10), and (3.12) we obtain that when $\tau T^2>\tau_2$ and $T<T_2$, for $u\in\mathcal{S}(\boldsymbol{R}^n)$

$$\begin{aligned}A_2(\Psi_1\,u)\leq&\,C_7(||Pu||_T^{(\tau)}+A_1(u))+C_{8,T}\,T^{-1}\,R(u)+C_5\,T^{1/2}\,A_3(u)\\&+C_2(T^{1/2}+T)\,A_2(\Psi_1\,u)+C_2\,A_1(\Psi_1\,u)\,.\end{aligned}\tag{3.13}$$

Here

$$\begin{aligned}C_7&=C_2\,C_4+\max(C_5,1)\,C_6\,,\\C_{8,T}&=\{C_2(C_4+2C_3)+\max(C_5,1)\,C_6\}\,T+C_6(T^{1/2}+1)\,.\end{aligned}\tag{3.14}$$

Next we estimate $A_3(\Psi_1\,u)$.

Lemma 3.9. (1) *Assume that (I) holds. Then for any distinct* $1\leq i,j\leq 3$

$$L_{01}\circ\psi_1=a_1\circ L_i\circ\psi_1+a_2\circ L_j\circ\psi_1+a_3+L_{01}\circ(1-\psi_0)\circ\psi_1 \tag{3.15}$$

with $a_1,a_2\in S^{0,0}_{\Phi,\varphi}$ *and* $a_3\in S^{0,-1}_{\Phi,\varphi}$. *And for any distinct* $1\leq i,j\leq 3$

$$c\circ\psi_1=a_1\circ L_i\circ\psi_1+a_2\circ L_j\circ\psi_1+a_3+c\circ(1-\psi_0)\circ\psi_1$$

with $a_1,a_2\in S^{0,0}_{\Phi,\varphi}$ *and* $a_3\in S^{0,-1}_{\Phi,\varphi}$.
(2) *Assume that (II) holds. Then for any distinct* $1\leq i,j\leq 3$, (3.15) *holds. And for any* $i\neq 1$

$$c\circ\psi_1=a_1\circ L_1\circ\psi_1+a_2\circ L_i\circ\psi_1+a_3+c\circ(1-\psi_0)\circ\psi_1$$

with $a_1,a_2\in S^{0,0}_{\Phi,\varphi}$ *and* $a_3\in S^{0,-1}_{\Phi,\varphi}$.

From Lemma 3.9 we see that there exist positive constants C_{10}, C_{11} such that for any τ, T, and $u\in\mathcal{S}(\boldsymbol{R}^n)$

$$\begin{aligned}||E_1\,L_{01}(x,D)\,\Psi_1\,u||_T^{(\tau)}\leq C_9(&\sum_{k=1}^{2}||E_1\,L_k(x,D)\,\Psi_1\,u||_T^{(\tau)}+||E_{3/2}\,u||_T^{(\tau)}\\&+||E_1\,L_{01}(x,D)\,(I-\Psi_0)\,\Psi_1\,u||_T^{(\tau)})\,,\end{aligned}$$

$$||E_1\, c(x, D')\, \Psi_1\, u||_T^{(\tau)} \leq C_{10}\big(\sum_{k=1}^{2} ||E_1\, L_k(x, D)\, \Psi_1\, u||_T^{(\tau)} + ||E_{3/2}\, u||_T^{(\tau)}$$
$$+ ||E_1\, c(x, D')\,(I-\Psi_0)\,\Psi_1\, u||_T^{(\tau)}\big)\,.$$

We obtain from these two inequalities that for any τ, T, and $u \in \mathcal{S}(\boldsymbol{R}^n)$

$$\begin{aligned} A_3(\Psi_1 u) &= T^{-1}(||E_1\, L_{01}(x, D)\, \Psi_1\, u||_T^{(\tau)} + ||E_1(L_{01}(x, D) - c(x, D'))\, \Psi_1\, u||_T^{(\tau)} \\ &\quad + ||E_1\, c(x, D')\, \Psi_1\, u||_T^{(\tau)}) \\ &\leq 2T^{-1}(||E_1\, L_{01}(x, D)\, \Psi_1\, u||_T^{(\tau)} + ||E_1\, c(x, D')\, \Psi_1\, u||_T^{(\tau)}) \\ &\leq 2\{(C_9 + C_{10})\,(\sum_{k=1}^{2} T^{-1}\, ||E_1\, L_k(x, D)\, \Psi_1\, u||_T^{(\tau)} + T^{-1}\, ||E_{3/2}\, u||_T^{(\tau)}) \qquad (3.16) \\ &\quad + C_9\, T^{-1}\, ||E_1\, L_{01}(x, D)\,(I-\Psi_0)\,\Psi_1\, u||_T^{(\tau)} \\ &\quad + C_{10}\, T^{-1}\, ||E_1\, c(x, D')\,(I-\Psi_0)\,\Psi_1\, u||_T^{(\tau)}\} \\ &\leq 2\{(C_9 + C_{10})\,(A_2(\Psi_1\, u) + T^{-1}\, R(u)) \\ &\quad + \max(C_9, C_{10})\, B((I-\Psi_0)\,\Psi_1\, u)\}\,. \end{aligned}$$

Substituting (3.13) and (3.11) into (3.16) we see that there exists a positive constant C_{11} such that when $\tau T^2 > \tau_2$, $T < T_2$, and $u \in \mathcal{S}_T(\boldsymbol{R}^n)$,

$$A_3(\Psi_1\, u) \leq C_{11}(||Pu||_T^{(\tau)} + A_1(u) + T^{1/2}\, A_3(u) + (T^{1/2} + T)\, A_2(\Psi_1\, u)$$
$$+ A_1(\Psi_1\, u)) + C_{12,T}\, T^{-1}\, R(u)\,. \qquad (3.17)$$

Here

$$C_{12,T} = 2\{C_{8,T} + C_9 + C_{10} + \max(C_9, C_{10})\, C_6\, T\}\,. \qquad (3.18)$$

Finally we estimate $A_1(\Psi_1\, u)$. First we have

$$A_1(\Psi_1\, u) \leq A_1(I-\Psi_0^2)\,\Psi_1\, u) + A_1(\Psi_0^2\,\Psi_1\, u)\,, \quad u \in \mathcal{S}(\boldsymbol{R}^n)\,. \qquad (3.19)$$

To estimate the second term on the right we need

Lemma 3.10. *Let $\chi \in S^{0,0}_{\Phi,\varphi}$ with supp $\chi \subseteq$ supp ψ_0. Then if $\alpha, \beta \in \boldsymbol{Z}^n_+$ with $|\alpha| + |\beta| = 1$ or 2,*

$$\langle \xi' \rangle^{(|\alpha| - |\beta|)/2} \circ p_{(\beta)}^{(\alpha)} \circ \chi = \sum_{i \neq j} a_{ij}^{\alpha\beta} \circ L_i \circ L_j + \sum_{i=1}^{3} a_i^{\alpha\beta} \circ L_i + \sum_{j=0}^{2} b_j^{\alpha\beta}\, \xi_1^{2-j}$$

with $a_{ij}^{\alpha\beta} \in S^{1/2}_{1/2,1/2}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$, $a_i^{\alpha\beta} \in S^{1}_{1/2,1/2}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$, $b_j^{\alpha\beta} \in S^{-(1/2)+j}_{1/2,1/2}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$.

From Lemma 3.10 we see that there exists a positive constant C_{13} such that for any τ, T, and $u \in \mathcal{S}(\boldsymbol{R}^n)$

$$T^{-1/2}\, A_1(\Psi_0\, u) \leq C_{13}(\sum_{i \neq j} T^{-1/2}\, ||E_{1/2}(L_i \circ L_j)\,(x, D)\, u||_T^{(\tau)}$$

$$+T^{-1/2}\sum_{i=1}^{3}\|E_1 L_i(x, D)u\|_T^{(\tau)}+T^{-1/2}R(u)) \tag{3.20}$$
$$\leq C_{13}(\max(1, T^{1/2})A_2(u)+T^{-1/2}R(u)).$$

Since $\Psi_0 \Psi_1 \in Op S^{0,0}_{\Phi,\varphi}$, from Lemma 3.4 there exists a positive constant C_{14} such that for any τ, T, and $u \in \mathcal{S}(\boldsymbol{R}^n)$

$$R(\Psi_0 \Psi_1 u) \leq C_{14} R(u). \tag{3.21}$$

Substituting $\Psi_0 \Psi_1 u$ into u in (3.20) and using (3.21) we get that for any τ, T, and $u \in \mathcal{S}(\boldsymbol{R}^n)$

$$T^{-1/2}A_1(\Psi_0^2 \Psi_1 u) \leq C_{13}(\max(1, T^{1/2})A_2(\Psi_0 \Psi_1 u)+C_{14}T^{-1/2}R(u)).$$

Substituting (3.10) into this inequality we see that there exists a positive constant C_{15} such that when $\tau T^2 > \tau_1$ and $T < T_1$,

$$T^{-1/2}A_1(\Psi_0^2 \Psi_1 u) \leq C_{15}\max(1, T^{1/2})(\|Pu\|_T^{(\tau)}+A_1(u)+(T^{1/2}+T)A_2(\Psi_1 u)$$
$$+A_1(\Psi_1 u)+(T^{1/2}+T)T^{-1}R(u)), \quad u \in \mathcal{S}_T(\boldsymbol{R}^n). \tag{3.22}$$

Since $(I-\Psi_0^2)\Psi_1=(I+\Psi_0)(I-\Psi_0)\Psi_1$, and since $(I-\Psi_0)\Psi_1 \in OpS^{-N,-N}_{\Phi,\varphi}$ for any $N>0$, we have

$$(I-\Psi_0^2)\Psi_1 \in OpS^{-N,-N}_{\Phi,\varphi} \quad \text{for any} \quad N>0.$$

Thus from Lemma 3.6 there exist positive constants $\tau_3 > \tau_2$ and $T_3 < T_2$ and C_{16} such that when $\tau T^2 > \tau_3$ and $T < T_3$, for $u \in \mathcal{S}_T(\boldsymbol{R}^n)$

$$T^{-1/2}A_1((I-\Psi_0^2)\Psi_1 u) \leq C_{16}(\|Pu\|_T^{(\tau)}+A_1(u)+T^{-1/2}R(u)).$$

From this inequality, (3.22), and (3.19) we see that when $\tau T^2 > \tau_3$ and $T < T_3$, for $u \in \mathcal{S}_T(\boldsymbol{R}^n)$

$$T^{-1/2}A_1(\Psi_1 u) \leq \max(1, T^{1/2})(C_{15}+C_{16})(\|Pu\|_T^{(\tau)}+A_1(u)+A_1(\Psi_1 u)$$
$$+(T^{1/2}+T)A_2(\Psi_1 u)+(T^{1/2}+T)T^{-1}R(u)). \tag{3.23}$$

Combining (3.13), (3.17), (3.23) we see that there exists a positive constant C_{17} such that when $\tau T^2 > \tau_3$ and $T < T_3$, for $u \in \mathcal{S}_T(\boldsymbol{R}^n)$

$$T^{-1/2}A_1(\Psi_1 u)+A_2(\Psi_1 u)+A_3(\Psi_1 u)$$
$$\leq C_{17}(\|Pu\|_T^{(\tau)}+A_1(u)+A_1(\Psi_1 u)+T^{1/2}A_2(\Psi_1 u)+T^{1/2}A_3(u)$$
$$+T^{-1}R(u)).$$

This completes the proof of (1).

Proof of (2). Set

$$p_0(x, \xi) = p(x, \xi) - g(x, \xi'),$$
$$P_0 = p_0(x, D), \quad (P_0)^{(\alpha)}_{(\beta)} = (p_0)^{(\alpha)}_{(\beta)}(x, D).$$

Then we have that

$$A_1(\Psi_1 u) \leq \sum_{1\leq|\alpha|+|\beta|\leq 2} (\|E_{(|\alpha|-|\beta|)/2}(P_0)^{(\alpha)}_{(\beta)} \Psi_1 u\|^{(\tau)}_T + \|E_{(|\alpha|-|\beta|)/2}\, g^{(\alpha)}_{(\beta)}(x, D')\, \Psi_1 u\|^{(\tau)}_T). \tag{3.24}$$

We use the next two lemmas to estimate the right hand side of the above inequality. The assumption (III) is used to estimate the second terms in the parenthesis.

Lemma 3.11. *Assume that (III) holds. Let* $\chi \in S^{0,0}_{\Phi,\varphi}$ *with supp* $\chi \subseteq$ *supp* ψ_0. *Then*

$$\langle\xi'\rangle^{(|\alpha|-|\beta|)/2} \circ g^{(\alpha)}_{(\beta)} \circ \chi \in S^{3/2}_{1/2,1/2}(\mathbf{R}^n \times \mathbf{R}^{n-1})$$

for $\alpha \in \mathbf{Z}^n_+$, $\beta \in \mathbf{Z}^{n-1}_+$ *with* $|\alpha| \leq 2$.

Lemma 3.12. *For* $\alpha, \beta \in \mathbf{Z}^n_+$ *with* $|\alpha|+|\beta|=1$ *or* 2 *we have that*

$$\langle\xi'\rangle^{(|\alpha|-|\beta|)/2} \circ (p_0)^{(\alpha)}_{(\beta)} = \sum_{k=1}^{2} a_k \circ L_{01} \circ L_{0k} + \sum_{k=1}^{2} b_k \circ L_{0k} + b_0$$

with some $a_k \in S^{1/2}_{1,0}(\mathbf{R}^n \times \mathbf{R}^{n-1})$, $b_k \in S^1_{1,0}(\mathbf{R}^n \times \mathbf{R}^{n-1})$ *for* $k \neq 0$, $b_0 \in S^{3/2}_{1,0}(\mathbf{R}^n \times \mathbf{R}^{n-1})$.

From Lemma 3.12 there exists a positive constant C_1 such that for any τ, T

$$\|E_{(|\alpha|-|\beta|)/2}(P_0)^{(\alpha)}_{(\beta)} u\|^{(\tau)}_T \leq C_1(\sum_{k=1}^{2} \|E_{1/2}(L_{01}\circ L_{0k})(x, D)\, u\|^{(\tau)}_T + \sum_{k=1}^{2} \|E_1 L_{0k}(x, D)\, u\|^{(\tau)}_T$$
$$+ \|E_{3/2}\, u\|^{(\tau)}_T), \quad u \in \mathcal{S}(\mathbf{R}^n).$$

Summing this for α, β with $|\alpha|+|\beta|=1, 2$ we get that for any τ, T, and $u \in \mathcal{S}(\mathbf{R}^n)$

$$T^{-1/2} \sum_{1\leq|\alpha|+|\beta|\leq 2} \|E_{(|\alpha|-|\beta|)/2}(P_0)^{(\alpha)}_{(\beta)} u\|^{(\tau)}_T \leq C_2(\sum_{k=1}^{2} T^{-1/2} \|E_{1/2}(L_{01}\circ L_{0k})(x, D)\, u\|^{(\tau)}_T$$
$$+ T^{1/2} \sum_{k=1}^{2} T^{-1} \|E_1 L_{0k}(x, D)\, u\|^{(\tau)}_T + T T^{-3/2} \|E_{3/2}\, u\|^{(\tau)}_T) \tag{3.25}$$
$$\leq C_2 \max(1, T)\, B(u).$$

Here $C_2 = C_1(2n + \binom{2n+1}{2})$. Thus from (3.24) and (3.25) we have that for any

τ, T, and $u \in \mathcal{S}(\boldsymbol{R}^n)$

$$T^{-1/2} A_1(\Psi_1 u) \leq C_2 \max(1, T)\, B(\Psi_1 u) + T^{-1/2} \sum_{1 \leq |\alpha|+|\beta| \leq 2} \|E_{(|\alpha|-|\beta|)/2}\, g_{(\beta)}^{(\alpha)}(x, D')\, \Psi_1 u\|_T^{(\tau)}. \tag{3.26}$$

From Lemma 3.11 there exists a positive constant C_3 such that for any τ, T, and $u \in \mathcal{S}(\boldsymbol{R}^n)$

$$\sum_{|\alpha|+|\beta| \leq 2} \|E_{(|\alpha|-|\beta|)/2}\, g_{(\beta)}^{(\alpha)}(x, D')\, \Psi_1 u\|_T^{(\tau)} \leq C_3 \|E_{3/2} u\|_T^{(\tau)}. \tag{3.27}$$

From (3.26) and (3.27) we get that for any τ, T, and $u \in \mathcal{S}(\boldsymbol{R}^n)$

$$T^{-1/2} A_1(\Psi_1 u) \leq C_2 \max(1, T)\, B(\Psi_1 u) + C_3 T^{-1/2} \|E_{3/2} u\|_T^{(\tau)}. \tag{3.28}$$

Since $c(x, \xi') = (L_{01} - L_{02})(x, \xi)$,

$$|||c(x, D') u|||_{T,2}^{(\tau)} \leq \sum_{k=1}^{2} |||L_{0k}(x, D) u|||_{T,2}^{(\tau)}.$$

Thus

$$B(u) \leq |||u|||_{T,6}^{(\tau)} + \sum_{i \neq j} \tau^{-1} |||(L_{0i} \circ L_{0j})(x, D) u|||_{T,2}^{(\tau)} + \tau^{-1} |||L_{01}(x, D) u|||_{T,2}^{(\tau)}$$
$$+ 2 \sum_{k=1}^{2} \tau^{-1/2} |||L_{0k}(x, D) u|||_{T,4}^{(\tau)}.$$

Using Proposition 3.2 and Lemma 3.2 we see that there exist positive constants τ_1, T_1, C_4 such that when $\tau T^2 > \tau_1$ and $T < T_1$,

$$B(u) \leq C_4 \|P_0 u\|_T^{(\tau)}, \quad u \in \mathcal{S}_T(\boldsymbol{R}^n). \tag{3.29}$$

From Lemma 3.3 there exists a positive constant C_5 such that for any τ, T, and $u \in \mathcal{S}(\boldsymbol{R}^n)$

$$\|P \Psi_1 u\|_T^{(\tau)} \leq C_5(\|Pu\|_T^{(\tau)} + A_1(u) + R(u)). \tag{3.30}$$

From (3.27), (3.29), and (3.30) we get that when $\tau T^2 > \tau_1$ and $T < T_1$, for $u \in \mathcal{S}_T(\boldsymbol{R}^n)$

$$\begin{aligned} B(\Psi_1 u) &\leq C_4(\|P \Psi_1 u\|_T^{(\tau)} + \|g(x, D')\, \Psi_1 u\|_T^{(\tau)}) \\ &\leq C_4\{C_5(\|Pu\|_T^{(\tau)} + A_1(u) + R(u)) + C_3 \|E_{3/2} u\|_T^{(\tau)}\} \\ &\leq C_4(C_3 + C_5)(\|Pu\|_T^{(\tau)} + A_1(u) + R(u)). \end{aligned} \tag{3.31}$$

Combining (3.28) and (3.31) we get that when $\tau T^2 > \tau_1$ and $T < T_1$, for $u \in {}_T\mathcal{S}(\boldsymbol{R}^n)$

$$\begin{aligned} T^{-1/2} A_1(\Psi_1 u) + B(\Psi_1 u) &\leq (C_2 \max(1, T) + 1)\, B(\Psi_1 u) + C_3 T^{-1/2} \|E_{3/2} u\|_T^{(\tau)} \\ &\leq C_6(\|Pu\|_T^{(\tau)} + A_1(u) + R(u)) + C_3 T^{-1/2} R(u) \end{aligned}$$

with $C_6=(C_2 \max(1, T)+1)\, C_4(C_3+C_5)$. This completes the proof of (2).

§4. Proof of Proposition 1.1

In this section we deduce Proposition 1.1 from Lemma 2.3 and Proposition 3.1. We define $p_l(x, \xi)$ from $P(x, \xi)$ by (2.11) in the same manner as in the beginning of section 2. We set

$$L_{01}^{(l)}(x, \xi) = \xi_1-\lambda_l(x, \xi'),$$
$$L_{02}^{(l)}(x, \xi) = \xi_1-\lambda_l(x, \xi')-c_l(x, \xi')$$

with the notations in (2.7) and (2.10). Then we define $A_i^{(l)}(u)$ for $i=1, 2$, $l=1, 2$, $u\in\mathcal{S}(\boldsymbol{R}^n)$, $\tau>1$, $T>0$ by

$$A_1^{(l)}(u) = \sum_{1\leq|\alpha|+|\beta|\leq 2} ||E_{(|\alpha|-|\beta|)/2}(p_l)_{(\beta)}^{(\alpha)}(x, D)\, u||_T^{(\tau)},$$

$$A_2^{(l)}(u) = T^{-1}\Big(\sum_{k=1}^{2} ||E_1\, L_{0k}^{(l)}(x, D)\, u||_T^{(\tau)}+||E_1\, c(x, D')\, u||_T^{(\tau)}\Big),$$

and we use the notations $A_1(u)$ and $R(u)$ in Proposition 3.1.

We use a family of C^∞-functions $\{\psi_{jk}\}_{k\in I}(j=0, 1)$ on $\boldsymbol{R}^n\times\boldsymbol{R}^{n-1}$ in Lemma 2.3 and we set

$$\Psi_k = \psi_{1k}(x, D').$$

Since for any $k\in I$ one can choose $l\in\{1, 2\}$ so that one of the conditions (I), (II), (III) in §3 holds with $\Phi=\Phi_l$, $\varphi=\varphi_l$, $\psi_i=\psi_{ik}$, $p=p_l$, $\lambda=\lambda_l$, $c=c_l$, $g=g_l$, it follows from Proposition 3.1 that for any $k\in I$ there exist $l(k)\in\{1, 2\}$ and positive constants $\tau^{(k)}$, $T^{(k)}$, $C^{(k)}$ such that the following condition holds: when $l=l(k)$ and $\tau T^2>\tau^{(k)}$ and $T<T^{(k)}$,

$$\begin{aligned}&T^{-1/2}\, A_1^{(l)}(\Psi_k\, u)+A_2^{(l)}(\Psi_k\, u)+|||\Psi_k\, u|||_{T,6}^{(\tau)}\\&\quad\leq C^{(k)}(||p_l(x, D)\, u||_T^{(\tau)}+A_1^{(l)}(u)+T^{1/2}\, A_2^{(l)}(u)+T^{-1}\, R(u)),\ u\in\mathcal{S}_T(\boldsymbol{R}^n).\end{aligned} \tag{4.1}$$

Since $L_{01}^{(1)}(x, \xi)-L_{01}^{(2)}(x, \xi)=-\dfrac{2}{3}\, c(x, \xi')$ and $L_{02}^{(1)}(x, \xi)-L_{02}^{(2)}(x, \xi)=\dfrac{4}{3}\, c(x, \xi')$, we have for any l, τ, T that

$$\sum_{m=1}^{2}\sum_{k=1}^{2} ||E_1\, L_{0k}^{(m)}(x, D)\, u||_T^{(\tau)}\leq 3TA_2^{(l)}(u), \quad u\in\mathcal{S}(\boldsymbol{R}^n).$$

This inequality and (4.1) imply that when $l=l(k)$ and $\tau>\tau^{(k)}$ and $T<T^{(k)}$,

$$T^{-1}\sum_{m=1}^{2}\sum_{s=1}^{2} ||E_1\, L_{0s}^{(m)}(x, D)\, \Psi_k\, u||_T^{(\tau)}$$

$$\leq 2C^{(k)}(||p_l(x, D)\, u||_T^{(\tau)} + A_1^{(l)}(u) + T^{1/2} A_2^{(l)}(u) + T^{-1} R(u)) \tag{4.2}$$
for $\quad u \in \mathcal{S}_T(\boldsymbol{R}^n)$.

To estimate $A_1(u)$ we need

Lemma 4.1. *There exists a positive constant C such that for any l, τ, T*

$$\sum_{1 \leq |\alpha| + |\beta| \leq 2} ||E_{(|\alpha| - |\beta|)/2}(P_{(\beta)}^{(\alpha)} - (p_l)_{(\beta)}^{(\alpha)})\,(x, D)\, u||_T^{(\tau)} \leq CR(u)\,, \quad u \in \mathcal{S}(\boldsymbol{R}^n)\,. \tag{4.3}$$

Proof. Using the equality (2.12) it can be easily checked that when $|\alpha| + |\beta| = 1$ or 2,

$$\langle \xi' \rangle^{(|\alpha| - |\beta|)/2}(P_{(\beta)}^{(\alpha)} - (p_l)_{(\beta)}^{(\alpha)})\,(x, \xi) = \sum_{k=0}^{2-\alpha_1} a_k(x, \xi')\, \xi_1{}^k$$

with some $a_k \in S_{1,0}^{(3/2)-k}(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$. This implies the lemma.

From Lemma 4.1 there exists a positive constant M_1 such that (4.3) with $C = M_1$ holds for any l, τ, T. From Lemma 3.4 there exists a positive constant M_2 such that we have for any $k \in I$ that

$$R(\Psi_k\, u) \leq M_2\, R(u)\,, \quad u \in \mathcal{S}(\boldsymbol{R}^n)\,. \tag{4.4}$$

Then we have for any $k \in I$ that when $l = l(k)$ and $\tau T^2 > \tau^{(k)}$ and $T < T^{(k)}$,

$$\begin{aligned} T^{-1/2} A_1(\Psi_k\, u) &\leq T^{-1/2} A_1^{(l)}(\Psi_k\, u) + T^{-1/2} M_1\, R(\Psi_k\, u) \\ &\leq C^{(k)}(||p_l(x, D)\, u||_T^{(\tau)} + A_1^{(l)}(u) + T^{1/2} A_2^{(l)}(u)) \\ &\quad + (C^{(k)} + M_1\, M_2\, T^{1/2})\, T^{-1}\, R(u) \end{aligned} \tag{4.5}$$
for $\quad u \in \mathcal{S}_T(\boldsymbol{R}^n)$.

To estimate $||(p_l - P)\,(x, D)\, u||_T^{(\tau)}$ we need

Lemma 4.2. *There exist positive constants τ_0, T_0, C_0 such that when $\tau T^2 > \tau_0$ and $T < T_0$, for any l and $u \in \mathcal{S}_T(\boldsymbol{R}^n)$*

$$\begin{aligned} ||(p_l - P)\,(x, D)\, u||_T^{(\tau)} \leq C_0(\tau^{-1/2}\, ||P(x, D)\, u||_T^{(\tau)} + R(u) + TA_2^{(l)}(u) \\ + T^2\, |||u|||_{T,4}^{(\tau)})\,. \end{aligned}$$

Proof. We recall (2.12). It is easy to see that

$$\begin{aligned} &[\sum_{j=1}^{2} b_{lj}(x, \xi')\, (\xi_1 - \lambda_l(x, \xi'))^j]\,(x, D) \\ &\quad = \sum_{j=1}^{2} b_{lj}(x, D)\, L_{01}^{(l)}(x, D)^j + r_{01}(x, D')\, D_1 + r_{l2}(x, D') \end{aligned}$$

with some $r_{l1} \in S_{1,0}^0(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$, $r_{l2} \in S_{1,0}^1(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$.

Thus there exists a positive constant C_1 such that for any τ, T, and $u\in\mathcal{S}(\boldsymbol{R}^n)$

$$\|(p_l-P)(x, D)u\|_T^{(\tau)}\le C_1\Big(\sum_{j=1}^{2}\|E_{2-j}L_{01}^{(l)}(x, D)^j u\|_T^{(\tau)}+\|D_1 u\|_T^{(\tau)}+\|E_1 u\|_T^{(\tau)}\Big). \tag{4.6}$$

Using Parseval's formula we can easily see that

$$\|E_s u\|_T^{(\tau)}\le\|E_{s'} u\|_T^{(\tau)} \quad \text{if} \quad s'\ge s. \tag{4.7}$$

So from (4.6) we have for any τ, T, and $u\in\mathcal{S}(\boldsymbol{R}^n)$ that

$$\begin{aligned}\|(p_l-P)(x, D)u\|_T^{(\tau)}&\le C_1(\|L_{01}^{(l)}(x, D)^2 u\|_T^{(\tau)}+\|E_1 L_{01}^{(l)}(x, D)u\|_T^{(\tau)}+\|E_{1/2} D_1 u\|_T^{(\tau)}\\&\qquad+\|E_{3/2} u\|_T^{(\tau)}) \qquad (4.8)\\&\le C_1(\|L_{01}^{(l)}(x, D)^2 u\|_T^{(\tau)}+TA_2^{(l)}(u)+R(u)).\end{aligned}$$

We have that

$$\begin{aligned}L_{01}^{(2)}(x, D)^2 &= L_{01}^{(1)}(x, D)^2+\frac{4}{3}c(x, D')L_{01}^{(1)}(x, D)+\frac{2}{3}(D_{x_1}c)(x, D')\\&\quad-\frac{2}{3}[\lambda(x, D'), c(x, D')]+\frac{4}{9}c(x, D')^2.\end{aligned}$$

Thus there exists a positive constant C_2 such that for any τ, T, and $u\in\mathcal{S}(\boldsymbol{R}^n)$

$$\begin{aligned}\|L_{01}^{(2)}(x, D)^2 u\|_T^{(\tau)}&\le\|L_{01}^{(1)}(x, D)^2 u\|_T^{(\tau)}+C_2(\|E_1 L_{01}^{(1)}(x, D)u\|_T^{(\tau)}\\&\quad+\|E_1 c(x, D')u\|_T^{(\tau)}+\|E_1 u\|_T^{(\tau)}).\end{aligned}$$

Using (4.7) we see from this inequality that for any τ, T, and $u\in\mathcal{S}(\boldsymbol{R}^n)$

$$\|L_{01}^{(2)}(x, D)^2 u\|_T^{(\tau)}\le\|L_{01}^{(1)}(x, D)^2 u\|_T^{(\tau)}+C_2(TA_2^{(l)}(u)+R(u)).$$

Combining this inequality and (4.8) we obtain that for any τ, T, and $u\in\mathcal{S}(\boldsymbol{R}^n)$

$$\|(p_l-P)(x, D)u\|_T^{(\tau)}\le C_1\|L_{01}^{(1)}(x, D)^2 u\|_T^{(\tau)}+C_1(C_2+1)(TA_2^{(l)}(u)+R(u)). \tag{4.9}$$

Note that $p(x, \xi)=L_{01}^{(1)}(x, \xi)^2 L_{01}^{(2)}(x, \xi)$. Thus using Proposition 3.1 and Lemma 3.2 we see that there exist positive constants τ_1, T_1, C_3 such that when $\tau T^2>\tau_1$ and $T<T_1$,

$$\|L_{01}^{(1)}(x, D)^2 u\|_T^{(\tau)}\le C_3\tau^{-1/2}\|p(x, D)u\|_T^{(\tau)}, \quad u\in\mathcal{S}_T(\boldsymbol{R}^n). \tag{4.10}$$

There exists a positive constant C_4 such that for any τ, T, and $u\in\mathcal{S}(\boldsymbol{R}^n)$

$$\|q(x, D)u\|_T^{(\tau)}\le C_4\sum_{k=0}^{2}\|E_{2-k}D_1{}^k u\|_T^{(\tau)}.$$

Since $P=p+q$, combining this inequality and (4.10) we obtain that when

$\tau T^2 > \tau_1$ and $T < T_1$,

$$||L_{01}^{(1)}(x, D)^2 u||_T^{(\tau)} \le C_3 \tau^{-1/2} ||P(x, D) u||_T^{(\tau)} + C_3 C_4 T^2 |||u|||_{T,4}^{(\tau)}, \ u \in \mathcal{S}_T(\boldsymbol{R}^n). \quad (4.11)$$

(4.9) and (4.11) imply the lemma.

From Lemma 4.2 there exist positive constants $\tau_1 > \max_{k\in I} \tau^{(k)}$, $T_1 < \min_{k\in I} T^{(k)}$, and M_3 such that when $\tau T^2 > \tau_1$ and $T < T_1$, for any l and $u \in \mathcal{S}_T(\boldsymbol{R}^n)$

$$||p_l(x, D) u||_T^{(\tau)} \le M_3(||P(x, D) u||_T^{(\tau)} + R(u) + T A_2^{(l)}(u) + T^2 |||u|||_{T,4}^{(\tau)}). \quad (4.12)$$

Combining (4.1), (4.2), (4.5), and (4.12) we see that there exists a positive constant M_4 such that when $\tau T^2 > \tau_1$ and $T < T_1$, for any l and $u \in \mathcal{S}_T(\boldsymbol{R}^n)$

$$\begin{aligned} &T^{-1/2} A_1(\Psi_k u) + \sum_{m=1}^{2} A_2^{(m)}(\Psi_k u) + |||\Psi_k u|||_{T,6}^{(\tau)} \\ &\le M_4(||P(x, D) u||_T^{(\tau)} + A_1^{(l(k))}(u) + T^{1/2} A_2^{(l(k))}(u) + T^{-1} R(u) \\ &\quad + T^2 |||u|||_{T,4}^{(\tau)}). \end{aligned} \quad (4.13)$$

Since $\sum_{k\in I} \Psi_{1k} u = u$ for $u \in \mathcal{S}(\boldsymbol{R}^n)$, we have that

$$\begin{aligned} &T^{-1/2} A_1(u) + \sum_{m=1}^{2} A_2^{(m)}(u) + |||u|||_{T,6}^{(\tau)} \\ &\le \sum_{k\in I} (T^{-1/2} A_1(\Psi_k u) + \sum_{m=1}^{2} A_2^{(m)}(\Psi_k u) + |||\Psi_k u|||_{T,6}^{(\tau)}), \quad u \in \mathcal{S}(\boldsymbol{R}^n). \end{aligned}$$

(4.13) and this inequality imply that when $\tau T^2 > \tau_1$ and $T < T_1$, for $u \in \mathcal{S}_T(\boldsymbol{R}^n)$

$$\begin{aligned} &T^{-1/2} A_1(u) + \sum_{m=1}^{2} A_2^{(m)}(u) + |||u|||_{T,6}^{(\tau)} \\ &\le M_4(\#(I) ||P(x, D) u||_T^{(\tau)} + \sum_{k\in I} A_1^{(l(k))}(u) + T^{1/2} \sum_{k\in I} A_2^{(l(k))}(u) + \#(I) T^{-1} R(u) \\ &\quad + \#(I) T^2 |||u|||_{T,4}^{(\tau)}). \end{aligned}$$

Since $A_1^{(l)}(u) \le A_1(u) + M_1 R(u)$ for any l and $u \in \mathcal{S}(\boldsymbol{R}^n)$, the above inequality implies that when $\tau T^2 > \tau_1$ and $T < T_1$, for $u \in \mathcal{S}_T(\boldsymbol{R}^n)$

$$\begin{aligned} &T^{-1/2} A_1(u) + \sum_{m=1}^{2} A_2^{(m)}(u) + |||u|||_{T,6}^{(\tau)} \\ &\le M_4 \#(I) \{||P(x, D) u||_T^{(\tau)} + A_1(u) + T^{1/2} \sum_{m=1}^{2} A_2^{(m)}(u) + (M_1 T + 1) T^{-1} R(u) \\ &\quad + T^2 |||u|||_{T,4}^{(\tau)}\}. \end{aligned} \quad (4.14)$$

To complete the proof of Proposition 1.1 we need

Lemma 4.3. *There exists a positive constant C such that we have for any τ, T that*

$$R(u)\leq\frac{2}{\tau T}\,||P(x, D)\,u||_T^{(\tau)}+CT^{3/2}\,|||u|||_{T,5}^{(\tau)}$$

for $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$.

Proof. An integration by parts gives that for any τ, T

$$||D_1\,u||_T^{(\tau)}\geq\frac{\tau T}{2}\,||u||_T^{(\tau)}\,,\quad u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)\,.$$

Substituting $E_{-1/2}\,D_1^2\,u$ into u yields that

$$||D_1^3\,E_{-1/2}\,u||_T^{(\tau)}\geq\frac{\tau T}{2}\,||E_{-1/2}\,D_1^2\,u||_T^{(\tau)}\,,\quad u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)\,. \tag{4.15}$$

We have that $P(x, D)=D_1^3+\sum_{i=0}^2 a_i(x, D')\,D_1^i$ with some $a_i\in S_{1,0}^{3-i}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$, then

$$\begin{aligned}D_1^3\,E_{-1/2}\,u &= E_{-1/2}\,D_1^3\,u\\ &= E_{-1/2}\,P(x, D)\,u-\sum_{i=0}^{2}E_{-1/2}\,a_i(x, D')\,D_1^i\,u \quad \text{for} \quad u\in\mathcal{S}(\boldsymbol{R}^n)\,.\end{aligned} \tag{4.16}$$

Using the fact that $\langle\xi'\rangle^{-1/2}\circ a_i\in S_{1,0}^{(5/2)-i}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$ and applying (4.7) to the first term in (4.16) we see that there exists a positive constant C such that for any τ, T, and $u\in\mathcal{S}(\boldsymbol{R}^n)$

$$||D_1^3\,E_{-1/2}\,u||_T^{(\tau)}\leq||P(x, D)||_T^{(\tau)}+C\sum_{i=0}^{2}||E_{(5/2)-i}\,D_1^i\,u||_T^{(\tau)}\,. \tag{4.17}$$

(4.15) and (4.16) imply that for any τ, T, and $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$

$$||E_{-1/2}\,D_1^2\,u||_T^{(\tau)}\leq\frac{2}{\tau T}\,||P(x, D)\,u||_T^{(\tau)}+2CT^{3/2}\,|||u|||_{T,5}^{(\tau)}\,.$$

Therefore, for any τ, T, and $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$

$$\begin{aligned}R(u) &= ||E_{-1/2}\,D_1^2\,u||_T^{(\tau)}+\sum_{i=1}^{2}||E_{-1/2+i}\,D_1^{2-i}\,u||_T^{(\tau)}\\ &\leq(\frac{2}{\tau T}\,||P(x, D)\,u||_T^{(\tau)}+2CT^{3/2}\,|||u|||_{T,5}^{(\tau)})+T^{3/2}\,|||u|||_{T,3}^{(\tau)}\\ &\leq\frac{2}{\tau T}\,||P(x, D)\,u||_T^{(\tau)}+(2C+1)\,T^{3/2}\,|||u|||_{T,5}^{(\tau)}\,.\end{aligned}$$

This completes the proof.

We take a positive number M_2 such that the inequality in Lemma 4.3

holds with $C=M_2$. Then (4.14) implies that when $\tau T^2>\tau_1$ and $T<T_1$, for $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$

$$\begin{aligned}
&T^{-1/2}A_1(u)+\sum_{m=1}^{2}A_2^{(m)}(u)+|||u|||_{T,6}^{(\tau)}\\
&\quad\le M_4\,\sharp(I)\,[\{1+(M_1T+1)\frac{1}{\tau T^2}\}\,||P(x,D)\,u||_T^{(\tau)}+A_1(u)+T^{1/2}\sum_{m=1}^{2}A_2^{(m)}(u)\\
&\qquad+(M_1T+1)\,T^{1/2}\,|||u|||_{T,5}^{(\tau)}+T^2\,|||u|||_{T,4}^{(\tau)}]\\
&\quad\le M_4\,\sharp(I)\,(1+\frac{M_1T_1+1}{\tau_1})\,||P(x,D)||_T^{(\tau)}+M_4\,\sharp(I)\{A_1(u)+T^{1/2}\sum_{m=1}^{2}A_2^{(m)}(u)\\
&\qquad+(M_1T_1+1)\,T^{1/2}\,|||u|||_{T,4}^{(\tau)}+T^2\,|||u|||_{T,5}^{(\tau)}\}\,.
\end{aligned}$$

The second term on the right hand side can be absorbed into the left hand side by decreasing T. Therefore, we have proved Proposition 1.1.

§5. Pseudodifferential Operators

In this section we collect the facts on the pseudodifferential operators which we use in this paper. In this paper we use the classes of symbols $S_{\Phi,\varphi}^{M,m}$ with (Φ,φ) stated after Lemma 2.2 which contain $S_{\rho,1-\rho}^{d}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$, $\frac{1}{2}\le\rho\le 1$.

Definition 5.1. *Let* (Φ,φ) *be a pair of weight functions satisfying* (2.31)~(2.23). *And let* $a\in S_{\Phi,\varphi}^{M,m}$. *We define an operator* $a(x,D')$ *on* $\mathcal{S}(\boldsymbol{R}^n)$ *by the standard formula*

$$a(x,D')u=(2\pi)^{-(n-1)}\int e^{ix'\cdot\xi'}a(x,\xi')\hat{u}(x_1,\xi')d\xi'$$

where $\hat{u}(x_1,\xi')$ denotes the partial Fourier transform of u in x'.

$a(x,D')$ transforms $\mathcal{S}(\boldsymbol{R}^n)$ into $\mathcal{S}(\boldsymbol{R}^n)$ and $\mathcal{S}_T(\boldsymbol{R}^n)$ into $\mathcal{S}_T(\boldsymbol{R}^n)$. If (Φ,φ) is a pair of weight functions, pairs of functions on $\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}$, $(\Phi(x_1,\cdot),\varphi(x_1,\cdot))\,(x_1\in\boldsymbol{R})$ satisfy uniformly the conditions for weight functions of Beals-Feefferman's class. This follows from (2.31)~(2.33) and the following lemma.

Lemma 5.1. *There exist positive constants* M,δ *satisfying the following condition.*

$$M^{-1}\le\Phi_t(x',\xi')/\Phi_t(y',\eta')\le M,\quad M^{-1}\le\varphi_t(x',\xi')/\varphi_t(y',\eta')\le M$$

for any $(y',\eta')\in U_t(x',\xi')$ *and* $t\in\boldsymbol{R}$ *where*

$$U_t(x,\xi')=\{(y',\eta')\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1};|x'-y'|<\delta\varphi_t((x',\xi'),|\xi'-\eta'|<\delta\Phi_t(x',\xi')\},$$
$$\Phi_t(x',\xi')=\Phi((t,x'),\xi'),\ \varphi_t(x',\xi')=\varphi((t,x'),\xi')\,.$$

Proof. Using Taylor's formula and (2.31), (2.32) we see that there exists a positive constant C_0 such that when $C\delta<\frac{1}{2}$ with the constant C in (2.31).

$$|\Phi_t(y', \eta')-\Phi_t(x', \xi')| \leq C_0\delta\Phi_t(x', \xi'),$$
$$|\varphi_t(y', \eta')-\varphi_t(x', \xi')| \leq C_0\delta\varphi_t(x', \xi')$$

for any $(y', \eta') \in U_t(x', \xi')$. This implies the lemma.

Remark 5.1. *We define semi-norms* $|\cdot|_N^{M,m}$ *in* $S_{\Phi,\varphi}^{M,m}$ *for* $N\in\boldsymbol{Z}_+$ *by*

$$a\mapsto|a|_N^{M,m} = \max_{|\alpha|+|\beta|\leq N} \sup_{(x,\xi')\in\boldsymbol{R}^n\times\boldsymbol{R}^{n-1}}[|a_{(\alpha)}^{(\beta)}(x, \xi')|(\Phi^{-M+|\beta|}\varphi^{-m+|\alpha|})(x, \xi')].$$

Then $S_{\Phi,\varphi}^{M,m}$ *becomes a Frechet space by the topology defined by these semi-norms.*

Lemma 5.2. *Let* $a\in S_{\Phi,\varphi}^{M,m}$ *and* $b\in S_{\Phi,\varphi}^{M',m'}$. *Then if we define* $a\circ b\in C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$ *by the formula*

$$(a\circ b)(x, \xi') = (2\pi)^{-(n-1)}OS-\iint e^{-i(y'-x')\cdot(\eta'-\xi')}a(x, \eta')b((x_1, y'), \xi')dy'\,d\eta', \tag{5.1}$$

we have that $a\circ b\in S_{\Phi,\varphi}^{M+M',m+m'}$ *and* $(a\circ b)(x, D') = a(x, D')b(x, D')$. *Moreover, we have an asymptotic expansion that for any* $N\in\boldsymbol{N}$

$$(a\circ b)(x, \xi') = \sum_{|\alpha|<N}\frac{1}{\alpha!}(\partial_{\xi'}^{\alpha}aD_{x'}^{\alpha}b)(x, \xi')+r_N[a, b](x, \xi') \tag{5.2}$$

with

$$r_N[a, b](x, \xi') = \int_0^1 r_{N\theta}[a, b](x, \xi')(1-\theta)^{N-1}d\theta, \tag{5.3}$$

$$r_{N\theta}[a, b](x, \xi') = N\sum_{|\alpha|=N}\frac{1}{\alpha!}(2\pi)^{-(n-1)}\iint e^{-i(y'-x')\cdot(\eta'-\xi')}(\partial_{\xi'}^{\alpha}a)(x, \xi'+\theta(\eta'-\xi'))\times D_{y'}^{\alpha}b((x_1, y'), \xi')dy'\,d\eta', \tag{5.4}$$

$$\{r_{N\theta}[a, b]\}_{\theta\in[0,1]} \text{ is a bounded set in } S_{\Phi,\varphi}^{M+M'-N,m+m'-N}. \tag{5.5}$$

Proof. When $a, b\in C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$, it is not hard to check (5.2) with the notations (5.1), (5.3), (5.4). Now let us consider an oscillatory integral

$$h_\theta[a, b](x, \xi') = (2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}a(x, \xi'+\theta\eta')b((x_1, y'+x'), \xi')dy'\,d\eta'$$

where $a\in S_{\Phi,\varphi}^{M,m}$, $b\in S_{\Phi,\varphi}^{M',m'}$, $\theta\in[0, 1]$.

Claim. (1) $h_\theta[a, b]\in S_{\Phi,\varphi}^{M+M',m+m'}$ for all θ and for any $L\in\boldsymbol{N}$ there exist positive constant C and $P\in\boldsymbol{N}$ depending only on $L, M, m, M', m', \Phi, \varphi$ such that

$$\sup_{\theta\in[0,1]} |\partial_x^\alpha \partial_{\xi'}^\beta h_\theta[a, b](x, \xi')| \leq |a|_P^{M,m} |b|_P^{M',m'}(\Phi^{M+M'-|\beta|}\varphi^{m+m'-|\alpha|})(x, \xi')$$

for any α, β with $|\alpha|+|\beta|\leq L$.

(2) Let $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ be bounded sets respectively in $S_{\Phi,\varphi}^{M,m}$ and $S_{\Phi,\varphi}^{M',m'}$ such that there exist $a\in S_{\Phi,\varphi}^{M,m}$ and $b\in S_{\Phi,\varphi}^{M',m'}$ such that $a_k\to a(k\to\infty)$ and $b_k\to b(k\to\infty)$ in $C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$. Then $h_\theta[a_k, b_k](x, \xi')\to h_\theta[a, b](x, \xi')(k\to\infty)$ for any θ and (x, ξ'), and $\{h_\theta[a_k, b_k]\}_{k\in N,\theta\in[0,1]}$ is bounded in $S_{\Phi,\varphi}^{M+M',m+m'}$.

Proof of claim. Set $f_\theta[a, b](x, y', \xi', \eta')=a(x, \xi'+\theta\eta')\, b((x_1, y'+x'), \xi')$. Then

$$\begin{aligned}|\partial_{y'}^\alpha \partial_{\eta'}^\beta f_\theta[a, b](x, y', \xi', \eta')| &\leq |a|_{|\beta|}^{M,m} |b|_{|\alpha|}^{M',m'}(\Phi^{M-|\beta|}\varphi^m)(x, \xi'+\theta\eta') \\ &\quad\times(\Phi^{M'}\varphi^{m'-|\alpha|})((x_1, y'+x'), \xi') \\ &\leq C_1|a|_{|\beta|}^{M,m}|b|_{|\alpha|}^{M',m'}\langle\xi'\rangle^{|M|+(|m|)/2+|M'|+(|m'|)/2+(|\alpha|)/2} \\ &\quad\times\langle\eta'\rangle^{|M|+(|m|)/2}\end{aligned} \tag{5.6}$$

with C_1 depending only on $M, m, M', m', \alpha, \beta, \Phi, \varphi$. From this inequality Leibniz rule shows that if $L, N\in\boldsymbol{N}$, one can find C_2 depending only on $L, N, M, m, M', m', \Phi, \varphi$ such that

$$\begin{aligned}&|\langle y'\rangle^{-2L}(1-\Delta_{\eta'})^L[\langle\eta'\rangle^{-2N}(1-\Delta_{y'})^N f_\theta[a, b](x, y', \xi', \eta')]| \\ &\leq C_2|a|_{2(N+L)}^{M,m}|b|_{2(N+L)}^{M',m'}\langle\xi'\rangle^{|M|+(|m|)/2+|M'|+(|m'|)/2+N+L} \\ &\qquad\times\langle\eta'\rangle^{-2N+|M|+(|m|)/2}\langle y'\rangle^{-2L}.\end{aligned} \tag{5.7}$$

It also follows from the estimate (5.6) that if $L,N\in N$ satisfy that

$$-2N+|M|+\frac{|m|}{2}<-(n-1), \quad -2L<-(n-1) \tag{5.8}$$

we have that

$$\begin{aligned}h_\theta[a, b](x, \xi') &= (2\pi)^{-(n-1)}\iint e^{-iy'\cdot\eta'}\langle y'\rangle^{-2L} \\ &\quad\times(1-\Delta_{\eta'})^L[\langle\eta'\rangle^{-2N}(1-\Delta_{y'})^N f_\theta[a, b](x, y', \xi', \eta')]dy'\,d\eta'.\end{aligned} \tag{5.9}$$

We shall show (2). From the estimates (5.7) with $a=a_k, b=b_k$, and L, N satisfying (5.8), and from the fact that $f_\theta[a_k, b_k](x, \cdot, \xi', \cdot)\to f_\theta[a, b](x, \cdot, \xi', \cdot)$ in $C^\infty(\boldsymbol{R}^{2(n-1)})(k\to\infty)$ for any fixed θ, (x, ξ') Lebesgue dominated convergence theorem shows the first assertion of (2), and the second one follows from (1).

To show (1) we use the following lemma which is Lemma 4.7 in [3].

Lemma 5.3. *Let Φ, φ be positive continuous functions on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ satisfying the following condition* (i)$\sim$(iv) *with some positive constants C, c, ε, C'*:

(i) $c \leq \Phi(x, \xi) \leq C(1+|\xi|)$, $C \geq \varphi(x, \xi) \geq c(1+|\xi|)^{\varepsilon-1}$;

(ii) $\Phi(x, \xi)\varphi(x, \xi) \geq c$;

(iii) *for any* $R>1$ *there exists* $M>1$ *such that* $R^{-1} \leq \frac{1+|\xi|}{1+|\eta|} \leq R$ *implies that* $M(R)^{-1} \leq \frac{\Phi(x, \xi)}{\varphi(x, \xi)}\left(\frac{\Phi(y, \eta)}{\varphi(y, \eta)}\right)^{-1} \leq M(R)$;

(iv) $C'^{-1} \leq \frac{\Phi(x, \xi)}{\Phi(y, \eta)} \leq C'$, $C'^{-1} \leq \frac{\varphi(x, \xi)}{\varphi(y, \eta)} \leq C'$ *whenever* $(y, \eta) \in U(x, \xi) = \{(y, \eta) \in \mathbf{R}^n \times \mathbf{R}^n;\ |y-x| < \varphi(x, \xi),\ |\eta-\xi| < \Phi(x, \xi)\}$.

Let $b(x, y, \xi, \eta)$ *be a* C^∞*-function in* (y, η) *for any fixed* (x, ξ) *satisfying the estimates*

$$|\partial_y^\alpha \partial_\eta^\beta b(x, y, \xi, \eta)| \leq C_{\alpha\beta} \sum_{\substack{r_j \in \mathbf{Z}_+ \\ r_1+\cdots+r_k=|\beta| \\ r_{k+1}+\cdots+r_{k+l}=|\alpha|}} \prod_{j=1}^{k} \sup_{Q \in K} (\Phi^{M_j-r_j}(Q)\varphi^{m_j}(Q)) \times \prod_{j=k+1}^{k+l} \sup_{Q \in K} (\Phi^{M_j}(Q)\varphi^{m_j-r_j}(Q))$$

where $K=$ *the covex hull of* $\{(x, \xi), (x, \eta), (y, \xi), (y, \eta)\}$, $M_j, m_j \in \mathbf{R}$, $k, l \in \mathbf{N}$. *Set*

$$a_{\Phi,\varphi}^{p,q}(x, y, \xi, \eta) = \sum_{\substack{r_j \in \mathbf{Z}_+ \\ r_1+\cdots+r_k=q \\ r_{k+1}+\cdots+r_{k+l}=p}} \prod_{j=1}^{k} \sup_{Q \in K} (\Phi^{M_j-r_j}(Q)\varphi^{m_j}(Q)) \times \prod_{j=k+1}^{k+l} \sup_{Q \in K} (\Phi^{M_j}(Q)\varphi^{m_j-r_j}(Q)),$$

$$a(x, \xi) = OS-\iint e^{-iy\cdot\eta} b(x, y+x, \xi, \eta+\xi)dyd\eta .$$

Define for $j \in \mathbf{Z}_+$

$$|b|_j^{\Phi,\varphi} = \max_{|\alpha|+|\beta| \leq j} \sup_{(x,y,\xi,\eta)} [|\partial_y^\alpha \partial_\eta^\beta b|(a_{\Phi,\varphi}^{|\alpha|,|\beta|})^{-1}](x, y, \xi, \eta) .$$

Then one can find $C_0>0$ *and* $L \in \mathbf{N}$ *depending only on* C, c, ε, C', $M(4)$, k, l, *and a permutation* $(M_1, \cdots, M_{k+l}, m_1, \cdots, m_{k+l})$ *of* $2(k+l)$ *real numbers such that*

$$|a(x, \xi)| \leq C_0 |b|_L^{\Phi,\varphi} (\Phi^{M_1+\cdots+M_{k+l}}\varphi^{m_1+\cdots+m_{k+l}})(x, \xi) .$$

For a proof of this lemma, see the appendix in [2]. This lemma will also be used in a later part of this paper.

Now we continue the proof of the claim. From (5.6)

$$|\partial_{y'}^{\alpha}\partial_{\eta'}^{\beta}[f_\theta[a, b]((t, x'), y'-x', \xi', \eta, -\xi')]| \leq |a|_{|\beta|}^{M,m}|b|_{|\alpha|}^{M',m'} \sup_{Q\in K}(\Phi_t^{M-|\beta|}\varphi_t^m)(Q) \times \sup_{Q\in K}(\Phi_t^{M'}\varphi_t^{m'-|\alpha|})(Q), \tag{5.10}$$

where $K=ch\{(x', \xi'), (x', \eta'), (y', \xi'), (y', \eta')\}$, and Φ_t, φ_t are as in Lemma 5.1. A remark just above Lemma 5.1 implies that if we take $n-1$, $\Phi_t(\cdot)$, $\varphi_t(\cdot)$ for n, Φ, φ in Lemma 5.3, the conditions (i)$\sim$(iv) in this lemma holds with uniform constants C, c, $\varepsilon=\frac{1}{2}$, C' in t. Thus taking $k=l=1$, $M_1=M$, $m_1=m$, $M_2=M'$, $m_2=m'$ in Lemma 5.3 we see from (5.10) that there exist constants $C_3>0$, $A\in\boldsymbol{N}$ depending only on Φ, φ, M, m, M', m' satisfying

$$|h_\theta[a, b](x, \xi')| \leq C_3|a|_A^{M,m}|b|_A^{M',m'}(\Phi^{M+M'}\varphi^{m+m'})(x, \xi'). \tag{5.11}$$

In view of the estimate (5.7) Lebesgue dominated covergence theorem shows that

$$\partial_{x_j}h_\theta[a, b] = h_\theta[\partial_{x_j}a, b]+h_\theta[a, \partial_{x_j}b],$$
$$\partial_{\xi_j}h_\theta[a, b] = h_\theta[\partial_{\xi_j}a, b]+h_\theta[a, \partial_{\xi_j}b].$$

Thus we see by induction that

$$\partial_x^{\alpha}\partial_{\xi'}^{\beta}h_\theta[a, b] = \sum_{\substack{\nu\leq\alpha\\ \mu\leq\beta}}\binom{\alpha}{\nu}\binom{\beta}{\mu}h_\theta[\partial_x^{\alpha-\nu}\partial_{\xi'}^{\beta-\mu}a, \partial_x^{\nu}\partial_{\xi'}^{\mu}b]. \tag{5.12}$$

From (5.11) and (5.12) there exist constants $C_4>0$ and $B\in\boldsymbol{N}$ depending only on Φ, φ, M, m, M', m', α, β satisfying

$$|\partial_x^{\alpha}\partial_{\xi'}^{\beta}h_\theta[a, b](x, \xi')| \leq C_4|a|_B^{M,m}|b|_B^{M',m'}(\Phi^{M+M'-|\beta|}\varphi^{m+m'-|\alpha|})(x, \xi')$$

Thus the assertion (1) has been proved.

Let us return to the proof of Lemma 5.2. $a\circ b\in S_{\Phi,\varphi}^{M+M',m+m'}$ follows from the fact that $a\circ b=h_\theta[a, b]$ when $\theta=1$ and the claim (1). We have that

$$r_{N\theta}[a, b] = N\sum_{|\alpha|=N}\frac{1}{\alpha!}h_\theta[\partial_{\xi'}^{\alpha}a, D_{x'}^{\alpha}b]. \tag{5.13}$$

Since $\partial_{\xi'}^{\alpha}a\in S_{\Phi,\varphi}^{M-N,m}$ and $D_{x'}^{\alpha}b\in S_{\Phi,\varphi}^{M',m'-N}$ on the right hand side, the claim (1) shows that the assertion (5.5) holds. Choose $\chi\in C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$ with $\chi(0, 0)=1$ and set $a_k=\chi_k a$, $b_k=\chi_k b$ with $\chi_k(x, \xi')=\chi\left(\frac{x}{k}, \frac{\xi'}{k}\right)$. Then $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ satisfy the conditions in the claim (2). This implies that for all $\alpha\in\boldsymbol{Z}_+$ and $\theta\in[0, 1]$

$$h_\theta[\partial_{\xi'}^{\alpha}a_k, D_{x'}^{\alpha}b_k](x, \xi')\to h_\theta[\partial_{\xi'}^{\alpha}a, D_{x'}^{\alpha}b](x, \xi') \quad \text{(pointwise)}$$

as $k\to\infty$ being bounded in θ for any fixed (x, ξ').

Since $h_\theta[a, b]=a\circ b$ for $\theta=1$, this implies that $\lim_{k\to\infty}(a_k\circ b_k)(x, \xi')=(a\circ b)(x, \xi')$ for any (x, ξ'). From (5.7) and (5.9) $h_\theta[a, b](x, \xi')$ is continuous function in θ for any fixed (x, ξ'). Thus from (5.13) and Lebesgue dominated convergence theorem, the above convergence also shows that $\lim_{k\to\infty}r_N[a_k, b_k](x, \xi')=r_N[a, b](x, \xi')$ for any (x, ξ'). Letting $k\to\infty$ in (5.2) with $a=a_k$, $b=b_k$ we see that (5.2) also holds for general a, b.

Finally we show that $(a\circ b)(x, D')u=a(x, D')b(x, D')u$ for all $u\in\mathcal{S}(\mathbb{R}^n)$. This is easily checked if $a, b\in C_0^\infty(\mathbb{R}^n\times\mathbb{R}^{n-1})$. Taking $\theta=1$, $a=a_k$, $b=b_k$, we see that $\{a_k\circ b_k\}_{k=1}^\infty$ is bounded in $S_{\Phi,\varphi}^{M+M', m+m'}$. Thus Lebesgue dominated convergence theorem shows that $(a_k\circ b_k)(x, D')u(x)\to(a\circ b)(x, D')u(x)$ pointwise as $k\to\infty$. Thus $(a_k\circ b_k)(x, D')u\to(a\circ b)(x, D')u$ in $\mathcal{S}(\mathbb{R}^n)$ since $\{(a_k\circ b_k)(x, D')u\}_{k=1}^\infty$ is bounded in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ is a Montel space. On the other hand $(a_k\circ b_k)(x, D')u-a(x, D')b(x, D')u=(a_k(x, D')-a(x, D'))b(x, D')u+a_k(x, D')(b_k(x, D')-b(x, D'))u$, and Lebesgue's theorem shows that $a_k(x, D')u\to a(x, D')u$ and $b_k(x, D')u\to b(x, D')u$ for all $u\in\mathcal{S}(\mathbb{R}^n)$. Thus on the right hand side of the above equality the first term converges to 0 in $\mathcal{S}(\mathbb{R}^n)$ and the second term does also because $\{a_k(x, D')\}_{k=1}^\infty$ is equicontinuous in the set of all cotinuous linear operators on $\mathcal{S}(\mathbb{R}^n)$ into itself. This completes the proof.

Lemma 5.4. *Let $a\in S_{\Phi,\varphi}^{M,m}$ and set*

$$a^\sharp(x, \xi') = (2\pi)^{-(n-1)}\, OS-\iint e^{-i(y'-x')\cdot(\eta'-\xi')}\overline{a((x_1, y'), \eta')}dy'd\eta'. \quad (5.14)$$

Then we have that

$$a^\sharp(x, \xi') = \sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_{\xi'}^\alpha D_{x'}^\alpha\bar{a}(x, \xi')+r_N[a](x, \xi') \quad (5.15)$$

where

$$r_N[a](x, \xi') = \int_0^1 r_{N\theta}[a](x, \xi')(1-\theta)^{N-1}d\theta\,, \quad (5.16)$$

$$r_{N\theta}[a](x, \xi') = N\sum_{|\alpha|=N}\frac{1}{\alpha!}(2\pi)^{-(n-1)}\, OS-\iint e^{-i(y'-x')\cdot(\eta'-\xi')}$$
$$(\partial_{\xi'}^\alpha D_{x'}^\alpha\bar{a})((x_1, y'), \xi'+\theta(\eta'-\xi'))dy'd\eta', \quad (5.17)$$

$$\{r_{N\theta}[a]\}_{\theta\in[0,1]} \text{ is bounded in } S_{\Phi,\varphi}^{M-N, m-N}\,. \quad (5.18)$$

Moreover we have that

$$(a((t, x'), D')u, v) = (u, a^\sharp((t, x'), D')v) \text{ for any } u, v\in\mathcal{S}(\mathbb{R}^n) \text{ and any fixed } t\in\mathbb{R} \quad (5.19)$$

where (,) *is the inner product of* $L^2(\boldsymbol{R}^{n-1}_{x'})$.

Proof. If $a\in C_0^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$, it is not difficult to check (5.15) using Taylor's formula and Fourier inversion formula. Now we set

$$h_\theta[a](x,\xi')=(2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}a((x_1,y'+x'),\xi'+\theta\eta')dy'd\eta'$$

for $a\in S^{M,m}_{\Phi,\varphi}$, $\theta\in[0,1]$.

Claim. (1) $h_\theta[a]\in S^{M,m}_{\Phi,\varphi}$ and for any $L\in\mathrm{N}$ there exist $C>0$ and $P\in\boldsymbol{N}$ depending only on Φ,φ,M,m,L such that

$$\sup_{\theta\in[0,1]}|\partial_x^\alpha\partial_{\xi'}^\beta h_\theta[a](x,\xi')|\le C|a|_P^{M,m}\Phi^{M-|\beta|}(x,\xi')\varphi^{m-|\alpha|}(x,\xi') \tag{5.20}$$

for all α,β with $|\alpha|+|\beta|\le L$.
(2) If $\{a_k\}_{k=1}^\infty$ be a bounded set in $S^{M,m}_{\Phi,\varphi}$ with $a_k\to a$ in $C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$, $\{h_\theta[a_k]\}_{\substack{k\in N\\ \theta\in[0,1]}}$ is bounded set in $S^{M,m}_{\Phi,\varphi}$ and $h_\theta[a_k](x,\xi')\to h_\theta[a](x,\xi')$ for any (x,ξ') and θ.

Proof of claim. We show (1) first. We set

$$f_\theta[a](x,y',\xi',\eta')=a((x_1,y'+x'),\xi'+\theta\eta').$$

Then we have that

$$\begin{aligned}|\partial_{y'}^\alpha\partial_{\eta'}^\beta f_\theta[a](x,y',\xi',\eta')|&\le|a|^{M,m}_{|\alpha|+|\beta|}\Phi^{M-|\beta|}((x_1,y'+x'),\xi'+\theta\eta')\\&\quad\times\varphi^{m-|\alpha|}((x_1,y'+x'),\xi'+\theta\eta')\\&\le C_1|a|^{M,m}_{|\alpha|+|\beta|}\langle\xi'\rangle^{|M|+(|m|)/2+(|\alpha|)/2}\langle\eta'\rangle^{|M|+(|m|)/2+(|\alpha|)/2},\end{aligned}\tag{5.21}$$

where C_1 depends only on $\Phi,\varphi,M,m,\alpha,\beta$. We also have that for any $L,N\in\boldsymbol{N}$

$$\begin{aligned}&|\langle y'\rangle^{-2L}(1-\Delta_{\eta'})^L|\langle\eta'\rangle^{-2N}(1-\Delta_{y'})^N f_\theta[a](x,y',\xi',\eta')]|\\&\le C_2|a|^{M,m}_{2(N+L)}\langle\xi'\rangle^{|M|+(|m|)/2+N}\langle\eta'\rangle^{|M|+(|m|)/2-N}\langle y'\rangle^{-2L}\end{aligned}\tag{5.22}$$

where C_2 depends only on Φ,φ,M,m,L,N. Thus, when $L,N\in\boldsymbol{N}$ satisfy

$$|M|+\frac{|m|}{2}-N<-(n-1),\quad -2L<-(n-1)$$

we have that

$$\begin{aligned}h_\theta[a](x,\xi')&=(2\pi)^{-(n-1)}\iint e^{-iy'\cdot\eta'}\langle y'\rangle^{-2L}\\&\quad\times(1-\Delta_{\eta'})^L[\langle\eta'\rangle^{-2N}(1-\Delta_{y'})^N f_\theta[a](x,y',\xi',\eta')]dy'd\eta'.\end{aligned}\tag{5.23}$$

From (5.21) and Lemma 5.3 we obtain the estimates (5.20) for $L=0$. Using the estimate (5.22) for first derivatives of a and Lebesgue dominated convergence theorem we see that $\partial_{x_j}h_\theta[a]=h_\theta[\partial_{x_j}a]$, $\partial_{\xi_j}h_\theta[a]=h_\theta[\partial_{\xi_j}a]$. Thus by induction we have that

$$\partial_x^\alpha\partial_{\xi'}^\beta h_\theta[a] = h_\theta[\partial_x^\alpha\partial_{\xi'}^\beta a] .$$

Thus from case that $L=0$ in (5.20) we also obtain the estimates (5.20) for all L.

Next we show (2). From the estimate (5.22) with a_k for a and (5.23) Lebesgue's theorem implies the second statement in (2) and the first one follows from (1). This completes the proof of the claim.

Now we return to the proof of the lemma. $a^\sharp\in S_{\Phi,\varphi}^{M,m}$ follows from the fact that $a^\sharp=h_\theta[\bar{a}]$ when $\theta=1$ and the claim (1). The boundedness of $\{r_{N\theta}\}_{\theta\in[0,1]}$ in $S_{\Phi,\varphi}^{M-N,m-N}$ follows from (5.24), because

$$r_{N\theta}[a] = N\sum_{|\alpha|=N}\frac{1}{\alpha!}h_\theta[\partial_{\xi'}^\alpha D_{x'}^\alpha\bar{a}] . \tag{5.24}$$

Note that $h_\theta[a](x,\xi')$ is a continuous function in θ for any fixed (x,ξ') from (5.23). Take $\{a_k\}_{k=1}^\infty\subset C_0^\infty(\mathbb{R}^n\times\mathbb{R}^{n-1})$ with the properties in the claim (2) as in the proof of Lemma 5.2. (5.24) and the claim (2) imply $\lim_{k\to\infty} r_N[a_k](x,\xi')$ $=r_N[a](x,\xi')$ pointwise from Lebesgue's theorem. Thus letting $k\to\infty$ in (5.15) with a_k for a we obtain (5.15) for a general $a\in S_{\Phi,\varphi}^{M,m}$. Finally we show (5.19). When $a\in C_0^\infty(\mathbb{R}^n\times\mathbb{R}^{n-1})$, this can be easily checked. Note that $\{a_k^\sharp\}_{k=1}^\infty$ is bounded in $S_{\Phi,\varphi}^{M,m}$ and $\lim_{k\to\infty} a_k^\sharp(x,\xi')=a^\sharp(x,\xi')$ pointwise from the claim (2). Thus noting Lebesgue's theorem and letting $k\to\infty$ in (5.19) with a_k for a we obtain (5.19) for general $a\in S_{\Phi,\varphi}^{M,m}$. This completes the proof.

Let $q(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ such that $q(x,\xi)=\sum_{j=0}^m a_j(x,\xi')\xi_1^j$ with $a_j\in\bigcup_{(P,p)\in\mathbb{R}^2}S_{\Phi,\varphi}^{P,p}$ and $a_m\not\equiv 0$. This expression is clearly unique. Then we define an operator $q(x,D)$ from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ by

$$q(x,D)u = \sum_{j=0}^m a_j(x,D')D_1^j u, \quad u\in\mathcal{S}(\mathbb{R}^n) .$$

$q(x,D)$ maps $\mathcal{S}_T(\mathbb{R}^n)$ into $\mathcal{S}_T(\mathbb{R}^n)$.

Lemma 5.5. *Let $q_i(x,D)=\sum_{j=0}^m a_{ij}(x,D')D_1^j$ $(i=1,2)$ be operators defined as above. Then*

$$(q_1\circ q_2)(x,D) = q_1(x,D)q_2(x,D) \tag{5.25}$$

with

$$(q_1\circ q_2)(x, D) = \sum_{j=0}^{m_1} \frac{(2\pi)^{-(n-1)}}{j!} OS-\iint e^{-iy'\cdot\eta'}(\partial_{\xi_1}^j q_1)(x, \xi+(0, \eta')) \times (D_{x_1}^j q_2)((x_1, y'+x'), \xi)dy'd\eta' \tag{5.26}$$

and for any $(N_0, \cdots, N_{m_1})\in \boldsymbol{N}^{m_1+1}$ *with* $N_j\leq 1$ *we have that*

$$(q_1\circ q_2)(x, \xi) = \sum_{j=0}^{m_1} \frac{1}{j!}\{\sum_{|\alpha|<N_j} \frac{1}{\alpha!}(\partial_{\xi_1}^j\partial_{\xi'}^\alpha q_1)(x, \xi)(D_{x_1}^j D_{x'}^\alpha q_2)(x, \xi)+r_j(x, \xi)\},$$

$$r_j(x, \xi) = \int_0^1 r_{j\theta}(x, \xi)(1-\theta)^{N_j-1}d\theta ,$$

$$r_{j\theta}(x, \xi) = N_j \sum_{|\alpha|=N_j} \frac{(2\pi)^{-(n-1)}}{\alpha!} OS-\iint e^{-iy'\cdot\eta'}(\partial_{\xi_1}^j\partial_{\xi'}^\alpha q_1)(x, \xi+\theta(0, \eta')) \times (D_{x_1}^j D_{x'}^\alpha q_2)((x_1, y'+x'), \xi)dy'd\eta'.$$

Proof. When $q_i(x, \xi)$ are monomials in ξ_1, the result follows from Lemma 5.2 and Leibniz rule. The general case follows from bilinearlity of (5.25), (5.26). The results in the remaining part of this section are used to prove Lemmas 3.7 and 3.10.

Lemma 5.6. *Let* $q\in C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ *be as before Lemma* 5.5, *and let* $a\in S_{\Phi,\varphi}^{M,m}$. *Let* $N\in\boldsymbol{N}$. *Then there exist sets of constants* $\{C_\beta\}_{\substack{\beta\in\boldsymbol{Z}_+^{n-1}\\|\beta|<N}}$ *and* $\{C_{\alpha l}\}_{\substack{(\alpha,l)\in\boldsymbol{Z}_+^{n-1}\times\boldsymbol{Z}_+\\|\alpha|=N, l\leq N-1}}$ *depending only on N such that*

$$(a\circ q)(x, \xi) = a(x, \xi')q(x, \xi)+\sum_{0<|\alpha|<N} C_\alpha(\partial_{\xi'}^\alpha a\circ D_{x'}^\alpha q)(x, \xi) + \sum_{|\alpha|=N}\int_0^1 q_\alpha(\theta, x, \xi)\sum_{l\leq N-1} C_{\alpha l}(1-\theta)^{N-1-l}d\theta ,$$

$$q_\alpha(\theta, x, \xi) = (2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}\partial_{\xi'}^\alpha a(x, \xi'+\theta\eta')D_{x'}^\alpha q((x_1, y' +x'), \xi)dy'd\eta'.$$

Proof. It suffices to prove by induction on K, $0\leq K<N$ that

$$(a\circ q)(x, \xi) = a(x, \xi')q(x, \xi)+\sum_{0<|\alpha|\leq K} C_{K\alpha}(\partial_{\xi'}^\alpha a\circ D_{x'}^\alpha q)(x, \xi) + \sum_{K<|\alpha|<N} C_{K\alpha}\partial_{\xi'}^\alpha a(x, \xi')D_{x'}^\alpha q(x, \xi) + \sum_{|\alpha|=N}\int_0^1 q_\alpha(\theta, x, \xi)\sum_{l\leq K} C_{K\alpha l}(1-\theta)^{N-1-l}d\theta .$$

This is trivial for $K=0$ and we assume that this is true for $K-1$. When $|\alpha|=K$, we have

$$\partial_{\xi'}^{\alpha}a(x,\xi')D_{x'}^{\alpha}q(x,\xi)=(\partial_{\xi'}^{\alpha}a\circ D_{x'}^{\alpha}q)(x,\xi)-\sum_{0<|\beta|<N-K}\frac{1}{\beta!}\partial_{\xi'}^{\alpha+\beta}a(x,\xi')D_{x'}^{\alpha+\beta}q(x,\xi)$$

$$-(N-k)\sum_{|\beta|=N-K}\frac{1}{\beta!}\int_0^1 q_{\alpha+\beta}(\theta,x,\xi)(1-\theta)^{N-K-1}d\theta\,.$$

Substituting this into the equality for $K-1$ we get one for K. This completes the proof.

Lemma 5.7. *Let $q(x,\xi)=\sum_{j=0}^{s}a_j(x,\xi')\xi_1^j$, $a_j\in\bigcup_{(P,p)\in R^2}S_{\Phi,\varphi}^{P,p}$, and let $a\in S_{\Phi,\varphi}^{M,m}$. Let $N,L\in N$ with $1\leq L<N$, $s<N$. Then we have that*

$$(q\circ a)(x,\xi)=q(x,\xi)a(x,\xi')+\sum_{\substack{0\leq j\leq s\\0<|\alpha|+j<L}}\frac{1}{\alpha!j!}\partial_{\xi'}^{\alpha}\partial_{\xi_1}^{j}q(x,\xi)D_{x'}^{\alpha}D_{x_1}^{j}a(x,\xi')$$

$$+\sum_{\substack{0\leq j\leq s\\L\leq|\alpha|+j<N}}\sum_{|\beta|<N-|\alpha|-j}C_{\alpha\beta j}(\partial_{\xi'}^{\beta}D_{x'}^{\alpha}D_{x_1}^{j}a\circ D_{x'}^{\beta}\partial_{\xi'}^{\alpha}\partial_{\xi_1}^{j}q)(x,\xi)$$

$$+\sum_{\substack{0\leq j\leq s\\|\alpha|+j=N}}\frac{N-j}{\alpha!j!}\int_0^1 q_{\alpha j}(\theta,x,\xi)(1-\theta)^{N-j-1}d\theta$$

$$+\sum_{\substack{0\leq j\leq s\\L\leq|\alpha|+j<N}}\sum_{|\beta|=N-|\alpha|-j}\int_0^1 q_{\alpha\beta j}(\theta,x,\xi)\sum_{l\leq|\beta|-1}C_{\alpha\beta jl}(1-\theta)^{|\beta|-1-l}d\theta$$

where $C_{\alpha\beta j}$, $C_{\alpha\beta jl}$ are constants depending only on its suffixes and N, L, and

$$q_{\alpha j}=(2\pi)^{-(n-1)}\iint e^{-iy'\cdot\eta'}\partial_{\xi'}^{\alpha}\partial_{\xi_1}^{j}q(x,\xi+\theta(0,\eta'))D_{x'}^{\alpha}D_{x_1}^{j}a((x_1,y'+x'),\xi')dy'd\eta',$$

$$q_{\alpha\beta j}=(2\pi)^{-(n-1)}\iint e^{-iy'\cdot\eta'}\partial_{\xi'}^{\beta}D_{x'}^{\alpha}D_{x_1}^{j}a(x,\xi'+\theta\eta')(D_{x'}^{\beta}\partial_{\xi'}^{\alpha}\partial_{\xi_1}^{j}q)((x_1,y'+x'),\xi)dy'd\eta'.$$

Proof. By Lemma 5.5

$$(q\circ a)(x,\xi)=\sum_{\substack{0\leq j\leq s\\|\alpha|+j<N}}\frac{1}{\alpha!j!}\partial_{\xi'}^{\alpha}\partial_{\xi_1}^{j}q(x,\xi)D_{x'}^{\alpha}D_{x_1}^{j}a(x,\xi')$$
$$+\sum_{\substack{0\leq j\leq s\\|\alpha|+j=N}}\frac{N-j}{\alpha!j!}\int_0^1 q_{\alpha j}(\theta,x,\xi)(1-\theta)^{N-j-1}d\theta\,. \tag{5.27}$$

Applying Lemma 5.6 with $D_{x'}^{\alpha}D_{x_1}^{j}a$, $\partial_{\xi'}^{\alpha}\partial_{\xi_1}^{j}q$ with $L\leq|\alpha|+j<N$, and $N-|\alpha|-j$ for a, q, N we have

$$\partial_{\xi'}^{\alpha}\partial_{\xi_1}^{j}q(x,\xi)D_{x'}^{\alpha}D_{x_1}^{j}a(x,\xi')=(D_{x'}^{\alpha}D_{x_1}^{j}a\circ\partial_{\xi'}^{\alpha}\partial_{\xi'_1}^{j}q)(x,\xi)$$
$$+\sum_{0<|\beta|<N-|\alpha|-j}C_{\alpha\beta j}(\partial_{\xi'}^{\beta}D_{x'}^{\alpha}D_{x_1}^{j}a\circ D_{x'}^{\beta}\partial_{\xi'}^{\alpha}\partial_{\xi_1}^{j}q)(x,\xi)$$

$$+\sum_{|\beta|=N-|\alpha|-j}\int_0^1 q_{\alpha\beta j}(\theta, x, \xi)\sum_{l=0}^{|\beta|-1} C_{\alpha\beta jl}(1-\theta)^{|\beta|-1-l}d\theta .$$

Substituting these equalities into (5.27) we get the second equality. This completes the proof.

Lemma 5.8. *Let* $q(x, \xi)=\sum_{j=0}^{3} a_j(x, \xi')\xi_1^j$ *with* a_3=*constant and* $a_j\in S_{1,0}^{m_j}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$ *for* $j\leq 2$, *and set* $m_3=0$. *Let* $a\in S_{\Phi,\varphi}^{M,m}$, *and set for* $\alpha, \beta\in\boldsymbol{Z}_+^n$

$$[a, q]_{\alpha\beta}(\theta, x, \xi) = (2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}a(x, \xi'+\theta\eta')$$
$$\times(\partial_\xi^\alpha\partial_x^\beta q)((x_1, y'+x'), \xi)dy'd\eta', \quad \theta\in[0, 1].$$

Then if $\alpha_1\leq 3$ *and* $|\alpha|+|\beta|>0$, *we have that*

$$[a, q]_{\alpha\beta}(\theta, x, \xi) = \sum_{j=0}^{\min(2,3-\alpha_1)} b_{j\theta}(x, \xi')\xi_1^j \tag{5.28}$$

where $\{b_{j\theta}\}_{\theta\in[0,1]}$ *is a bounded set in* $S_{\Phi,\varphi}^{M+(m_{j+\alpha_1}-|\alpha'|),m-(m_{j+\alpha_1}-|\alpha'|)}$.

Proof. With a notation in the proof of Lemma 5.2 we have that when $\alpha_1\leq 3$,

$$[a, q]_{\alpha\beta}(\theta, x, \xi) = \sum_{j=\alpha_1}^{3}\frac{j!}{(j-\alpha_1)!}h_\theta[a, \partial_{\xi'}^{\alpha'}\partial_x^\beta a_j](x, \xi')\xi_1^{j-\alpha_1}.$$

Since $\partial_{\xi'}^{\alpha'}\partial_x^\beta a_j\in S_{1,0}^{m_j-|\alpha'|}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})\subseteq S_{\Phi,\varphi}^{m_j-|\alpha'|,-(m_j-|\alpha'|)}$, and since from the assumption that a_3=*constant* the term for $j=3$ is dropped if $\alpha_1=0$ and $|\alpha|+|\beta|>0$, the assertion follows from the claim (1) in the proof of Lemma 5.2. The proof is complete.

Corollary 5.1. *Let* q be *as in the above* lemma *with* $m_j=3-j$, *and let* $a\in S_{1/2,1/2}^{l+(|\alpha|-|\beta|)/2}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$ *with* $\alpha, \beta\in\boldsymbol{Z}_+^n$ *satisfying the same assumption as above. Then* (5.28) *holds with* $\{b_{j\theta}\}_{\theta\in[0,1]}$ *bounded in* $S_{1/2,1/2}^{l+3-j-(|\alpha|+|\beta|)/2}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$.

Proof. Let $\Phi(x, \xi')=\langle\xi'\rangle^{1/2}$, $\varphi(x, \xi')=\langle\xi'\rangle^{-1/2}$, $M=-m=l+\dfrac{|\alpha|-|\beta|}{2}$. Then $a\in S_{\Phi,\varphi}^{M,m}$, and $M+m_{j+\alpha_1}-|\alpha'|=l+3-j-\dfrac{|\alpha|+|\beta|}{2}$. We apply Lemma 5.8. Since

$$S_{\Phi,\varphi}^{M+(m_{j+\alpha_1}-|\alpha'|),m-(m_{j+\alpha_1}-|\alpha'|)} = S_{\Phi,\varphi}^{M+m_{j+\alpha_1}-|\alpha'|,-(M+m_{j+\alpha_1}-|\alpha'|)}$$
$$= S_{1/2,1/2}^{M+m_{j+\alpha_1}-|\alpha'|}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$$

the conclusion follows. The proof is complete.

Lemma 5.9. *Let* q *and* a *be as in Lemma* 5.8, *and set for* $\alpha, \beta\in\boldsymbol{Z}_+^n$

$$[q, a]_{\alpha\beta}(\theta, x, \xi) = (2\pi)^{-(n-1)} OS-\iint e^{-iy'\cdot\eta'}(\partial_\xi^\alpha \partial_x^\beta q)(x, \xi+\theta(0, \eta'))$$
$$\times a((x_1, y'+x'), \xi')dy'd\eta', \quad \theta\in[0, 1].$$

Then if $\alpha_1 \leq 3$ *and* $|\alpha|+|\beta|>0$, *we have that*

$$[q, a]_{\alpha\beta}(\theta, x, \xi) = \sum_{j=0}^{\min(2,3-\alpha_1)} b_{j\theta}(x, \xi')\xi_1^j \tag{5.29}$$

where $\{b_{j\theta}\}_{\theta\in[0,1]}$ *is bounded set in* $S_{\Phi,\varphi}^{M+(m_{j+\alpha_1}-|\alpha'|), m-(m_{j+\alpha_1}-|\alpha'|)}$.

Proof. With a notation in the proof of Lemma 5.2 we have that if $\alpha_1\leq 3$.

$$[q, a]_{\alpha\beta}(\theta, x, \xi) = \sum_{j=\alpha_1}^{3} \frac{j!}{(j-\alpha)!} h_\theta[\partial_{\xi'}^{\alpha'}\partial_x^\beta a_j, a](x, \xi')\xi_1^j .$$

Now the proof is similar to that of Lemma 5.8. The proof is complete.

Corollary 5.2. *Under the same assumption as in Corollary* 5.1, (5.29) *holds with* $\{b_{j\theta}\}_{\theta\in[0,1]}$ *bounded in* $S_{1/2,1/2}^{l+3-j-(|\alpha|+|\beta|)/2}(\mathbb{R}^n\times\mathbb{R}^{n-1})$.

Proof. This is proved in the same way as in the proof of Corollary 5.1 from the above lemma. The proof is complete.

§6. Proof of Proposition 3.2

Proof of (1). Set $v=e^{1/2\tau(x_1-T)^2}u$. Then

$$e^{1/2\tau(x_1-T)^2}L(x, D)u = (L(x, D)+i\tau(x_1-T))v .$$

Set

$$A_1 = Re(a+b)(x,D'), \quad A_2 = Im(a+b)(x, D'),$$
$$L_1 = D_1-A_1, \quad L_2 = A_2+\tau(x_1-T) .$$

Then we have

$$(||L(x, D)u||_T^{(\tau)})^2 = ||L_1v||^2+2Im(L_1v, L_2v)+||L_2v||^2, \tag{6.1}$$

and

$$\begin{aligned} 2Im(L_1v, L_2v) &= \tau||v||^2+\frac{1}{i}\{(v, [D_1, A_2]v)+((A_1^*-A_1)v, L_2v) \\ &\quad -(v, [A_1, A_2]v)+((A_2^*-A_2)v, L_1v)\} \\ &= I+\cdots+V . \end{aligned} \tag{6.2}$$

From the proof of Lemma 5.4 with a notation in it we have

$$[Re(a+b)]^\sharp(x, \xi')-Re(a+b)(x, \xi') = \int_0^1 h_\theta[\sum_{|\alpha|=1} \partial_{\xi'}^\alpha D_{x'}^\alpha Re(a+b)](x, \xi')d\theta .$$

Since $\partial_{\xi'}^{\alpha} D_{x'}^{\alpha} Re(a+b) \in S_{\Phi,\varphi}^{0,-1}$ when $|\alpha|=1$, $A_1^* - A_1 \in OpS_{\Phi,\varphi}^{0,-1}$ from the claim (1) in Lemma 5.4. Similarly we have $A_2^* - A_2 \in OpS_{\Phi,\varphi}^{0,-1}$. Thus from (3.6) and Lemma 3.1 we have

$$|III| + |V| \leq C_1 \|E_{1/2} v\| (\|L_1 v\| + \|L_2 v\|) . \tag{6.3}$$

From the proof of Lemma 5.2 with a notation in it we have

$$\begin{aligned}&(Re(a+b)\circ Im(a+b) - Im(a+b)\circ Re(a+b))(x, \xi')\\ &= \int_0^1 \sum_{|\alpha|=1} (h_\theta[\partial_{\xi'}^{\alpha} Re(a+b), D_{x'}^{\alpha} Im(a+b)] - h_\theta[\partial_{\xi'}^{\alpha} Im(a+b), D_{x'}^{\alpha} Re(a+b)])(x, \xi') d\theta .\end{aligned}$$

Thus the claim (1) in Lemma 5.2 shows $[A_1, A_2] \in OpS_{\Phi,\varphi}^{1,-1}$. Thus noting that $[D_1, A_2] \in OpS_{\Phi,\varphi}^{1,-1}$, we see that

$$|II| + |IV| \leq C_2 \|v\| \, \|E_1 v\| . \tag{6.4}$$

Since $\|E_{1/2} v\|^2 \leq \|E_1 v\| \, \|v\|$, applying (6.3) and (6.4) to (6.2) and using Schwartz inequality we see that

$$2Im(L_1 v, L_2 v) \geq \tau \|v\|^2 - \frac{1}{4}(\|L_1 v\|^2 + \|L_2 v\|^2) - C_3 \|v\| \, \|E_1 v\| . \tag{6.5}$$

Now we shall use the assumption (3.3) of ellipticity to estimate the last term on the right of (6.5). To do so we prove

Lemma 6.1. *Let* $\lambda \in S_{\Phi,\varphi}^{1,-1}$ *with* $\partial_{x_j} \lambda \in S_{\Phi,\varphi}^{1,-1}$ *and* $\partial_{\xi_j} \lambda \in S_{\Phi,\varphi}^{0,0}$, *and with* $\inf_{|\xi'|\geq R} |\lambda(x, \xi')| \Phi^{-1}(x, \xi') \varphi(x, \xi') > 0$ *for some* $R > 0$. *Then there exists* $\mu \in S_{\Phi,\varphi}^{-1,1}$ *such that* $\mu(x, \xi') = \dfrac{1}{\lambda(x, \xi')}$ *when* $|\xi'| > 2R$ *and* $\mu \circ \lambda - 1 \in S_{\Phi,\varphi}^{-1,1}$.

Proof. Let $\psi \in C^\infty(\boldsymbol{R}^{n-1})$ with $\psi = 1$ when $|\xi'| > 2$, $\psi = 0$ when $|\xi'| < 1$. We define

$$\mu(x, \xi') = \begin{cases} \dfrac{1}{\lambda(x, \xi')} \psi(R^{-1}\xi') & |\xi'| > R \\ 0 & |\xi'| \leq R. \end{cases}$$

Then $\mu \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^{n-1})$ and

$$|\partial_x^\alpha \partial_{\xi'}^\beta \mu(x, \xi')| \leq C_{\alpha\beta} \frac{1}{|\lambda(x, \xi')|} (\Phi^{-|\beta|} \varphi^{-|\alpha|})(x, \xi') \quad \text{when} \quad |\xi'| > R.$$

This implies $\mu \in S_{\Phi,\varphi}^{-1,1}$. Moreover since

$$\partial_{\xi_j}\mu(x, \xi') = \frac{-\partial_{\xi_j}\lambda(x, \xi')}{\lambda(x, \xi')^2}\psi(R^{-1}\xi') + \frac{1}{\lambda(x, \xi')}R^{-1}(\partial_{\xi_j}\psi)(R^{-1}\xi') \quad \text{when} \quad |\xi'|>R,$$

from $\partial_{\xi_j}\lambda \in S^{0,0}_{\Phi,\varphi}$ and $\partial_{\xi_j}\psi \in C_0^\infty(\boldsymbol{R}^{n-1})$ we see that

$$|\partial_x^\alpha \partial_{\xi'}^\beta(\partial_{\xi_j}\mu(x, \xi'))| \le C_{\alpha\beta}\frac{1}{|\lambda(x, \xi')|^2}(\Phi^{-|\beta|}\varphi^{-|\alpha|})(x, \xi') \quad \text{when} \quad |\xi'|>R.$$

Thus $\partial_{\xi_j}\mu \in S^{-2,2}_{\Phi,\varphi}$. Thus from the proof of Lemma 5.2 with the notation in it we have

$$(\mu\circ\lambda)(x, \xi') - (\mu\lambda)(x, \xi') = \int_0^1 h_\theta[\mu, \lambda](x, \xi')d\theta \in S^{-1,1}_{\Phi,\varphi}.$$

Thus $\mu\circ\lambda - 1 \in S^{-1,1}_{\Phi,\varphi}$, for $\mu\lambda = 1$ when $|\xi'|>2R$. This completes the proof.

From Lemma 6.1 there exists $b_1 \in S^{-1,1}_{\Phi,\varphi}$ with $b_1\circ Im(a+b) - 1 \in S^{-1,1}_{\Phi,\varphi}$. Since $\langle\xi'\rangle \in S^{1,-1}_{\Phi,\varphi}$, $b_1\circ Im(a+b) - \langle\xi'\rangle \in S^{0,0}_{\Phi,\varphi}$ with $b_2 = \langle\xi'\rangle\circ b_1$. Thus

$$\begin{aligned} ||E_1v|| &\le C_4(||A_2v|| + ||v||) \\ &\le C_4(||L_2v|| + (\tau T+1)||v||)\,. \end{aligned} \tag{6.6}$$

Multiplying this inequality by $||v||$ and using Schwartz inequality we get that

$$||v||\;||E_1v|| \le C_5(\tau T+1)||v||^2 + \frac{1}{4}||L_2v||^2.$$

Substituting this into (6.5) we see that when $T+\frac{1}{\tau}<\delta$ for some $\delta>0$,

$$2Im(L_1v, L_2v) \ge \frac{1}{2}\tau||v||^2 - \frac{1}{2}(||L_1v||^2 + ||L_2v||^2)\,.$$

Substituting this into (6.1) we get that when $T+\frac{1}{\tau}<\delta$,

$$||L(x, D)u||_T^{(\tau)} \ge \frac{1}{\sqrt{6}}(||L_1v|| + ||L_2v|| + \tau^{1/2}||v||)\,. \tag{6.7}$$

From (6.6) and (6.7) we have that when $T+\frac{1}{\tau}<\delta$,

$$||E_1u||_T^{(\tau)} = ||E_1v|| \le C_5(1+\tau^{1/2}T)||L(x, D)u||_T^{(\tau)}\,.$$

Finally from inequalities that $||D_1u||_T^{(\tau)} \le ||L(x, D)u||_T^{(\tau)} + C_6||E_1u||_T^{(\tau)}$ and $T^{-1/2}||E_{1/2}u||_T^{(\tau)} \le \frac{1}{\sqrt{2}}(\tau^{-1/2}T^{-1}||E_1u||_T^{(\tau)} + \tau^{1/2}||u||_T^{(\tau)})$, this inequality and (6.7) imply that

$$\tau^{-1/2}T^{-1}\|D_1u\|_T^{(\tau)}+T^{-1/2}\|E_{1/2}u\|_T^{(\tau)}\leq C_7\|L(x, D)u\|_T^{(\tau)}$$

when $\tau^{1/2}T\geq 1$ and $T+\frac{1}{\tau}<\delta$. This completes the proof of (1).

Proof of (2). Let τ_0, T_0, C_0 be constants as in Proposition 3.1(1), and assume that $\tau T^2>\tau_0$, $T<T_0$. Then from (1) we have that

$$\begin{aligned}\sum_{1<i/2+j\leq 2}\tau^{1-i/2-j}T^{-i/2-j}\|E_{i/2}D_1^ju\|_T^{(\tau)}&\leq\tau^{-3/2}T^{-1}(\||D_1u\||_{T,2}^{(\tau)}+\||E_1u\||_{T,2}^{(\tau)})\\&\leq C_0\tau^{-1/2}T^{-1}(\|L_j(x, D)D_1u\|_T^{(\tau)}\\&\quad+\|L_j(x, D)E_1u\|_T^{(\tau)}),\end{aligned}\tag{6.8}$$

$$\sum_{i/2+j\leq 1}\tau^{1-i/2-j}T^{-i/2-j}\|E_{i/2}D_1^ju\|_T^{(\tau)}=\tau^{-1/2}\||u\||_{T,2}^{(\tau)}\leq C_0^2\|(L_i\circ L_j)(x, D)u\|_T^{(\tau)}.\tag{6.9}$$

From the proof of Lemma 5.2 we have with a notation in it that

$$(L_j\circ\langle\xi'\rangle)(x, \xi)=(\langle\xi'\rangle\circ L_j)(x, \xi)+\int_0^1\sum_{|\alpha|=1}h_\theta[\partial_{\xi'}^\alpha\langle\xi'\rangle, D_{x'}^\alpha(a_j+b_j)](x, \xi')d\theta.$$

Thus $[L_j(x, D), E_1]\in OpS_{\Phi,\varphi}^{1;-1}$ from the assumption. We also have $[L_j(x, D), D_1]\in OpS_{\Phi,\varphi}^{1;-1}$ from the assumption. Thus from (6.8) and (3.6) we have that

$$\begin{aligned}\sum_{1<i/2+j\leq 2}\tau^{1-i/2-j}T^{-i/2-j}\|E_{i/2}D_1^ju\|_T^{(\tau)}&\leq C_0\tau^{-1/2}T^{-1}(\|D_1L_j(x, D)u\|_T^{(\tau)}\\&\quad+\|E_1L_j(x, D)u\|_T^{(\tau)})+C_0C_1\tau^{-1/2}T^{-1}\|E_1u\|_T^{(\tau)})\\&\leq C_0^2(1+C_0C_1\tau^{-1/2})\|(L_i\circ L_j)(x, D)u\|_T^{(\tau)}).\end{aligned}$$

This inequality and (6.9) imply the first inequality in (2). Next, we show the second one. From the first inequality we have

$$\begin{aligned}\sum_{2<i/2+j\leq 3}\tau^{3/2-i/2-j}T^{-i/2-j}\|E_{i/2}D_1^ju\|_T^{(\tau)}&\leq\tau^{-1}T^{-1}(\||D_1u\||_{T,4}^{(\tau)}+\||E_1u\||_{T,4}^{(\tau)})\\&\leq C_2\tau^{-1/2}T^{-1}(\|(L_2\circ L_3)(x, D)D_1u\|_T^{(\tau)}\\&\quad+\|(L_2\circ L_3)(x, D)E_1u\|_T^{(\tau)}),\end{aligned}\tag{6.10}$$

$$\sum_{i/2+j\leq 2}\tau^{3/2-i/2-j}T^{-i/2-j}\|E_{i/2}D_1^ju\|_T^{(\tau)}=\||u\||_{T,4}^{(\tau)}\leq C_3\|(L_1\circ L_2\circ L_3)(x, D)u\|_T^{(\tau)}\tag{6.11}$$

if τT^2 and T^{-1} are large. We use identities that

$$[(L_2\circ L_3)(x, D), E_1] = [L_2(x, D), [L_3(x, D), E_1]]+\sum_{\{i,j\}=\{2,3\}}[L_i(x, D), E_1]L_i(x, D)\,,$$

$$[(L_2\circ L_3)(x, D), D_1] = [L_2(x, D), [L_3(x, D), D_1]]+\sum_{\{i,j\}=\{2,3\}}[L_i(x, D), D_1]L_j(x, D)\,.$$

If $r_j=L_j\circ\langle\xi'\rangle-\langle\xi'\rangle\circ L_j$, $r_j\in S^{1,-1}_{\Phi,\varphi}$ as showed above and $[L_i(x, D), r_j(x, D')]\in OpS^{1,-2}_{\Phi,\varphi}$ similarly from the proof of Lemma 5.2. We also have $[L_i(x, D), [L_j(x, D), D_1]]\in OpS^{1,-2}_{\Phi,\varphi}$, since $[L_j(x, D), D_1]\in OpS^{1,-1}_{\Phi,\varphi}$. Using these facts for above identities and noting (3.6) and Lemma 3.1 we get from (6.10) that

$$\begin{aligned}\sum_{2<i/2+j\leq 3}\tau^{3/2-i/2-j}T^{-i/2-j}\|E_{i/2}D_1^j u\|_T^{(\tau)}&\leq C_4\tau^{-1/2}T^{-1}(\|D_1(L_2\circ L_3)(x, D)u\|_T^{(\tau)}\\&\quad+\|E_1(L_2\circ L_3)(x, D)u\|_T^{(\tau)})\\&\quad+C_5\tau^{-1/2}T^{-1}(\|E_{3/2}u\|_T^{(\tau)}+\sum_{j=2,3}\|E_1L_j(x, D)u\|_T^{(\tau)})\\&\leq C_6\sum_{\{i,j\}=\{2,3\}}\|(L_1\circ L_i\circ L_j)(x, D)u\|_T^{(\tau)}\end{aligned}$$

if τT^2 and T^{-1} are large. This inequality and (6.11) proves the desired inequality. This completes the proof.

§7. Proofs of Lemmas in §3

Proof of Lemma 3.1. For some $C>0$, $\|a(x, D')u(x_1, \cdot)\|\leq C\|E_m u(x_1, \cdot)\|$ for any $x_1\in\mathbf{R}$, since $\{a(x_1, \cdot)\}_{x_1\in\mathbf{R}}$ is a bounded set in $S^m_{1/2,1/2}(\mathbf{R}^n\times\mathbf{R}^{n-1})$. Multiplying this inequality by $e^{\tau(x_1-T)^2}$ and integrating on $[0, T]$ in x_1 we get the desired inequality. Q.E.D.

Proof of Lemma 3.2. The proof needs three lemmas.

Lemma 7.1. (1) *Let $L=\xi_1-\lambda-\mu$ with $\lambda\in S^1_{1,0}(\mathbf{R}^n\times\mathbf{R}^{n-1})$ and $\mu\in S^{1,0}_{\Phi,\varphi}$. Let $a\in S^{M,m}_{\Phi,\varphi}$ (resp. $S^m_{1,0}(\mathbf{R}^n\times\mathbf{R}^{n-1})$). Then we have that*

$$a\circ L-aL,\ L\circ a-aL\in S^{M,m-1}_{\Phi,\varphi}(\text{resp. } S^{m,-m}_{\Phi,\varphi}).$$

(2) *Let $L_i(i=1, 2)$ be as L in* (1) *with λ_i, μ_i respectively for λ, μ. Let $a\in S^{M,m}_{\Phi,\varphi}$ (resp. $S^m_{1,0}(\mathbf{R}^n\times\mathbf{R}^{n-1})$). Then there exist $a_1, a_2\in S^{M,m-1}_{\Phi,\varphi}$ (resp. $S^{m,-m}_{\Phi,\varphi}$), $a_0\in S^{M,m-2}_{\Phi,\varphi}$ (resp. $S^{m,-m}_{\Phi,\varphi}$) such that*

$$a\circ(L_1L_2)-aL_1L_2=\sum_{i=1}^{2}a_i\circ L_i+a_0\,. \tag{7.1}$$

Proof. (1) We only prove that $a\circ L-aL\in S^{M,m-1}_{\Phi,\varphi}$ if $a\in S^{M,m}_{\Phi,\varphi}$. The others are proved similarly by using the fact that $S^m_{1,0}\subseteq S^{m,-m}_{\Phi,\varphi}$ in the case that $a\in S^{M,m}_{\Phi,\varphi}$. From Lemma 5.2 we have with a notation in its proof that

$$(a\circ(\lambda+\mu)-a(\lambda+\mu))(x, \xi')=\int_0^1 \sum_{|\alpha|=1} h_\theta[\partial_{\xi'}^\alpha a, D_{x'}^\alpha(\lambda+\mu)](x, \xi')d\theta .$$

Since $D_{x'}^\alpha(\lambda+\mu)\in S_{\Phi,\varphi}^{1,-1}$ for $|\alpha|=1$, the right hand side of the above equality belongs to $S_{\Phi,\varphi}^{M,m-1}$ from the proof of Lemma 5.2.

(2) We only prove the case that $a\in S_{\Phi,\varphi}^{M,m-1}$. The other case is proved similarly. Using Lemma 5.2 for $a\circ(L_1L_2)$ freezing the variable ξ_1, we have

$$a\circ(L_1L_2)(x, \xi)=(aL_1L_2)(x, \xi)+\int_0^1 r_\theta(x, \xi)d\theta ,$$

$$r_\theta(x, \xi)=(2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}\sum_{j=2}^n \partial_{\xi_j}a(x, \xi'+\theta\eta') \times \sum_{\{k,l\}=\{1,2\}} (D_{x_j}L_k\cdot L_l)((x_1, y'+x'), \xi)dy'd\eta'.$$

Applying Taylor's formula for $L_l((x_1, y'+x'), \xi)$ in y' and integrating by parts we see that

$$r_\theta(x, \xi)=r_{1\theta}(x, \xi)+r_{2\theta}(x, \xi')$$

where

$$\begin{aligned} &r_{1\theta}(x, \xi)\\ &=-\sum_{\{k,l\}=\{1,2\}}((2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}\sum_{j=2}^n \partial_{\xi_j}a(x, \xi'+\theta\eta')\\ &\qquad\times D_{x_j}(\lambda_k+\mu_k)((x_1, y'+x'), \xi')dy'd\eta')L_l(x, \xi) ,\end{aligned}$$

$$\begin{aligned} &r_{2\theta}(x, \xi')\\ &=-\sum_{\{k,l\}=\{1,2\}}(2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}\theta\sum_{s,j=2}^n \partial_{\xi_s}\partial_{\xi_j}a(x, \xi'+\theta\eta')\\ &\qquad\times D_{x_j}(\lambda_k+\mu_k)((x_1, y'+\eta'), \xi')\left(\int_0^1 \partial_{x_s}(\lambda_l+\mu_l)((x_1, x'+ty'), \xi')dt\right)dy'd\eta'.\end{aligned}$$

Thus using a notatoin in the proof of Lemma 5.2 we have

$$\int_0^1 r_{1\theta}(x, \xi)d\theta=-\sum_{\{k,l\}=\{1,2\}}\sum_{j=2}^n\int_0^1 h_\theta[\partial_{\xi_j}a, D_{x_j}(\lambda_k+\mu_k)](x, \xi')d\theta L_l(x, \xi) .$$

On the right of this equality, the coefficient of L_l belongs to $S_{\Phi,\varphi}^{M,m-1}$. Thus from (1), $\int_0^1 r_\theta(x, \xi)d\theta$ takes the form of the right of (7.1). Therefore, to complete the proof of (2), it suffices to show that $\{r_{2\theta}\}_{\theta\in[0,1]}$ is a bounded subset of $S_{\Phi,\varphi}^{M,m-2}$. This follows from the following Lemma 7.2 and Fact 7.1. Q.E.D.

Lemma 7.2. *Let* $(M_1, \cdots, M_N, m_1, \cdots, m_N)$ *be a permutation of* $2N$ *real numbers, and let* $a_j(x, y', \xi', \eta')\in C^\infty(\mathbf{R}_x^n\times\mathbf{R}_{(y',\xi',\eta')}^{3(n-1)})$ $(j=1, \cdots, N)$ *satisfying the* *estimates that*

$$|\partial_x^\alpha \partial_{y'}^\beta \partial_{\xi'}^\nu \partial_{\eta'}^\mu a_j(x, y', \xi', \eta')| \leq C_{\alpha\beta\nu\mu} \sup_{Q\in K} \Phi_{x_1}^{M_j-|\nu|-|\mu|}(Q)\varphi_{x_1}^{m_j-|\alpha|-|\beta|}(Q) \quad (7.2)$$

for all multi-indices where we use a notation K in (5.10). *Then if we set* $a= I[a_1, \cdots, a_N]$ *with*

$$I[a_1, \cdots, a_N](x, \xi') = (2\pi)^{-(n-1)} OS-\iint e^{-iy'\cdot\eta'} \prod_{j=1}^N a_j(x, y'+x', \xi', \eta'+\xi')dy'd\eta',$$

we have $a\in S_{\Phi,\varphi}^{M,m}$ *and* $|a|_l \leq C_l \prod_{j=1}^N |a_j|_{L_0+l}$ *where* $M=\sum_{j=1}^n M_j$, $m=\sum_{j=1}^N m_j$, $|a_j|_l = \max_{|(\alpha,\beta\nu,\mu)|\leq l}$ *{infimum of* $C_{\alpha\beta\nu\mu}$ *in* (7.2)*}, and the constants* C_0, L_0 *are depending only on l,* Φ, φ, *and a permutation given above.*

Fact 7.1. *Let* $a\in S_{\Phi,\varphi}^{M,m}$. *Then*

$$\sup_{\substack{0\leq t\leq 1\\0\leq s\leq 1}} |\partial_x^\alpha \partial_{y'}^\beta \partial_{\xi'}^\nu \partial_{\eta'}^\mu [a(x+t(0, y'-x'), \xi'+s(\eta'-\xi'))]|$$
$$\leq |a|_l (\Phi_{x_1}^{M-|\nu|-|\mu|}\varphi_{x_1}^{m-|\alpha|-|\beta|})((1-t)x'+ty', (1-s)\xi'+s\eta')$$

if $|\alpha|+|\beta|+|\nu|+|\mu| \leq l$.

Fact 7.1 is obvious. We shall prove Lemma 7.2.

Proof of Lemma 7.2. Set $b=\prod_{j=1}^N a_j$. Then we have the estimate

$$|\partial_{y'}^\alpha \partial_{\eta'}^\beta b(x, y', \xi', \eta')|$$
$$\leq N^{|\alpha|+|\beta|} \prod_{j=1}^N |a_j|_{|\alpha|+|\beta|} \sum_{\substack{\Sigma\alpha^j=\alpha\\ \Sigma\beta^j=\beta}} \prod_{j=1}^N (\sup_{Q\in K} \Phi_{x_1}^{M_j-|\beta^j|}(Q) \sup_{Q\in K} \varphi_{x_1}^{m_j-|\alpha^j|}(Q)).$$

Thus from Lemma 5.2 $|a(x, \xi')| \leq C_0 \prod_{j=1}^N |a_j|_{L_0} (\Phi^M \varphi^m)(x, \xi')$. Differentiating under integral sign we see that

$$\partial_x^\alpha \partial_{\xi'}^\beta a(x, \xi') = \sum_{\substack{\Sigma\nu^j+(0,\tilde\nu^j)=\alpha\\ \Sigma\mu^j+\tilde\mu^j=\beta}} \frac{\alpha!}{\nu^1!\cdots\nu^N!\tilde\nu^1!\cdots\tilde\nu^N!} \frac{\beta!}{\mu^1!\cdots\mu^N!\tilde\mu^1!\cdots\tilde\mu^N!}$$
$$\times I[\partial_x^{\nu^1}\partial_{y'}^{\tilde\nu^1}\partial_{\xi'}^{\mu^1}\partial_{\eta'}^{\tilde\mu^1}a_1, \cdots, \partial_x^{\nu^N}\partial_{y'}^{\tilde\nu^N}\partial_{\xi'}^{\mu^N}\partial_{\eta'}^{\tilde\mu^N}a_N].$$

Thus from above estimate we have

$$|\partial_x^\alpha \partial_{\xi'}^\beta a(x, \xi')| \leq C_0(2N)^{|\alpha|+|\beta|} \prod_{j=1}^N |a_j|_{L_0+|\alpha|+|\beta|} (\Phi^{M-|\beta|}\varphi^{m-|\alpha|})(x, \xi').$$

Q.E.D.

Lemma 3.2 is an immediate consequence of the second part of the next lemma and Proposition 3.2.

Lemma 7.3. *Let* $L_i=\xi_1-\lambda_i-\mu_i$ $(i=1, 2, 3)$ *with* $\lambda_i\in S^1_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$, $\mu_i\in S^{1,0}_{\Phi,\varphi}$. *Then we have the followings.*

(1) $L_1\circ L_2-L_1L_2\in S^{1,-1}_{\Phi,\varphi}$.

(2) *For some* $a_j\in S^{1,-1}_{\Phi,\varphi}(j=1, 2, 3)$ *and* $a_0\in S^{1,-2}_{\Phi,\varphi}$ *we have that*

$$L_1\circ L_2\circ L_3-L_1L_2L_3=\sum_{j=1}^{3} a_j\circ L_j+a_0\,. \tag{7.3}$$

Proof. (1) we have

$$\begin{aligned}&L_1\circ L_2-L_1L_2\\&\quad=-D_{x_1}(\lambda_2+\mu_2)+(\lambda_1\circ L_2-\lambda L_2)+(\mu_1\circ L_2-\mu_1L_2)\,.\end{aligned}$$

Each of terms on the right hand side belongs to $S^{1,-1}_{\Phi,\varphi}$ from the assumption and Lemma 7.1.

(2) In view of (1) and Lemma 7.1-(1) $L_1\circ L_2\circ L_3-L_1\circ(L_2L_3)$ takes the form of the right of (7.3). Thus it suffices to show that $L_1\circ(L_2L_3)-L_1L_2L_3$ does also. We have

$$\begin{aligned}L_1\circ(L_2L_3)-L_1L_2L_2&=\sum_{\{k,l\}=\{2,3\}}D_{x_1}L_k\cdot L_l\\&\quad-\{\lambda_1\circ(L_2L_3)-\lambda_1L_2L_3\}-\{\mu_1\circ(L_2L_3)-\mu_1L_2L_3\}\,.\end{aligned}$$

Each term on the right hand side takes the form of the right of (7.3) from Lemma 7.1. Q.E.D.

This completes the proof of Lemma 3.2.

Proof of Lemma 3.3. Take $a=\chi$, $q=p$ in Lemma 5.6 with $N=3$ and in Lemma 5.7. with $L=1$ and $N=3$. Then Lemma 3.3 follows from Corollary 5.1, 5.2. Q.E.D.

Proof of Lemma 3.4. This follows from Lebiniz rule. Q.E.D.

Proof of Lemma 3.5. From Proposition 3.2 and Lemma 3.2.

$$B(u)\le C_1||(p-q)(x, D)u||^{(\tau)}_T$$

if τT^2, $\dfrac{1}{T}$ are large. Since $g\circ\chi\in S^{0,-3}_{\Phi,\varphi}\subset S^{3/2}_{1/2,1/2}$, we have

$$||g\circ\chi(x, D')u||^{(\tau)}_T\le C_2||E_{3/2}u||^{(\tau)}_T\,.$$

From these two inequalities we see that for large τT^2, $\frac{1}{T}$

$$||B(\chi(x, D')u)||_T^{(\tau)}+||g\circ\chi(x, D')u||_T^{(\tau)}\leq C_1||p\circ\chi(x, D')u||_T^{(\tau)}+C_2(1+C_1)||E_{3/2}u||_T^{(\tau)}.$$

Applying Lemma 3.3 to the first term on the right we get the desired inequality. Q.E.D.

Proof of Lemma 3.6. Taking $\Phi(x, \xi')=\langle\xi'\rangle, \varphi(x, \xi')=1$ in Lemma 7.1, 7.3-(1), we have that if $|\alpha|+|\beta|=1$ (*resp.* 2)

$$\partial_\xi^\alpha\partial_x^\beta(p-g) = \sum_{i\leq j} a_{ij}\circ L_{0i}\circ L_{0j} + \sum_{i=1}^{2} a_i\circ L_{0i}+a_0$$

with $a_{ij}\in S_{1,0}^{1-|\alpha|}$, $a_i\in S_{1,0}^{1-|\alpha|}$ (*resp.* $S_{1,0}^{2-|\alpha|}$) for $i\neq 0$, $a_0\in S_{0,1}^{2-|\alpha|}$

From this we have that when $|\alpha|+|\beta|=1$ or 2,

$$\langle\xi'\rangle^{(|\alpha|-|\beta|)/2}\circ\partial_\xi^\alpha\partial_x^\beta(p-g) = \sum_{i\leq j} a_{ij}\circ L_{0i}\circ L_{0j} + \sum_{i=1}^{2} a_i\circ L_{0i}+a_0$$

with $a_{ij}\in S_{1,0}^{1/2}$, $a_i\in S_{1,0}^{1}$ $(i\neq 0)$, $a_0\in S_{1,0}^{3/2}$.

Thus Proposition 3.2 implies that if τT^2 and $\frac{1}{T}$ are large

$$T^{-1/2}\sum_{|\alpha|+|\beta|=1,2}||E_{(|\alpha|-|\beta|)/2}(p-g)_{(\beta)}^{(\alpha)}(x, D)u||_T^{(\tau)}\leq C_1||(p-g))(x, D)u||_T^{(\tau)}.$$

Thus using $\langle\xi'\rangle^{(|\alpha|-|\beta|)/2}\circ g_{(\beta)}^{(\alpha)}\chi\in S_{\Phi,\varphi}^{0,-3}$ and Lemma 3.2 we get the desired inequality. Q.E.D.

Proof of Lemma 3.7. As in the proof of Lemma 3.3 we take $q=p, a=\chi$, $N=3$ in Lemma 5.6 and apply Corollary 5.1. Then we have with some $b_i\in S_{1/2,1/2}^{i-1/2}$ and constants C_α

$$\chi\circ p = \chi p+\sum_{|\alpha|=1,2} C_\alpha\partial_{\xi'}^\alpha a\circ\partial_{x'}^\alpha p+\sum_{i=0}^{2} b_i\xi_1^{2-i}.$$

Thus Lemma 3.7 follows from the following lemma and Lemma 7.1, 7.3-(1). Q.E.D.

Lemma 7.4. *Let* $L_i(i=1, 2, 3)$ *be as in Lemma* 7.3 *and* $a\in S_{\Phi,\varphi}^{M,m}$. *Then*

$$(L_1L_2L_3)\circ a = aL_1L_2L_3+\sum_{i\neq j} a_{ij}L_iL_j+\sum_{i=1}^{3} a_iL_i+a_0$$

with some $a_{ij}\in S^{M,m-1}_{\Phi,\varphi}$, $a_i\in S^{M,m-2}_{\Phi,\varphi}$ $(i\neq 0)$, $a_0\in S^{M,m-3}_{\Phi,\varphi}$.

Proof. We have

$$
\begin{aligned}
&(L_1L_2L_3)\circ a(x,\xi)\\
&=(2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}(L_1L_2L_3)(x,\xi+(0,\eta'))a(x+0,y'),\xi')dy'd\eta'\\
&+\sum_{l>j}(2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}(L_lL_j)(x,\xi+(0,\eta'))D_{x_1}a(x+(0,y'),\xi')dy'd\eta'\\
&+\sum_{l=1}^{3}(2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}L_l(x,\xi+(0,\eta'))D_{x_1}^2a(x+(0,y'),\xi')dy'd\eta'\\
&+(2\pi)^{-(n-1)}\iint e^{-iy'\cdot\eta'}D_{x_1}^3a(x+(0,y'),\xi')dy'd\eta'\\
&=I+\cdots+IV.
\end{aligned}
$$

Since $(2\pi)^{-(n-1)}\ OS-\iint e^{-iy'\cdot\eta'}a(x+(0,y'),\xi')dy'd\eta'=a(x,\xi')$ by Fourier inversion formula and a limiting argument, in the term I we see using Taylor's formula for $L_l(x,\xi+(0,\eta'))$ in η' and the integration by parts

$$
\begin{aligned}
I=&\prod_{l=1}^{3}L_l(x,\xi)a(x,\xi')\\
&-\sum_{l=1}^{3}(2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}\sum_{|\alpha|=1}\int_0^1\partial_{\xi'}^{\alpha}(\lambda_l+\mu_l)(x,\xi'+\theta\eta')\\
&\times D_{x'}^{\alpha}a(x+(0,y'),\xi')dy'd\eta'\prod_{j\neq l}L_j(x,\xi)\\
&-\sum_{\substack{j_1,j_2,j_3:\,distinct\\ j_1>j_2}}(2\pi)^{-(n-1)}OS-\iint e^{iy'\cdot\eta'}\\
&\times\sum_{|\alpha^1|,|\alpha^2|=1}\{\prod_{k=1}^{2}\int_0^1\partial_{\xi'}^{\alpha^k}(\lambda_k+\mu_k)(x,\xi')+\theta\eta')d\theta\}D_{x'}^{\alpha^1+\alpha^u}a(x+(0,y'),\xi')dy'd\eta'\\
&\times L_{j_3}(x,\xi)\\
&-(2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta}\sum_{|\alpha^1|,|\alpha^2|,|\alpha^3|=1}\{\prod_{k=1}^{3}\int_0^1\partial_{\xi'}^{\alpha^k}(\lambda_k+\mu_k)(x,\xi'+\theta\eta')d\theta\}\\
&\times D_{x'}^{\alpha^1+\alpha^2+\alpha^3}a(x+(0,y'),\xi')dy'd\eta'.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
II=&\sum_{l>j}D_{x_1}a(x,\xi')(L_lL_j)(x,\xi)\\
&-\sum_{l\neq j}(2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}\sum_{|\alpha|=1}\int_0^1\partial_{\xi'}^{\alpha}(\lambda_l+\mu_l)(x,\xi'+\theta\eta')d\theta\\
&\times D_{x_1}D_{x'}^{\alpha}a(x+(0,y'),\xi')dy'd\eta'\cdot L_j(x,\xi)
\end{aligned}
$$

$$+\sum_{j_1>j_2}(2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}\sum_{|\alpha^1|,|\alpha^2|=1}\{\prod_{k=1}^{2}\int_0^1\partial_{\xi'}^{\alpha^k}(\lambda_k+\mu_k)(x,\xi'+\theta\eta')d\theta\}$$

$$\times D_{x_1}D_{x'}^{\alpha^1+\alpha^2}a(x+(0,y'),\xi')dy'd\eta',$$

$$III=\sum_{l=1}^{3}D_{x_1}^{3}a(x,\xi')L_l(x,\xi)$$

$$-\sum_{l=1}^{3}(2\pi)^{-(n-1)}OS-\iint e^{-iy'\cdot\eta'}\sum_{|\alpha|=1}\int_0^1\partial_{\xi'}^{\alpha}(\lambda_l+\mu_l)(x,\xi'+\theta\eta')d\theta$$

$$\times D_{x_1}^{2}D_{x'}^{\alpha}a(x+(0,y'),\xi')dy'd\eta',$$

$$IV=D_{x_1}^{3}a(x,\xi').$$

As in the proof of Lemma 7.1-(2) applying Lemma 7.2 to oscillaroty integrals in I∼III we see that $I+\cdots+IV$ is the form of the right of the equality in Lemma 7.4. Q.E.D.

Proof of Lemma 3.8. This follows easily from the following lemma and Lemma 7.3-(1). Q.E.D.

Lemma 7.5. *Let* $a\in S_{\Phi,\varphi}^{M,m}$, *and let* L_i $(i=1,2)$ *be as in Lemma* 7.3. *Set* $L_{0i}=\xi_1-\lambda_i$. *Then*

$$(L_1L_2)\circ a=(L_{01}L_{02})\circ a+\sum_{i=1}^{2}b_i\circ L_{0i}+b_0$$

with $b_i\in S_{\Phi,\varphi}^{M+1,m}$ $(i=1,2)$ *and* $b_0\in S_{\Phi,\varphi}^{M+2,m}$, *and*

$$L_1\circ a=L_{01}\circ a+a_1,$$

with $a_1\in S_{\Phi,\varphi}^{M+1,m}$.

Proof. We have

$$L_1L_2=L_{01}L_{02}-\mu_1L_{02}-\mu_2L_{01}+\mu_1\mu_2.$$

Thus from Lemma 7.1-(1) we have with some $b\in S_{\Phi,\varphi}^{1,-1}$

$$(L_1L_2)\circ a=(L_{01}L_{02})\circ a-L_{02}\circ(\mu_1\circ a)-L_{01}\circ(\mu_2\circ a)+(\mu_1\mu_2+b)\circ a.$$

Hence the last term on the right is in $S_{\Phi,\varphi}^{M+2,m}+S_{\Phi,\varphi}^{M+1,m-1}\subset S_{\Phi,\varphi}^{M+2,m}$. Thus applying Lemma 7.1-(1) to the middle two terms on the right we obtain the first statement. The second statement is trivial. Q.E.D.

Proof of Lemma 3.9. We prove (1) first. We need a lemma.

Lemma 7.6. *Assume the notations in* §3. *Assume* (*I*) *holds.*
(1) For any distinct $1 \leq k, l \leq 3$

$$L_{01}\psi_0 = a_1 L_k + a_2 L_l$$

with some $a_i \in S^{0,0}_{\Phi,\varphi}(i=1,2)$ *with* $supp a_i \subset supp \psi_0$.
(2) *For any distinct* $1 \leq k, l \leq 3$ *we have with some* a_1, a_2 *as in* (1)

$$\Phi\psi_0 = a_1 L_k + a_2 L_l\,.$$

Proof. (1) Set

$$a_1 = \psi_0 \Lambda_l/(\Lambda_l - \Lambda_k),\ a_2 = -\psi_0 \Lambda_k/(\Lambda_l - \Lambda_k)$$

with a trivial convention outside $supp \psi_0$. These have required properties.
(2) Set

$$a_1 = \psi_0 \Phi/(\Lambda_l - \Lambda_k),\ a_2 = -\psi_0 \Phi/(\Lambda_l - \Lambda_k)\,.$$

Then $a_i \in S^{0,0}_{\Phi,\varphi}(i=1,2)$ and satisfy the equality in (2). Q.E.D.

If $k \neq l$, from Lemma 7.6-(1) and Lemma 7.1-(1) we have with some $a_1, a_2 \in S^{0,0}_{\Phi,\varphi}$, $a_0 \in S^{0,-1}_{\Phi,\varphi}$

$$L_{01} \circ \psi_0 = a_1 \circ L_k + a_2 \circ L_l + a_0\,.$$

Since

$$L_{01} \circ \psi_1 = L_{01} \circ (1 - \psi_0) \circ \psi_1 + L_{01} \circ \psi_0 \circ \psi_1\,,$$

the first statement in (1) follows. Next the proof of Lemma 5.2 shows that $c \circ \psi_0 - c\psi_0 \in S^{0,-1}_{\Phi,\varphi}$. From this and that $c\psi_0/\Phi \in S^{0,0}_{\Phi,\varphi}$ Lemma 5.2 shows that with $b_0 = c\psi_0/\Phi$ and some $b_1 \in S^{0,-1}_{\Phi,\varphi}$

$$c \circ \psi_0 \circ \psi_1 = b_0 \circ (\Phi\psi_0) \circ \psi_1 + b_1\,.$$

Now, the second statement immediately follows from Lemmas 7.6-(2), 7.1-(1). This completes the proof of (1)

(2) can be proved similarly by using the following lemma Q.E.D.

Lemma 7.7. *Assume the notations in* §3. *Assume* (*II*) *holds.*
(1) The statement of (1) in the previous lemma holds.
(2) If $k \neq 1$, we have with some $a_i \in S^{0,0}_{\Phi,\varphi}(i=1,2)$

$$c\psi_0 = a_1 L_1 + a_2 L_k\,.$$

Proof. (1) If $k=2, l=3$, the same proof as that of Lemma 7.6-(1) works well. In another cases we set

$$a_1 = -\psi_0\, A_k/(c+A_1-A_k),\, a_2 = \psi_0(c+A_1)/(c+A_1-A_k)\,.$$

We have

$$|c_{(\alpha)}^{(\beta)}(x,\xi')| \leq C_{\alpha\beta}|c(x,\xi')|(\Phi^{-|\beta|}\varphi^{-|\alpha|})(x,\xi') \quad \text{for any } \alpha, \beta \tag{7.4}$$

because

$$|c_{(\alpha)}^{(\beta)}(x,\xi')| \leq C_{\alpha\beta}(\Phi^{1-|\beta|}\varphi^{-|\alpha|})(x,\xi') \quad \text{if} \quad |\alpha|+|\beta|>0\,.$$

Using (7.4) we can easily check $a_i \in S_{\Phi,\varphi}^{0,0}$ and the desired equality clearly holds.

(2) Set

$$a_1 = -\psi_0\, c/(c+A_1-A_k)\,, \quad a_2 = \varphi_0\, c/(c+A_1-A_k)\,.$$

Then these have required properties. Q.E.D.

Proof of Lemma 3.10. We need a lemma.

Lemma 7.8. *Let* L_i $(i=1, 2, 3)$ *be as in Lemma* 7.3 *and set* $q=\prod_{i=1}^{3} L_i$. *Then we have for* α, β *with* $1\leq|\alpha|+|\beta|\leq 2$

$$q_{(\beta)}^{(\alpha)} = \sum_{i>j} a_{ij} \prod_{j\in\{i,j\}} L_l \tag{7.5}$$

with some $a_{ij} \in S_{\Phi,\varphi}^{1-|\alpha|,-|\beta|}(|\alpha|+|\beta|=1)$,

$$q_{(\beta)}^{(\alpha)} = \sum_{i>j} a_{ij} \prod_{l\in\{i,j\}} L_l + \sum_{i=1}^{3} a_i\, L_i \tag{7.6}$$

with some $a_{ij} \in S_{\Phi,\varphi}^{1-|\alpha|,-|\beta|}$, $a_i \in S_{\Phi,\varphi}^{2-|\alpha|,-|\beta|}(|\alpha|+|\beta|=2)$.

Proof. (i) Assume $|\alpha|+|\beta|=1$. If $\alpha_1=1$, (7.5) is clear. If $\alpha_1=0$, we have

$$q_{(\beta)}^{(\alpha)} = \sum_{\substack{i,j,k\,:\,distinct \\ i>j}} -(\lambda_k+\mu_k)_{(\beta)}^{(\alpha)} \prod_{l\in\{i,j\}} L_l\,.$$

Here $(\lambda_k)_{(\beta)}^{(\alpha)} \in S_{1,0}^{1-|\alpha|} \subset S_{\Phi,\varphi}^{1-|\alpha|,-(1-|\alpha|)} = S_{\Phi,\varphi}^{1-|\alpha|,-|\beta|}$. Thus (7.5) holds.

(ii) Assume $|\alpha|+|\beta|=2$. If $\alpha_1=2$, (7.6) is clear. If $\alpha_1=1$, we have

$$q_{(\beta)}^{(\alpha)} = \sum_{i\neq j} -(\lambda_i+\mu_i)_{(\beta)}^{(\alpha')} L_j\,.$$

Since $|\alpha'|+|\beta|=1$, $(\lambda_i+\mu_i)_{(\beta)}^{(\alpha')} \in S_{\Phi,\varphi}^{1-|\alpha'|-|\beta|} = S_{\Phi,\varphi}^{2-|\alpha|,-|\beta|}$. Thus (7.6) holds. If $\alpha_1=0$, we have

$$q_{(\beta)}^{(\alpha)} = \sum_{\substack{i,j,k\,:\,distinct \\ i>j}} \{-(\lambda_k+\mu_k)_{(\beta)}^{(\alpha')} \prod_{l\in\{i,j\}} L_l$$

$$+\sum_{\substack{\mu\leq\alpha'\\ \nu\leq\beta\\ |\mu+\nu|=1}}\binom{\alpha'}{\mu}\binom{\beta}{\nu}(\lambda_i+\mu_i)^{(\mu)}_{(\nu)}(\lambda_j+\mu_j)^{(\alpha'-\mu)}_{(\beta-\nu)}L_k\}\ .$$

Here in the first term in the parenthis $(\lambda_k)^{(\alpha')}_{(\beta)}\in S^{1-|\alpha|,-|\beta|}_{\Phi,\varphi}$ because $-1+|\alpha|\geq-|\beta|$, and in the second term $(\lambda_i+\mu_i)^{(\mu)}_{(\nu)}\in S^{1-|\mu|,-|\nu|}_{\Phi,\varphi}$, $(\lambda_j+\mu_j)^{(\alpha'-\mu)}_{(j-\nu)}\in S^{1-|\alpha'|+|\mu|,-|\beta|+|\nu|}_{\Phi,\varphi}$ for the same reason so that their product belongs to $S^{2-|\alpha'|,-|\beta|}_{\Phi,\varphi}$. This shows (7.6). Q.E.D.

Now we shall prove Lemma 3.10. From Lemma 5.7 we have with the notations in Lemma 5.7, 5.8, 5.9

$$\begin{aligned}
p^{(\alpha)}_{(\beta)}\circ\chi &= \sum_{\substack{0\leq j\leq 2\\ |\gamma|+j\leq L-1}}\frac{1}{\gamma!j!}\partial_\xi^{\alpha+(j,\gamma)}\partial_x^{(0,\beta)}pD_x^{(j,\gamma)}\chi\\
&+\sum_{\substack{0\leq j\leq 2\\ L\leq|\gamma|+j<N}}\sum_{|\mu|<N-|\gamma|-j}C_{\gamma\mu j}(-i)^{|\mu|}[\partial^\mu_{\xi'}D_x^{(j,\gamma)}\chi,p]_{\alpha+(j,\gamma)(0,\beta+\mu)}(1,x,\xi)\\
&+\sum_{\substack{0\leq j\leq 2\\ L\leq|\gamma|+j<N}}\frac{N-j}{\gamma!j!}\int_0^1[p,D_x^{(j,\gamma)}\chi]_{\alpha+(j,\gamma)(0,\beta)}(\theta,x,\xi)(1-\theta)^{N-j-1}d\theta\\
&+\sum_{\substack{0\leq j\leq 2\\ L\leq|\gamma|+j<N}}\sum_{|\mu|=N-|\gamma|-j}(-i)^{|\mu|}\int_0^1[\partial^\mu_{\xi'}D_x^{(j,\gamma)}\chi,p]_{\alpha+(j,\gamma)(0,\beta)}(\theta,x,\xi)\\
&\times\sum_{L\leq|\mu|-1}C_{\gamma\mu jl}(1-\theta)^{|\mu|-1-l}d\theta\\
&=I+II+III+IV\,.
\end{aligned}$$

Assume $1\leq|\alpha|+|\beta|\leq 2$ in the following. From Collorary 5.1 we have for any j, γ, μ

$$[\partial^\mu_{\xi'}D_x^{(j,\gamma)}\chi,p]_{\alpha+(j,\gamma)(0,\beta+\mu)}(\theta,x,\xi)=\begin{cases}\sum_{j=0}^{\min(2,3-\alpha_1-j)}b'_{k\theta}\,\xi_1^k & (\alpha_1+j\leq 3)\\ 0 & (\alpha_1+j>3)\end{cases}\tag{7.7}$$

with some $\{b_{k\theta}\}_{\theta\in[0,1]}$ being bounded in $S^{3-k-|\alpha|-(|(j,\gamma)|+|\mu|)/2}_{1/2,1/2}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$. From Corollary 5.2 we have for any j, γ

$$[p,D_x^{(j,\gamma)}\chi]_{(\alpha+(j,\gamma)(0,\beta)}(\theta,x,\xi)=\begin{cases}\sum_{k=0}^{\min(2,3-\alpha_1-j)}b'_{k\theta}\,\xi_1^k & (\alpha_1+j\leq 3)\\ 0 & (\alpha_1+i>3)\end{cases}\tag{7.8}$$

with some $\{b'_{k\theta}\}_{\theta\in0,1}$ being bounded in $S^{3-k-|\alpha|-|(j,\gamma)|/2}_{1/2,1/2}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$.

We devide our argument into two cases.

(i) Assume $|\alpha|+|\beta|=1$. Take $L=2$, $N=3$. Then from (7.7), (7.8) we have

$$II+III+IV=\sum_{k=0}^{\min(2,3-\alpha_1)}b_k\,\xi_1^k\quad\text{with some}\quad b_k\in S^{2-k-|\alpha|}_{1/2,1/2}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})\,.$$

Next from Lemma 7.8 and that $supp\ \chi \subset supp\ \psi_0$ we have

$$I = \sum_{i>k} \chi a_{ik} \prod_{l\in\{i,k\}} L_l + \sum_{\substack{0\le j\le 2\\ |\gamma|+j=1}} D_x^{(j,\gamma)} \chi \Big(\sum_{i>k} a_{i,k}^{\gamma,j} \prod_{l\in\{i,k\}} L_l + \sum_{i=1}^{3} a_i^{\gamma,j} L_i\Big)$$

with some $a_{ik}\in S_{\Phi,\varphi}^{1-|\alpha|,-|\beta|}$, $a_{i,k}^{\gamma,j}\in S_{\Phi,\varphi}^{1-|\alpha|-j-|\gamma|,-|\beta|}$, $a_i^{\gamma,j}\in S_{\Phi,\varphi}^{2-|\alpha|-j-|\gamma|,-|\beta|}$. Here we have $D_x^{(j,\gamma)} \chi\circ a_{i,k}^{\gamma,j}\in S_{\Phi,\varphi}^{1-|\alpha|,-|\beta|}$, $D_x^{(j,\gamma)} \chi\circ a_i^{\gamma,j}\in S_{\Phi,\varphi}^{1-|\alpha|,-|\beta|-1}$.

(ii) Assume $|\alpha|+|\beta|=2$. From Corollary 5.2 we have

$$II+III+IV = \sum_{k=0}^{\min(2,3-\alpha_1)} b_k \xi_1^k \quad \text{with some} \quad b_k\in S_{1/2,1/2}^{5/2-k-|\alpha|}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})\,.$$

Similarly as in (i)

$$I = \chi\Big(\sum_{i>k} a_{ik} \prod_{l\in\{i,k\}} L_l + \sum_{i=1}^{3} a_i L_i\Big)$$

with $a_{ik}\in S_{\Phi,\varphi}^{1-|\alpha|,-|\beta|}$, $a_i\in S_{\Phi,\varphi}^{2-|\alpha|,-|\beta|}$. From (i) and (ii) we see using Lemma 7.1, 7.3-(1), and that $\langle\xi'\rangle^{(|\alpha|-|\beta|)/2}\in S_{\Phi,\varphi}^{(|\alpha|-|\beta|)/2,(|\beta|-|\alpha|)/2}$ that

$$\begin{aligned}&\langle\xi'\rangle^{(|\alpha|-|\beta|)/2}\circ p_{(\beta)}^{(\alpha)}\circ\chi\\ &\quad= \sum_{i>j} A_{ij}\circ L_i\circ L_j + \sum_{i=1}^{3} A_i\circ L_i + R + \sum_{i=0}^{\min(2,3-\alpha_1)} B_i \xi_1^i\end{aligned}$$

with some

$$A_{ij}\in S_{\Phi,\varphi}^{1-(|\alpha|+|\beta|)/2,-(|\alpha|+|\beta|)/2},\quad R\in S_{\Phi,\varphi}^{2-(|\alpha|+|\beta|)/2,-1-(|\alpha|+|\beta|)/2}$$

$$A_i\in\begin{cases} S_{\Phi,\varphi}^{1/2,-3/2} & (|\alpha|+|\beta|=1)\\ S_{\Phi,\varphi}^{1,-1} & (|\alpha|+|\beta|=2)\end{cases},\quad B_i\in S_{1/2,1/2}^{3/2-i}\,.$$

This implies Lemma 3.10 in view of (3.6). Q.E.D.

Proof of Lemma 3.11. Let α, β as in the assertion. From Lemma 5.2 we have with a notation in the claim (1) in its proof

$$\begin{aligned}(g_{(\beta)}^{(\alpha)}\circ\chi)(x,\xi') &= \sum_{|\gamma|<3-|\alpha|} \frac{1}{\gamma!}(\partial_{\xi'}^{\gamma} g_{(\beta)}^{(\alpha)} D_{x'}^{\gamma}\chi)(x,\xi')\\ &\quad+ \sum_{|\gamma|=3-|\alpha|} \frac{3-|\alpha|}{\gamma!}\int_0^1 h_\theta[\partial_{\xi'}^{\gamma} g_{(\beta)}^{(\alpha)}, D_{x'}^{\gamma}\chi](x,\xi')(1-\theta)^{2-|\alpha|}\,d\theta\,.\end{aligned}$$

The second summation on the right belongs to $S_{\Phi,\varphi}^{0,-(3-|\alpha|)}\subseteq S_{1/2,1/2}^{(3-|\alpha|)/2}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$ from the claim (1) in the proof of Lemma 5.2, because $\partial_{\xi'}^{\gamma} g_{(\beta)}^{(\alpha)}\in S_{1,0}^{0}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})\subseteq S_{\Phi,\varphi}^{0,0}$ if $|\gamma|+|\alpha|=3$. Next in the first summation we have $\partial_{\xi'}^{\gamma} g_{(\beta)}^{(\alpha)} D_{x'}^{\gamma}\chi\in S_{\Phi,\varphi}^{3-|\gamma|-|\alpha|,-|\gamma|-|\beta|}$, so it belongs to $S_{\Phi,\varphi}^{3/2-|\gamma|-|\alpha|,-3/2-|\gamma|-|\mu|}$ since $\Phi(x,\xi')^3\le$

$C\langle\xi'\rangle^{3/2}\leq C'(\Phi/\varphi)^{3/2}$ on *supp* χ. Thus the second and first summations multiplied by $\langle\xi'\rangle^{(|\alpha|-|\beta|)/2}$ from the left in the operator product sence belong to respectively to $S^{3/2-|\beta|/2}_{1/2,1/2}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$ and $S^{3/2-|\gamma|-(|\alpha|+|\beta|)/2,-3/2-|\gamma|-(|\alpha|+|\beta|)/2}_{\Phi,\varphi}\subseteq S^{3/2}_{1/2,1/2}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})$. This proves the assertion. Q.E.D.

Proof of Lemma 3.12. This has been proved in the proof of Lemma 3.6. Q.E.D.

§8. Invariance of the Assumption of Theorem 1.1

Let $\Omega_1=\Omega_{11}\times\Omega_{12}$ where Ω_{11}, Ω_{12} are open sets in $\boldsymbol{R}$, $\boldsymbol{R}^{n-1}$ containing the origin respectively. Let $\varphi\in C^\infty(\Omega_{12})$ with $\varphi(0)=0$, $d\varphi(0)=0$ and set $\Phi(x)=(x_1-\varphi(x'), x')$ where $x'=(x_2,\cdots,x_n)$. Then $\Phi(0)=0$ and Φ is a diffeomorphism from Ω_1 onto some open neighbourhood of the origin. Set $\Psi=\Phi^{-1}$ and let $\tilde{P}(y, D)$ be a differential operator on Ω_2 with the symbol $\tilde{P}(y,\eta)$ defined by

$$\tilde{P}(y, D)\,u(y)=[P(x, D)\,(u\circ\Phi)]\,(\Psi(y)),\ u\in C^\infty(\Omega_2)\,.$$

Then we have

$$\tilde{P}(\Phi(x_0),\eta)=\sum_\alpha\frac{1}{\alpha\,!}P^{(\alpha)}(x_0, {}^t\Phi'(x_0)\,\eta)\,D_x^\alpha\,[e^{i\langle f(x,x_0),\eta\rangle}]\,|_{x=x_0},\ x_0\in\Omega_1$$

where $f(x, x_0)=\Phi(x)-\Phi(x_0)-\Phi'(x_0)\,(x-x_0)$. If $\tilde{P}(y,\eta)=\tilde{P}_m(y,\eta)+\cdots+\tilde{P}_0(y,\eta)$ with $\tilde{P}_j$ homogeneous of degree j in ξ,

$$P_m(x, {}^t\Phi'(x)\,\eta)=\tilde{P}_m(\Phi(x),\eta)$$

$$P_{m-1}(x,\Phi'(x)\,\eta)+\sum_{|\alpha|=2}P_m^{(\alpha)}(x, {}^t\Phi'(x)\,\eta)\,D_x^\alpha\langle i\Phi(x),\eta\rangle/\alpha\,!=\tilde{P}_{m-1}(\Phi(x),\eta)\,. \quad (8.1)$$

The aim of this section is to prove the following.

Lemma 8.1. *$\tilde{P}(y, D)$ satisfies the assumptions* (i), (ii) *in Theorem* 1.1.

Proof. From the assumption (i) in Theorem 1.1, for any $\xi_0'\in\boldsymbol{R}^{n-1}\backslash(0)$ there exist an open neighbourhood U of the origin in $\boldsymbol{R}^n$ and an open conic neighbourhood Γ of ξ_0' in $\boldsymbol{C}^{n-1}\backslash(0)$ such that

$$Q_i(x,\xi)=Q_i(x, e_1)\prod_{l=1}^{m_i}(\xi_1-\lambda_{il}(x,\xi'))\quad(i=1,2)$$

for $(x,\xi')\in U\times\Gamma$ as polynomials in ξ_1 where $\lambda_{il}\in C^\infty(U\times\Gamma)$ which is holomorphic in ξ' and satisfies that $\lambda_{il}(x,\xi')\neq\lambda_{is}(x,\xi')$ for all (x,ξ') when $l\neq s$.

Since ${}^t\Phi'(0)=id$ and $\Phi(0)=0$, it is trivial that (i) also holds for $\tilde{P}(y, D)$. Thus we shall show that (ii) holds for $\tilde{P}(y, D)$. Assume that $\tilde{P}_m=\partial_{\eta_1}\tilde{P}_m=\partial^2_{\eta_1}$

$\tilde{P}_m=0$ at $(0, \xi_0)\in \boldsymbol{R}^n\times(\boldsymbol{C}\setminus\boldsymbol{R})\times(\boldsymbol{R}^{n-1}\setminus(0))$. Then $P_m=\partial_{\xi_1} P_m=\partial^2_{\xi_1} P_m=0$ at $(0, \xi_0)$. This implies that $\lambda_{1l_1}(0, \xi'_0)=\lambda_{2l_2}(0, \xi'_0)=\xi_{01}$ for some $l_i\in\{1, \cdots, m_i\}$ $(i=1, 2)$. Set

$$\lambda_{1l_1}=\lambda\,,\quad \lambda_{1l_1}-\lambda_{2l_2}=c\,,$$
$$q(x, \xi)=P_m(x, e_1)\prod_{l\neq l_1}(\xi_1-\lambda_{1l}(x, \xi'))^2\prod_{l\neq l_2}(\xi_1-\lambda_{2l}(x, \xi'))\,.$$

Then

$$P_m=(\xi_1-\lambda)^2(\xi_1-\lambda+c)\,q,\ (x, \xi')\in U\times\Gamma,\ \xi_1\in\boldsymbol{C}\,,$$
$$\lambda(0, \xi'_0)=\xi_{01},\ c(0, \xi'_0)=0,\ q(0, \xi_0)\neq 0\,.$$

Let $\psi\in C^\infty(\Omega_1)$ with $d\psi(0)=(1, 0, \cdots, 0)$. Then

$$\begin{aligned}\{P_m, \psi\}&=3(\psi'_{x_1}-\{\lambda, \psi\})(\xi_1-\lambda)(\xi_1-\lambda+\frac{2}{3}c)\,q\\&+(\xi_1-\lambda)^2(\xi_1-\lambda+c)\{q, \psi\}\\&+(\xi_1-\lambda)^2\{c, \psi\}\,q\,.\end{aligned}\tag{8.2}$$

Here, by definition, $\{f, g\}(x, \xi)=\sum_{j=1}^n(\partial_{\xi_j}f\partial_{x_j}g-\partial_{x_j}f\partial_{\xi_j}g)(x, \xi)$ for C^∞-functions f, g in an open set of $\boldsymbol{R}^n\times\boldsymbol{C}^n$, which are holomorphic in ξ. Set

$$\begin{aligned}F(\sigma, z, u, (x, \xi'))&=3(\psi'_{x_1}(x)-\{\lambda, \psi\}(x, \xi'))(\sigma+\frac{2}{3})\,q(x, (\lambda(x, \xi')+\sigma z, \xi'))\\&+\sigma(\sigma+1)\,z\{q, \psi\}(x, (\lambda(x, \xi')+\sigma z, \xi'))\\&+q(x, (\lambda(x, \xi')+\sigma z, \xi'))\,u\sigma\end{aligned}\tag{8.3}$$

for $(\sigma, z, u)\in\boldsymbol{C}^3,\ (x, \xi')\in U\times\Gamma$.

Then

$$F(-\frac{2}{3}, 0, 0, (x, \xi'))=0 \text{ on } U\times\Gamma\,,$$
$$\partial_\sigma F(-\frac{2}{3}, 0, 0, (0, \xi'_0))\neq 0\,.$$

Thus from the implicit function theorem and the uniqueness of the implicit function, there exists a C^∞-function $\sigma(z, u, (x, \xi'))$ on an open set $V=V_1\times V_2$ in $\boldsymbol{C}^2\times U\times\Gamma$ with $V_1\subset\boldsymbol{C}^2$, $V_2\subset U\times\Gamma$ containing $(0, 0, (0, \xi'_0))$ such that

$$F(\sigma(z, u, (x, \xi')), z, u, (x, \xi'))=0 \text{ on } V_1\,,\tag{8.4}$$

$$\sigma(0, 0, (x, \xi'))=-\frac{2}{3} \text{ on } V_2\,.\tag{8.5}$$

We may assume that

$$(c(x, \xi'), \{c, \psi\}(x, \xi')) \in V_1 \text{ when } (x, \xi') \in V_2. \tag{8.6}$$

Noting this we set

$$a(x, \xi') = \sigma(c(x, \xi'), \{c, \psi\}(x, \xi'), (x, \xi')) \quad \text{for} \quad (x, \xi') \in V_2.$$

Then (8.2)∼(8.4) and (8.6) imply that

$$\{P_m, \psi\}(x, ((\lambda + ac)(x, \xi'), \xi')) = 0 \text{ on } V_2. \tag{8.7}$$

Since $\sigma(z, u, (x, \xi'))$ is holomorphic in (z, u), (8.5) implies that there exists an open subset W of V_2 containing $(0, \xi_0')$ such that with some $a_i \in C^\infty(W)$ $(i=1, 2)$

$$a(x, \xi') = -\frac{2}{3} + c(x, \xi')\, a_1(x, \xi') + \{c, \psi\}(x, \xi')\, a_2(x, \xi') \text{ on } W. \tag{8.8}$$

Since $a(0, \xi_0') = -\frac{2}{3}$, we may assume that

$$\left| a(x, \xi') + \frac{2}{3} \right| < \frac{1}{10} \text{ on } W. \tag{8 9}$$

Since $\{P_m, \psi\}(0, (\xi_1, \xi_0')) = \partial_{\xi_1} P_m(0, (\xi_1, \xi_0'))$, the degree of a polynomial $\{P_m, \psi\}(x, \xi)$ in ξ_1 is constant for (x, ξ') in an open subset W_1 of W containing $(0, \xi_0')$ where both of $\xi_1 = \lambda(x, \xi')$ and $\xi_1 = (\lambda + ac)(x, \xi')$ are solutions of the equation $\{P_m, \psi\}(x, \xi) = 0$ from (8.2) and (8.7). Since $\xi_1 = \lambda(x, \xi')$ is a double root of this equation for $(x, \xi') \in W_1$ with $c(x, \xi') = 0$, and since $\lambda(x, \xi')$ and $(\lambda + ac)(x, \xi')$ are distinct for $(x, \xi') \in W_1$ with $c(x, \xi') \neq 0$ because of (8.9), we have that

$$\{P_m, \psi\}(x, \xi') = (\xi_1 - (\lambda + ac)(x, \xi'))(\xi_1 - \lambda(x, \xi'))\, q_1(x, \xi), \ (x, \xi') \in W_1 \tag{8.10}$$

as polynomials in ξ_1 where q_1 is a polynomial in ξ_1 with coefficients in $C^\infty(W_1)$.

Let $\hat{W}_1$ be the intersection of W_1 and $\{(x, \xi') \in \boldsymbol{R}^n \times \boldsymbol{C}^{n-1};\ |\xi'| = |\xi_0'|\}$ and let $\tilde{W}_1$ be an open cone generated by $\hat{W}_1$. We extend the restrictions of functions a, a_1, a_2 to $\hat{W}_1$ to functions on $\tilde{W}_1$ being homogeneous degree 0, -1, 0 in ξ' respectively and we also extend the restriction of q_1 to $\boldsymbol{C} \times \hat{W}_1$ to function on $\boldsymbol{C} \times \tilde{W}_1$ being homogeneous degree $m-3$ in ξ. Then using homogeneity of c, $\{c, \psi\}$, $\{P_m, \psi\}$ we see that (8.8)∼(8.10) also hold on $\tilde{W}_1$ when we replace a, a_1, a_2, q_1 by their extentions in the above. Moreover since multiplicities of the characteristic roots of P_m are at most triple, we see from (8.10) $q_1(0, (\lambda(0, \xi_0'), \xi_0')) \neq 0$.

Thus taking $\psi(x) = \psi_0(x) \equiv x_1 - \varphi(x')$ and $\psi(x) = x_1$ we see that there exist an open subset U_1 of U containing the origin, an open conic subset Γ_1 of Γ containing ξ_0', and an open conic subset $\tilde{\Gamma}$ of $C\times\Gamma$ containing $(\lambda(0,\xi_0'),\xi_0')$ such that the following factorization of $\partial_{\xi_1}P_m$ and $\{P_m,\psi_0\}$ holds:

$$\begin{aligned}\partial_{\xi_1}P_m(x,\xi)&=(\xi_1-(\lambda+a_0c)(x,\xi'))(\xi_1-\lambda(x,\xi'))\tilde{q}(x,\xi)\,,\\ \{P_m,\psi_0\}(x,\xi)&=(\xi_1-(\lambda+bc)(x,\xi'))(\xi_1-\lambda(x,\xi'))\tilde{\tilde{q}}(x,\xi)\end{aligned}\tag{8.11}$$

for $(x,\xi')\in U_1\times\Gamma_1$ as polynomials in ξ_1 where a_0, $b\in C^\infty(U_1\times\Gamma_1)$ and $\tilde{\tilde{q}},\tilde{q}\in C^\infty(U_1\times\Gamma)$ satisfying that

$$\begin{aligned}&|a_0+\frac{2}{3}|<\frac{1}{10}\,;\\ &a_0=-\frac{2}{3}+ca_{01}\,,\\ &b=-\frac{2}{3}+cb_1+\{c,\psi_0\}\,b_2\end{aligned}\tag{8.12}$$

with some a_{01}, b_1, $b_2\in C^\infty(U_1\times\Gamma_1)$ which are homogeneous degree -1, -1, 0 in ξ' respectively; q and $\tilde{q}$ are homogeneous degree $m-3$, and

$$C|\tilde{q}(x,\xi)|\geq|\xi|^{m-3}\quad\text{and}\quad C|\tilde{\tilde{q}}(x,\xi)|\geq|\xi|^{m-3}\text{ on }U_1\times\tilde{\Gamma}\tag{8.13}$$

for some positive constant C.

We may assume, decreasing U_1 and $\tilde{\Gamma}$ if necessary, that

the inequality in the assumption (ii) in Theorem 1.1 holds when $(x,\xi)\in U_1\times\tilde{\Gamma}$ and $\partial_{\xi_1}P_m(x,\xi)=0$; (8.14)

$$C|q(x,\xi)|\geq|\xi|^{m-3}\text{ on }U_1\times\tilde{\Gamma}\quad\text{for some}\quad C>0\,.\tag{8.15}$$

We define τ_0, $\tau\in C^\infty(U_1\times\Gamma_1)$ by

$$\tau_0(x,\xi')=(\lambda+a_0c)(x,\xi')\text{ and }\tau(x,\xi')=(\lambda+bc)(x,\xi')\text{ for }(x,\xi')\in U_1\times\Gamma_1\,.$$

To prove that the assumption (ii) holds for $\tilde{P}(x,D)$ we must show that there exists an open conic neighbourhood $\tilde{\tilde{\Gamma}}\subset\tilde{\Gamma}$ of $(0,\xi_0)$ in $\mathbf{R}^n\times\mathbf{C}^n$ such that

$$\begin{aligned}&(|(\partial_\eta\tilde{P}_m)(y,\eta)||\eta|+|(\partial_y\tilde{P}_m)(y,\eta)|)|\tilde{P}_{m-1}(y,\eta)|\\ &\quad\leq C|\tilde{P}_m(y,\eta)|^{2/3}(|P_m(y,\eta)|^{1/3}|\eta|^{m-1}+|(\tilde{P}_m+\tilde{P}_{m-1})(y,\eta)||\eta|^{m/3}\\ &\qquad+|\eta|^{(4m/3)-(3/2)}+1)\end{aligned}$$

when $(y,\eta)=(\Phi(x),{}^t\Phi'(x)^{-1}\xi)$ with some $(x,\xi)\in\tilde{\tilde{\Gamma}}$ satisfying

$$\{P_m,\psi_0\}(x,\xi)=0\,.\tag{8.16}$$

From (8,1) it is easy to see that (8.16) follows if we prove that there exists an open conic neighbourhood $\tilde{\tilde{\Gamma}}\subset\tilde{\Gamma}$ of $(0, \xi_0)\in \boldsymbol{R}^n\times\boldsymbol{C}^n$ such that

$$\begin{aligned}&(|(\partial_\xi P_m)(x,\xi)|\,|\xi|+|(\partial_x P_m)(x,\xi)|)(|P_{m-1}(x,\xi)|+\sum_{|\alpha|=2}|P_m^{(\alpha)}(x,\xi)|\,|\xi|)\\&\quad\leq|P_m(x,\xi)|^{2/3}(|P_m(x,\xi)|^{1/3}|\xi|^{m-1}\\&\qquad+|(P_m+P_{m-1})(x,\xi)+\sum_{|\alpha|=2}P_m^{(\alpha)}(x,\xi)\,D_x^\alpha<i\Phi(x),\,{}^t\Phi'(x)^{-1}\xi>/\alpha!|\,|\xi|^{m/3}\\&\qquad+|\xi|^{(4m/3)-(3/2)}+1)\end{aligned}\tag{8.17}$$

when $(x, \xi)\in\tilde{\tilde{\Gamma}}$ and $\{P_m, \psi_0\}(x, \xi)=0$. Note that (8.11), (8.13), and the definition of $\tau(x, \xi')$ imply that

$$\{P_m, \psi_0\}=0 \text{ if and only if } \xi_1=\lambda(x, \xi') \text{ or } \xi_1=\tau(x, \xi')$$

when $(x, \xi')\in\tilde{\Gamma}$.

Thus since the inequality in (8.17) is trivial when $(x, \xi)\in\tilde{\Gamma}$ and $\xi_1=\lambda(x, \xi')$, it suffices for us to show that

there exists an open neighbourhood $U_0\subset U_1$ of the origin in $\boldsymbol{R}^n$ an open conic neighbourhood $\Gamma_0\subset\Gamma_1$ of ξ_0' in $\boldsymbol{C}^{n-1}$ such that the inequality in (8.17) holds when $(x, \xi')\in U_0\times\Gamma_0$ and $\xi_1=\tau(x, \xi')$. (8.18)

Indeed, if (8.18) is proved, (8.17) holds with $\tilde{\tilde{\Gamma}}=\tilde{\Gamma}\cap(U_0\times(\boldsymbol{C}\times\Gamma_0))$.

We shall show (8.18). Let us choose an open neighbourhood U_2 of the origin in $\boldsymbol{R}^n$ and an open conic neighbourhood Γ_2 of ξ_0' in $\boldsymbol{C}^{n-1}\backslash(0)$ so that

$$U_2\subset\subset U_1,\ \Gamma_2\cap\{\xi'\in\boldsymbol{C}^{n-1};\ |\xi'|=1\}\subset\subset\Gamma_1\ ;\tag{8.19}$$

$$(\tau_0(x, \xi'), \xi'),\ (\tau(x, \xi'), \xi')\in\tilde{\Gamma} \text{ when } (x, \xi')\in U_2\times\Gamma_2.\tag{8.20}$$

Sublemma 8.1. *The following estimates holds on* $U_2\times\Gamma_2$.

$$C|P_m|\,|_{\xi_1=\tau_0}\geq|c|^3|\xi'|^{m-3}\geq C^{-1}|P_m|\,|_{\xi_1=\tau_0}\tag{8.21}$$

$$|(P_m)^{(\beta)}_{(\alpha)}|_{\xi_1=\tau}-(P_m)^{(\beta)}_{(\alpha)}|_{\xi_1=\tau_0}|\leq C(|c|\,|\xi'|^{-1}+|\{c, \psi_0\}|)|c|^{3-|\alpha|-|\beta|}|\xi'|^{m-3+|\alpha|}\tag{8.22}$$

if $|\alpha|+|\beta|\leq 2$.

$$|(P_m)^{(0,\beta)}_{(\alpha)}|\,|_{\xi_1=\tau_0}\geq C_1|c^{(\beta)}_{(\alpha)}|\,|c|^2|\xi'|^{m-3}-C_2|c|^3|\xi'|^{m-3-|\beta|}\tag{8.23}$$

if $\alpha\in\boldsymbol{Z}^n_+$, $\beta\in\boldsymbol{Z}^{n-1}_+$ *with* $|\alpha|+|\beta|=1$. *Here, constants* C, C_1, C_2 *are all positive.*

Proof of Sublemma 8.1. (8.21) immediately follows from (8.12). To show the next two inequalities we observe that

$$|(P_m)^{(\beta)}_{(\alpha)}|\,|_{\xi_1=\tau_0} \leq C|\xi'|^{m-3+|\alpha|}|c|^{3-|\alpha|-|\beta|} \text{ on } U_2\times\Gamma_2 \tag{8.24}$$

if $|\alpha|+|\beta|\leq 2$.

$$|\tau-\tau_0|\leq C(|c|\,|\xi'|^{-1}+|\{c, \psi_0\}|)|c| \text{ on } U_2\times\Gamma_2\,. \tag{8.25}$$

Then we obtain (8.22) by Tayolr expansion of $(P_m)^{(\beta)}_{(\alpha)}$ in ξ_1 at $\xi_1=\tau_0$, substituting τ for ξ_1, and estimating each term in the expansion by (8.24) and (8.25) except for $(P_m)^{(\beta)}_{(\alpha)}|_{\xi_1=\tau_0}$. Finally (8.23) immediately follows from (8.12) and the equality that for α, β as in (8.23).

$$(P_m)^{(0,\beta)}_{(\alpha)} = -3\lambda^{(\beta)}_{(\alpha)}(\xi_1-\lambda)(\xi_1-\lambda+\frac{2}{3}c)q+c^{(\beta)}_{(\alpha)}(\xi_1-\lambda)^2 q$$
$$+(\xi_1-\lambda)^2(\xi_1-\lambda+c)\,q^{(0,\beta)}_{(\alpha)}\,. \qquad \text{Q.E.D.}$$

Since $c=\{c, \psi_0\}=0$ at $(0, \xi_0')$, from (8.21), (8.22), and (8.25) there exists an open nighbourhood $U_3\subset U_2$ of the origin in $\boldsymbol{R}^n$ and an open conic neighbourhood $\Gamma_3\subset\Gamma_2$ of ξ_0' in $\boldsymbol{C}^n$ such that for some positive constant C

$$C|P_m|\,|_{\xi_1=\tau}\geq|c|^3|\xi'|^{m-3}\geq C^{-1}|P_m|\,|_{\xi_1=\tau} \text{ on } U_3\times\Gamma_3\,. \tag{8.26}$$

From (8.25) we have that

$$|P_{m-1}|_{\xi_1=\tau}-P_{m-1}|_{\xi_1=\tau_0}|$$
$$\leq C(|c|\,|\xi'|^{-1}+|\{c, \psi_0\}|)|c|\,|\xi'|^{m-2} \text{ on } U_2\times\Gamma_2\,. \tag{8.27}$$

Using (8.21), (8.22), (8.24), (8.26), (8.27) we see that on $U_3\times\Gamma_3$

$$(|\partial_\xi P_m|\,|\xi|+|\partial_x P_m|)(|P_{m-1}|+\sum_{|\alpha|=2}|P^{(\alpha)}_m|\,|\xi|)|_{\xi_1=\tau}$$
$$\leq C\{(|\partial_\xi P_m|\,|\xi|+|\partial_x P_m|)|P_{m-1}|+|P_m|\,|\xi|^{m-1}\}|_{\xi_1=\tau_0} \tag{8.28}$$

$$|P_m|^{2/3}\sum_{|\alpha|=2}|P^{(\alpha)}_m|\,|\xi|^{(m/3)+1}|_{\xi_1=\tau}\leq C|P_m|\,|\xi|^{m-1}|_{\xi_1=\tau}\,. \tag{8.29}$$

In the same way one can deduce that on $U_3\times\Gamma_3$

$$|P_m|^{2/3}|P_m+P_{m-1}|\,|\xi|^{m/3}|_{\xi_1=\tau_0}$$
$$\leq C_1|P_m|^{2/3}|P_m+P_{m-1}|\,|\xi|^{m/3}|_{\xi_1=\tau}$$
$$+C_2(|c|\,|\xi'|^{-1}+\{c, \psi_0\}|)(|c|^3|\xi'|^{m-3})^{2/3}(|P_m|\,|\xi|^{m/3})|_{\xi_1=\tau}$$
$$+C_3(|c|\,|\xi'|^{-1}+|\{c, \psi\}|)|\xi'|^{m-1}|P_m|\,|_{\xi_1=\tau}$$
$$\leq C_4|P_m|^{2/3}|P_m+P_{m-1}|\,|\xi|^{m/3}|_{\xi_1=\tau}$$
$$+C_2(|c|\,|\xi'|^{-1}+|\{c, \psi_0\}|)|c|^2|\xi'|^{m-2}|P_{m-1}|\,|_{\xi_1=\tau}+C_5|P_m|\,|\xi|^{m-1}|_{\xi_1=\tau}\,.$$

From (8.23) and (8.26) the middle term on the right of the second inequlaity can be dominated on $U_3\times\Gamma_3$ by a constant multiple of

$$|P_m||\xi|^{m-1}|_{\xi_1=\tau}+|d\psi_0||\xi||\partial_\xi P_m||P_{m-1}||_{\xi_1=\tau_0}.$$

Thus we get with another constants

$$\begin{aligned}&|P_m|^{2/3}|P_m+P_{m-1}||\xi|^{m/3}|_{\xi_1=\tau_0}\\&\leq C_1(|P_m|^{2/3}|P_m+P_{m-1}||\xi|^{m/3}+|P_m||\xi|^{m-1})|_{\xi_1=\tau}\\&\quad+C_2|d\psi_0||\xi||\partial_\xi P_m||P_{m-1}||_{\xi_1=\tau_0}\end{aligned}\tag{8.30}$$

on $U_3\times\Gamma_3$.

From (8.11) and (8.14) the inequality in (ii) in Theorem 1.1 holds when $(x, \xi')\in U_3\times\Gamma_3$ and $\xi_1=\tau_0$. Thus conbining this inequality, (8.21), (8.26) ~ (8.30) we obtain

$$\begin{aligned}&(|\partial_\xi P_m||\xi|+|\partial_x P_m|)(|P_{m-1}|+\sum_{|\alpha|=2}|P_m^{(\alpha)}||\xi|)|_{\xi_1=\tau}\\&\leq C_1\{(|\partial_\xi P_m||\xi|+|\partial_x P_m|)|P_{m-1}|+|P_m||\xi|^{m-1}\}|_{\xi_1=\tau_0}\\&\leq C_2\{|P_m|^{2/3}(|P_m|^{1/3}|\xi|^{m-1}+|P_m+P_{m-1}||\xi|^{m/3}+|\xi|^{4m/3-3/2}+1\}|_{\xi_1=\tau}\\&\quad+C_3|d\psi||\xi||\partial_\xi P_m||P_{m-1}||_{\xi_1=\tau_0}\end{aligned}\tag{8.31}$$

on $U_3\times\Gamma_3$.

Moreover we see from (8.29) that on $U_3\times\Gamma_3$

$$\begin{aligned}&|P_m|^{2/3}|P_m+P_{m-1}||\xi|^{m/3}|_{\xi_1=\tau}\\&\leq C_1|P_m|^{2/3}|P_m+P_{m-1}+\sum_{|\alpha|=2}P_m^{(\alpha)}D_x^\alpha\langle i\Phi(x), {}^t\Phi'(x)^{-1}\xi\rangle/\alpha!||\xi|^{m/3}|_{\xi_1=\tau}\\&\quad+C_2|P_m||\xi|^{m-1}|_{\xi_1=\tau}.\end{aligned}$$

This inequality, (8.31), and that $d\psi(0)=0$ immediately imply that there exists an open neighbourhood $U_4\subset U_3$ of the origin such that the inequality in (8.17) holds when $(x, \xi')\in U_4\times\Gamma_3$ and $\xi_1=\tau$. Thus, the proof of Lemma 8.1 is complete.

§9. Proof of Theorem 1.1

From Lemma 8.1, to prove Theorem 1.1 it suffices to show the existence of an open neighbourhood $\Omega'\subset\Omega$ of the origin such that every $u\in C^\infty(\Omega)$ satisfying $P(x, D)u=0$ in Ω and $u|_{x_1\leq|x'|^2}=0$ vanishes in Ω, where $P(x, D)$ and Ω are as in Theorem 1.1.

In case that $m_1=0$ or $m_2=0$ Theorem 1.1 was proved by Calderón [2], Mizohata [5], and Hörmander [4]. Thus we only have to prove Theorem 1.1 in case that $m_1\geq 1$ and $m_2\geq 1$. In this case the theorem follows from the following.

Lemma 9.1. *Let $P(x, D)$ and Ω be as in Theorem 1.1. Assume that $m_i \geq 1$ $(i=1, 2)$. Then there exist positive constants δ_0, τ_0, C_0 such that when $\tau T^2 > \tau_0$ and $T^{-1} > \tau_0$,*

$$||u||^{(\tau)}_{m,T} \leq C_0 ||P(x, D)\, u||^{(\tau)}_T, \quad u \in C_0^\infty(B_{\delta_0}(0)) \cap \mathcal{S}_{T/2}(\mathbf{R}^n)\,.$$

Here, $B_r(0)$ is the open ball with the center at the origin in $\mathbf{R}^n$ and the radius r, and by definition

$$||u||^{(\tau)}_{s,T} = \sum_{\substack{i+2j \leq 2s \\ i,j \in \mathbf{Z}_+}} \tau^{m-3/2-i/2-j}\, T^{m-3-i/2-j}\, ||E_{i/2}\, D_1^j\, u||^{(\tau)}_T$$
$$\textit{for} \quad u \in \mathcal{S}(\mathbf{R}^n),\ s = 0, \cdots, m\,.$$

Proof. We may assume that $Q_i(x, e_1)=1$ $(i=1, 2)$, for $P_m(x, e_1)^{-1} P(x, D)$ also satisfies the assumption in Theorem 1.1.

Let $\xi_0' \in \mathbf{R}^{n-1} \backslash (0)$. Then the possible cases are the following (i), (ii).

Case (i). Two equations $Q_i(0, \tau, \xi_0')=0$ $(i=1, 2)$ have no common root.

Then there exist an open neighbourhood U of the origin in $\mathbf{R}^n$ and an open conic neighbourhood Γ of ξ_0' in $\mathbf{R}^{n-1} \backslash 0$ such that in $U \times \Gamma$ we can write

$$P_m(x, \xi) = \prod_{j=1}^{m_1} (\xi_1 - \lambda_j(x, \xi'))^2 \prod_{j=m_1+1}^{m_1+m_2} (\xi_1 - \lambda_j(x, \xi'))\,. \tag{9.1}$$

Here $\lambda_j \in C^\infty(U \times \Gamma)$, $(1 \leq j \leq m_1 + m_2)$ are homogeneous degree 1 in ξ' being at every point, pairwise distinct and non-real. Choose a C^∞-mapping $\Xi(\xi')$ from $\mathbf{R}^{n-1}$ to Γ such that $\Xi(\xi')=\xi'$ if ξ' lies in a conic nieghbourhood of ξ_0' and $|\xi'|>1$, and such that $\Xi(\xi')$ is homogeneous degree 1 in ξ' when $|\xi'|>1$ and satisfies that $|\Xi(\xi')| \geq C(1+|\xi'|)$. Let $\psi \in C^\infty(\mathbf{R}^n)$ with *supp* $\psi \subset U$, $\psi=1$ in a neighbourhood of 0, $0 \leq \psi \leq 1$. Set $\Psi(x)=\psi(x)\, x$. We set $\tilde{P}_m(x, \xi) = P_m(\Psi(x), (\xi_1, \Xi(\xi')))$, $\tilde{\lambda}_j(x, \xi')=\lambda_j(\Psi(x), \Xi(\xi'))$. Then $\tilde{\lambda}_j \in S^1_{1,0}(\mathbf{R}^n \times \mathbf{R}^{n-1})$, $C\,|Im\, \tilde{\lambda}_j(x, \xi')| \geq 1+|\xi'|$, and $\lambda_j(x, \xi')=\tilde{\lambda}_j(x, \xi')$ and $\tilde{P}_m(x, \xi)=P_m(x, \xi)$ for (x, ξ') in a conic neighbourhood of $(0, \xi_0')$.

Case (ii). Two euqations $Q_i(0, \tau, \xi_0')=0$ $(i=1, 2)$ have exactly $r(\geq 1)$ common roots.

Then there exist an open neighboruhood U of the origin in $\mathbf{R}^n$ and an open conic neighbourhood Γ of ξ_0' in $\mathbf{R}^{n-1} \backslash (0)$ such that in $U \times \Gamma$ we can write

$$P_m(x, \xi) = \prod_{j=1}^{m_1+m_2-r} p_j(x, \xi)$$

where

$$p_j(x,\xi)=\begin{cases}(\xi_1-\lambda_j(x,\xi'))^2\,(\xi_1-\lambda_j(x,\xi')+c_j(x,\xi')) & (1\le j\le r)\\ (\xi_1-\lambda_j(x,\xi'))^2 & (r<j\le m_1)\\ \xi_1-\lambda_j(x,\xi') & (m_1<j\le m_1+m_2-r)\end{cases}$$

with $\lambda_j, c_j\in C^\infty(U\times\Gamma)$ which are homogeneous degree 1 and satisfy that $\lambda_i(1\le j\le m_1+m_2-r)$ are non-real and distinct everywhere, λ_j and λ_i-c_i are distinct everywhere if $i\neq j$, and $|Im\,\lambda_j|\ge 2|c_j|$ for $j=1,\cdots,r$. Since equations $p_j(x,\xi)=0$ in ξ_1 have no common zero for any $(x,\xi')\in U\times\Gamma$, we can write

$$P_{m-1}(x,\xi)=\sum_{i=1}^{m_1+m_2-r}q_i(x,\xi)\prod_{j\neq i}p_j(x,\xi)\,,$$

$$(P_m+P_{m-1})(x,\xi)=\prod_{i=1}^{m_1+m_2-r}(p_i(x,\xi)+q_i(x,\xi))+s(x,\xi) \tag{9.2}$$

where with the notation that l_i=the degree of p_i as polynomial in ξ_1, q_i is a polynomial in ξ_1 of degree l_i-1 with coefficients in $C^\infty(U\times\Gamma)$ and homogeneous degree l_i-1 in ξ, and s is a polynomial in ξ_1 of degree $m-2$ such that the coefficient of ξ_1^k is a sum of functions in $C^\infty(U\times\Gamma)$ which are homogeneous in ξ' of degree $k,\cdots,k+2-\min(r,k+2)$.

Then there exist an open subset U_1 of U containing the origin and an open conic subset Γ_1 of Γ containing ξ_0' such that for any $i=1,\cdots,r$ we have that

$$\begin{aligned}&|q_i(x,\xi)|(|\partial_x p_i(x,\xi)|+|\partial_\xi p_i(x,\xi)|\,|\xi|)\\ &\quad\le C|p_i(x,\xi)|^{2/3}(|p_i(x,\xi)|^{1/3}|\xi|^2+|(p_i+q_i)(x,\xi)|\,|\xi|+|\xi|^{5/2}+1)\\ &\text{if}\quad (x,\xi)\in U_1\times(\boldsymbol{C}\times\Gamma_1)\quad\text{and}\quad \partial_{\xi_1}p_i(x,\xi)=0\,.\end{aligned} \tag{9.3}$$

Indeed, from the proof of Lemma 8.1 there exist an open neighbourhood U_1 of the origin in $\boldsymbol{R}^n$ with $U_1\subset\subset U$ and an open conic neighbourhood Γ_1 of ξ_0' in $\boldsymbol{R}^{n-1}\backslash(0)$ with $\Gamma_1\cap S^{n-2}\subset\subset\Gamma$ such that for any $i=1,\cdots,r$ there exists $a_i\in C^\infty(U_1\times\Gamma_1)$ which is homogeneous degree 0 in ξ' and satisfying the following:

$$a_i(0,\xi_0')=-\frac{2}{3}\,; \tag{9.4}$$

with the notation that $\tau_i=\lambda_i+a_i c_i$, the inequality in the assumption (ii) in Theorem 1.1 holds when $(x,\xi')\in U_1\times\Gamma_1$ and $\xi_1=\tau_i(x,\xi')$, and the inequalities (8.21) and (8.23) hold on $U_1\times\Gamma_1$ with instead of τ_0. (9.5)

From (9.4) and (9.5) we may assume that on $U_1\times\Gamma_1$

$$C|p_i||_{\xi_1=\tau_j}\geq|\xi'|^{l_i} \text{ if } i\neq j\,,$$
$$C|c_i|^3\geq|p_i||_{\xi_1=\tau_i}\geq C^{-1}|c_i|^3\,. \tag{9.6}$$

Using (9.5) and (9.6) one can easily see that for any $i=1,\cdots,r$ on $U_1\times\Gamma_1$

$$\begin{aligned}&(|\partial_x c_i|+|\partial_{\xi'} c_i|\,|\xi'|)|c_i|^2|q_i||_{\xi_1=\tau_i}\\&\leq C|c_i|^2(|c_i|\,|\xi'|^2+|p_i+q_i||_{\xi_1=\tau_i}|\xi'|+(1+|\xi'|)^{5/2})\,.\end{aligned} \tag{9.7}$$

Note that with notation that $\tau_{i0}=\lambda_i-\dfrac{2}{3}c_i$

$$\partial_{\xi_1}p_i=3(\xi_1-\lambda_i)(\xi_1-\tau_{i0})\,.$$

Then using this inequality and that $(p_i)^{(\alpha)}_{(0,\beta)}|_{\xi_1=\tau_{i0}}=\dfrac{4}{9}(c_i)^{(\alpha)}_{(\beta)}c_i^2$ if $\alpha\in\boldsymbol{Z}^n_+$, $\beta\in\boldsymbol{Z}^{n-1}_+$ with $|\alpha|+|\beta|=1$, and using Taylor expansion of p_i and q_i in ξ_1 at $\xi_1=\tau_{i0}$ we obtain (9.3) from (9.7). Let us choose mappings $\Xi(\xi')$ and $\Psi(x)$ as in (i) for Γ_1 and U_1 instead of Γ and U. Then we define $\tilde{P}_m$, $\tilde{P}_{m-1}$, $\tilde{\lambda}_j$, $\tilde{c}_j$, $\tilde{p}_j$, $\tilde{q}_j$, $\tilde{s}$ as in the same way in case (i). Then (9.2) holds for (x,ξ') in a conic neighbourhood of $(0,\xi_0')$ with $\tilde{p}_j$, $\tilde{q}_j$, $\tilde{s}$ instead of p_j, q_j, s, and $\tilde{p}_j$ and $\tilde{q}_j$ satisfy the assumption for p and q respectively in Proposition 1.1.

Now we prove the following lemma.

Lemma 9.2. *Assume notations in the above arguments. Then we have the following estimate in the above cases* (i), (ii),

Case (i). *If τT^2 and T^{-1} are large, for $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$*

$$\begin{aligned}&\tau^{1/2}T||u||^{(\tau)}_{m,T}+T^{-1/2}\sum_{|\alpha|=1}(||E_{1/2}(\tilde{P}_m)^{(\alpha)}(x,D)u||^{(\tau)}_T+||E_{-1/2}(\tilde{P}_m)_{(\alpha)}(x,D)u||^{(\tau)}_T)\\&\leq C||\tilde{P}_m(x,D)u||^{(\tau)}_T\,.\end{aligned}$$

Case (ii). *If τT^2 and T^{-1} are large, for $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$*

$$\begin{aligned}&||u||^{(\tau)}_{m,T}+T^{-1/2}\sum_{|\alpha|=1}(||E_{1/2}Q^{(\alpha)}(x,D)u||^{(\tau)}_T+||E_{-1/2}Q_{(\alpha)}(x,D)u||^{(\tau)}_T)\\&\leq C||Q(x,D)u||^{(\tau)}_T\,.\end{aligned}$$

where $Q(x,\xi)=\prod_{i=1}^{m_1+m_2-r}(\tilde{p}_i+\tilde{q}_i)(x,\xi)$.

Proof of Lemma 9.2. First we prove the estimate in case (i). The inequlaity for $||u||^{(\tau)}_{m,T}$ is well-known. (See [8]). We shall show that the one for $(\tilde{P}_m)_{(\alpha)}$. This also contains nothing new.

Set $\tilde{Q}_j(x,\xi)=(\xi_1-\lambda_j(x,\xi'))^2$ for $1\leq j\leq m_1$, $\tilde{Q}_j(x,\xi)=\xi_1-\lambda_j(x,\xi')$ for

$m_1+1\leq j\leq m_1+m_2$.

From Proposition 3.2 we have that if τT^2 and T^{-1} are large,

$$T^{-1/2}\sum_{|\alpha|+|\beta|=1}||E_{(|\alpha|-|\beta|)/2}(\tilde{Q}_i)^{(\alpha)}_{(\beta)}(x, D)\, u||^{(\tau)}_T\leq C||\tilde{Q}_i(x, D)\, u||^{(\tau)}_T,\ u\in\mathcal{S}_T(\boldsymbol{R}^n) \tag{9.8}$$

for any i.

We denote by $A_{s,k}(s\in\boldsymbol{R},\ k\in\boldsymbol{Z}_+)$ the set of functions $R(x, \xi)$ in $C^\infty(\boldsymbol{R}^n\times\boldsymbol{R}^n)$ of the form

$$R(x, \xi)=a_0(x, \xi')\,\xi_1^k+\cdots+a_k(x, \xi'),\ a_i\in S^{s-k+i}_{1,0}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-1})\,.$$

Then if $R_i\in A_{s_i,k_i}$ $(i=1, 2)$, we have that $R_1\circ R_2$, $R_1 R_2\in A_{s_1+s_2,k_1+k_2}$, and if $R\in A_{s,k}$ and α, $\beta\in\boldsymbol{Z}^n_+$ with $\alpha_1\leq k$, we have that $R^{(\alpha)}_{(\beta)}\in A_{s-|\alpha|,k-\alpha_1}$.

Using partial fraction decomposition we see that if $|\alpha|+j=m$ and $j\leq m-1$ we can write

$$\xi'^\alpha\,\xi_1^j=\sum_{k=1}^m R_k(x, \xi)\prod_{i\neq k}\tilde{Q}_i(x, \xi)$$

where $R_k\in A_{l_k,l_k-1}$.

Next using that for $|\alpha|=1$

$$\langle\xi'\rangle^{-1/2}\circ(\tilde{P}_m)_{(\alpha)}-\sum_{k=1}^m\langle\xi'\rangle^{-1/2}\circ(\tilde{Q}_k)_{(\alpha)}\circ[\prod_{i\neq k}\tilde{Q}_i]\in A_{m-(3/2),m-2}$$

and (9.8) we get that for large τT^2 and T^{-1} and $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$

$$\begin{aligned}T^{-1/2}||E_{-1/2}(\tilde{P}_m)_{(\alpha)}(x, D)\, u||^{(\tau)}_T&\leq C_1(\sum_{k=1}^m||[\tilde{Q}_k\circ\prod_{i\neq k}\tilde{Q}_i]\,(x, D)\, u||^{(\tau)}_T+T\,||u||^{(\tau)}_{m-1,T})\\&\leq C_2(||\tilde{P}_m(x, D)\, u||^{(\tau)}_T+((\tau T^2)^{1/2}+T)\,||u||^{(\tau)}_{m-1,T})\\&\leq C_3\,||\tilde{P}_m(x, D)\, u||^{(\tau)}_T\,.\end{aligned}$$

The inequality for $(\tilde{P}_m)^{(\alpha)}$ can be deduced similarly. This completes the proof in case (i).

Next, we consider case (ii). We note that $C\,|Im\,\tilde{\lambda}_i|\geq 1+|\xi'|$ and $|Im\,\tilde{\lambda}_i|\geq 2|\tilde{c}_i|$. In case (ii) we shall use Proposition 1.1.

If $m_1+m_2-r=1$, if follows that $m=3$. Thus the desired estimate is nothing but Proposition 1.1. So we may assume that $m_1+m_2-r\geq 2$.

We set $\tilde{Q}_i=\tilde{p}_i+\tilde{q}_i$ and $Q^{(k)}=\prod_{i\neq k}\tilde{Q}_i$. Since for any (x, ξ') equations $\tilde{p}_i(x, \xi)=0$ in ξ_1 $(i=1, \cdots, m_1+m_2-r)$ have no common root, from partial fraction decomposition we have that if $|\alpha|+j=m$ and $j\leq m-1$

$$\xi'^\alpha\,\xi_1^j-\sum_{k=1}^{m_1+m_2-r}R_k\circ Q^{(k)}\in A_{m-1,m-1}$$

for some $R_k \in A_{l_k, l_k - 1}$. Then from Proposition 1.1, we have that for large τT^2 and T^{-1}, and $u \in \mathcal{S}_{T/2}(\boldsymbol{R}^n)$

$$\tau^{-3/2} T^{-3} \sum_{0<|\alpha|\le m} ||D'^{\alpha} D_1^{m-|\alpha|} u||_T^{(\tau)}$$
$$\le C\Big(\sum_{k=1}^{m_1+m_2-r} ||Q_k \circ Q^{(k)}(x, D) u||_T^{(\tau)} + (\tau T)^{-1} ||u||_{m-1,T}^{(\tau)} \Big) . \tag{9.9}$$

On the other hand as in the proof of Lemma 4.3 we see that for $u \in \mathcal{S}(\boldsymbol{R}^n)$

$$||D_1^m u||_T^{(\tau)} \le ||Q(x, D) u||_T^{(\tau)} + C\Big((\tau T)^{-1} ||D_1^m u||_T^{(\tau)} + \sum_{\alpha \ne 0} ||D'^{\alpha} D_1^{m-|\alpha|} u||_T^{(\tau)}\Big) .$$

This inequality, (9.9), and that $||u||_{m,T}^{(\tau)} \le C(\tau T^2)^{-3/2} \sum_{|\alpha|\le m} ||D'^{\alpha} D_1^{m-|\alpha|} u||_T^{(\tau)}$ for $u \in \mathcal{S}_{T/2}(\boldsymbol{R}^n)$ which follows from an interpolation on Sobolev norms on x' and the first inequality in the proof of Lemma 4.3 imply that if τT^2 and T^{-1} are large

$$||u||_{m,T}^{(\tau)} \le C\Big((\tau T^2)^{-3/2} ||Q(x, D) u||_T^{(\tau)} + \sum_{k=1}^{m_1+m_2-r} ||\tilde{Q}_k \circ Q^{(k)}(x, D) u||_T^{(\tau)}\Big) \tag{9.10}$$

for $u \in \mathcal{S}_{T/2}(\boldsymbol{R}^n)$.

We have to estimate the summation on the right in (9.10). We set $Q_{ik} = \prod_{j \ne i,k} \tilde{Q}_j$. Since

$$\tilde{Q}_k \circ Q^{(k)} - Q + \sqrt{-1} \sum_{|\alpha|=1} \sum_{i \ne k} \langle \xi' \rangle^{1/2} \circ (\tilde{Q}_k)^{(\alpha)} \circ \langle \xi' \rangle^{-1/2} \circ (\tilde{Q}_i)_{(\alpha)} \circ Q_{ik} \in A_{m-2,m-2}$$

we have from Proposition 1.1 that for large τT^2 and T^{-1}, and $u \in \mathcal{S}_{T/2}(\boldsymbol{R}^n)$

$$||\tilde{Q}_k \circ Q^{(k)}(x, D) u||_T^{(\tau)} \le ||Q(x, D) u||_T^{(\tau)}$$
$$+ C\Big(\sum_{|\alpha|=1} \sum_{i \ne k} T^{1/2} ||\tilde{Q}_k(x, D) E_{-1/2} [(\tilde{Q}_i)_{(\alpha)} \circ Q_{ik}] (x, D) u||_T^{(\tau)} \tag{9.11}$$
$$+ \sum_{j \le m-2} ||E_j D_1^{m-2-j} u||_T^{(\tau)}\Big) .$$

Setting $Q_{i\alpha} = \langle \xi' \rangle^{-1/2} \circ (\tilde{Q}_i)_{(\alpha)}$ we have that $(\tilde{Q}_k \circ Q_{i\alpha} - Q_{i\alpha} \circ \tilde{Q}_k) \circ Q_{ik}$ and $Q_{i\alpha} \circ \tilde{Q}_k \circ Q_{ik} - Q_{i\alpha} \circ Q^{(i)}$ are in $A_{m-(3/2),m-2}$. Thus from Proposition 1.1 a term in the first summation on the right of (9.11) is dominated by a constant multiple of $T ||\tilde{Q}_i \circ Q^{(i)}(x, D) u||_T^{(\tau)} + \sum_{j \le m-2} ||E_{m-(3/2)-j} D_1^j u||_T^{(\tau)}$ if τT^2 and T^{-1} are large and $u \in \mathcal{S}_{T/2}(\boldsymbol{R}^n)$. Thus applying this estimate to (9.11) and summing up it in k we obtain that for large τT^2 and T^{-1}

$$||u||_{m,T}^{(\tau)} + \sum_{k=1}^{m_1+m_2-r} ||\tilde{Q}_k \circ Q^{(k)}(x, D) u||_T^{(\tau)} \le C ||Q(x, D) u||_T^{(\tau)}, \ u \in \mathcal{S}_{T/2}(\boldsymbol{R}^n) . \tag{9.12}$$

Finally using $\langle \xi' \rangle^{-1/2} \circ Q_{(\alpha)} - \langle \xi' \rangle^{-1/2} \circ [\sum_{k=1}^{m_1+m+2-r} (\tilde{Q}_k)_{(\alpha)} \circ Q^{(k)}] \in A_{m-(3/2),m-2}$

we have from Proposition 1.1 that for large τT^2 and T^{-1}, and $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$

$$||E_{-1/2}\,Q_{(\alpha)}(x, D)\,u||_T^{(\tau)}\leq C(\sum_{k=1}^{m_1+m_2-r} T^{1/2}\,||\tilde{Q}_k\circ Q^{(k)}(x, D)\,u||_T^{(\tau)}+T^{3/2}\,||u||_{m-1,T}^{(\tau)})\,.$$

Similarly $||E_{1/2}\,Q^{(\alpha)}(x, D)\,u||_T^{(\tau)}$ is dominated by the same expression as the right of the above inequality. Combining (9.12) and these estimate we obtain the desired inequlity. This completes the proof in case (ii) and therefore the proof of Lemma 9.2.

Now we can complete the proof of Lemma 9.1 by patching the estimates in Lemma 9.2. We assume the notation $A_{s,k}$ in the proof of Lemma 9.2. Choose C^∞-functions $\chi_j(\xi')$ on $\boldsymbol{R}^{n-1}$ $(j=1, \cdots, s)$ which is homogeneous degree 0 for $|\xi'|>1$ and satisfy $\sum_{j=1}^s \chi_j(\xi')=1$ for $|\xi'|>1$ so that there exists an open neighbourhood V of the origin in $\boldsymbol{R}^n$ such that for any $j\in\{1, \cdots, s\}$ there exists $R_j\in A_{m,m}$ with $R_j-\xi_1^m\in A_{m,m-1}$ such that $P(x, \xi)=R_j(x, \xi)$ on $V\times supp\ \chi_j$ and the inequality in case (ii) of Lemma 9.2 holds with R_j for Q.

Let us choose $\delta_0>0$ with $\overline{B_{2\delta_0}(0)}\subset V$. Let $\phi\in C^\infty(\boldsymbol{R}^n)$ with $\phi(t)=1$ when $|t|\leq 1$, $\phi(t)=0$ when $|t|\geq\frac{3}{2}$, and set $\tilde{P}(x, \xi)=P(\phi(\delta_0^{-1}\,|x|)\,x, \xi)$. Let $\chi_0\in C_0^\infty(B_{\delta_0}(0))$ with $\chi_0=1$ on $\overline{B_{\delta_0/2}(0)}$. We set $\varphi_j(x, \xi')=\chi_0(x)\,\chi_j(\xi')$ $(j=1, \cdots, s)$, $\varphi_0(x, \xi')=\chi_0(x)\,(1-\sum_{j=1}^s \chi_j(\xi'))+(1-\chi_0(x))$.

Then we have that $\sum_{j=0}^s \varphi_j(x, \xi')=1$, $\tilde{P}(x, \xi)=R_j(x, \xi)$ for $(x, \xi')\in supp\ \varphi_j$ $(j=1, \cdots, s)$.

Thus we have for large τT^2 and T^{-1}, and $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$ that

$$||u||_{m,T}^{(\tau)}\leq\sum_{j=0}^{s}||\varphi_j(x, D')\,u||_{m,T}^{(\tau)}\leq||\varphi_0(x, D')\,u||_{m,T}^{(\tau)}+C\sum_{j=1}^{s}||(R_j\circ\varphi_j)\,(x, D)\,u||_{m,T}^{(\tau)}\,. \tag{9.13}$$

Since $R_j\circ\varphi_j-\varphi_j\circ\tilde{P}-\sqrt{-1}\sum_{|\alpha|=1}(\langle\xi'\rangle^{-1/2}\circ R_{j(\alpha)}\circ[\varphi_j^{(\alpha)}\langle\xi'\rangle^{1/2}]-\langle\xi'\rangle^{1/2}\circ R_j^{(\alpha)}\circ[\varphi_{j(\alpha)}\langle\xi'\rangle^{-1/2}])$ is in $A_{m-2,m-1}$, we have that for any τ, T, $u\in\mathcal{S}_T(\boldsymbol{R}^n)$

$$\begin{aligned}||(R_j\circ\varphi_j)\,(x, D)\,u||_T^{(\tau)}&\\ \leq C(||\tilde{P}(x, D)\,u||_T^{(\tau)}&+T^{1/2}\sum_{|\alpha|=1}||[R_j\circ(\varphi_j^{(\alpha)}\langle\xi'\rangle^{1/2}]\,(x, D)\,u||_T^{(\tau)}\\ &+T^{1/2}\sum_{|\alpha|=1}||[R_j\circ(\varphi_{j(\alpha)}\langle\xi'\rangle^{-1/2})]\,(x, D)\,u||_T^{(\tau)}\\ &+\tau^{-1/2}\,T\,||u||_{m-2,T}^{(\tau)}+||E_{-1}\,D_1^{m-1}\,u||_T^{(\tau)})\,.\end{aligned}$$

We set $\tilde{\varphi}_{i\alpha}=\varphi_j^{(\alpha)}\langle\xi'\rangle^{1/2}$, $\tilde{\tilde{\varphi}}_{j(1)}=(\varphi_j)_{(\alpha)}\langle\xi'\rangle^{-1/2}$. Then we have that $R_j\circ\tilde{\varphi}_{j\alpha}-\tilde{\varphi}_{j\alpha}\circ\tilde{P}$ and $R_j\circ\tilde{\tilde{\varphi}}_{j\alpha}-\tilde{\tilde{\varphi}}_{j\alpha}\circ\tilde{P}$ are in $A_{m-(3/2),m-1}$. Thus we have from the above equality that for large τT^2 and T^{-1}, and $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$

$$\begin{aligned}&||(R_j\circ\varphi_j)(x, D)\,u||_T^{(\tau)}\\&\quad\leq C(||\tilde{P}(x, D)\,u||_T^{(\tau)}+T^2\,||u||_{m-1,T}^{(\tau)}+||E_{-1/2}\,D_1^{m-1}\,u||_T^{(\tau)})\,.\end{aligned}$$

As in the proof of Lemma 4.3 the last term on the right of the above inequality can be dominated by $C'\{(\tau T)^{-1}\,||\tilde{P}(x, D)\,u||_T^{(\tau)}+T^{3/2}\,||u||_{m,T}^{(\tau)}\}$ for any τ, T, $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$. Thus we get from (9.13) and the above inequality that for large τT^2 and T^{-1}, and $u\in S_{T/2}(\boldsymbol{R}^n)$

$$||u||_{m,T}^{(\tau)}\leq||\varphi_0(x, D')\,u||_{m,T}^{(\tau)}+C\,||\tilde{P}(x, D)\,u||_T^{(\tau)}\,. \tag{9.14}$$

Finally using Leibniz rule we see that for any τ, T, $u\in\mathcal{S}_T(\boldsymbol{R}^n)\cap C_0^\infty(B_{\delta_0/8}(0))$

$$||\varphi_0(x, D')\,u||_{m,T}^{(\tau)}\leq C(\tau^{-1/2}\,T\,||u||_{m-2,T}^{(\tau)}+||E_{-1}\,D_1^{m-1}\,u||_T^{(\tau)}+||E_{-2}\,D_1^m\,u||_T^{(\tau)})$$

because $\varphi_0(x, D')\,u=\varphi_0(x, D')\,(\chi_1\,u)$ for $u\in C_0^\infty(B_{\delta_0/8}(0))$ and $\varphi_0\circ\chi_1\in S^{-\infty}$ with a notation that $\chi_1(x)=\chi_0(4x)$. Again, the latter two terms on the right of the above inequality can be dominated by $C'\{(1+(\tau T)^{-1})\,||\tilde{P}(x, D)\,u||_T^{(\tau)}+\tau^{-1/2}\,T\,||u||_{m-1,T}^{(\tau)}\}$ for any τ, T, $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)$. Thus we have that for any τ, T, $u\in\mathcal{S}_{T/2}(\boldsymbol{R}^n)\cap C_0^\infty(B_{\delta_0/8}(0))$

$$||\varphi_0(x, D')\,u||_{m,T}^{(\tau)}\leq C\,\{(1+(\tau T)^{-1})\,||\tilde{P}(x, D)\,u||_T^{(\tau)}+\tau^{-1/2}\,T||u||_{m-1,T}^{(\tau)}\}\,.$$

Substituting this inequality into (9.14) we get the desired result with $\frac{\delta_0}{8}$ for δ_0 in the lemma because $\tilde{P}(x, D)\,u=P(x, D)\,u$ for $u\in C_0^\infty(B_{\delta_0}(0))$. The proof is complete.

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