Publ. RIMS, Kyoto Univ.
28 (1992), 437–453

Index for von Neumann Algebras with Finite Dimensional Centers

By

Tamotsu TERUYA*

Introduction

Extending Jones' index [J], Kosaki [Ko] defined index, denoted by Index E, for a (normal faithful) conditional expectation E of an arbitrary factor onto a subfactor, which is based on Connes' spatial theory [Co] and Haagerup's theory on operator-valued weights [Ha1, 2]. For a pair $N \subset M$ of von Neumann algebras, let $\mathcal{E}(M, N)$ denote the set of all faithful normal conditional expectations from M onto N. When $N \subset M$ are factors, Kosaki's index of $E \in \mathcal{E}(M, N)$ is defined by Index $E = E^{-1}(1)$ where E^{-1} is the operator valued weight from N' to M' determined by the equation of spatial derivatives

$$\frac{d(\phi \circ E)}{d\psi} = \frac{d\phi}{d(\psi \circ E^{-1})}$$

with faithful normal semifinite weights ϕ on N and ψ on M'. When $N \subset M$ are factors, the minimum index $[M: N]_0$ is defined by

$$[M: N]_0 = \min\{\text{Index } E;\ E \in \mathcal{E}(M, N)\}$$

(see [Hi1], [Lo], [Hav]). Furthermore, Hiai [Hi2] (also Kawakami [Kk]) defined the entropy $K_\varphi(M \mid N)$ of an arbitrary von Neumann algebra M relative to its subalgebra N and a faithful normal state φ on M such that $E \in \mathcal{E}(M, N)$ with $\varphi \circ E = \varphi$ exists, which is an extension of the entropy $H(M \mid N)$ developed by Pimsner and Popa [PP] for finite von Neumann algebras. He established the relation between the minimum index $[M: N]_0$ and the entropy $K_\varphi(M \mid N)$, including the characterization of $E \in \mathcal{E}(M, N)$ with Index $E = [M: N]_0$ by means of the entropy. On the other hand, the index theory in the non-factor case

Communicated by H. Araki, May 16, 1991.
1991 Mathematics Subject Classifications: 46L10
* Department of Mathematics, Faculty of Science, Hokkaido University, Sappoor 060, Japan

was discussed in several ways (see [BDH], [Jol], [Kk], [Wa] for instance).

In this paper, following [Ko], we shall introduce Index E of $E \in \mathcal{E}(M, N)$ for von Neumann algebras $N \subset M$ with finite dimensional centers and give a formula of Index E which is an element of the extended positive part of the center of M. Havet [Hav] also gave the same formula of the index independently, while his method is different from ours. When $N \subset M$ is a connected inclusion, we shall uniquely minimiz $\|\text{Index } E\|$ for $E \in \mathcal{E}(M, N)$ and define the minimum index $[M: N]_0$. Moreover we shall establish several characterizations of $E \in \mathcal{E}(M, N)$ with Index $E = [M: N]_0$ extending those by Hiai.

§1. Preliminaries

In this section, we recall definitions of the minimum index and the entropy $K_\varphi(M|N)$.

Let $N \subset M$ be a factor and a subfactor. If there exists $E \in \mathcal{E}(M, N)$ such that Index $E < \infty$, then Index $E < \infty$ for all $E \in \mathcal{E}(M, N)$ and we have (see [Hi1]):

(M1) There exists a unique $E_0 \in \mathcal{E}(M, N)$ such that

$$\text{Index } E_0 = \min\{\text{Index } E;\ E \in \mathcal{E}(M, N)\}\ .$$

(M2) For $E \in \mathcal{E}(M, N)$, $E = E_0 \Leftrightarrow E^{-1}|_{N' \cap M} = (\text{Index } E)\, E|_{N' \cap M}$.

Definition 1.1. ([Hi1]) For a pair $N \subset M$ of factors, we define the *minimum index* $[M: N]_0$ by $[M: N]_0 = \min\{\text{Index } E;\ E \in \mathcal{E}(M, N)\}$ where $[M: N]_0 = \infty$ if $\mathcal{E}(M, N) = \emptyset$ or Index $E = \infty$ $(E \in \mathcal{E}(M, N))$.

Now, let M be a von Neumann algebra and N its von Neumann subalgebra. Let $\varphi \in \mathcal{E}(M)\ (= \mathcal{E}(M, \mathbf{C}))$ be such that $E \in \mathcal{E}(M, N)$ with $\varphi \circ E = \varphi$ exists. Taking account of Pimsner and Popa's estimate of $H(M|N)$ in the type II_1 case [PP], Hiai [Hi2] introduced the entropy $K_\varphi(M|N)$ of M relative to φ and N as follows. Set $\omega = \varphi|_{N' \cap M}$ and $\hat{\omega} = \varphi \circ (E^{-1}|_{N' \cap M})$. Then since $E^{-1}((N' \cap M)_+)$ is contained in the extended positive part of $\mathcal{Z}(M)\ (= M \cap M')$, $\hat{\omega}$ is well-defined as a faithful normal weight on $N' \cap M$. But $\hat{\omega}$ is not necessarily bounded (possibly not semifinite). So the relative entropy $S(\hat{\omega}, \omega)$ of ω and $\hat{\omega}$ is given by

$$S(\hat{\omega}, \omega) = \inf\{S(\omega', \omega);\ \omega' \in (N' \cap M)_*^+,\ \omega' \leq \hat{\omega}\}$$

where $S(\omega', \omega)$ is Araki's relative entropy [A1, 2].

Definition 1.2. ([Hi2, 3.1]) We define *the entropy* $K_\varphi(M|N)$ *of* M *relative* *to φ and N* by

$$K_{\varphi}(M|N)=-S(\hat{\omega},\omega).$$

Moreover we define

$$K_{E}(M|N)=\sup\{K_{\varphi}(M|N);\ \varphi\in\mathcal{E}(M),\ \varphi\circ E=\varphi\}.$$

Note that $K_{\varphi}(M|N)$ does not depend on the choice of $\mathcal{H}$. When $N\subset M$ are type II_1 von Neumann lagebras with atomic centers and $\tau\in\mathcal{E}(M)$ is a trace, we can show $K_{\tau}(M|N)=H(M|N)$ by arguing as in [KY1, 2], [Hi2].

Finally we recall relation between the minimum index and the entropy $K_{\varphi}(M|N)$. Let $N\subset M$ be a pair of factors such that $[M:N]_0=\mathrm{Index}\,E_0<\infty$. Since $E|_{N'\cap M}$ and $E^{-1}|_{N'\cap M}$ are scalar-valued for each $E\in\mathcal{E}(M,N)$, the entropy $K_{\varphi}(M|N)$ is independent of the choice of $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$, so that $K_{E}(M|N)=K_{\varphi}(M|N)$ for any such $\varphi\in\mathcal{E}(M)$.

Theorem 1.3. ([Hi2, 6.1, 6.3]) *Let $N\subset M$ be a pair of factors. For $E\in\mathcal{E}(M,N)$, $K_E(M|N)\leq\log[M:N]_0$ and the following conditions are equivalent*:

(i) $\mathrm{Index}\,E=[M:N]_0$, *i.e.*, $E=E_0$;

(ii) $K_E(M|N)=\log[M:N]_0$;

(iii) $K_E(M|N)=\log\mathrm{Index}\,E$;

(iv) *for every nonzero projection $e\in N'\cap M$*, $\mathrm{Index}\,E_e=E(e)^2\,\mathrm{Index}\,E$;

(v) *for every nonzero projections $e_1,\cdots,e_n\in N'\cap M$ with $\sum_i e_i=1$*,

$$\sum_{i=1}^{n}E(e_i)\log\frac{\mathrm{Index}\,E_{e_i}}{E(e_i)^2}=\log\mathrm{Index}\,E.$$

§2. Index Formula

Let $N\subset M$ be a pair of σ-finite von Neumann algebras with finite dimensional centers and let $\{p_1,\cdots,p_m\}$, and $\{q_i,\cdots,q_n\}$ be the minimal central projections of M and N respectively with $\sum_i p_i=1$ and $\sum_j q_j=1$. Put $N_{ij}=N_{p_iq_j}\subset M_{ij}=M_{p_iq_j}$ (factors) if $p_iq_j\neq 0$. Let Λ denote the set of m-by-n matrices $[\lambda_{ij}]$ such that $\lambda_{ij}>0$ if $p_iq_j\neq 0$, $\lambda_{ij}=0$ if $p_iq_j=0$, and $\sum_i\lambda_{ij}=1$ for any j. Throughout this paper we shall consider only the pairs (i,j) with $p_iq_j\neq 0$. Consider the three-step inclusion

$$N\subseteq N\vee\{p_i\}''\subseteq M\cap\{q_j\}'\subseteq M.$$

The two intermediate algebras have the same center (the minimal central projections are $\{p_iq_j\}$), and the joint central decompositions are

$$\bigoplus_{i,j} N_{ij} \subseteq \bigoplus_{i,j} M_{ij} .$$

When $E_{ij} \in \mathcal{E}(M_{ij}, N_{ij})$ for all (i, j) and $[\lambda_{ij}] \in \Lambda$ are given, we define the maps F: $\bigoplus_{i,j} N_{ij} \to N$, G: $\bigoplus_{i,j} M_{ij} \to \bigoplus_{i,j} N_{ij}$ and H: $M \to \bigoplus_{i,j} M_{ij}$ as follows:

$$F(\sum_{i,j} y_{ij} p_i q_j) = \sum_j (\sum_i \lambda_{ij} y_{ij}) q_j \qquad \text{for} \quad y_{ij} \in N ,$$
$$G(\sum_{i,j} x_{ij}) = \sum_{i,j} E_{ij}(x_{ij}) \qquad \text{for} \quad x_{ij} \in M_{ij} ,$$
$$H(x) = \sum_{i,j} p_i q_j x p_i q_j \qquad \text{for} \quad x \in M .$$

Proposition 2.1. *If we define the map E: $M \to N$ by $E = F \circ G \circ H$, then $E \in \mathcal{E}(M, N)$. Conversely if $E \in \mathcal{E}(M, N)$, then there exist unique $E_{ij} \in \mathcal{E}(M_{ij}, N_{ij})$ for any (i, j) and $[\lambda_{ij}] \in \Lambda$ such that $E = F \circ G \circ H$ where F and G are defined by $[\lambda_{ij}]$ and E_{ij} as above.*

Proof. Suppose $E_{ij} \in \mathcal{E}(M_{ij}, N_{ij})$ and $[\lambda_{ij}] \in \Lambda$. Since the central support of $p_i q_j$ in N is q_j, F is well-defined. Since $F(y) = y$ for $y \in N$, we get $F \in \mathcal{E}(\bigoplus_{i,j} N_{ij}, N)$. It is clear that $G \in \mathcal{E}(\bigoplus_{i,j} M_{ij}, \bigoplus_{i,j} N_{ij})$ and $H \in \mathcal{E}(M, \bigoplus_{i,j} M_{ij})$. Hence $E = F \circ G \circ H \in \mathcal{E}(M, N)$.

Conversely let E be in $\mathcal{E}(M, N)$. For a faithful $\phi \in \mathcal{E}(N)$ we have

$$\sigma_t^{\phi \circ E}(p_i) = p_i \quad (\text{since } p_i \text{ is central in } M) ,$$
$$\sigma_t^{\phi \circ E}(q_j) = \sigma_t^{\phi}(q_j) = q_j \quad (\text{since } q_j \text{ is central in } N)$$

so that $\sigma_t^{\phi \circ E}$ leaves $N \vee \{p_i\}''$ and $M \cap \{q_j\}'$ (and of course N) globally invariant. Thus it follows from Takesaki's theorem that there exist unique $F' \in \mathcal{E}(\bigoplus_{i,j} N_{ij}, N)$, $G' \in \mathcal{E}(\bigoplus_{i,j} M_{ij}, \bigoplus_{i,j} N_{ij})$ and $H' \in \mathcal{E}(M, \bigoplus_{i,j} M_{ij})$ such that $E = F' \circ G' \circ H'$. Since $M \cap (\bigoplus_{i,j} M_{ij})' = \mathcal{Z}(\bigoplus_{i,j} M_{ij})$, $\mathcal{E}(M, \bigoplus_{i,j} M_{ij})$ consist of only one element. Thus $H' = H$. Since $E(p_i q_j) = E(p_i) q_j \in \mathcal{Z}(N) q_j$, there exists $\lambda_{ij} > 0$ such that $E(p_i q_j) = \lambda_{ij} q_j$. Since $\sum_i E(p_i q_j) = q_j$, we get $\sum_i \lambda_{ij} = 1$ and hence $[\lambda_{ij}] \in \Lambda$. Put $E_{ij} = E_{p_i q_j}$, i.e., $E_{ij}(x) = E(x) E(p_i q_j)^{-1} p_i q_j = \lambda_{ij}^{-1} E(x) p_i q_j$ for $x \in M_{ij}$. Then for any $x \in M$ we have

$$F \circ G \circ H(x) = F(\sum_{i,j} E_{ij}(p_i q_j x p_i q_j)) = \sum_j (\sum_i \lambda_{ij} \lambda_{ij}^{-1} E(x p_i)) q_j = E(x) ,$$

and thus by the uniqueness of F' and G', we have $F = F'$ and $G = G'$. ∎

Let us define Inedx E of $E \in \mathcal{E}(M, N)$ as in [Ko].

Definition 2.2. For $E \in \mathcal{E}(M, N)$, we define Index $E = E^{-1}(1)$. N is said to be of *finite index* in M if there exists a conditional expectation $E \in \mathcal{E}(M, N)$

such that Index E is bounded.

Since $uE^{-1}(1)u^*=E^{-1}(1)$ for any unitary $u\in M'$, Index E is an element of the extended positive part of $\mathcal{Z}(M)$. Since M is not a factor, Index E is not necessarily a scalar multiple of the identity.

Proposition 2.3. (1) *$\mathcal{E}(M, N)\neq\emptyset$ if and only if $\mathcal{E}(M_{ij}, N_{ij})\neq\emptyset$ for any i, j.*

(2) *There exists $E\in\mathcal{E}(M, N)$ such that* Index E *is bounded if and only if for any i, j there exists $E_{ij}\in\mathcal{E}(M_{ij}\ N_{ij})$ such that* Index $E_{ij}<\infty$.

(3) *If there exists $E\in\mathcal{E}(M, N)$ such that* Index E *is bounded, then for any $E\in\mathcal{E}(M, N)$,* Index E *is bounded.*

(4) *If $E\in\mathcal{E}(M, N)$ is defined by $[\lambda_{ij}]\in\Lambda$ and $E_{ij}\in\mathcal{E}(M_{ij}, N_{ij})$ as in Proposition* 2.1, *then*

$$\text{Index}\, E=\sum_i\Big(\sum_j \lambda_{ij}^{-1}\,\|\text{Index}\, E_{ij}\|\Big)\,p_i\,. \tag{2.1}$$

This formula does not depend on the chosen Hilbert space.

Havet [Hav] also obtained the same formula as (2.1) independently. His presentation is based on a Pimsner-Popa type basis. We give a different proof.

Proof. If we obtain the formula (2.1), then by Proposition 2.1 and [Hi1] we can get (1), (2) and (3). Since $H^{-1}(p_iq_j)=H^{-1}(q_j)p_i\in\mathcal{Z}(M)p_i$, there exists $\alpha_{ij}>0$ such that $H^{-1}(p_iq_j)=\alpha_{ij}p_i$. Since $H_{p_iq_j}=id_{M_{ij}}$, we have $(H_{p_iq_j})^{-1}=id_{M'_{ij}}$. By [Hi2, 1.4],

$$p_iq_j=(H_{p_iq_j})^{-1}(p_iq_j)=H^{-1}(p_iq_j)\,p_iq_j=\alpha_{ij}p_iq_j$$

and hence $\alpha_{ij}=1$. Thus we get

$$H^{-1}\Big(\sum_{i,j}x'_{ij}p_iq_j\Big)=\sum_i\Big(\sum_j x'_{ij}\Big)p_i \qquad \text{for} \quad x'_{ij}\in M'\,. \tag{2.2}$$

If for $\varphi\in P(\oplus_{i,j}N_{ij})$ and $\psi\in P(\oplus_{i,j}M'_{ij})$ we set $\varphi_{ij}=\varphi|_{N_{ij}}$ and $\psi_{ij}=\psi|_{M'_{p_iq_j}}$, then

$$\frac{d\varphi_{ij}\circ E_{ij}}{d\psi_{ij}}=\frac{d(\varphi\circ G)|_{M_{ij}}}{d\psi|_{M'_{p_iq_j}}}=\frac{d\varphi\circ G}{d\psi}\bigg|_{p_iq_j\mathfrak{H}}=\frac{d\varphi}{d\psi\circ G^{-1}}\bigg|_{p_iq_j\mathfrak{H}}$$
$$=\frac{d\varphi|_{N_{ij}}}{d(\psi\circ G^{-1})|_{N'_{p_iq_j}}}=\frac{d\varphi_{ij}}{d\psi_{ij}\circ(G^{-1}|_{N'_{p_iq_j}})}$$

and hence $G^{-1}|_{N'_{p_iq_j}}=E_{ij}^{-1}$. Thus

$$(2.3)\qquad G^{-1}(\sum_{i,j} p_i q_j y'_{ij} p_i q_j) = \sum_{i,j} E_{ij}^{-1}(p_i q_j y'_{ij} p_i q_j) \qquad \text{for} \quad y'_{ij} \in N'.$$

For any $y' \in N'$, we get $F^{-1}(p_i q_j y' p_k q_l)=0$ if $(i, j) \neq (k, l)$ and hence

$$F^{-1}(y') = \sum_{i,j} F^{-1}(p_i q_j y' p_i q_j) = \sum_{i,j} p_i q_j y' p_i q_j F^{-1}(p_i q_j).$$

Since $F_{p_i q_j}=id_{N_{ij}}$, we have $(F_{p_i q_j})^{-1}=id_{N'_{p_i q_j}}$. Again by [Hi2, 1.4],

$$p_i q_j = (F_{p_i q_j})^{-1}(p_i q_j) = F^{-1}(F(p_i q_j) p_i q_j) p_i q_j = \lambda_{ij} F^{-1}(p_i q_j) p_i q_j .$$

So we get $F^{-1}(p_i q_j) p_i q_j = \lambda_{ij}^{-1} p_i q_j$. Thus

$$(2.4)\qquad F^{-1}(y') = \sum_{i,j} \lambda_{ij}^{-1} p_i q_j y' p_i q_j \qquad \text{for} \quad y' \in N'.$$

By (2.2)-(2.4), we have

$$\begin{aligned} E^{-1}(1) &= H^{-1} \circ G^{-1} \circ F^{-1}(1) = H^{-1} \circ G^{-1}(\sum_{i,j} \lambda_{ij}^{-1} p_i q_j) \\ &= H^{-1}(\sum_{i,j} \lambda_{ij}^{-1} E_{ij}^{-1}(p_i q_j)) = H^{-1}(\sum_{i,j} \lambda_{ij}^{-1} \|\mathrm{Index}\, E_{ij}\| p_i q_j) \\ &= \sum_i (\sum_j \lambda_{ij}^{-1} \|\mathrm{Index}\, E_{ij}\|) p_i . \end{aligned}$$

So we get the formula (2.1). ∎

Definition 2.4. The pair $N \subset M$ is said to be a *connected inclusion* if $\mathcal{Z}(N) \cap \mathcal{Z}(M) = \boldsymbol{C}$.

If $z_k \in \mathcal{Z}(M) \cap \mathcal{Z}(N)$ with $\sum_k z_k = 1$, then Index $E = \oplus_k$ Index E_{z_k}. So we can assume without loss of generality that $N \subset M$ is connected.

Let N be of finite index in M and $p_i q_j \neq 0$. If $p_i M$ is a finite factor, then so M_{ij} is since $M_{ij} \subset p_i M$. Since N is of finite index in M, N_{ij} is also of finite index in M_{ij}. Thus $N_{ij} \cong q_j N$ is a finite factor. Conversely if $q_j N$ is finite, then M_{ij} is also finite since N_{ij} is of finite index in M_{ij}. So $p_i = \sum_j p_i q_j$ is a finite projection, i.e., $p_i M$ is finite. Hence if $N \subset M$ is a connected inclusion, then either all of M_{ij} and N_{ij} are finite or they are infinite.

In the rest of this section we shall fix $E_{ij} \in \mathcal{E}(M_{ij}, N_{ij})$ for any (i, j) and define for $[\lambda_{ij}] \in \Lambda$

$$(2.5)\qquad f_i([\lambda_{ij}]) = \sum_j \lambda_{ij}^{-1} \|\mathrm{Index}\, E_{ij}\| \quad \text{and} \quad f([\lambda_{ij}]) = \max_{1 \le i \le m} f_i([\lambda_{ij}]).$$

Note that if $E \in \mathcal{E}(M, N)$ is given by $[\lambda_{ij}]$ and E_{ij} as in Proposition 2.1, then (2.1) implies $\|\mathrm{Index}\, E\| = f([\lambda_{ij}])$.

Lemma 2.5. *Let $N \subset M$ be connected and $[\lambda_{ij}] \in \Lambda$. If $f_i([\lambda_{ij}])$ are not con-*

stant for i, then there exists $[\lambda'_{ij}]\in\Lambda$ *such that* $f([\lambda'_{ij}])<f([\lambda_{ij}])$.

Proof. We set

$$I=\{i; f_i([\lambda_{ij}])=f([\lambda_{ij}])\} \quad\text{and}\quad J=\{j; p_i q_j\neq 0 \text{ for some } i\in I\}\ .$$

By the hypothesis, $I\neq\{1,\cdots,m\}$. Since $N\subset M$ is connected, there exists $i_0\notin I$ and $j_0\in J$ such that $p_{i_0}q_{j_0}\neq 0$. By the definition of J, there exists $i_1\in I$ such that $p_{i_1}q_{j_0}\neq 0$. For $\varepsilon>0$, we define $[\lambda'_{ij}]$ by $\lambda'_{i_0j_0}=\lambda_{i_0j_0}-\varepsilon$, $\lambda'_{i_1j_0}=\lambda_{i_1j_0}+\varepsilon$ and $\lambda'_{ij}=\lambda_{ij}$ for others. Taking a small $\varepsilon>0$, we get $[\lambda'_{ij}]\in\Lambda$ with

$$f_{i_0}([\lambda'_{ij}])<f([\lambda_{ij}]) \quad\text{and}\quad f_{i_1}([\lambda'_{ij}])<f([\lambda_{ij}])\ .$$

If $I=\{i_1\}$ then $f([\lambda'_{ij}{}'])<f([\lambda_{ij}])$. If $I\neq\{i_1\}$ then we can do the same argument for $I\setminus\{i_1\}$ instead of I. We get the statement by induction. ∎

Proposition 2.6. *Let $N\subset M$ be connected. There exists a unique matrix* $[\lambda^0_{ij}]\in\Lambda$ *such that*

$$f([\lambda^0_{ij}])=\min_{[\lambda_{ij}]\in\Lambda} f([\lambda_{ij}])\ .$$

Moreover $f_i([\lambda^0_{ij}])$ *are constant for i.*

Proof. The existence of such $[\lambda^0_{ij}]\in\Lambda$ is obvious. Let $c=\min_{[\lambda_{ij}]\in\Lambda} f([\lambda_{ij}])$. Suppose $f([\lambda^1_{ij}])=f([\lambda^2_{ij}])=c$ for $[\lambda^1_{ij}],[\lambda^2_{ij}]\in\Lambda$. If $[\lambda^1_{ij}]\neq[\lambda^2_{ij}]$, there exists i_0, j_0 such that $\lambda^1_{i_0j_0}\neq\lambda^2_{i_0j_0}$. Define $[\lambda^3_{ij}]\in\Lambda$ by $\lambda^3_{ij}=\dfrac{\lambda^1_{ij}+\lambda^2_{ij}}{2}$. Since f_i are strictly convex, we have

$$f_i([\lambda^3_{ij}])\leq\frac{f_i([\lambda^1_{ij}])+f_i([\lambda^2_{ij}])}{2}\leq c\ .$$

In particular

$$f_{i_0}([\lambda^3_{ij}])<\frac{f_{i_0}([\lambda^1_{ij}])+f_{i_0}([\lambda^2_{ij}])}{2}\leq c\ .$$

Thus by Lemma 2.5 there exists $[\lambda'_{ij}]\in\Lambda$ such that $f([\lambda'_{ij}])<f([\lambda^3_{ij}])\leq c$. This is a contradiction and hence there uniquely exists $[\lambda^0_{ij}]\in\Lambda$ such that $f([\lambda^0_{ij}])=c$. Lemma 2.5 implies that $f_i([\lambda^0_{ij}])$ are constant for i. ∎

§3. Minimum Index and Entropy $K_E(M|N)$

Let $N\subset M$ be as in the previous section. In this section we shall introduce the minimum index for $N\subset M$ and characterize $E_0\in\mathcal{E}(M,N)$ having the minimum index.

Proposition 3.1. *If $N\subset M$ is connected and N is of finite index in M, then there exists a unique expectation $E_0\in\mathcal{E}(M, N)$ such that*

$$\|\mathrm{Index}\, E_0\| = \min\{\|\mathrm{Index}\, E\|;\ E\in\mathcal{E}(M, N)\}\ .$$

Moreover Index E_0 *is a scalar multiple of the identity and* $\|\mathrm{Index}\,(E_0)_{p_iq_j}\|=[M_{ij}, N_{ij}]_0$ *for any* (i, j).

Proof. For $\{E_{ij}\}$ with $E_{ij}\in\mathcal{E}(M_{ij}, N_{ij})$ and $[\lambda_{ij}]\in\Lambda$, we set

$$f([\lambda_{ij}],\ \{E_{ij}\}) = \max_{1\leq i\leq m}\sum_j \lambda_{ij}^{-1}\ \|\mathrm{Index}\, E_{ij}\|\ .$$

Let $c_0=\inf_{E\in\mathcal{E}(M,N)}\|\mathrm{Index}\, E\|$. Then by Propositions 2.1 and 2.3,

$$c_0 = \inf_{\{E_{ij}\}}\min_{[\lambda_{ij}]\in\Lambda} f([\lambda_{ij}],\ \{E_{ij}\})\ .$$

By [Hi1], for any (i, j), there exists...$E_{ij}^0\in\mathcal{E}(M_{ij}, N_{ij})$ such that $\|\mathrm{Index}\, E_{ij}^0\|=[M_{ij}\colon N_{ij}]_0$. If $\{E_{ij}\}\neq\{E_{ij}^0\}$, then for any $[\lambda_{ij}]\in\Lambda, f([\lambda_{ij}]),\ \{E_{ij}^0\})<f([\lambda_{ij}],\ \{E_{ij}\})$. So by Proposition 2.6, there exists...$[\lambda_{ij}^0]\in\Lambda$ such that

$$c_0 = \min_{[\lambda_{ij}]\in\Lambda} f([\lambda_{ij}],\ \{E_{ij}^0\}) = f([\lambda_{ij}^0],\ \{E_{ij}^0\})\ .$$

Thus if $E_0\in\mathcal{E}(M, N)$ is determined by $[\lambda_{ij}^0]$ and $\{E_{ij}^0\}$ as in Proposition 2.1, then

$$\|\mathrm{Index}\, E_0\| = c_0 = \min_{E\in\mathcal{E}(M,N)}\|\mathrm{Index}\, E\|$$

and by Proposition 2.6, Index E_0 is a scalar multiple of the identity. ∎

Definition 3.2. For a connected inclusion $N\subset M$, we define the *minimum index* $[M\colon N]_0$ as follows: $[M\colon N]_0 = \|\mathrm{Index}\, E_0\|$ if N is of finite index in M where $E_0\in\mathcal{E}(M, N)$ is defined in the preceding proposition, and $[M\colon N]_0=\infty$ if N is not of finite index in M.

Theorem 3.3. *Let $N\subset M$ be a connected inclusion such that N is of finite index in M. If $E\in\mathcal{E}(M, N)$, then the following conditions are equivalent*:

(a) $E=E_0$, *i.e.*, $\|\mathrm{Index}\, E_0\|=\min\{\|\mathrm{Index}\, E\|;\ E\in\mathcal{E}(M, N)\}$;

(b) *If E is determined by $[\lambda_{ij}]\in\Lambda$ and E_{ij} as in Proposition* 2.1, *then*

  (i) $\|\mathrm{Index}\, E_{ij}\|=[M_{ij}\colon N_{ij}]_0$ *for any* (i, j),

  (ii) Index E *is a scalar*,

  (iii) *there exist* $\mu_i>0$ $(i=1, \cdots, m)$ *and* $\nu_j>0$ $(j=1, \cdots, n)$ *such that* $\lambda_{ij}^{-2}\|\mathrm{Index}\, E_{ij}\|=\mu_i\,\nu_j$ *for any* (i, j);

(c) *There exists* $\varphi\in\mathcal{E}(M)$ *with* $\varphi\circ E=\varphi$ *such that* $\varphi\circ(E^{-1}|_{N'\cap M})=c\cdot\varphi|_{N'\cap M}$ *for some constant c, in fact* $c=[M:N]_0$;

(d) $K_E(M|N)=\log\|\text{Index } E\|$.

Proof. (a)⇒(b). By Proposition 3.1, we can see that (a) implies (i) and (ii). So we shall show that (a) implies (iii). We set $\alpha_{ij}=\lambda_{ij}^{-2}\|\text{Index } E_{ij}\|$. We shall prove that if $p_{i_1}q_{j_1}\neq 0$, $p_{i_2}q_{j_1}\neq 0$, $p_{i_2}q_{j_2}\neq 0$, $\cdots$, $p_{i_k}q_{j_k}\neq 0$, $p_{i_1}q_{j_k}\neq 0$, then

$$\alpha_{i_1j_1}^{-1}\alpha_{i_1j_2}\alpha_{i_2j_2}^{-1}\alpha_{i_3j_2}\cdots\alpha_{i_kj_k}^{-1}\alpha_{i_1j_k}=1\,.$$

We can assume without loss of generality that $i_k\neq i_l$ if $k\neq l$. Suppose that

$$\alpha_{i_1j_1}^{-1}\alpha_{i_1j_1}\alpha_{i_2j_2}^{-1}\cdots\alpha_{i_kj_k}^{-1}\alpha_{i_1j_k}<1\,.$$

For $\varepsilon>0$ and $\gamma>1$, we define

$$\begin{aligned}
\varepsilon_1 &= \varepsilon\,,\\
\varepsilon_2 &= \gamma\varepsilon_1\alpha_{i_2j_1}\alpha_{i_2j_2}^{-1}\,,\\
\varepsilon_3 &= \gamma\varepsilon_2\alpha_{i_3j_2}\alpha_{i_3j_3}^{-1}=\gamma^2\varepsilon\alpha_{i_2j_1}\alpha_{i_2j_2}^{-1}\alpha_{i_3j_2}\alpha_{i_3j_3}^{-1}\,,\\
&\;\vdots\\
\varepsilon_k &= \gamma\varepsilon_{k-1}\alpha_{i_kj_{k-1}}\alpha_{i_kj_k}^{-1}=\gamma^{k-1}\varepsilon\alpha_{i_2j_1}\alpha_{i_2j_2}^{-1}\cdots\alpha_{i_kj_{k-1}}\alpha_{i_kj_k}^{-1}\,.
\end{aligned}$$

and

$$\begin{aligned}
\lambda'_{i_1j_k} &= \lambda_{i_1j_k}-\varepsilon_k\,, & \lambda'_{i_1j_1} &= \lambda_{i_1j_1}+\varepsilon_1\,,\\
\lambda'_{i_2j_1} &= \lambda_{i_2j_1}-\varepsilon_1\,, & \lambda'_{i_2j_2} &= \lambda_{i_2j_2}+\varepsilon_2\,,\\
&\;\vdots & &\;\vdots\\
\lambda'_{i_kj_{k-1}} &= \lambda_{i_kj_{k-1}}-\varepsilon_{k-1}\,, & \lambda'_{i_kj_k} &= \lambda_{i_kj_k}+\varepsilon_k\,,\\
& & \lambda'_{ij} &= \lambda_{ij}\quad\text{for others.}
\end{aligned}$$

Taking γ close to 1, we have

$$\begin{aligned}
\varepsilon_k\alpha_{i_1j_k}-\varepsilon_1\alpha_{i_1j_1} &= \varepsilon\alpha_{i_1j_1}(\gamma^{k-1}\alpha_{i_1j_1}^{-1}\alpha_{i_2j_1}\cdots\alpha_{i_kj_k}^{-1}\alpha_{i_1j_k}-1)<0\,,\\
\varepsilon_1\alpha_{i_2j_1}-\varepsilon_2\alpha_{i_2j_2} &= (1-\gamma)\,\varepsilon_1\alpha_{i_2j_1}<0\,,\\
&\;\vdots\\
\varepsilon_{k-1}\alpha_{i_kj_{k-1}}-\varepsilon_k\alpha_{i_kj_k} &= (1-\gamma)\,\varepsilon_{k-1}\alpha_{i_kj_{k-1}}<0\,.
\end{aligned}$$

Taking a small $\varepsilon>0$, we get $[\lambda'_{ij}]\in\Lambda$. For $0<s<1$, we have

$$\left.\frac{df_i((1-s)\,[\lambda_{ij}]+s[\lambda'_{ij}])}{ds}\right|_{s=0}=-\sum_j\lambda_{ij}^{-2}(\lambda'_{ij}-\lambda_{ij})\,\|\text{Index } E_{ij}\|\,.$$

In particular,

$$\left.\frac{df_{i_1}((1-s)[\lambda_{ij}]+s[\lambda'_{ij}])}{ds}\right|_{s=0} = \varepsilon_k\,\alpha_{i_1j_k}-\varepsilon_1\,\alpha_{i_1j_1}<0\,,$$

$$\vdots$$

$$\left.\frac{df_{i_k}((1-s)[\lambda_{ij}]+s[\lambda'_{ij}])}{ds}\right|_{s=0} = \varepsilon_{k-1}\,\alpha_{i_kj_{k-1}}-\varepsilon_k\,\alpha_{i_kj_k}<0\,.$$

Moreover if $i\notin\{i_1,\cdots,i_k\}$, then $f_i((1-s)[\lambda_{ij}]+s[\lambda'_{ij}])=f_i([\lambda_{ij}])$. When $\{i_1,\cdots,i_k\}=\{1,\cdots,m\}$, there exists $s\in(0,1)$ such that $f((1-s)[\lambda_{ij}]+s[\lambda'_{ij}])<f([\lambda_{ij}])$. This contradicts $E=E_0$. When $\{i_1,\cdots,i_k\}\neq\{1,\cdots,m\}$, there exists $s\in(0,1)$ such that $f_i((1-s)[\lambda_{ij}]+s[\lambda'_{ij}])$ are not constant and $f((1-s)[\lambda_{ij}]+s[\lambda'_{ij}]=f([\lambda_{ij}])$. Then by Lemma 2.5, there exists $[\lambda''_{ij}]\in\Lambda$ such that $f([\lambda''_{ij}])<f([\lambda_{ij}])$, contradicting $E=E_0$ again. So we have $\alpha^{-1}_{i_1j_1}\alpha_{i_2j_1}\cdots\alpha^{-1}_{i_kj_k}\alpha_{i_1j_k}\geq 1$. But we can do the same argument for $\alpha^{-1}_{i_1j_k}\alpha_{i_kj_k}\cdots\alpha^{-1}_{i_2j_1}\alpha_{i_1j_1}$, and hence we get $\alpha^{-1}_{i_1j_1}\alpha_{i_2j_1}\cdots\alpha^{-1}_{i_kj_k}\alpha_{i_1j_k}=1$. We shall from now fix i_1. Since $N\subset M$ is connected, for any i there is a path from i_1 to i, i.e.,

$$i_1\to j_1\to i_2\to\cdots\to i_k\to j_k\to i$$

where $i_k\to j_l$ (also $j_l\to i_k$) means $p_{i_k}q_{j_l}\neq 0$. If there is another path from i_1 to i

$$i_1\to j'_1\to i'_2\to\cdots\to i'_l\to j'_l\to i\,,$$

then by the preceding remark, $\alpha^{-1}_{i_1j_1}\alpha_{i_2j_1}\cdots\alpha^{-1}_{i_kj_k}\alpha_{ij_k}\alpha^{-1}_{ij'_l}\alpha_{i'_lj'_l}\cdots\alpha^{-1}_{i'_2j'_1}\alpha_{i_1j'_1}=1$, and hence $\alpha^{-1}_{i_1j_1}\alpha_{i_2j_1}\cdots\alpha^{-1}_{i_kj_k}\alpha_{ij_k}=\alpha^{-1}_{i_1j'_1}\alpha_{i'_2j'_1}\cdots\alpha^{-1}_{i'_lj'_l}\alpha_{ij'_l}$. So $\mu_i=\alpha^{-1}_{i_1j_1}\alpha_{i_2j_1}\cdots\alpha^{-1}_{i_kj_k}\alpha_{ij_k}$ is well-defined. Also for any j, there is a path from i_1 to j

$$i_1\to j_1\to i_2\to j_2\to\cdots\to j_{k-1}\to i_k\to j\,.$$

Then by the similar argument, $\nu_j=\alpha_{i_1j_1}\alpha^{-1}_{i_2j_1}\alpha_{i_2j_2}\cdots\alpha^{-1}_{i_kj_{k-1}}\alpha_{i_kj}$ is well-defined. If $p_iq_j\neq 0$, then there is a path such that

$$i_1\to j_1\to i_2\to j_2\to\cdots\to i_k\to j_k\to i\to j\,,$$

and then

$$\mu_i=\alpha^{-1}_{i_1j_1}\alpha_{i_2j_1}\cdots\alpha^{-1}_{i_kj_k}\alpha_{ij_k}\,,\quad \nu_j=\alpha_{i_1j_1}\alpha^{-1}_{i_2j_1}\cdots\alpha_{i_kj_k}\alpha^{-1}_{ij_k}\alpha_{ij}\,.$$

Thus $\mu_i\nu_j=\alpha_{ij}$. We get (iii) of (b).

(b)$\Rightarrow$(a). Assume that E satisfies (b) with $[\lambda_{ij}]\in\Lambda$ and E_{ij}. If there eixsts $[\lambda'_{ij}]\in\Lambda$ such that $f([\lambda'_{ij}])<f([\lambda_{ij}])$, then by (ii) we have

$$f_i([\lambda'_{ij}])\leq f([\lambda'_{ij}])<f([\lambda_{ij}])=f_i([\lambda_{ij}])\,,$$

i.e., $f_i([\lambda'_{ij}])<f_i([\lambda_{ij}])$ for any i. Since f_i are convex,

$$0>\frac{\partial f_i}{\partial_s}((1-s)[\lambda'_{ij}]+s[\lambda_{ij}])\bigg|_{s=0}=\sum_j(\lambda_{ij}-\lambda'_{ij})\,\mu_i\,\nu_j\quad\text{by (iii).}$$

Since $\mu_i>0$, $0>\sum_j(\lambda_{ij}-\lambda'_{ij})\,\nu_j$ for any i. Hence

$$0>\sum_i\sum_j(\lambda_{ij}-\lambda'_{ij})\,\nu_j=\sum_j(1-1)\,\nu_j=0\,.$$

This is a contradiction. Since $||\text{Index }E||=f([\lambda_{ij}])$, we get $E=E_0$ by (i).

(b)$\Rightarrow$(c). By taking $\frac{\nu_j}{\sum_k\nu_k}$ if necessary, we can assume $\sum_j\nu_j=1$ and hence we can get $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$ such that $\varphi(q_j)=\nu_j$. If $x\in(N'\cap M)_{p_iq_j}$, then by the proof of Proposition 2.3 we have

$$\begin{aligned}\varphi(E^{-1}(x))&=\varphi(H^{-1}\circ G^{-1}\circ F^{-1}(x))\\&=\varphi(H^{-1}(\lambda_{ij}^{-1}\,||\text{Index }E_{ij}||\,E_{ij}(x)))\\&\qquad\qquad\text{by (i) and (M2) in §1}\\&=\varphi(H^{-1}(\lambda_{ij}^{-2}\,||\text{Index }E_{ij}||\,E(x)\,p_iq_j))\,.\end{aligned}$$

Since $E(x)=E(xq_j)=E(x)\,q_j\in\mathcal{Z}(N)\,q_j$, there exists $\alpha\in\boldsymbol{C}$ such that $E(x)=\alpha q_j$. Then $\varphi(x)=\varphi(E(x))=\alpha\varphi(q_j)=\alpha\nu_j$. Also $\varphi(p_i)=\sum_k\varphi(E(p_iq_k))=\sum_k\lambda_{ik}\nu_k$. So we have

$$\begin{aligned}\varphi(E^{-1}(x))&=\varphi(H^{-1}(\lambda_{ij}^{-2}\,||\text{Index }E_{ij}||\,\alpha p_i\,q_j))\\&=\lambda_{ij}^{-2}\,||\text{Index }E_{ij}||\,\alpha\varphi(p_i)\\&=\mu_i\,\nu_j\,\alpha\sum_k\lambda_{ik}\,\nu_k&&\text{by (iii)}\\&=\varphi(x)\sum_k\lambda_{ik}^{-1}\,||\text{Index }E_{ik}||&&\text{by (iii)}\\&=||\text{Index }E||\,\varphi(x)&&\text{by (ii) and (2.1).}\end{aligned}$$

Hence for any $x\in N'\cap M$, we get

$$\varphi(E^{-1}(x))=\sum_{i,j}\varphi(E^{-1}(xp_i\,q_j))=||\text{Index }E||\,\varphi(x)\,.$$

(c)$\Rightarrow$(b). Assume that $\varphi\circ(E^{-1}|_{N'\cap M})=c\cdot\varphi|_{N'\cap M}$ for some constant c. Let $\varphi(q_j)=\nu_j$ and $x\in(N'\cap M)_{p_iq_j}$. Since $E_{ij}^{-1}(x)\in\mathcal{Z}(M_{ij})$, there eixts $\alpha\in\boldsymbol{C}$ such that $E_{ij}^{-1}(x)=\alpha p_i\,q_j$. Then we have since $H^{-1}(p_i\,q_j)=p_i$

$$\varphi(E^{-1}(x))=\varphi(H^{-1}(\lambda_{ij}^{-1}\,E_{ij}^{-1}(x)))=\lambda_{ij}^{-1}\,\alpha\varphi(p_i)=\lambda_{ij}^{-1}\,\alpha\sum_k\lambda_{ik}\,\nu_k\,.$$

Let $E_{ij}(x)=\beta p_iq_j$ with $\beta\in\boldsymbol{C}$. Then $\varphi(x)=\varphi(E(x))=\lambda_{ij}\,\beta\nu_j$. So by the hypothesis, we have $\lambda_{ij}^{-1}\sum_k\lambda_{ik}\nu_k\,\alpha=c\cdot\lambda_{ij}\nu_j\,\beta$. This shows that $E_{ij}^{-1}(x)=c'E_{ij}(x)$ for any $x\in N'_{ij}\cap M_{ij}$ where $c'=\frac{c\lambda_{ij}^2\,\nu_j}{\sum_k\lambda_{ik}\,\nu_k}$. By (M2) in Section 1, $||\text{Index }E_{ij}||$

$=[M_{ij}: N_{ij}]_0$ and $||\text{Index } E_{ij}||=\frac{c\lambda_{ij}^2 \nu_j}{\sum_k \lambda_{ik}\nu_k}$. Thus if we put $\mu_i=\frac{c}{\sum_k \lambda_{ik}\nu_k}$, then $\lambda_{ij}^{-2}\, ||\text{Index } E_{ij}||=\mu_i \nu_j$ and

$$\sum_j \lambda_{ij}^{-1}\, ||\text{Index } E_{ij}|| = \sum_j \frac{c\lambda_{ij}\nu_j}{\sum_k \lambda_{ik}\nu_k} = c \quad \text{for any } i.$$

So we get (b).

(d)$\Rightarrow$(b). For $\alpha_{ij}\geq 0$, $\beta_{ij}\geq 0$ ($i=1, \cdots m; j=1, \cdots n$) such that $\alpha_{ij}>0 \Leftrightarrow \beta_{ij}>0$, we have

$$\sum_{i,j} \alpha_{ij} \log \frac{\alpha_{ij}}{\beta_{ij}} \geq \sum_i \left(\sum_j \alpha_{ij}\right) \log \frac{\sum_j \alpha_{ij}}{\sum_j \beta_{ij}} \tag{3.1}$$

where $0\cdot\log\frac{0}{0}=0$. Moreover the equality holds in the above if and only if $\frac{\alpha_{ij}}{\beta_{ij}}$ are constant for j. (This is a special case of the monotonicity of the relative entropy.)

If we set $\nu_j=\varphi(q_j)$ for $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$, then $\nu_j>0$ and $\sum_j \nu_j=1$. By [Hi2, 4.1, 4.2], letting $\eta(t)=-t\log t$, $t\geq 0$, we have

$$K_\varphi(M\,|\,N) = \sum_i \varphi(\eta E(p_i)) + \sum_i \varphi(p_i)\, K_{\varphi_i}(M_{p_i}\,|\,N_{p_i}),$$
$$K_{\varphi_i}(M_{p_i}\,|\,N_{p_i}) = \sum_j \eta(\varphi_i(p_i q_j)) + \sum_j \varphi_i(p_i q_j)\, K_{\varphi_{ij}}(M_{ij}\,|\,N_{ij}),$$

where $\varphi_i=\varphi(p_i)^{-1}\varphi\,|_{M_{p_i}}$ and $\varphi_{ij}=\varphi_i(p_iq_j)^{-1}\varphi_i\,|_{M_{ij}}$. Since $E(p_i)=\sum_j \lambda_{ij}q_j$, $\eta E(p_i)=\sum_j \eta(\lambda_{ij})\, q_j$ and $\varphi(\eta E(p_i))=\sum_j \eta(\lambda_{ij})\,\nu_j$. Since $\varphi(p_iq_j)=\varphi(E(p_iq_j))=\varphi(\lambda_{ij}q_j)=\lambda_{ij}\nu_j$, $\varphi(p_i)=\sum_j \lambda_{ij}\nu_j$ and hence we have $\varphi_i(p_iq_j)=\frac{\lambda_{ij}\nu_j}{\sum_k \lambda_{ik}\nu_k}$. Since $\varphi_{ij}\circ E_{ij}=\varphi_{ij}$, we note (see the remark before Theorem 1.3) that $K_{\varphi_{ij}}(M_{ij}\,|\,N_{ij}) =K_{E_{ij}}(M_{ij}\,|\,N_{ij})$. If we put $\alpha_{ij}=\exp K_{E_{ij}}(M_{ij}\,|\,N_{ij})$, then we get

$$\begin{aligned} K_\varphi(M\,|\,N) &= -\sum_{i,j} \lambda_{ij}\nu_j \log \lambda_{ij} - \sum_{i,j} \lambda_{ij}\nu_j \log \frac{\lambda_{ij}\nu_j}{\sum_k \lambda_{ik}\nu_k} \\ &\quad + \sum_{i,j} \lambda_{ij}\nu_j\, K_{E_{ij}}(M_{ij}\,|\,N_{ij}) \\ &= -\sum_{i,j} \lambda_{ij}\nu_j \log \frac{\lambda_{ij}\nu_j}{(\sum_k \lambda_{ik}\nu_k)\,\lambda_{ij}^{-1}\alpha_{ij}}. \end{aligned} \tag{3.2}$$

Let $F(\nu_1, \cdots, \nu_n)=-\sum_{i,j} \lambda_{ij}\nu_j \log \frac{\lambda_{ij}\nu_j}{(\sum_k \lambda_{ik}\nu_k)\,\lambda_{ij}^{-1}\alpha_{ij}}$. If $\nu_j>0$ and $\sum_j \nu_j=1$, then there exists $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$ such that $\varphi(q_j)=\nu_j$. So we get

$$K_E(M\,|\,N) = \sup\{F(\nu_1, \cdots, \nu_n);\ \nu_j>0,\ \sum_j \nu_j = 1\}.$$

Since F is continuous, there exists $(\nu_j^0)_{j=1}^n$ with $\nu_j^0 \geq 0$, $\sum_j \nu_j^0 = 1$ such that

$$K_E(M\,|\,N) = F(\nu_1^0, \cdots, \nu_n^0)\,.$$

Since for any j

$$\frac{\partial F}{\partial \nu_j}(\nu_1, \cdots, \nu_j, \cdots, \nu_n)\,|_{\nu_j=0} = \infty\,,$$

we have $\nu_j^0 > 0$ for any j. For simplicity we denote $\nu_j^0 = \nu_j$. Then

$$\begin{aligned}
K_E(M\,|\,N) &= -\sum_{i,j} \lambda_{ij}\,\nu_j \log \frac{\lambda_{ij}\,\nu_j}{(\sum_{ik} \lambda_{ik}\,\nu_k)\,\lambda_{ij}^{-1}\,\alpha_{ij}} \\
(3.3)\qquad &\leq -\sum_i (\sum_j \lambda_{ij}\,\nu_j) \log \frac{\sum_j \lambda_{ij}\,\nu_j}{(\sum_k \lambda_{ik}\,\nu_k) \sum_j \lambda_{ij}^{-1}\,\alpha_{ij}} \quad \text{by (3.1)} \\
&= \sum_i (\sum_j \lambda_{ij}\,\nu_j) \log \sum_j \lambda_{ij}^{-1}\,\alpha_{ij} \\
(3.4)\qquad &\leq \sum_i (\sum_j \lambda_{ij}\,\nu_j) \log \sum_j \lambda_{ij}^{-1}\,||\mathrm{Index}\,E_{ij}|| \quad \text{by Theorem 1.3} \\
(3.5)\qquad &\leq \sum_{i,j} \lambda_{ij}\,\nu_j \log ||\mathrm{Index}\,E|| \\
&= \log ||\mathrm{Index}\,E||\,.
\end{aligned}$$

If $K_E(M\,|\,N) = \log ||\mathrm{Index}\,E||$, then (3.3)-(3.5) are not inequalities but equalities. By the equality in (3.5), Index E is a scalar. By the equality in (3.4), we get $\alpha_{ij} = ||\mathrm{Index}\,E_{ij}||$ for any i, j, i.e., $||\mathrm{Index}\,E_{ij}|| = [M_{ij} \colon N_{ij}]_0$ for any i, j by Theorem 1.3. By the equality in (3.3), $\dfrac{\lambda_{ij}\,\nu_j}{\lambda_{ij}^{-1}\,\alpha_{ij}}$ are constant for j (see the remark after (3.1)). So we can get $\mu_i > 0$ such that $\lambda_{ij}^{-2}\,\alpha_{ij} = \mu_i\,\nu_j$. Thus we get (b).

(b)$\Rightarrow$(d). Since we can assume $\sum_j \nu_j = 1$, we can get $\varphi \in \mathcal{E}(M)$ with $\varphi \circ E = \varphi$ such that $\varphi(q_j) = \nu_j$. By the equation (3.2),

$$\begin{aligned}
K_\varphi(M\,|\,N) &= -\sum_{i,j} \lambda_{ij}\,\nu_j \log \frac{\lambda_{ij}\,\nu_j}{(\sum_k \lambda_{ik}\,\nu_k)\,\lambda_{ij}^{-1}\,\alpha_{ij}} \\
&= \sum_{i,j} \lambda_{ij}\,\nu_j \log \frac{(\sum_k \lambda_{ik}\,\nu_k)\,\lambda_{ij}^{-2}\,||\mathrm{Index}\,E_{ij}||}{\nu_j} \quad \text{by (i)} \\
&= \sum_{i,j} \lambda_{ij}\,\nu_j \log \frac{(\sum_k \lambda_{ik}\,\nu_k)\,\mu_i\,\nu_j}{\nu_j} \quad \text{by (iii)} \\
&= \sum_{i,j} \lambda_{ij}\,\nu_j \log \sum_k \lambda_{ik}^{-1}\,||\mathrm{Index}\,E_{ik}|| \quad \text{by (iii)} \\
&= \sum_{i,j} \lambda_{ij}\,\nu_j \log ||\mathrm{Index}\,E|| \quad \text{by (ii)} \\
&= \log ||\mathrm{Index}\,E||\,.
\end{aligned}$$

So we have $K_E(M\,|\,N) \geq \log ||\mathrm{Index}\,E||$. But we know that $K_E(M\,|\,N) \leq$

log $\|\text{Index } E\|$ in general (see [Hi2]). Thus $K_E(M|N)=\log\|\text{Index } E\|$. ∎

Remark 3.4. When $N\subset M$ is not connected, we can define the *minimum index* $[M: N]_0$ by $[M: N]_0=\sum_{k=1}^{l}[M_{z_k}: N_{z_k}]_0 z_k$ where $z_1, \cdots, z_l$ are minimal projections of $\mathcal{Z}(M)\cap\mathcal{Z}(N)$ with $\sum_k z_k=1$. If N is of finite index in M, then there exists a unique $E_0\in\mathcal{E}(M, N)$ such that Index $E_0=[M: N]_0$. Then conditions (a)-(c) of Theorem 3.3 are equivalent when we modify (ii) of (b) and (c) as follows:

(ii) Index $E\in\mathcal{Z}(M)\cap\mathcal{Z}(N)$;

(c) There exists $\varphi\in\mathcal{E}(M)$ with $\varphi\circ E=\varphi$ scuh that $\varphi\circ(E^{-1}|_{N'\cap M})=\varphi(c\,\cdot)|_{N'\cap M}$ for some $c\in\mathcal{Z}(M)\cap\mathcal{Z}(N)$.

§4. Corollaries

Let $N_i\subset M_i$ $(i=1, 2)$ and $N\subset M$ be von Neumann algebras with finite dimensional centers. We consider the minimum index as in Remark 3.4.

Corollary 4.1. *If N_i is of finite index in M_i, then*

$$[M_1\otimes M_2: N_1\otimes N_2]_0 = [M_1: N_1]_0\otimes[M_2: N_2]_0$$

Proof. Let $E_i\in\mathcal{E}(M_i, N_i)$ be such that Index $E_i=[M_i: N_i]_0$. By Theorem 3.3 (Remark 3.4), there exists $\varphi_i\in\mathcal{E}(M_i)$ with $\varphi_i\circ E_i=\varphi_i$ such that $\varphi_i\circ(E_i^{-1}|_{N_i'\cap M_i})=\varphi_i(c\,\cdot)|_{N_i'\cap M_i}$ where $c_i=[M_i: N_i]_0$. Since by [Hi2, 1.7]

$$(E_1\otimes E_2)^{-1} = E_1^{-1}\otimes E_2^{-1}$$

and

$$(N_1\otimes N_2)'\cap(M_1\otimes M_2) = (N_1'\cap M_1)\otimes(N_2'\cap M_2)\,,$$

we have

$$\begin{aligned}(\varphi_1\otimes\varphi_2)&\circ((E_1\otimes E_2)^{-1}|_{(N_1\otimes N_2)'\cap(M_1\otimes M_2)})\\ &= (\varphi_1\circ(E_1^{-1}|_{N_1'\cap M_1}))\otimes(\varphi_2\circ(E_2^{-1}|_{N_2'\cap M_2}))\\ &= (\varphi_1(c_1\,\cdot)|_{N_1'\cap M_1})\otimes(\varphi_2(c_2\,\cdot)|_{N_1'\cap M_2})\\ &= (\varphi_1\otimes\varphi_2)(c_1\otimes c_2\,\cdot)|_{(N_1\otimes N_2)'\cap(M_1\otimes M_2)}\,.\end{aligned}$$

Since $c_1\otimes c_2\in\mathcal{Z}(M_1\otimes M_2)\cap\mathcal{Z}(N_1\otimes N_2)$, Theorem 3.3 implies the conclusion. ∎

Corollary 4.2. *Let α_g be an action of a finite group G on M such that $\alpha_g(N)=N$ for all $g\in G$, and $M\rtimes_\alpha G$ (resp. $N\rtimes_\alpha G$) denote the crossed product of M (resp. N) by α (resp. $\alpha|_N$). If N is of finite index in M and H is a* *subgroup of G, then*

$$[M \rtimes_\alpha G : N \rtimes_\alpha H]_0 = \pi_\alpha([M:N]_0)\,[G:H]$$

where π_α *is the usual representation of M associated with* α. *In particular*

$$[M \rtimes_\alpha G : N \rtimes_\alpha G]_0 = \pi_\alpha([M:N]_0)\,,$$
$$[M \rtimes_\alpha G : M \rtimes_\alpha H]_0 = [G:H]\,.$$

Proof. Let $E_0 \in \mathcal{E}(M, N)$ and $\varphi \in \mathcal{E}(M)$ with $\varphi \circ E_0 = \varphi$ be such that $\varphi \circ (E_0^{-1}|_{N' \cap M}) = \varphi(c\,\cdot)|_{N' \cap M}$ where $c = [M:N]_0$. Regarding $M \rtimes_\alpha G$ as a subalgebra of $M \otimes B(l^2(G))$, we set $\tilde{E}_0 = E_0 \otimes id_{B(l^2(G))}|_{M \rtimes_\alpha G}$. Then $\tilde{E}_0 \in \mathcal{E}(M \rtimes_\alpha G, N \rtimes_\alpha G)$ as in [Hi3]. By the same argument as in the factor case ([PP], [Ko], [Wa]) we can show that there exists a basis $\{m_1, \cdots, m_n\}$ in M for E_0, i.e., $x = \sum_{j=1}^n m_j\, E_0(m_j^* x)$ for $x \in M$. Then $\{\pi_\alpha(m_1), \cdots, \pi_\alpha(m_n)\}$ is a basis in $M \rtimes_\alpha G$ for $\tilde{E}_0$, so that (cf. [Wa])

$$\tilde{E}_0^{-1}(X) = \sum_{j=1}^n \pi_\alpha(m_j)\, X\, \pi_\alpha(m_j)^* \qquad \text{for} \quad X \in (N \rtimes_\alpha G)'\,.$$

Also define $F \in \mathcal{E}(N \rtimes_\alpha G, N \rtimes_\alpha H)$ by

$$F\Big(\sum_{g \in G} \pi_\alpha(x_g)\, \lambda(g)\Big) = \sum_{g \in H} \pi_\alpha(x_g)\, \lambda(g)$$

where $x_g \in N$, $\lambda(g) = 1 \otimes \lambda_g$ and λ_g is the left regular representation of G on $l^2(G)$. Let $G = \bigcup_{i=1}^m H g_i$ be the decomposition of G into the left cosets with $[G:H] = m$. Then it follows that $\{\lambda(g_1)^*, \cdots, \lambda(g_m)^*\}$ is a basis in $N \rtimes_\alpha G$ for F. Hence

$$F^{-1}(X) = \sum_{i=1}^m \lambda(g_i)^*\, X \lambda(g_i) \qquad \text{for} \quad X \in (N \rtimes_\alpha H)'\,.$$

Furthermore, define $\phi \in \mathcal{E}(M)$ by $\phi(x) = |G|^{-1} \sum_{g \in G} \varphi(\alpha_g(x))$ for $x \in M$, and $\tilde{\varphi} \in \mathcal{E}(M \rtimes_\alpha G)$ by $\tilde{\varphi}(\sum_{g \in G} \pi_\alpha(x_g)\, \lambda(g)) = \phi(x_e)$. By the uniqueness of E_0, we see that $\alpha_g \circ E_0 \circ \alpha_g^{-1} = E_0$ and hence $\alpha_g \circ E_0^{-1} \circ \alpha_g^{-1}|_{N' \cap M} = E_0^{-1}|_{N' \cap M}$ (cf. [Hi2, 3.2]). This implies that $c = E_0^{-1}(1)$ is α-invariant. Hence $\pi_\alpha(c) \in \mathcal{Z}(M \rtimes_\alpha G) \cap \mathcal{Z}(N \rtimes_\alpha H)$. Since

$$\begin{aligned}
\tilde{\varphi} \circ F \circ \tilde{E}_0\Big(\sum_{g \in G} \pi_\alpha(x_g)\, \lambda(g)\Big) &= \tilde{\varphi}\Big(\sum_{g \in H} \pi_\alpha(E_0(x_g))\, \lambda(g)\Big) \\
&= \phi(E_0(x_e)) = \frac{1}{|G|} \sum_{g \in G} \varphi(E_0(\alpha_g(x_e))) \\
&= \tilde{\varphi}\Big(\sum_{g \in G} \pi_\alpha(x_g)\, \lambda(g)\Big)\,,
\end{aligned}$$

we have $\tilde{\varphi}\circ F\circ\tilde{E}_0=\tilde{\varphi}$. If $X=\sum_{g\in G}\pi_\alpha(x_g)\,\lambda(g)\in(N\rtimes_\alpha H)'\cap(M\rtimes_\alpha G)$, then we have

$$\begin{aligned}\tilde{\varphi}\circ(F\circ E_0)^{-1}(X) &= \tilde{\varphi}\circ\tilde{E}_0^{-1}\circ F^{-1}(X)\\ &= \tilde{\varphi}(\sum_{g\in G}\sum_{i=1}^{m}\sum_{j=1}^{n}\pi_\alpha(m_j)\,\lambda(g_i)^*\,\pi_\alpha(x_g)\,\lambda(g)\,\lambda(g_i)\,\pi_\alpha(m_j^*))\\ &= \tilde{\varphi}(\sum_{g\in G}\sum_{i=1}^{m}\sum_{j=1}^{n}\pi_\alpha(m_j\,\alpha_{g_i^{-1}}(x_g)\,\alpha_{g_i^{-1}gg_i}(m_j^*)\,\lambda(g_i^{-1}\,gg_i))\\ &= \phi(\sum_{i=1}^{m}\sum_{j=1}^{n}m_j\,\alpha_{g_i^{-1}}(x_e)\,m_j^*)\\ &= \phi(\sum_{i=1}^{m}\alpha_{g_i^{-1}}\circ E_0^{-1}(x_e))\\ &= m\phi(E_0^{-1}(x_e)) = m\phi(cx_e) = m\tilde{\varphi}(\pi_\alpha(c)\,X)\,,\end{aligned}$$

because $x_e\in N'\cap M$ and so $\alpha_g(x_e)\in N'\cap M$. Finally we have (see [BDH, 3.18]) that $M\rtimes_\alpha G$ and $N\rtimes_\alpha H$ have finite dimensional centers. So we get the conclusion. ∎

Acknowledgements

I am greatly indebted to Professor F. Hiai for his many useful suggestions, and to Professor Y. Watatani for his helpful discussions. I would like to thank Professors T. Ando and Y. Nakamura for their interest and encouragement.

References

[A1] Araki, H., Relative entropy of states of von Neumann algebras, *Publ. RIMS, Kyoto Univ.*, **11** (1976), 809–833.

[A2] ————, Relative entropy of states of von Neumann algebras II, *Publ. RIMS, Kyoto Univ.*, **13** (1977), 173–192.

[BDH] Baillet, M., Denizeau, Y. and Havet, J.-F., Indice e'une esperance conditionnelle, *Comp. Math.*, **66** (1988), 199–236.

[Co] Connes, A., Spatial theory of von Neumann algebras, *J. Funct. Anal.*, **35** (1980), 153–164.

[GH] Goodman, F., P. de la Harpe and V.F.R. Jones, *Coxeter Graphs and Towers of Algebras*, MSRI Publ. 14, Springer-Verlag, New-York, 1989.

[Ha1] Haagerup, U., Operator valued weights in von Neumann algebras I, *J, Funct. Anal.*, **32** (1979), 175–206.

[Ha2] ————, Operator valued weights in von Neumann algebras II, *J, Funct. Anal.*, **33** (1979), 339–361.

[Hav] Havet, J.-F., *Esperance conditionnelle minimale*, preprint, Orleans.

[Hi1] Hiai, F., Minimizing indices of conditional expectation onto a subfactor, *Publ. RIMS, Kyoto Univ.*, **24** (1988), 673–678.

[Hi2] ————, Minimum index for subfactors and entropy, to appear in *J. Operator Theory*.

[Hi3] ———, Minimum index for subfactors and entropy II, to appear in *J. Math. Soc. Japan* (1990).

[Jo1] Jolissaint, P., Index for Pairs of Finite von Neumann Algebras, *Pacific J. Math.*, **146** (1990), 43–70.

[J] Jones, V.F.R., Index for subfactors, *Invesnt. Math.*, **72** (1983), 1–15.

[Kk] Kawakami, S., Some remarks on index and entropy for von Neumann algebras, *Proc. Japan Acad., Ser*, A **65** (1989), 323–325.

[KY1] Kawakam, S. and Yoshida, H., *Actions of finite von Neumann algebras and relative entropy, J. Math. Soc. Japan.*, **39** (1987), 609–626.

[KY2] ———, Reduction theory on the relative entropy, *Math. Japan.*, **33** (1988), 975–990.

[Ko] Kosaki, H., Extension of Jones' theory on index to arbitrary factors, *J. Funct. Anal.*, **66** (1986), 123–140.

[Lo] Longo, R., Index of subfactors and statistics of quantum fields. I, *Commun. Math. Phys.*, **126** (1989), 217–247.

[PP] Pimsner, M. and Popa, S., Entropy and index for subfactors, *Ann. Sci. Ec. Norm. Sup.*, **19** (1986), 57–106.

[Wa] Watatani, Y., Index for C^*-subalgebras, *Memoirs Amer. Math. Soc.*, **424**, 1990.