Publ. RIMS, Kyoto Univ.
28 (1992), 455–494

On Algebraic $\sharp$-Cones in Topological Tensor-Algebras, I. Basic Properties and Normality

By

Gerald HOFMANN*

Abstract

The concept of algebraic $\sharp$-cones (alg-$\sharp$ cones) in topological tensor-algebras $E_{\otimes}[\tau]$ is introduced. It seems to be useful because the well-known cones such as the cone of positivity $E_{\otimes}^{+}$, the cone of reflection positivity (Osterwalder-Schrader cone), and some cones of α-positivity in QFT with an indefinite metric are examples of alg-$\sharp$ cones.

It is investigated whether or not the known properties of $E_{\otimes}^{+}$ (e.g., $E_{\otimes}^{+}$ is a proper and generating cone not satisfying the decomposition property) apply to alg-$\sharp$ cones. For proving deeper results, the structure of the elements of alg-$\sharp$ cones is analyzed, and certain estimations between the homogeneous components of those elements are proven. Using them, a detailed investigation of the normality of alg-$\sharp$ cones is given.

Furthermore, the convex hull of finitely many alg-$\sharp$ cones is also considered.

§0. Introduction

The motivation of the present investigations comes from axiomatic quantum field theory (QFT). Within the so-called nonlinear program of the algebraic approach to QFT there are considered several cones in tensor-algebras $E_{\otimes}$. Such cones are the cone of positivity $E_{\otimes}^{+}$, [5], [30], the cone of reflection positivity (Osterwalder-Schrader cone), [25], [29], and the convex hull of both, [9]. Furthermore, indefinite inner product QFT and gauge field theories in local (renormalizable) gauges demand some positivity conditions that lead us to the investigation of the cone of α-positivity, [3], [16], [17], [18]. As a generalization of all of these cones the concept of algebraic $\sharp$-cones (alg-$\sharp$ cones) is introduced (see Examples 2.4).

This series of two papers is devoted to a systematic investigation of the

Communicated by H. Araki, June 26, 1991.
1991 Mathematics Subject Classifications: 46K05
* Universität Leipzig, Fachb. Mathematik, Augustusplatz 10, D-0-7010 Leipzig, Germany

structure of such cones. It is shown that these cones share some important properties which lead to some interesting applications such as i) characterization of normal topologies on tensor-algebras, ii) explicite description of the closed hull of alg-# cones, iii) representation of alg-# cones as the convex hull of their extremal rays, iv) extension of linear functionals to positive ones. It is worth mentioning that all the results apply also to the convex hull of finitely many alg-# cones.

More precisely, the well-known results concerning the algebraic structure of $E_{\otimes}^{+}$ (such as $E_{\otimes}^{+}$ is a proper and generating cone for the hermitian part of $E_{\otimes}$, [32]) extent to some families of alg-# cones. However, in order to prove deeper results, which are especially related to the topological structure of $E_{\otimes}$, the structure of the elements k of alg-# cones has to be investigated in more detail. There are certain relations between the homogeneous components of such an element k. If one considers functionals which satisfy some special properties (Definition 3.1), then certain estimations between those homogeneous components are implied, see Theorem 3.3. This generalizes some results given in [23].

Those estimations are the key for solving the above given problems i)$\cdots$ iv). Concerning i), sufficient conditions are provided for locally convex (l.c.) topologies such that a given alg-# cone is normal (Theorem 4.2). It follows further that these conditions are also necessary for the normality of a wide and important class of l.c. topologies (Lemma 4.3, Corollary 4.4). This is a generalization of the known results for $E_{\otimes}^{+}$, see [10], [11], [28], [7]. Problems ii), iii), iv) will be considered in the second paper of this series.

The pattern of the present paper is as follows. For the convenience of the reader, we will first recall, in Section 1, the for the further considerations needed definitions and facts from the theory of topological tensor-algebras. The definition of the class of alg-# cones and some properties connecting the algebraic structure of $E_{\otimes}$ with the semi-ordering defined by an alg-# cone are given in Section 2, see Theorem 2.3a),$\cdots$d). The aim of Section 3 is to prove explicite estimations between the homogeneous components of the elements of alg-# cones, see Theorem 3.3. Further, there are discussed some interesting examples such as positive linear functionals, ε- and σ-semi-norms on tensor-algebras. The results of Section 3 are used, in Section 4, for a systematic investigation of the normality of alg-# cones and of the convex hull of such cones.

§1. Preliminaries

For the following let us be given a vector-space E over the field of complex numbers $\boldsymbol{C}$, and let

$$E_n = E \otimes E \otimes \cdots \otimes E$$

stand for the n-fold (algebraic) tensor product of E by itself, $n \in \boldsymbol{N}$. The *tensor-algebra* $E_{\otimes}$ over the basic space E is then defined by

$$E_{\otimes} = \boldsymbol{C} \oplus E_1 \oplus E_2 \oplus \cdots \quad \text{(direct sum)},$$

i.e., the elements $f \in E_{\otimes}$ are terminating sequences

$$f = (f_0, f_1, \cdots, f_N, 0, 0, \cdots),$$

where $f_n \in E_n$, $n=0, 1, 2, \cdots$ ($E_0 = \boldsymbol{C}$, $E_1 = E$). Further, f_n will be called the *n-th homogeneous component* of f.

Defining componentwise algebraic operations

$$\begin{aligned}
(f+g)_n &= f_n + g_n, \\
(\mu f)_n &= \mu f_n, \\
(fg)_n &= \sum_{r+s=n} f_r \otimes g_s, \quad (f_0 \otimes g_n = g_n \otimes f_0 = f_0 g_n),
\end{aligned}$$

for $f, g \in E_{\otimes}$, $\mu \in \boldsymbol{C}$ ($n=0, 1, 2, \cdots$), $E_{\otimes}$ becomes an (associative) algebra with unity $\boldsymbol{1} = (1, 0, 0, \cdots)$. If an involution "$*$" is given on E, then $E_{\otimes}$ becomes a $*$-algebra by setting

$$\begin{aligned}
f^* &= (f_0^*, f_1^*, \cdots, f_N^*, 0, 0, \cdots), \\
f_n^* &= h^{(n)*} \otimes h^{(n-1)*} \otimes \cdots \otimes h^{(1)*}
\end{aligned}$$

for $f_n = h^{(1)} \otimes \cdots \otimes h^{(n)} \in E_n$ ($n \in \boldsymbol{N}$), $f_0^* = \bar{f}_0$, and using antilinearity of $*$. Then, the *cone of positive elements* $E_{\otimes}^+$ is defined by

$$E_{\otimes}^+ = \sum_{i=1}^{M} f^{(i)*} f^{(i)},$$

$f^{(i)} \in E_{\otimes}$, $M \in \boldsymbol{N}$.

Let us be given some $f = (0, \cdots, 0, f_{N_1}, \cdots, f_{N_2}, 0, 0, \cdots) \in E_{\otimes}$, where $f_{N_1} \neq 0$, $f_{N_2} \neq 0$, $N_1, N_2 \in \boldsymbol{N}^*$ ($\boldsymbol{N}^* = \{0, 1, 2, \cdots\}$). Then put

$$\begin{aligned}
&\operatorname{Grad}(f) = N_2, \operatorname{grad}(f) = N_1 \quad \text{for} \quad f \neq \boldsymbol{0}, \\
&\operatorname{Grad}(\boldsymbol{0}) = -\infty, \operatorname{grad}(\boldsymbol{0}) = \infty,
\end{aligned}$$

where $\boldsymbol{0} = (0, 0, \cdots) \in E_{\otimes}$. For $f, g \in E_{\otimes}$, it follows

$$\mathrm{Grad}\,(fg) = \mathrm{Grad}\,(f)+\mathrm{Grad}\,(g)\,, \tag{1}$$
$$\mathrm{grad}\,(fg) = \mathrm{grad}\,(f)+\mathrm{grad}\,(g)\,, \tag{1'}$$
$$\mathrm{Grad}\,(f+g)\leqslant \max\{\mathrm{Grad}\,(f),\,\mathrm{Grad}\,(g)\}\,, \tag{2}$$
$$\mathrm{grad}\,(f+g)\geqslant \min\{\mathrm{grad}\,(f),\,\mathrm{grad}\,(g)\}\,, \tag{2'}$$

where (1), (1′) are based on the fact that $f_n, f_m \neq 0$ imply $f_n \otimes f_m \neq 0$, [19, §9.6(4)].

For the following let $Q_n\colon E_\otimes \to \bigoplus_{i=0}^{n} E_i$ denote the canonical projections, where $\bigoplus_{i=0}^{n} E_i$ is considered as a subspace of $E_\otimes$, i.e., for $f=(f_0, f_1, \cdots, f_N, 0, 0, \cdots) \in E_\otimes$ it follows

$$Q_n(f) = (f_0, \cdots, f_n, 0, 0, \cdots)\,.$$

Further, a subset $M \subset E_\otimes$ will be called *filtrated,* if

$$Q_n(M) \subset M$$

for all $n=0, 1, 2, \cdots$. Furthermore, let

$$\check{f}_n = (0, \cdots, 0, f_n, 0, 0, \cdots) \in E_\otimes\,,$$

$f_n \in E_n$, and $\check{F}_n = \{\check{f}_n, f_n \in F_n\}$, $F_n \subset E_n$.

For any two l.c. topologies τ, τ', let $\tau < \tau'$ and $\tau \lneqq \tau'$ mean that τ' is finer (stronger) than τ and that τ' is strictly finer than τ, respectively.

Assume now that $E[t]$ is an l.c. space. On E_n let us consider the class of l.c. topologies that are *compatible* (with the tensor product), see [19; §44.1]. Recall that for every compatible topology t_n on E_n, $\varepsilon_n < t_n < \iota_n$ follow, where ε_n and ι_n denote the injective and inductive topology on E_n, respectively ($n=2, 3, 4, \cdots$). Further let π_n denote the projective topology on E_n. Concerning the equivalence of some of these topologies ε_n, π_n and ι_n, we refer to [12]. Let us further mention that the importance of the inductive topology for applications to axiomatic QFT was first discussed by J. Alcántara ([1]).

Let us be given $E_n[t_n]$ with $\varepsilon_n < t_n < \iota_n$, $n=2, 3, 4, \cdots$. An l.c. topology τ on $E_\otimes$ is called an *intermediate* one, if

$$\tau{\restriction}_{E_m} = t_m \tag{3}$$

for $m=0, 1, 2, \cdots$ ($t_1=t$, t_0 denotes the Euclidean topology on $\boldsymbol{C}$). In order to define intermediate topologies τ on $E_\otimes$ the algebraic structure of $E_\otimes$ defines the weakest l.c. topology $\tau_{P,(t_n)}$ and the finest l.c. topology $\tau_{\otimes,(t_n)}$ such that (3) holds. Recall that $\tau_{\otimes,(t_n)}$ is the topology of the direct sum $\bigoplus_{m=0}^{\infty} E_m[t_m]$, and $\tau_{P,(t_n)}$ is the topology which is induced by the topology of the direct product $\prod_{m=0}^{\infty} E_m[t_m]$ on its subspace $E_\otimes$. Hence an l.c. topology τ on $E_\otimes$ is an intermediate one, if and only if

$$\tau_{P,(t_n)} < \tau < \tau_{\otimes,(t_n)} .$$

Let the topologies t_m be defined by the following systems of semi-norms

$$\mathfrak{P}(t_m) = \{f_m \to p_{\alpha_m}^{(m)}(f_m);\ \alpha_m \in A_m\} ,$$

where $f_m \in E_m$, A_m is a set of indices, $m=1, 2, 3, \cdots$, and

$$\mathfrak{P}(t_0) = \{f_0 \to p_0^{(0)}(f_0)\} ,$$

$f_0 \in E_0$, $p_0^{(0)}(f_0) = |f_0|$. Put $A_0 = \{0\}$. Then let us introduce a semi-ordering "$<$" in A_m by setting $\alpha_m < \alpha'_m$ if there is some constant $c>0$ such that

$$p_{\alpha_m}^{(m)}(f_m) \leqslant c\, p_{\alpha'_m}^{(m)}(f_m)$$

for all $f_m \in E_m$.

In the following let respectively, $\boldsymbol{R}_+^{N^*}$ and A denote the set of all sequences $(\gamma_n)_{n=0}^{\infty}$ and $(\alpha_n)_{n=0}^{\infty}$ such that $\gamma_n \geqslant 0$, $\alpha_n \in A_n$. A semi-norm q on $E_\otimes$ is called *graded*, if there are t_n-continuous semi-norms $q^{(n)}$ on $E_n (n=0, 1, 2, \cdots)$ such that

$$q(f) = \sum_{n=0}^{\infty} q^{(n)}(f_n)$$

for all $f=(f_0, f_1, \cdots, f_N, 0, 0, \cdots) \in E_\otimes$. Further, an l.c. topology τ on $E_\otimes$ will be called *graded*, if there is a τ-defining system $\mathfrak{P}(\tau)$ that consists only of graded semi-norms.

The following definition introduces graded topologies on $E_\otimes$.

Definition 1.1. Let us be given two sets $\Gamma \subset \boldsymbol{R}_+^{N^*}$ and $B \subset A$ such that

i) for each $\mu \in \boldsymbol{N}^*$ there is a $(\gamma_n) \in \Gamma$ with $\gamma_\mu > 0$,

ii) for each $\mu \in \boldsymbol{N}^*$ and $\alpha_\mu \in A_\mu$ there is a $(\beta_n) \in B$ with $\alpha_\mu < \beta_\mu$. The l.c. topology that is defined by the system of semi-norms

$$\{f \to p_{(\gamma_n)(\alpha_n)}(f);\ (\gamma_n) \in \Gamma,\ (\alpha_n) \in B\} ,$$

$p_{(\gamma_n)(\alpha_n)}(f) = \sum_{n=0}^{\infty} \gamma_n p_{\alpha_n}^{(n)}(f_n)$, $f=(f_0, \cdots, f_N, 0, 0, \cdots) \in E_\otimes$, will be denoted by $\tau(\Gamma, B; \mathfrak{P}(t_n))$.

Remarks. a) Conditions i), ii) imply that for each $f \neq \mathbf{0}$, $f \in E_\otimes$, there are sequences $(\gamma_n) \in \Gamma$, $(\alpha_n) \in B$ such that

$$p_{(\gamma_n)(\alpha_n)}(f)>0\,.$$

Hence $\tau(\Gamma, B; \mathfrak{P}(t_n))$ exists by [19; §18.1(3)]. Further, i) and ii) yield that $\tau(\Gamma, B; \mathfrak{P}(t_n))$ induces the topologies t_m on the subspaces $E_m \subset E_\otimes$ $(m=0,1,2,\cdots)$.

b) If $\Gamma_i \subset \mathbf{R}_+^{N^*}$, $B_i \subset A$ $(i=1, 2)$ satisfy i), ii) and also $\Gamma_1 \subset \Gamma_2$, $B_1 \subset B_2$, then

$$\tau(\Gamma_1, B_1; \mathfrak{P}(t_n)) < \tau(\Gamma_2, B_2; \mathfrak{P}(t_n))$$

is immediately implied.

c) Let

$\Gamma_P = \{(\gamma_n) \in \mathbf{R}_+^{N^*},\ \gamma_n \neq 0$ holds only for finite many $n \in \mathbf{N}^*\}$. Then

$$\begin{aligned}\tau_{P,(t_n)} &= \tau(\Gamma_P, A; \mathfrak{P}(t_n))\,,\\ \tau_{\otimes,(t_n)} &= \tau(\mathbf{R}_+^{N^*}, A; \mathfrak{P}(t_n))\end{aligned}$$

hold true. Note also that $\tau(\Gamma_P, A; \mathfrak{P}(t_n))$, $\tau(\mathbf{R}_+^{N^*}, A; \mathfrak{P}(t_n))$ do not depend on the choice of the t_n-defining systems $\mathfrak{P}(t_n)$ of semi-norms $(n=0, 1, 2, \cdots)$.

d) If $t_m=\varepsilon_m (m=2, 3, 4, \cdots)$, then let us write ε_P and $\varepsilon_\otimes$ instead of $\tau_{P,(t_n)}$ and $\tau_{\otimes,(t_n)}$, respectively. Analogously, π_P, ι_P, $\pi_\otimes$, $\iota_\otimes$ are defined.

e) Let $\mathfrak{P}(t_1)=\{p_{\alpha_1}^{(1)};\ \alpha_1 \in A_1\}$ be given. Further let $A^n=A_1 \times \cdots \times A_1$ (n factors), $n=1, 2, 3, \cdots$, denote the set of all multi-indices

$$\alpha^n = (\alpha^{(1,n)}, \alpha^{(2,n)}, \cdots, \alpha^{(n,n)})\,,$$

$\alpha^{(i,n)} \in A_1$, $i=1, 2, \cdots, n$. Set $A^0=\{0\}$. Then consider

$$\begin{aligned}A_\infty = \{(\alpha^j)_{j=0}^\infty;\ \alpha^j \in A^j, &\text{ there exists an } \alpha_1 \in A_1 \text{ such that}\\ &\alpha^{(i,n)} = \alpha_1,\ i=1, 2, \cdots, n \text{ and } n=1, 2, 3, \cdots\}\,.\end{aligned}$$

Further note that the systems of semi-norms

$$\mathfrak{P}(\varepsilon_m) = \{p_{\alpha^{(1,m)}}^{(1)} \underset{\varepsilon}{\otimes} \cdots \underset{\varepsilon}{\otimes} p_{\alpha^{(m,m)}}^{(1)};\ \alpha^{(i,m)} \in A_1,\ \mathrm{i}=1, 2, \cdots, m\}$$

define the injective topologies ε_m on E_m $(m=2, 3, \cdots)$, where

$$\begin{aligned}p_{\alpha_1}^{(1)} \underset{\varepsilon}{\otimes} \cdots \underset{\varepsilon}{\otimes} p_{\alpha_1}^{(1)}(f_m) = \sup\{|T^{(1)} \otimes \cdots \otimes T^{(m)}(f_m)|\,;\ T^{(i)} \in U^0_{p_{\alpha_1}^{(1)}}\\ \alpha_1 \in A_1,\ i=1, 2, \cdots, m\}\,,\end{aligned}$$

$U^0_{p_{\alpha_1}^{(1)}} = \{T \in E';\ |T(f_1)| \leq p_{\alpha_1}^{(1)}(f_1)$ for all $f_1 \in E\}$. Then the l.c. topology

$$\varepsilon_\infty = \tau(\mathbf{R}_+^{N^*}, A_\infty; \mathfrak{P}(\varepsilon_n))$$

is considered, where $\mathfrak{P}(\varepsilon_1)=\mathfrak{P}(t_1)$, $\mathfrak{P}(\varepsilon_0)=\{p_0^{(0)}\}$. Analogously, the l.c. topology π_∞ is introduced. It is straightforward to prove that the topologies ε_∞, π_∞ do not depend on the choice of the t_1-defining system $\mathfrak{P}(t_1)$. Let us mention that ε_∞ was introduced by G. Lassner in the case of the tensor-algebra $\mathcal{S}_\otimes$, [20].

f) For further investigations on regular tensor-algebras and of l.c. topologies that do not depend on the special choice of $\mathfrak{P}(t_1)$, we refer to [7].

g) Note that the following order-relations between the topologies introduced above are valid:

$$\begin{array}{ccccc} t_p & & \lneqq & & t_\otimes \\ \vee & & & & \vee \\ \pi_p & \lneqq & \pi_\infty & < & \pi_\otimes \\ \vee & & & & \vee \\ \varepsilon_p & \lneqq & \varepsilon_\infty & < & \varepsilon_\otimes \, . \end{array} \tag{4}$$

For a characterization of some of the topological properties of the basic-space $E[t]$ in terms of the equivalence of some of the topologies given in (4), we refer to [12].

§2. Definition and Some Properties of alg-$\#$ Cones

Let us be given a subspace

$$F = \bigoplus_{n=0}^{\infty} F_n \tag{5}$$

of $E_\otimes$, where $F_n \subset E_n$. Further, let us consider an antilinear mapping $\#: E_\otimes \to E_\otimes$ which satisfies

$$f^{\#\#} = f\,, \tag{6}$$

$$(Q_n f)^{\#} = Q_n(f^{\#}) \tag{6'}$$

for all $f \in E_\otimes$, $n=0, 1, 2, \cdots$. Notice that $\#$ is bijective. Let us put

$$\{F, \#\} = \{\sum_{i=1}^{M} f^{(i)\#} f^{(i)};\, f^{(i)} \in F,\ M \in N\}\ .$$

It is immediate that $\{F, \#\}$ is a convex cone (containing its apex $\mathbf{0}$). In the following such cones $\{F, \#\}$ will be called *alg-$\#$ cones*. If $\#$ satisfies additionally

$$(fg)^{\#} = g^{\#} f^{\#} \tag{7}$$

for all $f, g \in E_\otimes$, then $\{F, \#\}$ will be called *involutive cone*.

For every subset $H\subset E_{\otimes}$ let us define the *$\sharp$-hermitian part* of H by

$$h(H,\sharp)=\{f\in H; f^{\sharp}=f\}\ .$$

If H is a subspace, then $h(H,\sharp)$ is a real vector space. Further, for every involutive cone $\{F,\sharp\}$ it follows

$$\{F,\sharp\}\subset h(E_{\otimes},\sharp)\ . \tag{7'}$$

For every subspace $H\subset E_{\otimes}$ let $L(H,\boldsymbol{C})$ (resp. $L(H,\boldsymbol{R})$) denote the set of complex-valued (resp. real-valued) linear functionals on H. If $H=H^{\sharp}$ $(=\{h^{\sharp}; h\in H\})$, then let us consider the set of *$\sharp$-hermitian linear functionals* which is given by

$$L^{\sharp}(H,\sharp)=\{T\in L(H,\boldsymbol{C});\ T(f^{\sharp})=\overline{T(f)}, f\in H\}\ .$$

Finally, $T\in L(E_{\otimes},\boldsymbol{C})$ is called *$\{F,\sharp\}$-positive*, if $T(k)\geqslant 0$ for each $k\in\{F,\sharp\}$.

As in the well-known theory of *-algebras ([24]) the antilinear bijection $\sharp$ implies a decomposition of $E_{\otimes}$ into a "real" and an "imaginary" part. More precisely, the following hold true.

Lemma 2.1. *If the antilinear bijection $\sharp$ satisfies* (6), (6′), *then the following are satisfied.*

a) *For every subspace $H\subset E_{\otimes}$ with $H=H^{\sharp}$, it follows*

i) $h(H,\sharp)=\{f+f^{\sharp}; f\in H\}=\{i(f^{\sharp}-f); f\in H\}$,

ii) $H=h(H,\sharp)+i\,h(H,\sharp)$, (*where* $i^2=-1$).

b) *For every $T\in L(E_{\otimes},\boldsymbol{C})$,*

$$T=T^{(1)}+i\,T^{(2)}$$

follows, where $T^{(1)}(x)=\frac{1}{2}(T(x)+T(x^{\sharp}))$, $T^{(2)}(x)=\frac{1}{2i}(T(x)-T(x^{\sharp}))$, $x\in E_{\otimes}$, *and* $T^{(1)}, T^{(2)}\in L^{\sharp}(E_{\otimes},\boldsymbol{C})$. *Furthermore, there is a linear isomorphism*

$$\mu\colon L^{\sharp}(E_{\otimes},\boldsymbol{C})\to L(h(E_{\otimes},\sharp),\boldsymbol{R})$$

given by $\mu(T)=L$, where $L(y)=T^{(1)}(y)$, $y\in h(E_{\otimes},\sharp)$.

c) *If $T\in L(E_{\otimes},\boldsymbol{C})$ is $\{F,\sharp\}$-positive, then*

i) $T(g^{\sharp}f)=\overline{T(f^{\sharp}g)}$,

ii) $|T(f^{\sharp}g)|^2\leqslant T(f^{\sharp}f)\,T(g^{\sharp}g)$ (*Cauchy-Schwarz' inequality*) *hold for all* $f,g\in F$.

d) *If $\sharp$ satisfies* (7) *in addition, then* $\mathbf{1}^{\sharp}=\mathbf{1}$.

e) *If $\{F,\sharp\}$ is an involutive cone with $F=F^{\sharp}$ and $\mathbf{1}\in F$, then every $\{F,\sharp\}$-*

positive linear functional T satisfies $T\in L^{\#}(F, \boldsymbol{C})$.

f) *The antilinear mapping $\#$ is graded (i.e., $f_n^{\#}\in E_n$ for all $f_n\in E_n$), and $(E_n)^{\#}=E_n$ are satisfied, $n=0, 1, 2, \cdots$.*

Proof. All the proofs of a)$\cdots$f) are straightforward and analogous to the corresponding ones from the theory of *-algebras, see [24; § 10. 1, 2].

The following lemma collects some of the central and in the following frequently used properties of the elements of alg-$\#$ cones. For the *proof* we refer to [13].

Lemma 2.2. *Let us be given an alg-$\#$ cone $\{F, \#\}$ and an element $k\in\{F, \#\}$ with $k=(k_0, k_1, \cdots, k_m, 0, 0, \cdots)=\sum_{i=1}^{M} f^{(i)\#}f^{(i)}\neq\boldsymbol{0}, f^{(i)}\in F, i=1, 2, \cdots, M (M\in\boldsymbol{N})$. Then, the following hold.*

a) *It is* $\mathrm{Grad}(k)=2N$ (*resp.* $\mathrm{grad}(k)=2n$), *if and only if* $\max\{\mathrm{Grad}(f^{(i)}); i=1, 2, \cdots, M\}=N$ (*resp.* $\min\{\mathrm{grad}(f^{(i)}); i=1, \cdots, M\}=n$).

b) *If* $\mathrm{grad}(k)=2n$ *and* $\mathrm{Grad}(k)=2N$ *are satisfied, then $(T_n\otimes T_n)(k_{2n})\geqslant 0$, $(T_N\otimes T_N)(k_{2N})\geqslant 0$ hold for each $T_n\in L^{\#}(E_n, \boldsymbol{C})$, $T_N\in L^{\#}(E_N, \boldsymbol{C})$.*

c) *If the assumptions of* b) *are satisfied, then there are $T_n^0\in L^{\#}(E_n, \boldsymbol{C})$, $T_N^0\in L^{\#}(E_N, \boldsymbol{C})$ such that $(T_n^0\otimes T_n^0)(k_{2n})>0$, $(T_N^0\otimes T_N^0)(k_{2N})>0$.*

Some of the immediate consequences of Lemma 2.2a) are collected in a), b) of the following remark. In c), d) it will be shown by some examples that the representation of $k\in\{F, \#\}$ as a finite sum $\sum_{i=1}^{M} f^{(i)\#}f^{(i)}$ is not unique, and it is not necessarily implied that $f^{(i)}\in F$ $(i=1, 2, \cdots, M)$.

Remark. a) If $k\in\{F, \#\}$ has two decompositions

$$k=\sum_{i=1}^{M} f^{(i)\#}f^{(i)}=\sum_{j=1}^{M'} g^{(j)\#}g^{(j)},$$

$f^{(i)}, g^{(j)}\in F$, then

$$\max\{\mathrm{Grad}(f^{(i)}); i=1, 2, \cdots, M\}=\max\{\mathrm{Grad}(g^{(j)}); j=1, 2, \cdots, M'\},$$
$$\min\{\mathrm{grad}(f^{(i)}); i=1, 2, \cdots, M\}=\min\{\mathrm{grad}(g^{(j)}); j=1, 2, \cdots, M'\}$$

are implied.

b) If $\boldsymbol{0}\neq k\in\{F, \#\}$, then $\mathrm{grad}(k)$, $\mathrm{Grad}(k)$ are even numbers.

c) Assuming $\dim(F)\geqslant 2$, there is the following example for different decompositions of elements $k\in\{F, \#\}$: Choose linearly independent $f, g\in F$, and consider

$$k = 2f^{\#}f + 2g^{\#}g = (f+g)^{\#}(f+g) + (f-g)^{\#}(f-g)\,. \tag{8}$$

Noticing that $f^{\#}f$, $(f+g)^{\#}(f+g)$, and $(f-g)^{\#}(f-g)$ are linearly independent, (8) yields different decompositions of k.

d) Consider $(\mathbb{C}^2)_{\otimes}$, $F_0=\mathbb{C}$, $F_1=\text{span}\{z_1\}$, $F_2=\text{span}\{z_2\}$, where $z_1=x+iy$, $z_2=i(y\otimes x-x\otimes y)$, $x, y\in\mathbb{R}^2$, and x, y are linearly independent. Put $F_n=0$ for $n=3, 4, \cdots$. Let us define an antilinear bijection "$\#$" on $(\mathbb{C}^2)_{\otimes}$ by $\mathbb{1}^{\#}=\mathbb{1}$, $(u+iv)^{\#}=u-iv$ for $u, v\in\mathbb{R}^2$, and

$$(w^{(1)}\otimes\cdots\otimes w^{(n)})^{\#} = w^{(n)\#}\otimes\cdots\otimes w^{(1)\#}$$

for $w^{(j)}\in\mathbb{C}^2$ $(j=1, 2, \cdots, n)$. Setting

$$f^{(1)} = (1, z_1, 0, 0, \cdots)\,, \quad f^{(2)} = (1, 0, \frac{1}{2}z_2, 0, 0, \cdots)\,,$$

$$g^{(1)} = (i, y+ix, 0, 0, \cdots)\,, \quad g^{(2)} = (1, 0, -\frac{1}{2}z_2, 0, 0, \cdots)\,,$$

it follows that

$$\sum_{j=1}^{2} f^{(j)\#}f^{(j)} = (2, 2x, x\otimes x+y\otimes y, 0, \frac{1}{4}z_2^{\#}\otimes z_2, 0, 0, \cdots) = \sum_{j=1}^{2} g^{(j)\#}g^{(j)}\,. \tag{8$'$}$$

Noticing $g^{(1)}\notin F$, (8′) shows that $k=\sum_{j=1}^{M} f^{(j)\#}f^{(j)}\in\{F, \#\}$ does not necessarily imply $f^{(j)}\in F$ for all $j=1, 2, \cdots, M$.

Some of the basic properties connecting the semi-ordering which is induced by an alg-$\#$ cone $\{F, \#\}$ with the algebraic structure of the vector space $\{F, \#\}-\{F, \#\}$ are collected in the following theorem. Let us mention that in the case of $\mathcal{S}_{\otimes}^{+}$ (see the following example a)) the assertions b), c), e) of the following theorem are due to W. Wyss, [32].

Theorem 2.3. *Let us be given an alg-$\#$ cone $\{F, \#\}$. Then the following are satisfied.*

a) *The following are equivalent:*

i) *There is a certain $n\in\mathbb{N}^*$ with $F\subset\check{E}_n$,*

ii) *$\{F, \#\}$ is a filtrated set.*

b) *$\{F, \#\}$ is a proper cone.*

c) *If $\mathbb{1}\in F$ and $\mathbb{1}^{\#}=\mathbb{1}$ are satisfied, then*

$$h(F, \#)\subset\{F, \#\}-\{F, \#\}\,.$$

If additionally $\{F, \#\}\subset h(F, \#)$, then

$$h(F, \sharp) = \{F, \sharp\} - \{F, \sharp\} .$$

d) *The following are equivalent:*

i) $\dim(F)=1$,

ii) *$\{F, \sharp\}$ is a lattice cone in the real vector space $\{F, \sharp\}-\{F, \sharp\}$,*

iii) *$\{F, \sharp\}$ satisfies the decomposition property.*

e) *If $\{F, \sharp\} \subset h(E_\otimes, \sharp)$, then $\{F, \sharp\}$ does not contain any topologically interior points with respect to every $\tau'=\tau_{\restriction h(E_\otimes,\sharp)}$, where τ is an arbitrary l.c. topology on $E_\otimes$ with $\varepsilon_P<\tau<\iota_\otimes$.*

Proof. a) i)⇒ii) is obvious. ii)⇒i): Let us assume that $\{F, \sharp\}$ is a filtrated set, and i) is not satisfied. Hence, there are $0 \neq f_n \in F_n$, $0 \neq f_m \in F_m$, $n<m$, such that $f=\check{f}_n+\check{f}_m \in F$, $f'=i\check{f}_n+\check{f}_m \in F$. Thus,

$$f_n^\sharp \otimes f_m + f_m^\sharp \otimes f_n \neq 0 \tag{9}$$

or

$$(if_n)^\sharp \otimes f_m + f_m^\sharp \otimes (if_n) = i(f_m^\sharp \otimes f_n - f_n^\sharp \otimes f_m) \neq 0 \tag{9'}$$

are fulfilled. Using ii), it follows that $Q_{n+m}(f^\sharp f) \in \{F, \sharp\}$, $Q_{n+m}(f'^\sharp f') \in \{F, \sharp\}$, and

$$\mathrm{Grad}(Q_{n+m}(f^\sharp f)) = n+m \tag{10}$$

or

$$\mathrm{Grad}(Q_{n+m}(f'^\sharp f') = n+m ,$$

due to (9) and (9′). Assuming (10) for definiteness, Lemma 2.2a) yields the existence of $r \in \boldsymbol{N}$ and certain $g^{(i)} \in F$ $(i=1, 2, \cdots, M)$ such that $n+m=2r$ and

$$Q_{n+m}(f^\sharp f) = \sum_{i=1}^{M} g^{(i)\sharp} g^{(i)} ,$$
$$\max\{\mathrm{Grad}(g^{(i)});\ i=1, 2, \cdots, M\} = r ,$$

$M \in \boldsymbol{N}$. Choose $i_0 \in \{1, \cdots, M\}$ with $\mathrm{Grad}(g^{(i_0)})=r$, and consider

$$f^{(1)} = \check{g}_r^{(i_0)} + \check{f}_m \in F .$$

It follows that $f^{(1)\sharp} f^{(1)} \in \{F, \sharp\}$, $Q_{r+m}(f^{(1)\sharp} f^{(1)}) \in \{F, \sharp\}$. Arguing as above, there are $r_1 \in \boldsymbol{N}^*$ with $m+r=2r_1$ and $0 \neq \check{h}_{r_1} \in F$. It is

$$r_1 = (3m+n)/4 .$$

Applying this procedure k-times, one gets the existence of $0 \neq \check{h}_{r_k} \in F$ with

$$r_k = ((2^{k+1}-1)\, m+n)/2^{k+1} = m-(m-n)/2^{k+1} . \tag{11}$$

If $k>\log_2(m-n)$, (11) implies a contradiction to $r_k \in N^*$.

b) follows readily from Lemma 2.2b), c).

c) If $g \in \{F, \sharp\}$, then

$$g = \frac{1}{2}(g+g^\sharp) = \frac{1}{4}((\mathbb{1}+g)^\sharp(\mathbb{1}+g)-(\mathbb{1}-g)^\sharp(\mathbb{1}-g)) \in \{F, \sharp\} - \{F, \sharp\}$$

yields the first assertion of c). The second is a consequence of

$$h(F, \sharp) \subset \{F, \sharp\} - \{F, \sharp\} \subset h(F, \sharp) - h(F, \sharp) \overset{(*)}{=} h(F, \sharp),$$

where it is used in $(*)$ that $h(F, \sharp)$ is a (real) vector space.

e) is a consequence of Lemma 2.2a).

d) ii)$\Rightarrow$iii) holds because of [26; V.1.1].

i)$\Rightarrow$ii): If $\dim(F)=1$, then $\{F, \sharp\} - \{F, \sharp\}$ is isomorphic to $\mathbb{R}$ (furnished with its canonical ordering) as ordered vector-spaces. Hence, $\{F, \sharp\}$ is a lattice cone.

iii)$\Rightarrow$i): For the following assume that $\dim(F) \geqslant 2$. Let us consider the alternative:

I) There are elements $0 \neq g_n \in F \cap E_n$, $0 \neq h_m \in F \cap E_m$, $n<m$.

II) There is an $n \in N$ such that $\check{F}_n = F$.

I): Consider $a=\check{g}_n+\check{h}_m$, $b=\check{g}_n-\check{h}_m \in F$. Then,

$$(a^\sharp a)_{n+m} = g_n^\sharp \otimes h_m + h_m^\sharp \otimes g_n \neq 0 \tag{12}$$

(otherwise, replace h_m by $-ih_m \in F_m$). It follows that

$$a^\sharp a + b^\sharp b = 2(\check{g}_n^\sharp \check{g}_n + \check{h}_m^\sharp \check{h}_m),$$

$$a^\sharp a \in [0, 2(\check{g}_n^\sharp \check{g}_n + \check{h}_m^\sharp \check{h}_m)], \tag{13}$$

where $[x, y]$ denotes the order intervall between x and y. Let $c \in [0, 2\, \check{g}_n^\sharp \check{g}_n]$. Then,

$$\operatorname{grad}(c) = \operatorname{Grad}(c) = 2n \tag{14}$$

are implied. (Otherwise, in the case of $2s=\operatorname{grad}(c)<\operatorname{Grad}(c)=2t$ it follows that $s \neq n$ or $t \neq n$. If $s \neq n$, then there is a certain $T \in L^\sharp(E_s, \mathbb{C})$ such that $(T\otimes T)(c_{2s})>0$ and

$$(T\otimes T)((2\,\check{g}_n^\sharp \check{g}_n - c)_{2s}) = -(T\otimes T)(c_{2s})<0.$$

This is a contradiction to Lemma 2.2b) and $2\,\check{g}_n^\sharp \check{g}_n - c \in \{F, \sharp\}$. If $s=n$, then $t \neq n$, and the above given applies to t instead of s). Analogously, $d \in [0, 2\,\check{g}_m^\sharp \check{g}_m]$

implies

$$\operatorname{grad}(d) = \operatorname{Grad}(d) = 2m\,. \tag{14'}$$

Using (12), (14) and (14′), it follows that

$$a^{\#} a \notin [0, 2\, \check{g}_n^{\#}\, \check{g}_n] + [0, 2\, \check{h}_m^{\#}\, \check{h}_m]\,.$$

Hence, the decomposition property is not satisfied.

II) Using $\dim(F) \geqslant 2$, there are linearly independent $g_n, h_n \in F_n$ such that

$$g_n^{\#} \otimes h_n + h_n^{\#} \otimes g_n \neq 0\,. \tag{15}$$

Setting $a_n = g_n + h_n$, $b_n = g_n - h_n$, it follows that

$$\begin{aligned} a_n^{\#} \otimes a_n + b_n^{\#} \otimes b_n &= 2(g_n^{\#} \otimes g_n + h_n^{\#} \otimes h_n)\,, \\ \check{a}_n^{\#}\, \check{a}_n \in [0, 2(\check{g}_n^{\#}\, \check{g}_n &+ \check{h}_n^{\#}\, \check{h}_n)]\,. \end{aligned} \tag{16}$$

Let

$$c \in [0, 2\, \check{g}_n^{\#}\, \check{g}_n]\,. \tag{16'}$$

Applying (14) and Lemma 2.2a), there are $u_n^{(1)}, \cdots, u_n^{(M)} \in E_n$ such that

$$c_{2n} = \sum_{i=1}^{M} u_n^{(i)\#} \otimes u_n^{(i)}\,,$$

$M \in \boldsymbol{N}$. Choose a linearly independent system $\{v_n^{(1)}, \cdots, v_n^{(N)}\}$, $v_n^{(j)} \in E_n$, and an (M, N)-matrix A such that

$$\underline{u} = A\, \underline{v}\,,$$

where $\underline{u} = (u_n^{(1)}, \cdots, u_n^{(M)})^t$, $\underline{v} = (v_n^{(1)}, \cdots, v_n^{(N)})^t$, $\operatorname{rank}(A) = N$ ($(\,.\,)^t$ denotes the transposed of a vector.). Taking a unitary (N, N)-matrix $U = (u_{ij})$ with $U(A^*A)\, U^* = \operatorname{diag}[\mu_1, \cdots, \mu_N]$, $\mu_j > 0$, it follows that

$$\begin{aligned} \sum_{i=1}^{M} u_n^{(i)\#} \otimes u_n^{(i)} &= (\underline{u}^{\#})^t \otimes \underline{u} = (\underline{v}^{\#})^t \otimes (A^*A)\, \underline{v} \\ &= (U\underline{v}^t)^{\#} \otimes \operatorname{diag}[\mu_1, \cdots, \mu_N]\, (U\underline{v}) = \sum_{i=1}^{N} \mu_i\, w_n^{(i)\#} \otimes w_n^{(i)}\,, \end{aligned}$$

where $\underline{u}^{\#} = (u_n^{(1)\#}, \cdots, u_n^{(M)\#})^t$, $\underline{v}^{\#} = (v_n^{(1)\#}, \cdots, v_n^{(N)\#})^t$, $w_n^{(j)} = (U\underline{v})_j = \sum_{i=1}^{N} u_{ji}\, v_n^{(i)}$, $j = 1, 2, \cdots, N$.

Now, if $\operatorname{span}\{w_n^{(1)}, \cdots, w_n^{(N)}\} \neq \operatorname{span}\{g_n\}$, then choose $T \in L^{\#}(E_n, \boldsymbol{C})$ with $T(g_n) = 0$ and $T(w_n^{(j_0)}) \neq 0$ for a certain $j_0 \in \{1, \cdots, N\}$. Then,

$$(T \otimes T)\,(2\, g_n^{\#} \otimes g_n - c_{2n}) \leqslant -\,|\,T(w_n^{(j_0)})\,|^2 < 0$$

yields a contradiction to (16′) and Lemma 2.2b). Consequently, there is a

$0 \leqslant \mu \leqslant 2$ such that

$$c = \check{c}_{2n} = \mu \check{g}_n^{\sharp} \check{g}_n . \tag{17}$$

Analogously, $d \in [0, 2 \check{h}_m^{\sharp} \check{h}_m]$ implies that there is an $0 \leqslant \lambda \leqslant 2$ with

$$d = \check{d}_{2m} = \lambda \check{h}_m^{\sharp} \check{h}_m . \tag{17'}$$

Because of (15), (17) and (17′), it follows that

$$\check{a}_n^{\sharp} \check{a}_n \notin [0, 2 \check{g}_n^{\sharp} \check{g}_n] + [0, 2 \check{h}_m^{\sharp} \check{h}_m] .$$

Hence, the decomposition property is not satisfied. This completes the proof.

Examples 2.4. a) *Cone of positivity* $E_{\otimes}^{+}$.
If "$\sharp$" coincides with the involution "$*$" given in §1, then "$\sharp$" satisfies (6), (6′), (7). Hence, $E_{\otimes}^{+} = \{E_{\otimes}, *\}$ is an involutive cone. Applying Theorem 2.3 and (7′), it follows that $E_{\otimes}^{+}$ is generating for $h(E_{\otimes}, *)$ (i.e., $h(E_{\otimes}, *) = E_{\otimes}^{+} - E_{\otimes}^{+}$). Further, $E_{\otimes}^{+}$ is neither a filtrated set nor a lattice cone.

b) *Cone of reflection positivity* (Osterwalder-Schrader cone). Let us consider the Schwartz-space $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$, $d \in \mathbb{N}$, $d \geqslant 2$, of basic (repidly diminishing) functions. Put $\mathcal{S}_n = \mathcal{S} \otimes \cdots \otimes \mathcal{S}$ (n-times). Further, let $\mathcal{F} = \bigoplus_{m=0}^{\infty} \mathcal{F}_m$, $\mathcal{F}_0 = \mathbb{C}$ and

$$\mathcal{F}_n = \{f_n \in \mathcal{S}_n ; \operatorname{supp}(f_n) \subset \{x \in \mathbb{R}^{dn} ; 0 < x_1^0 < x_2^0 < \cdots < x_n^0\}\} ,$$

where $x_i = (x_i^0, x_i^1, \cdots, x_i^{d-1}) \in \mathbb{R}^d$ $(i = 1, 2, \cdots, n)$, $n \in \mathbb{N}$. Let us introduce an antilinear bijection $\sharp$ on $\mathcal{S}_{\otimes}$ by setting

$$\begin{aligned} f^{\sharp} &= (f_0^{\sharp}, f_1^{\sharp}, \cdots) , \\ f_0^{\sharp} &= \bar{f}_0 , \\ (f_n)^{\sharp}(x_1, \cdots, x_n) &= \bar{f}_n(\tilde{x}_n, \cdots, \tilde{x}_1) , \end{aligned}$$

where $\tilde{x}_i = (-x_i^0, x_i^1, \cdots, x_i^{d-1}) \in \mathbb{R}^d$, $\bar{\cdot}$ denotes the conjugate complex value of $\cdot$. Notice that $\sharp$ satisfies (6), (6′), (7). It follows that

i) $h(\mathcal{F}, \sharp) = \{f \in \mathcal{S}_{\otimes} ; f_0 \in \mathbb{R}, f_n = 0, n = 1, 2, \cdots\}$,
ii) $\mathcal{F}^{\sharp} \neq \mathcal{F}$, $\mathbb{1}^{\sharp} = \mathbb{1}$, $\mathbb{1} \in \mathcal{F}$,
iii) $h(\mathcal{F}, \sharp) \subsetneq \{\mathcal{F}, \sharp\} - \{\mathcal{F}, \sharp\}$,
iv) $\{\mathcal{F}, \sharp\} \subset h(\mathcal{S}_{\otimes}, \sharp)$.

Furthermore, the assertions of Theorem 2.3b), e) do not apply to $\{\mathcal{F}, \sharp\}$. $\{\mathcal{F}, \sharp\}$ is also neither a filtrated set nor a lattice cone.

Let us mention that $\{\mathcal{F}, \sharp\}$ is the cone of reflection positivity of the Euclidean approach to axiomatic QFT (e.g., see [25], [29]).

c) *Cone of α-positivity of free QED in local gauges.*
Let us consider $E_1=\mathcal{S}(\boldsymbol{R}^4)\otimes\boldsymbol{C}^4$ and introduce an antilinear bijection $\#$ on $E_\otimes$ by setting

$$f_0^\# = \bar{f}_0\,,$$
$$(f^{(1)}\otimes\cdots\otimes f^{(n)})^\# = f^{(n)\#}\otimes\cdots\otimes f^{(1)\#}\,,$$
$$f^{(i)\#} = (f^{(i,0)},\cdots,f^{(i,3)})^\# = (-\bar{f}^{(i,0)},\bar{f}^{(i,1)},\cdots,\bar{f}^{(i,3)})\,,$$

$f^{(i)}\in E_1$, $f^{(i,j)}\in\mathcal{S}(\boldsymbol{R}^4)$, $i=1,2,\cdots,n$, $j=0,1,2,3$, $n\in\boldsymbol{N}$. Then, $\{E_\otimes,\#\}$ is the cone of α-positivity of the Gupta-Bleuler formulation of free QED, see [4], [17]. Obviously, $\#$ satisfies (6), (6′), (7), and thus $\{E_\oplus,\#\}$ is an involutive cone. Further, $\{E_\otimes,\#\}$ is generating for $h(E_\otimes,\#)$, Theorem 2.3b), e) apply to $\{E_\otimes,\#\}$, and $\{E_\otimes,\#\}$ is neither a filtrated set nor a lattice cone. For further discussions of the concept of α-positivity we refer to [3], [15], [18].

§3. Estimations Between Homogeneous Components of alg-# Cones

To begin with let us discuss the following simple example.

Example. Let us be given the tensor-algebra $\boldsymbol{C}_\otimes$ over the basic space $\boldsymbol{C}$, its cone of positive elements $\boldsymbol{C}_\otimes^+$, an integer $N\in\boldsymbol{N}$, and a sequence $(f^{(n)})_{n=1}^\infty$, $f^{(n)}\in\boldsymbol{C}_\otimes^+$ such that $\mathrm{Grad}(f^{(n)})=2N$, $f_{2N-1}^{(n)}\neq 0$ $(n=1,2,\cdots)$ and

$$\lim_{n\to\infty}|f_{2N}^{(n)}| = 0\,.$$

Using now Lemma 2.2a), $\lim_{n\to\infty}|f_{2N-1}^{(n)}|=0$ is implied.

The example given above indicates that there are certain relations between the homogeneous components of the elements of alg-# cones. The aim of the present chapter is to prove explicite estimations between those homogeneous components.

For the following let us be given a sequence $(\omega_i)_{i=0}^\infty$, $\omega_i>0$, an integer $n\in\boldsymbol{N}$, and a constant $c>0$. Set

$$\hat{\omega}_m = \max\{\omega_i\,\omega_j;\ i+j=m,\ i\neq j\}\ ,$$

$m=1,2,3,\cdots$. Let us then define the *estimation-sequence* $(\mathrm{B}_m^{(n)}(c,\omega_i))_{m=n}^\infty$ by

$$\mathrm{B}_n^{(n)} = \min\left\{\left(\frac{c}{(n+1)\,\hat{\omega}_n}\right)^2,\ (2n\,\hat{\omega}_n)^2\right\}\ , \tag{18}$$

$$\text{ß}_{m+1}^{(n)} = \min\{\left(\frac{\text{ß}_m^{(n)}}{2m\,\hat{\omega}_m}\right)^2, (2\,(m+1)\,\hat{\omega}_{m+1})^2\}\;, \tag{18'}$$

$m=n,\ n+1,\ n+2, \cdots,$ (If it is clear from the context, then let us write $\text{ß}_m^{(n)}$ instead of $\text{ß}_m^{(n)}(c, (\omega_i))$.)

Using (18), (18′),

$$\text{ß}_n^{(n)} \geqslant \left(\frac{\text{ß}_n^{(n)}}{2n\,\hat{\omega}_n}\right)^2 \geqslant \text{ß}_{n+1}^{(n)}\;, \tag{19}$$

$$\text{ß}_{m+1}^{(n)} \geqslant \left(\frac{\text{ß}_{m+1}^{(n)}}{2(m+1)\,\hat{\omega}_{m+1}}\right)^2 \geqslant \text{ß}_{m+2}^{(n)}\;,$$

$m=n, n+1, n+2, \cdots$, are implied. Hence, $(\text{ß}_m^{(n)})_{m=n}^{\infty}$ is a monotonously decreasing sequence. Notice also that if $c \leqslant 1$, $\omega_i \geqslant 1$ $(i=0, 1, 2, \cdots)$, then

$$\text{ß}_m^{(n)} = c^{2^{m-n+1}}[((n+1)\,\hat{\omega}_n)^{2^{m-n+1}} \prod_{i=n}^{m-1} (2i\,\hat{\omega}_i)^{2^{m-i}}]^{-1}$$

(m, $n \in \boldsymbol{N}$, $m \geqslant n$) follows from (18), (18′), where $\prod_{i=n}^{n-1} \cdots = 1$.

Further, let us define the *uniform estimation-sequence* $(\text{ß}_m)_{m=1}^{\infty}$, which does not depent on the upper index "n", by setting

$$\text{ß}_m = \min\{\text{ß}_m^{(n)};\ n=1, 2, \cdots, m\}\;. \tag{20}$$

Let us be given a functional $\mathcal{L}: E_{\otimes} \rightarrow \boldsymbol{C}$ and an alg-$\#$ cone $\{F, \#\}$. Put $\mathcal{L}_n(f_n) = \mathcal{L}(\check{f}_n)$, $f_n \in E_n$, $n=0, 1, 2, \cdots$.

Definition 3.1. The functional $\mathcal{L}$ is called to satisfy *condition* (A) (*concerning* $\{F, \#\}$ *and* $(\omega_i)_{i=0}^{\infty}$), if the following are fulfilled:

$(\mathrm{A_i})$ $\quad |\mathcal{L}_n(a_n+b_n)| \leqslant |\mathcal{L}_n(a_n)| + |\mathcal{L}_n(b_n)|$, $a_n, b_n \in E_n$, $\mathcal{L}(0) = 0$,

$(\mathrm{A_{ii}})$ $\quad \mathcal{L}_{2n}(\sum_{i=1}^{M} f_n^{(i)\#} \otimes f_n^{(i)}) \geqslant 0$, $f_n^{(i)} \in F_n$, $M \in \boldsymbol{N}$,

$(\mathrm{A_{iii}})$ $\quad |\mathcal{L}_n(\pm \sum_{\substack{r+s=n \\ r \neq s}} \sum_{i=1}^{M} f_r^{(i)\#} \otimes f_s^{(i)})|$

$$\leq \sum_{\substack{r+s=n \\ r \neq s}} \omega_r\, \omega_s (\mathcal{L}_{2r}(\sum_{i=1}^{M} f_r^{(i)\#} \otimes f_r^{(i)})\, \mathcal{L}_{2s}(\sum_{i=1}^{M} f_s^{(i)\#} \otimes f_s^{(i)}))^{1/2}\,,$$

$f_r^{(i)} \in F_r$, $f_s^{(i)} \in F_s$, $n, r, s = 0, 1, 2, \cdots$.

In the following let us put

$$L_n = (\mathcal{L}_{2n}(\sum_{i=1}^{M} f_n^{(i)\#} \otimes f_n^{(i)}))^{1/2}$$

for $n=0, 1, 2, \cdots$, and $L_\mu=0$ for $\mu\notin N^*$. For every $\sum_{i=1}^{M} f^{(i)\#}f^{(i)}\in\{F, \#\}$, $f^{(i)}\in F$ $(i=1, 2, \cdots, M)$, let us consider the matrix

$$\mathfrak{A} = (a_{rs})_{r,s=0}^{\infty},$$

where $a_{rs}=\sum_{i=1}^{M} f_r^{(i)\#}\otimes f_s^{(i)}$. Obviously, $a_{rs}\neq 0$ is satisfied only for finitely many $r, s\in N^*$.

Immediate consequences of (A_i), (A_{iii}) are the estimations

$$(L_{n/2})^2+\sum_{\substack{r+s=n\\ r\neq s}}\omega_r\,\omega_s\,L_r\,L_s\geqslant|\mathcal{L}_n(\sum_{r+s=n}a_{rs})|$$
$$\geqslant(L_{n/2})^2-\sum_{\substack{r+s=n\\ r\neq s}}\omega_r\,\omega_s\,L_r\,L_s, \tag{21}$$

$n=0, 1, 2, \cdots$.

Lemma 3.2. *Let us be given* $k=\sum_{i=1}^{M} f^{(i)\#}f^{(i)}\in\{F, \#\}$, $f^{(i)}\in F(i=1,2,\cdots,M)$, *and a functional* $\mathcal{L}$ *satisfying condition* (A). *Then,* $\text{Grad}(k)=2N$ *implies* $L_n=0$ *for* $n>N$.

Proof. Assume that there is an index $n'\in N$ such that $n'>N$ and $L_{n'}\neq 0$. Using (A_i),

$$\sum_{i=1}^{M} f_{n'}^{(i)\#}\otimes f_{n'}^{(i)}\neq 0$$

is implied. Hence, $\max\{\text{Grad}(f^{(i)});\ i=1, 2, \cdots, M\}\geqslant n'$. Applying Lemma 2.2a), $\text{Grad}(k)\geqslant 2n'>2N$ follow. But this is a contradiction to the assumption of the lemma under consideration.

Let us now state and prove the main theorem of this section.

Theorem 3.3. *Let us be given an alg-# cone* $\{F, \#\}$ *and a functional* $\mathcal{L}$ *satisfying condition* (A) *with respect to* $\{F, \#\}$ *and a certain sequence* $(\omega_i)_{i=0}^{\infty}$, $\omega_i>0$. *Further, let there be an element* $\sum_{i=1}^{M} f^{(i)\#}f^{(i)}\in\{F, \#\}$, $f^{(i)}\in F$ $(i=1, 2, \cdots, M)$ *such that* $L_n\leqslant 1$ *for all* $n=0, 1, 2, \cdots$.

a) *If there is an odd index* $n_0\in N$ *with*

$$|\mathcal{L}_{n_0}(\sum_{r+s=n_0}a_{rs})| = c>0, \tag{22}$$

then there exists an even index $2m>n_0$ *such that*

$$|\mathcal{L}_{2m}(\sum_{r+s=2m}a_{rs})|>\frac{1}{2}\,\mathrm{B}_{2m}^{(n_0)}(c, (\omega_i))\geqslant\frac{1}{2}\,\mathrm{B}_{2m}(c, (\omega_i)).$$

b) *If there is an even index* $n_0=2s_0$ *such that* (22) *is satisfied, and some constant* $\rho>0$ *with*

$$(L_{s_0})^2 \leq \rho \,|\mathcal{L}_{n_0}(\sum_{\substack{r+s=n_0 \\ r\neq s}} a_{rs})| \,, \tag{22'}$$

then there exists an even index $2m>n_0$ *fulfilling*

$$|\mathcal{L}_{2m}(\sum_{r+s=2m} a_{rs})| > \frac{1}{2}\, \mathrm{B}_{2m}^{(n_0)}\left(\frac{c}{\rho+1}, (\omega_i)\right) \geq \frac{1}{2}\, \mathrm{B}_{2m}\left(\frac{c}{\rho+1}, (\omega_i)\right).$$

c) *If there are an even index* $n_0=2s_0$ *and two constants* $c>0$, $1>\delta>0$ *such that*

$$c=(L_{s_0})^2 \geq \delta^{-1} |\mathcal{L}_{2s_0}(\sum_{r+s=2s_0} a_{rs})| \,, \tag{22''}$$

then there is an even index $2m>n_0$ *with*

$$|\mathcal{L}_{2m}(\sum_{r+s=2m} a_{rs})| > \frac{1}{2}\, \mathrm{B}_{2m}^{(n_0)}((1-\delta)\, c, (\omega_i)) \geq \frac{1}{2}\, \mathrm{B}_{2m}((1-\delta)\, c, (\omega_i)) \,.$$

Proof. a) Assume that the assertion of a) is not valid. Using (22), $(\mathrm{A}_{\mathrm{iii}})$, and $L_n \leqslant 1$, it follows that there is an index j_1 with $n_0/2<j_1\leqslant n_0$ such that

$$\begin{aligned} 0<c &= |\mathcal{L}_{n_0}(\sum_{r+s=n_0} a_{rs})| \leqslant \sum_{r+s=n_0} \omega_r\, \omega_s\, L_r\, L_s \\ &\leqslant (n_0+1)\, \hat{\omega}_{n_0} \max\{L_r\, L_s;\ r+s=n_0\} \leqslant (n_0+1)\, \hat{\omega}_{n_0} L_{j_1} \,. \end{aligned}$$

Hence,

$$L_{j_1} \geqslant \frac{c}{(n_0+1)\, \hat{\omega}_{n_0}} \geqslant (\mathrm{B}_{n_0}^{(n_0)})^{1/2} \geqslant (\mathrm{B}_{2j_1}^{(n_0)})^{1/2} \,, \tag{23}$$

where (18), (19) were applied.

Now, let us be given r indices $j_1, j_2, \cdots, j_r \in N$ such that $1\leqslant j_i \leqslant |\tilde{j}_{i-1}|$, $i=1, 2, \cdots, r$ $(r\in N)$ and

$$L_{|\tilde{j}_r|} \geqslant (\mathrm{B}_{2|\tilde{j}_r|}^{(n_0)})^{1/2}$$

are satisfied, where $|\tilde{j}_r|=j_1+j_2+\cdots+j_r$, $|j_0|=n_0$.
The following inequalities are valid:

$$\begin{aligned} &\frac{1}{2}\, \mathrm{B}_{2|\tilde{j}_r|}^{(n_0)} - \sum_{i=1}^{|\tilde{j}_r|} \omega_{|\tilde{j}_r|-1}\, \omega_{|\tilde{j}_r|+1}\, L_{|\tilde{j}_r|-1}\, L_{|\tilde{j}_r|+1} \\ &\leq \frac{1}{2}\, (L_{|\tilde{j}_r|})^2 \sum_{j=1}^{|\tilde{j}_r|} \omega_{|\tilde{j}_r|-1}\, \omega_{|\tilde{j}_r|+1}\, L_{|\tilde{j}_r|-1}\, L_{|\tilde{j}_r|+1} \\ &\overset{(21)}{\leq} \frac{1}{2}\, |\mathcal{L}_{2|\tilde{j}_r|}(\sum_{r+s=2|\tilde{j}_r|} a_{rs})| \leqslant \frac{1}{4}\, \mathrm{B}_{2|\tilde{j}_r|}^{(n_0)} \,, \end{aligned}$$

where the last inequality holds because of the assumption of the proof. Hence, there is an index j_{r+1} with $1\leqslant j_{r+1}\leqslant |\tilde{j}_r|$ such that

$$\frac{1}{4}\,\text{ß}^{(n_0)}_{2|\tilde{j}_r|}\leqslant\sum_{i=1}^{|\tilde{j}_r|}\omega_{|\tilde{j}_r|-1}\,\omega_{|\tilde{j}_r|+1}\,L_{|\tilde{j}_r|-1}\,L_{|\tilde{j}_r|+1}$$
$$\leqslant|\tilde{j}_r|\;\hat{\omega}_{2|\tilde{j}_r|}\,L_{|\tilde{j}_r|+j_{r+1}}\,.$$

Consequently,

$$L_{|\tilde{j}_{r+1}|}\geqslant(4\,|\tilde{j}_r|\;\hat{\omega}_{2|\tilde{j}_r|})^{-1}\,\text{ß}^{(n_0)}_{2|\tilde{j}_r|}\geqslant(\text{ß}^{(n_0)}_{2|\tilde{j}_{r+1}|})^{1/2}\,,$$

where (18′), (19) were applied. Hence there is a sequence $(s_r)_{r=1}^{\infty}$ $s_r=|\tilde{j}_r|\in\boldsymbol{N}$, $s_1<s_2<\cdots$, such that

$$L_{s_r}\geqslant(\text{ß}^{(n_0)}_{2s_r})^{1/2}>0\,. \tag{23′}$$

(A_i) and Lemma 2.2a) imply now $a_{s_r s_r}\neq 0$ and

$$\operatorname{Grad}\Big(\sum_{i=1}^{M}f^{(i)\#}f^{(i)}\Big)\geqslant 2s_r\,, \tag{23″}$$

$r=1,2,3,\cdots$. But (23″) is a contradiction to $\sum_{i=1}^{M}f^{(i)\#}f^{(i)}\in E_{\otimes}$.

b) For $n_0=2s_0$ there is an index $1\leqslant j_0\leqslant s_0$ such that

$$c\leqslant(L_{s_0})^2+\sum_{\substack{i+j=2s_0\\ i\neq j}}\omega_i\,\omega_j\,L_i\,L_j\leqslant(\rho+1)\sum_{\substack{i+j=2s_0\\ i\neq j}}\omega_i\,\omega_j\,L_i\,L_j$$
$$\leqslant(\rho+1)\,2s_0\,\hat{\omega}_{2s_0}\,L_{s_0+j_0}\,,$$

where (21), (22′) were used. Setting $j_1=s_0+j_0$ and using (18), (19),

$$\begin{aligned}L_{j_1}&\geqslant\frac{c}{(\rho+1)\,2s_0\,\hat{\omega}_{2s_0}}\geqslant\Big(\text{ß}^{(n_0)}_{n_0}\Big(\frac{c(n_0+1)}{(\rho+1)\,n_0},(\omega_i)\Big)\Big)^{1/2}\\&\geqslant\Big(\text{ß}^{(n_0)}_{2j_1}\Big(\frac{c}{\rho+1},(\omega_i)\Big)\Big)^{1/2}\end{aligned}\tag{24}$$

are implied. If one considers (24) instead of (23), then the further proof is analogously to that of a).

c) (21) and (22″) imply

$$c=(L_{s_0})^2\geqslant\delta^{-1}\,|\,\pounds_{2s_0}\Big(\sum_{r+s=2s_0}a_{rs}\Big)|\geqslant\delta^{-1}\Big((L_{s_0})^2-\sum_{\substack{r+s=2s_0\\ r\neq s}}\omega_r\,\omega_s\,L_r\,L_s\Big)\,.$$

Hence, there is an index $j_1>s_0$ such that

$$(1-\delta)\,(L_{s_0})^2\leqslant\sum_{\substack{r+s=2s_0\\ r\neq s}}\omega_r\,\omega_s\,L_r\,L_s\leqslant 2s_0\,\hat{\omega}_{2s_0}\,L_{j_1}\,.$$

It follows further that

$$L_{j_1}\geqslant(1-\delta)\,c(2s_0\,\hat{\omega}_{2s_0})^{-1}\geqslant(\mathrm{B}_{n_0}^{(n_0)}((1-\delta)\,c,\,(\omega_g)))^{1/2}\geqslant(\mathrm{B}_{2j_1}^{(n_0)}((1-\delta)\,c,\,(\omega_g)))^{1/2}. \quad (24')$$

Considering (24′) instead of (23), the further proof is analogously to that of a). The proof is completed.

Let us consider some examples of functionals $\mathcal{L}$ which satisfy condition (A).

Examples 3.4. a) Let us be given an $\{F, \#\}$-positive and linear functional $T\neq 0$ on $E_{\otimes}$. Then, T satisfies (A) with respect to $\{F, \#\}$ and $(\omega_g)_{i=0}^{\infty}$, $\omega_g=1$.

Proof. (A_i) and (A_{ii}) are obviously satisfied. For $f_r^{(i)}\in F_r, f_s^{(i)}\in F_s$, $i=1, 2, \cdots, M$, $(r, s, n\in N^*)$, the estimations

$$\begin{aligned}|T_n(\pm\sum_{\substack{r+s=n\\ r\neq s}}\sum_{i=1}^{M}f_r^{(i)\#}\otimes f_s^{(i)})| &\leqslant \sum_{\substack{r+s=n\\ r\neq s}}\sum_{i=1}^{M}|T_n(f_r^{(i)\#}\otimes f_s^{(i)})|\\ &\leqslant \sum_{\substack{r+s=n\\ r\neq s}}\sum_{i=1}^{M}(T_{2r}(f_r^{(i)\#}\otimes f_r^{(i)})\,T_{2s}(f_s^{(i)\#}\otimes f_s^{(i)}))^{1/2}\\ &\leqslant \sum_{\substack{r+s=n\\ r\neq s}}(T_{2r}(\sum_{i=1}^{M}f_r^{(i)\#}\otimes f_r^{(i)})\,T_{2s}(\sum_{i=1}^{M}f_s^{(i)\#}\otimes f_s^{(i)}))^{1/2}\end{aligned}$$

yield (A_{iii}), where Lemma 2.1c) ii) and the Cauchy-Schwarz inequality were used.

b) Let us be given a semi-scalar product $\langle\cdot, \cdot\rangle$ and an antilinear bijection $\#$ on E_1 such that

$$\langle f_1, g_1\rangle = \overline{\langle f_1^\#, g_1^\#\rangle}$$

for all $f_1, g_1\in E_1$. For each $n\in N$, $n\geqslant 2$, let us be given a permutation $\pi(\cdot)$ of $\{1, 2, \cdots, n\}$. Then, let us define an antilinear bijection $\#$ on E_n by setting

$$f_n^\# = \sum_{i=1}^{r}f^{(\pi(1),i)\#}\otimes\cdots\otimes f^{(\pi(n),i)\#} \quad (25)$$

for $f_n=\sum_{i=1}^{r}f^{(1,i)}\otimes\cdots\otimes f^{(n,i)}\in E_n, f^{(j,i)}\in E_1$. Notice that $\#$ satisfies (6), (6′). Let us put $s_0(f_0)=|f_0|$, $s_1(f_1)=\langle f_1, f_1\rangle^{1/2}$, $f_0\in E_0$, $f_1\in E_1$, and

$$\begin{aligned}s_n(f_n) &= (s_1\otimes_\sigma\cdots\otimes_\sigma s_1)\,(f_n) = (\langle f_n, f_n\rangle)^{1/2}\\ &= (\sum_{i=1}^{r}\sum_{j=1}^{r}\langle f^{(1,i)}, f^{(1,j)}\rangle\cdots\langle f^{(n,i)}, f^{(n,j)}\rangle)^{1/2},\end{aligned}$$

$n=2, 3, 4, \cdots$. Further, let us be given a subspace $F\subset E_\otimes$ satisfying (5). Then, the semi-norm

$$f\to s(f)=\sum_{n=0}^{\infty} s_n(f_n)$$

satisfies condition (A) with respect to $\{F, \#\}$ and $(\omega_i)_{i=0}^{\infty}$, $\omega_i=1$.

Proof. (A_i) and (A_{ii}) are obviously fulfilled. (A_{iii}) follows from the following estimations:

$$\begin{aligned}(s_{n+m}(\sum_{i=1}^{M} f_n^{(i)\#}\otimes f_m^{(i)}))^2 &= \sum_{i=1}^{M}\sum_{j=1}^{M} \overline{\langle f_n^{(i)}, f_n^{(j)}\rangle}\langle f_m^{(i)}, f_m^{(j)}\rangle \\ &\leqslant (\sum_{i=1}^{M}\sum_{j=1}^{M} |\langle f_n^{(i)}, f_n^{(j)}\rangle|^2)^{1/2}(\sum_{i=1}^{M}\sum_{j=1}^{M} |\langle f_m^{(i)}, f_m^{(j)}\rangle|^2)^{1/2} \\ &= s_{2n}(\sum_{i=1}^{M} f_n^{(i)\#}\otimes f_n^{(i)})\, s_{2m}(\sum_{i=1}^{M} f_m^{(i)\#}\otimes f_m^{(i)}),\end{aligned}$$

where the Cauchy-Schwarz inequality is used, $n, m\in \boldsymbol{N}^*$.

c) Let us be given a semi-norm $f_1\to r_1(f_1)$ on E_1 such that $r_1(f_1^\#)=r_1(f_1)$, $f_1\in E_1$. Further, let the anti-linear bijection $\#$ satisfy (6), (6′), (25). Let us put $r_n(f_n)=(r_1\otimes_\varepsilon\cdots\otimes_\varepsilon r_1)(f_n)$, $f_n\in E_n$ $(n=2, 3, \cdots)$, $r_0(f_0)=|f_0|$, $f_0\in E_0$, and

$$r(f)=\sum_{n=0}^{\infty} r_n(f_n),$$

$f\in E_\otimes$. Then, the semi-norm $f\to r(f)$ satisfies condition (A) with respect to $\{F, \#\}$ and $(\omega_i)_{i=0}^{\infty}$, $\omega_i=1$.

Proof. The validity of (A_i), (A_{ii}) is evident. (A_{iii}) is a consequence of the following estimations:

$$\begin{aligned}&(r_{n+m}(\sum_{i=1}^{M} f_n^{(i)\#}\otimes f_m^{(i)}))^2 \\ &= (\sup\{|\sum_{i=1}^{M} T_n(f_n^{(i)\#})\, S_m(f_m^{(i)})|\,;\, T_n\in U_{r_n}^0,\, S_m\in U_{r_m}^0\})^2 \\ &\leqslant \sup\{\sum_{i=1}^{M} |T_n(f_n^{(i)\#})|^2;\, T_n\in U_{r_n}^0\}\, \sup\{\sum_{i=1}^{M} |S_m(f_m^{(i)})|^2;\, S_m\in U_{r_m}^0\} \\ &\overset{(*)}{=} r_{2n}(\sum_{i=1}^{M} f_n^{(i)\#}\otimes f_n^{(i)})\, r_{2m}(\sum_{i=1}^{M} f_m^{(i)\#}\otimes f_m^{(i)}),\end{aligned}$$

where $U_{r_n}^0$ denotes the polar set concerning the semi-norm r_n in E_n'. ((*): Setting $T_1^\#(f_1)=\overline{T_1(f_1^\#)}$ and using $r_1(f_1)=r_1(f_1^\#)$, $f_1\in E_1$, it follows that $T_1\in U_{r_1}^0$ if and only if $T_1^\#\in U_{r_1}^0$. This implies that $T_n\in U_{r_n}^0$ if and only if $T_n^\#\in U_{r_n}^0$ for $n=$

2, 3, 4, $\cdots$, $T_n^\sharp(f_n)=\overline{T_n(f_n^\sharp)}$, $f_n\in E_n$. It follows that

$$\begin{aligned} r_{2n}(\sum_{i=1}^{M} f_n^{(i)\sharp}\otimes f_n^{(i)}) &= \sup\{|\sum_{i=1}^{M} T_n(f_n^{(i)\sharp})\,\tilde{T}_n(f_n^{(i)})|\,;\ T_n,\ \tilde{T}_n\in U_{r_n}^0\} \\ &= \sup\{|\sum_{i=1}^{M} \overline{T_n^\sharp(f_n^{(i)})}\,T_n(f_n^{(i)})|\,;\ T_n^\sharp,\ T_n\in U_{r_n}^0\} \\ &= \sup\{\sum_{i=1}^{M} |T_n(f_n^{(i)})|^2;\ T_n\in U_{r_n}^0\}\ . \end{aligned}$$

Treating the second factor on the left-hand side of (*) in the same way, (*) is implied.)

Now, let us be given two sequences $(\omega_n)_{n=0}^\infty$, $(d_n)_{n=0}^\infty$ of reals with ω_n, $d_n>0$. Further, let us consider the diagonalized matrix $D=\mathrm{diag}\,[d_0, d_1, d_2, \cdots]$. Let $\mathfrak{A}_{(\omega,d)}$ denote the set of all the sequences $(\alpha_n)_{n=0}^\infty$ with $\alpha_n\geqslant 0$, $\alpha_{2s+1}=0$ $(s=0, 1, 2, \cdots)$ such that the inequality of matrices $G\geqslant D$ is satisfied, where

$$G=\begin{bmatrix} \alpha_0 & 0 & -\omega_0\,\omega_2\,\alpha_2 & 0 & -\omega_0\,\omega_4\,\alpha_4 & 0 & \cdots \\ 0 & \alpha_2 & 0 & -\omega_1\,\omega_3\,\alpha_4 & 0 & -\omega_1\,\omega_5\,\alpha_6 & \cdots \\ -\omega_0\,\omega_2\,\alpha_2 & 0 & \alpha_4 & 0 & -\omega_2\,\omega_4\,\alpha_6 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{26}$$

(In 3.6 there it will be shown that $\mathfrak{A}_{(\omega,d)}\neq\emptyset$.)
For every functional $S\colon E_\otimes\to \boldsymbol{C}$ let us set

$$||f||_{(S,(\alpha_n))} = \sum_{n=0}^\infty \alpha_{2n}\,|S_{2n}(f_{2n})|\,, \tag{26'}$$

where $S_n(f_n)=S(\check{f}_n)$, $f=(f_0, \cdots, f_N, 0, 0, \cdots)\in E_\otimes$. (If it is clear from the context, then let us drop the index of $||\cdot||$.) If S satisfies (A_1) ,then

$$||f+g||\leqslant ||f||+||g|| \tag{27}$$

is immediately implied.

Further, let us put

$$A_{n,\mu} = \sum_{r=1}^\infty r^{-\mu}+\sum_{r=1}^n r^\mu \tag{28}$$

for $0<\mu<\infty$ and $n\in \boldsymbol{N}^*$, where $\sum_{r=1}^0 r^\mu=0$. Obviously, $\mu>1$ yields $A_{n,\mu}<\infty$ for each $n\in \boldsymbol{N}^*$. It holds also $\sum_{r=1}^\infty r^{-\mu}=\xi(\mu)$ (Riemann's ξ-function), and

$$\sum_{r=1}^n r^\mu = \frac{n^{\mu+1}}{\mu+1}+\frac{n^\mu}{2}+\frac{\mu n^{\mu-1}}{12}+R_\mu(n)\,, \tag{28'}$$

where $R_\mu(n)=\sum_{r=2}^{\infty}\frac{1}{2r}\binom{\mu}{2r-1}B_{2r}\,n^{\mu-2r+1}$, and B_{2r} denote Bernoulli's numbers (see [8; 0.233, 0.121, 9.71]).

If $1<\mu\leqslant 2$, then $\frac{\pi^2}{6}\leqslant\xi(\mu)<\infty$ and $|R_\mu(n)|\to 0$ as $n\to\infty$. Hence, there is an $n'\in\boldsymbol{N}$ such that

$$|R_\mu(n)|\leqslant 1 \tag{28''}$$

for all $n>n'$.

Theorem 3.5. *Let us be given an alg-# cone $\{F, \#\}$ in a tensor-algebra $E_\otimes$. Let (ω_n), (d_n) be sequences as described above. Further, let the functional $\pounds$: $E_\otimes\to\boldsymbol{C}$ satisfy* (A) *with respect to $\{F, \#\}$ and (ω_n).*

a) *If $(\alpha_n)_{n=0}^{\infty}\in\mathfrak{A}_{(\omega,d)}$, then*

$$\sum_{n=0}^{\infty}d_n(L_n)^2\leqslant\|\sum_{i=1}^{M}f^{(i)\#}f^{(i)}\|_{(\pounds,(\alpha_n))}$$

for $f^{(i)}\in F$, $i=1, 2, \cdots, M$ $(M\in\boldsymbol{N})$.

b) *It is*

$$|\sum_{n=0}^{\infty}\pounds_n(\sum_{r+s=n}\sum_{i=1}^{M}f_r^{(i)\#}\otimes f_s^{(i)})|\leqslant\sum_{n=0}^{\infty}(1+A_{n,\mu}\,\omega_n^2)\,(L_n)^2$$

where $1<\mu<\infty$, $f^{(i)}\in F$, $i=1, 2, \cdots, M$.

Proof. a) Let us put $x=(L_0, L_1, L_2, \cdots)^t$. Lemma 3.2 implies that there are only finitely many $n\in\boldsymbol{N}^*$ with $L_n\neq 0$. The proof of a) is now a consequence of

$$\sum_{n=0}^{\infty}\alpha_{2n}|\pounds_{2n}(\sum_{r+s=2n}\sum_{i=1}^{M}f_r^{(i)\#}\otimes f_s^{(i)})|\geqslant x^t\,G\,x\geqslant x^t\,D\,x=\sum_{n=0}^{\infty}d_n(L_n)^2\,,$$

where (21), (26) were used.

b) follows from

$$\begin{aligned}
&|\sum_{n=0}^{\infty}\pounds_n(\sum_{r+s=n}\sum_{i=1}^{M}f_r^{(i)\#}\otimes f_s^{(i)})|\\
&\leqslant 2\sum_{n=1}^{\infty}\sum_{\substack{r+s=n\\ r<s}}(s-r)^{-\mu/2}\,\omega_r\,L_r(s-r)^{\mu/2}\,\omega_s\,L_s+\sum_{n=0}^{\infty}(L_n)^2\\
&\leqslant\sum_{n=0}^{\infty}(L_n)^2+\sum_{n=1}^{\infty}\sum_{\substack{r+s=n\\ r<s}}((s-r)^{-\mu}(\omega_r\,L_r)^2+(s-r)^{\mu}(\omega_s\,L_s)^2)\\
&\leqslant\sum_{n=0}^{\infty}(L_n)^2+\sum_{n=0}^{\infty}A_{n,\mu}(\omega_n\,L_n)^2=\sum_{n=0}^{\infty}(1+A_{n,\mu}\,\omega_n^2)\,(L_n)^2\,.
\end{aligned}$$

3.6. Construction of a sequence $(\alpha_n)_{n=0}^{\infty} \in \mathfrak{A}_{(\omega,d)}$

At first let us prove the following.

Lemma. *Let us be given an hermitian matrix* $H=(h_{ij})_{i,j=0}^{\infty}$, $h_{i,j}\in \boldsymbol{C}$, *and a sequence* $(c_n)_{n+0}^{\infty}$ *of reals with* $c_0=0$, $c_m \geqslant 0$, $m=1, 2, 3, \cdots$. *Let us put*

$$x_n = \begin{cases} c_n^{-1} \sum_{j=0}^{n-1} |h_{nj}|^2 & \text{if } \sum_{j=0}^{n-1} |h_{nj}|^2 \neq 0 \ , \quad c_n > 0 \, , \\ \infty & \text{if } \quad - \ '' \ - \qquad , \quad c_n = 0 \, , \\ 0 & \text{if } \sum_{j=0}^{n-1} |h_{nj}|^2 = 0 \, , \end{cases}$$

$n=1, 2, 3, \cdots$. *If* $\sum_{n=1}^{\infty} x_n < \infty$, *then the matrix-inequality* $K \leqslant H$ *follows, where* $K=(k_{ij})_{i,j=0}^{\infty}$,

$$k_{ij} = \begin{cases} h_{ii} - c_i - \sum_{r=i+1}^{\infty} x_r & \text{if } \ i=j \, , \ \ i=0, 1, 2, \cdots \, , \\ 0 & \text{if } \ i \neq j \, . \end{cases}$$

Proof. Assume that $\sum_{m=1}^{\infty} x_m < \infty$. For every sequence $(a_i)_{i=0}^{\infty}$, $a_i \in \boldsymbol{C}$, and each $n \in \boldsymbol{N}$ the following estimations are satisfied:

$$\begin{aligned} \sum_{i,j=0}^{n} h_{ij}\, \bar{a}_i\, a_j &= \sum_{i,j=0}^{n-1} h_{ij}\, \bar{a}_i\, a_j + h_{nn} |a_n|^2 + 2\mathrm{Re}\Big(\sum_{j=0}^{n-1} h_{nj}\, \bar{a}_n\, a_j\Big) \\ &\overset{(*)}{\geqslant} \sum_{i,j=0}^{n-1} h_{ij}\, \bar{a}_i\, a_j + h_{nn} |a_n|^2 - c_n |a_n|^2 - c_n^{-1} \Big|\sum_{j=0}^{n-1} h_{nj}\, a_j\Big|^2 \\ &\geqslant \sum_{i,j=0}^{n-1} h_{ij}\, \bar{a}_i\, a_j - x_n \Big(\sum_{j=0}^{n-1} |a_j|^2\Big) + (h_{nn} - c_n) |a_n|^2 \, . \end{aligned} \tag{29}$$

Let us be given $\underline{b}=(b_0, b_1, \cdots)^t$, where $b_i \in \boldsymbol{C}$ and $b_i=0$ for all but finitely many $i \in \boldsymbol{N}^*$. Then, there is an $N \in \boldsymbol{N}$ such that $b_j=0$ for $j>N$, and

$$\begin{aligned} \underline{b}^* H \underline{b} &= \sum_{i,j=0}^{N} h_{ij}\, b_i\, \bar{b}_j \\ &\geqslant \sum_{i,j=0}^{N-1} h_{ij}\, b_i\, \bar{b}_j - x_N \Big(\sum_{j=0}^{N-1} |b_j|^2\Big) + (h_{NN} - c_N) |b_N|^2 \geqslant \cdots \\ &\geqslant - \sum_{i=1}^{N} x_i \Big(\sum_{j=0}^{i-1} |b_j|^2\Big) + \sum_{i=1}^{N} (h_{ii} - c_i) |b_i|^2 + h_{00} |b_0|^2 \\ &= |b_0|^2 \Big(h_{00} - \sum_{i=1}^{N} x_i\Big) + \sum_{j=1}^{N-1} |b_j|^2 \Big(h_{jj} - c_j - \sum_{r=j+1}^{N} x_r\Big) + |b_N|^2 (h_{NN} - c_N) \\ &\geqslant |b_0|^2 \Big(h_{00} - \sum_{i=1}^{\infty} x_i\Big) + \sum_{j=1}^{N} |b_j|^2 \Big(h_{jj} - c_j - \sum_{r=j+1}^{\infty} x_r\Big) \\ &= \underline{b}^* K \underline{b} \, , \end{aligned}$$

where $\underline{b}^* = (\bar{b}_0, \bar{b}_1, \cdots)$, and (29) were applied.

(*Proof* of (*). If $c_n > 0$, then (*) is a consequence of

$$|2\mathrm{Re}(\bar{a}_n \sum_{j=0}^{n-1} h_{nj} a_j)| \le 2|(\sqrt{c_n}\, \bar{a}_n)(\sqrt{c_n}^{-1} \sum_{j=0}^{n-1} h_{nj} a_j)|$$
$$\le c_n |a_n|^2 + c_n^{-1} |\sum_{j=0}^{n-1} h_{nj} a_j|^2 .$$

If $c_n = 0$, then $\sum_{n=1}^{\infty} x_n < \infty$ yields

$$|\sum_{j=0}^{n-1} h_{nj} a_j|^2 \le \sum_{j=0}^{n-1} |h_{nj}|^2 \sum_{i=0}^{n-1} |a_i|^2 = 0 ,$$

where $c_n^{-1} |\sum_{j=0}^{n-1} h_{nj} a_j|^2 = 0$.)

Let us be given sequences $(\omega_n)_{n=0}^{\infty}$, $(d_n)_{n=0}^{\infty}$ as above. Further let $\mu > 1$. Let us put

$$\omega_{2s}^* = \omega_{2s} \max\{\omega_{2(t-s)};\ t = s, s+1, \cdots, 2s-1\}, \qquad s = 1, 2, 3, \cdots ,$$
$$\omega_{2r-1}^* = \omega_{2r-1} \max\{\omega_{2(t-r)+1};\ t = r, r+1, \cdots, 2r-2\}, \quad r = 2, 3, 4, \cdots .$$

Let us then recursively define sequences $(\alpha_n)_{n=0}^{\infty}$, $(c_n)_{n=0}^{\infty}$ by

$$\begin{aligned} c_0 &= c_1 = 0,\ \alpha_0 = d_0 + \xi(\mu) - 1,\ \alpha_{2s-1} = 0 ,\\ \alpha_{2s} &= d_s + c_s + \sum_{m=s+1}^{\infty} m^{-\mu} ,\\ c_{2s} &= (2s)^{\mu} (\omega_{2s}^*)^2 \sum_{j=2s}^{4s-2} |\alpha_j|^2 ,\\ c_{2r-1} &= (2r-1)^{\mu} (\omega_{2r-1}^*)^2 \sum_{j=2r}^{4r-4} |\alpha_j|^2 , \end{aligned} \tag{30}$$

$s = 1, 2, 3, \cdots$, $r = 2, 3, 4, \cdots$.

For showing $(\alpha_n)_{n=0}^{\infty} \in \mathfrak{A}_{(\omega,d)}$ let us consider the matrix G from (26) instead of H in the above given Lemma. It follows that $x_1 = 0$ and

$$x_{2s} = (c_{2s})^{-1} \sum_{j=2s}^{4s-2} |\omega_{j-2s}\, \omega_{2s}\, \alpha_j|^2 \le c_{2s}^{-1}\, \omega_{2s}^* \sum_{j=2s}^{4s-2} |\alpha_j|^2 = (2s)^{-\mu} ,$$
$$x_{2r-1} = (c_{2r-1})^{-1} \sum_{j=2r}^{4r-4} |\omega_{j-2r+1}\, \omega_{2r-1}\, \alpha_j|^2$$
$$\le (c_{2r-1})^{-1}\, \omega_{2r-1}^* \sum_{j=2r}^{4r-4} |\alpha_j|^2 = (2r-1)^{-\mu} ,$$

$s = 1, 2, 3, \cdots$, $r = 2, 3, 4, \cdots$. Hence $\sum_{n=1}^{\infty} x_n < \infty$, and the above given Lemma implies $K \leqslant G$, where $K=(k_{ij})_{i,j=0}^{\infty}$,

$$k_{ij} = \begin{cases} \alpha_{2i} - c_i - \sum_{r=i+1}^{\infty} x_r & \text{if } \; i=j, i=0,1,2,\cdots, \\ 0 & \text{if } \; i \neq j. \end{cases}$$

Using

$$k_{ii} \geqslant \alpha_{2i} - c_i - \sum_{r=i+1}^{\infty} r^{-\mu} \overset{(30)}{=} d_i ,$$

$i=0, 1, 2, \cdots$, it follows that $\operatorname{diag}[d_0, d_1, \cdots] \leq K$. Hence, $\operatorname{diag}[d_0, d_1, \cdots] \leq G$. This proves $(\alpha_n)_{n=0}^{\infty} \in \mathfrak{A}_{(\omega,d)}$.

Now, let us be given l alg-$\#$ cones $\{F^{(s)}, \#_s\}$, $s=1, 2, \cdots, l$, in a tensor-algebra $E_{\otimes}$. The following is aimed at an extension of the results given above to the convex hull

$$\sum_{s=1}^{l} \{F^{(s)}, \#_s\} . \tag{31}$$

For a given functional $\mathcal{L}$: $E_{\otimes} \to \boldsymbol{C}$, and $\sum_{i=1}^{M_s} a^{(i,s)\#_s} a^{(i,s)} \in \{F^{(s)}, \#_s\}$, $a^{(i,s)} \in F^{(s)}$, $i=1, 2, \cdots, M_s$ ($M_s \in \boldsymbol{N}$) set

$$L_n^{(s)} = (\mathcal{L}_{2n}(\sum_{i=1}^{M_s} a_n^{(i,s)\#_s} \otimes a_n^{(i,s)}))^{1/2} ,$$

$$L_n^{(1,\cdots,l)} = (\mathcal{L}_{2n}(\sum_{s=1}^{l} \sum_{i=1}^{M_s} a_n^{(i,s)\#_s} \otimes a_n^{(i,s)}))^{1/2} .$$

Let us modify condition (A) for the convex hull (31) by replacing ($\mathrm{A_{ii}}$), ($\mathrm{A_{iii}}$) by

($\mathrm{A'_{ii}}$) $\mathcal{L}_{2n}(\sum_{s=1}^{l} \sum_{i=1}^{M_s} a_n^{(i,s)\#_s} \otimes a_n^{(i,s)}) \geqslant 0$ for all $a^{(i,s)} \in F^{(s)}$,

($\mathrm{A'_{iii}}$) $$|\mathcal{L}_n(\sum_{\substack{r+k=n \\ r \neq k}} \sum_{s=1}^{l} \sum_{i=1}^{M_s} a_r^{(i,s)\#_s} \otimes a^{(i,s)})| \leq \sum_{\substack{r+k=n \\ r \neq k}} \omega_r \, \omega_k \, L_r^{(1,\cdots,l)} \, L_s^{(1,\cdots,l)} ,$$

$$a_r^{(i,s)} \in F_r^{(s)}, \; a_k^{(i,s)} \in F_k^{(s)} \quad (s=1, 2, \cdots, l) , \quad n, r, k = 0, 1, 2, \cdots .$$

If $\mathcal{L}$ satisfies (A′) with respect to the convex hull (31), then the following analogous statement to (21) holds true:

$$|(L_{n/2}^{(1,\cdots,l)})^2 - |\mathcal{L}_n(\sum_{r+k=n} \sum_{s=1}^{l} \sum_{i=1}^{M_s} a_r^{(i,s)\#_s} \otimes a^{(i,s)}||$$

$$\geqslant \sum_{\substack{r+k=n \\ r \neq k}} \omega_r \, \omega_k \, L_r^{(1,\cdots,l)} \, L_k^{(1,\cdots,l)} , \tag{32}$$

$n=0, 1, 2, \cdots$. Furthermore, the corresponding assertions to Theorem 3.3 apply also to (31).

A discussion of (A′) is given by the following.

Lemma 3.6. *Let us be given l alg-# cones $\{F^{(s)}, \#_s\}$, $s=1, 2, \cdots, l$, in a tensoralgebra $E_\otimes$. If $\pounds$ satisfies* (A) *for each $\{F^{(s)}, \#_s\}$, and if further there is a sequence $(e_n)_{n=0}^\infty$, $e_n>0$, such that*

$$\sum_{s=1}^{l} L_n^{(s)} \leq e_n \, L_n^{(1,\cdots,l)} ,$$

$n=0, 1, 2, \cdots$, then the convex hull (31) *satisfies* (A′).

Proof. Assuming that (A) is satisfied for each $\{F^{(s)}, \#_s\}$ with $(\omega_\mu^{(s)})_{\mu=0}^\infty$, let us put

$$\omega_\mu = e_\mu \sup\{\omega_\mu^{(s)};\ s=1, 2, \cdots, l\} ,$$

$\mu \in N^*$. The assertion to be shown is now a consequence of the following estimations:

$$\begin{aligned} &|\pounds(\sum_{\substack{r+k=n \\ r \neq k}} \sum_{s=1}^{l} \sum_{i=1}^{M_s} a_r^{(i,s)\#_s} \otimes a_k^{(i,s)})| \\ &\leq \sum_{s=1}^{l} |\pounds(\sum_{\substack{r+k=n \\ r \neq k}} \sum_{i=1}^{M_s} a_r^{(i,s)\#_s} \otimes a_k^{(i,s)})| \\ &\leq \sum_{s=1}^{l} \sum_{\substack{r+k=n \\ r \neq k}} \omega_r^{(s)} L_r^{(s)} \omega_k^{(s)} L_k^{(s)} \\ &\leq \sum_{\substack{r+k=n \\ r \neq k}} (\sum_{s=1}^{l} \omega_r^{(s)} L_r^{(s)}) (\sum_{s=1}^{l} \omega_k^{(s)} L_k^{(s)}) \\ &\leq \sum_{\substack{r+k=n \\ r \neq k}} \omega_r \, \omega_k \, L_r^{(1,\cdots,l)} \, L_k^{(1,\cdots,l)} . \end{aligned}$$

Example 3.7. Let us consider the convex hull of the cone of positivity $\mathcal{S}_\otimes^+$ and that of reflection positivity $\{\mathcal{F}, \#\}$ in $\mathcal{S}_\otimes$, see Example 2.4. Let us mention that this convex hull is used in the axiomatic approach to Euclidean QFT given by G. Hegerfeldt ([9]).

For each $m \in N^*$ let us consider the norm

$$f \rightarrow p^{(m)}(f) = \sum_{n=0}^{\infty} p_n^{(m)}(f_n) ,$$

$f \in \mathcal{S}_\otimes$, where

$$p_n^{(m)}(f_n) = \sup_{x \in \mathbf{R}^{dn}} \max_{r_\mu^j \leqslant m} |W_n^{(m,r_\mu^j)} f_n(x_1, \cdots, x_n)| ,$$

$$W_n^{(m,r_\mu^j)} = \prod_{\mu=1}^{n} \prod_{j=1}^{d-1} (1+(x_\mu^j)^2)^m (\partial/\partial x_\mu^j)^{r_\mu^j}, \; r_\mu^j \in \mathbf{N}^*, \; x_\mu = (x_\mu^0, \cdots, x_\mu^{d-1}) \in \mathbf{R}^d .$$

(In the following let us abbreviate $W_n^{(m,r_\mu^j)}$ by W_n.) Recall also that the system of norms $\{p_n^{(m)}; m=0, 1, 2, \cdots\}$ defines the well-known topology on the Schwartz-space $\mathcal{S}_n$. Further, let $g_r \otimes f_s$, $g_r \in \mathcal{S}_r$, $f_s \in \mathcal{S}_s$, be identified with the function $g_r(x_1, \cdots, x_r) f_s(x_{r+1}, \cdots, x_{r+s}) \in \mathcal{S}_{r+s}$.

For given $a^{(i)} \in \mathcal{S}_\otimes$, $b^{(j)} \in \mathcal{F}$, $m \in \mathbf{N}^*$ let us put

$$L_n^{(1)} = (p_{2n}^{(m)} (\sum_{i=1}^{M} a_n^{(i)*} \otimes a_n^{(i)})^{1/2} ,$$

$$L_n^{(2)} = (p_{2n}^{(m)} (\sum_{j=1}^{M'} b_n^{(j)\#} \otimes b_n^{(j)})^{1/2} ,$$

$$L_n^{(1,2)} = (p_{2n}^{(m)} (\sum_{i=1}^{M} a_n^{(i)*} \otimes a_n^{(i)} + \sum_{j=1}^{M'} b_n^{(j)\#} \otimes b_n^{(j)}))^{1/2} ,$$

$n=0, 1, 2, \cdots$. Setting

$$\Delta_{2n} = \{x \in \mathbf{R}^{2dn}; \; x_1 = x_{2n}, \; x_2 = x_{2n-1}, \cdots, x_n = x_{n+1}\} ,$$
$$Z_{2n} = \{x \in \mathbf{R}^{2dn}; \; x_n^0 < \cdots < x_1^0 < 0 < x_{n+1}^0 < \cdots < x_{2n}^0\} ,$$

it follows

$$L_n^{(1)} = (\sup_{x \in \Delta_{2n}} \max_{r_\mu^j \leq m} | W_{2n}(\sum_{i=1}^{M} a_n^{(i)}(x_n, \cdots, x_1) \, a_n^{(i)}(x_{n+1}, \cdots, x_{2n}))|)^{1/2} ,$$

$$\operatorname{supp} (\sum_{j=1}^{M'} b_n^{(j)\#} \otimes b_n^{(j)}) \subset Z_{2n} . \tag{33}$$

Then,

$$p_n^{(m)} (\sum_{\substack{r+s=n \\ r \neq s}} \sum_{i=1}^{M} a_r^{(i)*} \otimes a_s^{(i)})$$
$$\overset{(*)}{\leqslant} \sum_{\substack{r+s=n \\ r \neq s}} (\sup_{x \in \mathbf{R}^{dn}} \max_{r_\mu^j \leqslant m} (\sum_{i=1}^{M} | W_r \, a_r^{(i)} (x_1, \cdots, x_r)|^2)$$
$$(\sum_{i=1}^{M} | W_s \, a_s^{(i)} (x_{r+1}, \cdots, x_n)|^2))^{1/2} = \sum_{\substack{r+s=n \\ r \neq s}} L_r^{(1)} L_s^{(1)}$$

imply that (A_{iii}) is satisfied for $\mathcal{S}_\otimes^+$, where the triangle and the Cauchy-Schwarz inequality are used in (*). Analogously,

$$p_n^{(m)}\Big(\sum_{\substack{r+s=n\\ r\neq s}}\sum_{j=1}^{M'} b_r^{(j)\#}\otimes b_s^{(j)}\Big)\leqslant \sum_{\substack{r+s=n\\ r\neq s}} L_r^{(2)} L_s^{(2)}$$

follows, i.e., (A_{iii}) is also satisfied for $\{\mathcal{F},\#\}$, $n, m=0, 1, 2, \cdots$. Hence, Condition (A) is satisfied for both $\mathcal{S}_\otimes^+$ and $\{\mathcal{F},\#\}$.

The following lemma shows that Lemma 3.6 applies to the convex hull $\mathcal{S}_\otimes^+ + \{\mathcal{F},\#\}$.

Lemma. *It holds* $L_0^{(1)}+L_0^{(2)}\leqslant\sqrt{2}\,L_0^{(1,2)}$, $L_n^{(1)}+L_n^{(2)}\leqslant(1+\sqrt{2})\,L_n^{(1,2)}$, $n=1, 2, 3, \cdots$.

Proof. If $n=0$, then the assertion under consideration is implied by the inequality between arithmetic and geometric mean. Noticing $Z_{2n}\cap\Delta_{2n}=\emptyset$, (33) implies

$$L_n^{(1)}\leqslant L_n^{(1,2)}, \tag{34}$$

$n=1, 2, 3, \cdots$. Applying the triangle inequality to the definition of $L_n^{(1,2)}$, it follows

$$L_n^{(1,2)}\geqslant|(L_n^{(2)})^2-(L_n^{(1)})^2|^{1/2}. \tag{34'}$$

Let us consider i) $L_n^{(2)}\leq L_n^{(1)}$, ii) $L_n^{(2)}<L_n^{(1)}$. If i) applies, then (34) yields the assertion to be shown. In case of ii), the inequalities

$$(L_n^{(1,2)})^2\geqslant(L_n^{(2)})^2-(L_n^{(1)})^2\geqslant(L_n^{(2)})^2-(L_n^{(1,2)})^2,$$
$$L_n^{(1)}+L_n^{(2)}\leqslant L_n^{(1)}+\sqrt{2}\,L_n^{(1,2)}\leqslant(1+\sqrt{2})\,L_n^{(1,2)}$$

follow from (34), (34′). This completes the proof.

Let us mention that there are examples such that the equality signs apply to the estimations in the above given lemma.

§4. On the Normality of alg-# Cones

This section is devoted to a systematic investigation of the normality of alg-# cones and of the convex hull of such cones in topological tensor-algebras $E_\otimes[\tau]$. Recall that the concept of normality is a central one in the theory of semi-ordered topological vector-spaces (since it connects the semi-ordering induced by the cone under consideration with the topological structure of the underlying vector-space), and it has some interesting applications such as i) characterization of AO^*-algebras ([27], [11], [22]), ii) decomposition of continuous linear functionals into the difference of positive and continuous ones ([26], [33]), iii) characterization of state-related ideals ([34], [14]).

For a given alg-$\#$ cone $\{F, \#\}$, let us introduce conditions which prove to be sufficient for the normality of the cone under consideration. Concerning a very large class of l.c. topologies it will be shown that these conditions are also necessary for the normality of $\{F, \#\}$.

Definition 4.1. Let us be given an alg-$\#$ cone in a topological tensor-algebra $E_\otimes[\tau]$. Then, τ will be called to satisfy *condition* (B) *w.r.t.* $\{F, \#\}$, if there is a system of semi-norms $\mathfrak{P}(\tau)$ that defines τ, and for each $p \in \mathfrak{P}(\tau)$ there are two sequences of reals $(\omega_n)_{n=0}^\infty$, $(d_n)_{n=0}^\infty$, $\omega_n > 0$, $d_n \geqslant 1$, and a constant λ, $1 < \lambda \leqslant 2$, such that

(B_i) p satisfies condition (A) *w.r.t.* $\{F, \#\}$ and (ω_n),

(B_ii) $p((\sum_{i=1}^{M} f_n^{(i)\#} \otimes f_n^{(i)})^\vee) \leqslant d_n\, p((\sum_{i=1}^{M'} f_n^{(i)\#} \otimes f_n^{(i)})^\vee)$

for all $\check{f}_n^{(i)} \in F$ $(i = 1, 2, \cdots, M')$, $M \leqslant M'$ $(M, M' \in N)$, $n = 0, 1, 2, \cdots$,

(B_iii) for $(\beta_n)_{n=0}^\infty$, $\beta_n = d_n(1 + (1 + n^{\lambda+1})\, \omega_n^2)$, there is a sequence $(\alpha_n)_{n=0}^\infty \in \mathfrak{A}_{(\omega, \beta)}$ such that $f \to \|f\|_{(p, (\alpha_n))}$, $f \in E_\otimes$, is τ-continuous.

Theorem 4.2. *Let us be given an alg-$\#$ cone $\{F, \#\}$ in a topological tensor-algebra $E_\otimes[\tau]$. If τ satisfies* (B)*, then $\{F, \#\}$ is τ-normal.*

Proof. Consider any semi-norm $p \in \mathfrak{P}(\tau)$, where $\mathfrak{P}(\tau)$ is given in Definition 4.1. It will be shown that for

$$U = \{g \in E_\otimes;\, p(g) < 1\}$$

there are a 0-neighborhood V of τ and a constant $\rho > 0$ such that $[V] \subset \rho\, U$, where $[V] = (V + \{F, \#\}) \cap (V - \{F, \#\})$ denotes the $\{F, \#\}$-saturated hull of V.

Setting $\tilde{p}(g) = \max\{p(g), \|g\|_{(p, (\alpha_n))}\}$, $g \in E_\otimes$, and using (B_iii) it follows that

$$V = \{g \in E_\otimes;\, \tilde{p}(g) < 1\}$$

is a 0-neighborhood of τ. Let us be given any $f \in [V]$. Then, there are $k^{(i)} \in \{F, \#\}$ and $v^{(i)} \in V$, $i = 1, 2$, such that

$$f = k^{(1)} + v^{(1)} = v^{(2)} - k^{(2)}\,.$$

For $k^{(1)} = \sum_{i=1}^{M} a^{(i)\#} a^{(i)}$, $k^{(2)} = \sum_{i=M+1}^{M'} a^{(i)\#} a^{(i)}$, $M' > M$, $a^{(i)} \in F$, let us put

$$L_n = (p((\sum_{i=1}^{M} a_n^{(i)\sharp} \otimes a_n^{(i)})^\vee))^{1/2},$$

$$L_n' = (p((\sum_{i=1}^{M'} a_n^{(i)\sharp} \otimes a_n^{(i)})^\vee))^{1/2},$$

$n=0, 1, 2, \cdots$. (B_{ii}) implies

$$L_n \leqslant \sqrt{d_n}\, L_n'.$$

For λ given in Definition 4.1 let us set

$$\mathcal{H}_\lambda = \sup\{|R_\lambda(n)|\,;\, n=0,1,2,\cdots\},$$

where $R_\lambda(n)$ is taken from (28′). (28″) implies $\mathcal{H}_\lambda < \infty$. The following estimations are valid:

$$\begin{aligned}
p(f) &\leqslant p(k^{(1)}) + p(v^{(1)}) \overset{(*)}{\leqslant} 1 + p(k^{(1)}) \leqslant 1 + \sum_{n=0}^{\infty} p(k_n^{(1)}) \\
&\overset{(**)}{\leqslant} 1 + \sum_{n=0}^{\infty} (1 + A_{n,\lambda}\, \omega_n^2)\, L_n^2 \\
&\leqslant 1 + \sum_{n=0}^{\infty} (1 + (\xi(\lambda) + \frac{n^{\lambda+1}}{\lambda+1} + \frac{n^\lambda}{2} + \frac{\lambda n^{\lambda-1}}{12})\, \omega_n^2)\, L_n^2 \\
&\quad + \sum_{n=0}^{\infty} |R_\lambda(n)|\, \omega_n^2\, L_n^2 \\
&\leqslant 1 + (1+\mathcal{H}_\lambda) \sum_{n=0}^{\infty} (1 + (\xi(\lambda) + \frac{n^{\lambda+1}}{\lambda+1} + \frac{n^\lambda}{2} + \frac{\lambda n^{\lambda-1}}{12})\, \omega_n^2)\, L_n^2 \\
&\overset{(+)}{\leqslant} 1 + (1+\mathcal{H}_\lambda) \sum_{n=0}^{\infty} (1 + (\xi(\lambda) + \frac{3}{2} n^{\lambda+1})\, \omega_n^2)\, L_n^2 \\
&\overset{(++)}{\leqslant} 1 + (1+\mathcal{H}_\lambda)\, \xi(\lambda) \sum_{n=0}^{\infty} (1 + (1 + n^{\lambda+1})\, \omega_n^2)\, L_n^2) \\
&\leqslant 1 + (1+\mathcal{H}_\lambda)\, \xi(\lambda) \sum_{n=0}^{\infty} d_n (1 + (1 + n^{\lambda+1})\, \omega_n^2)\, (L_n')^2 \\
&\overset{(\sharp)}{\leqslant} 1 + (1+\mathcal{H}_\lambda)\, \xi(\lambda)\, \|k^{(1)} + k^{(2)}\|_{(p,(\alpha_n))} \\
&\overset{(\sharp\sharp)}{\leqslant} 1 + 2(1+\mathcal{H}_\lambda)\, \xi(\lambda).
\end{aligned}$$

Hence, $[V] \subset (1 + 2(1+\mathcal{H}_\lambda)\, \xi(\lambda))\, U$. Thus, the theorem under consideration is shown.

((*) follows from $p(v^{(1)}) \leqslant \tilde{p}(v^{(1)}) < 1$. (**) is a consequence of Theorem 3.5b). (+) is yielded by $\frac{n^{\lambda+1}}{\lambda+1} + \frac{n}{2} + \frac{n^{\lambda-1}}{12} \leqslant \frac{3}{2} n^{\lambda+1}$, where $n \in \boldsymbol{N}^*$, $1 < \lambda \leqslant 2$. (++) is a consequence of $\frac{\pi^2}{6} < \xi(\lambda) < \infty$ and $\xi(\lambda)(1 + n^{\lambda+1}) > \xi(\lambda) + \frac{3}{2} n^{\lambda+1}$. (#) follows from Theorem 3.5a). (##) is implied by

$$||k^{(1)}+k^{(2)}|| = ||v^{(2)}-v^{(1)}|| \leqslant ||v^{(2)}||+||v^{(1)}|| < 2.)$$

The following is aimed at a discussion of (B_i) and (B_{ii}). It will be shown that for a large class of alg-$\sharp$ cones, (B_i) and (B_{ii}) are also necessary for the normality of these cones. Furthermore, a special class of l.c. topologies will be distinguished such that (B_i) and (B_{ii}) are even equivalent to the normality of the cones under consideration.

Lemma 4.3. *Let us be given a topological tensor-algebra $E_{\otimes}[\tau]$ and an alg-$\sharp$ cone $\{F, \sharp\}$ such that $\{F, \sharp\} \subset h(E_{\otimes}, \sharp)$, and the mapping $\sharp$ is τ-continuous. If $\{F, \sharp\}$ is -normal, then* (B_i) *and* (B_{ii}) *are satisfied.*

Proof. The proof is based on some ideas from [28; Theorem 3]. Using Lemma 2.1a), the τ-continuity of $\sharp$ implies the topological direct decomposition

$$E_{\otimes}[\tau] = h(E_{\otimes}, \sharp)\,[\tau'] \oplus i\,h(E_{\otimes}, \sharp)\,[\tau'] \,,$$

(for a definition of τ', see Theorem 2.3e)).
The τ-normality of $\{F, \sharp\}$ yields that τ' is defined on $h(E_{\otimes}, \sharp)$ by the system of semi-norms $\mathfrak{P}=\{p_{\mathcal{M}};\ \mathcal{M} \in \mathfrak{M}\}$, where

$$p_{\mathcal{M}}(f) = \sup\{|T(f)|\,;\ T \in \mathcal{M}\} \,,$$

$f \in h(E_{\otimes}, \sharp)$, and $\mathfrak{M}$ denotes the set of all the weakly bounded subsets of $\{F, \sharp\}$-positive and τ'-continuous, real linear functionals on $h(E_{\otimes}, \sharp)$, see [26; V.3.3, Cor. 1].

Consider $f_r^{(i)} \in F_r, f_s^{(i)} \in F_s$ $(i=1, 2, \cdots, M,\ M \in N)$, and put

$$a_{rs} = \sum_{i=1}^{M} f_r^{(i)\sharp} \otimes f_s^{(i)} \,,$$

$r, s \in N^*$. For each $\mathcal{M} \in \mathfrak{M}$,

$$\begin{aligned}
p_{\mathcal{M}}(a_{rs}) &= \sup\{|\sum_{i=1}^{M} T((f_r^{(i)\sharp} \otimes f_s^{(i)})^{\vee})|\,;\ T \in \mathcal{M}\} \\
&\overset{(+)}{\leqslant} \sup\{\sum_{i=1}^{M} (T((f_r^{(i)\sharp} \otimes f_r^{(i)})^{\vee})\ T((f_s^{(i)\sharp} \otimes f_s^{(i)})^{\vee}))^{1/2};\ T \in \mathcal{M}\} \\
&\overset{(*)}{\leqslant} \sup\{(\sum_{i=1}^{M} T(f_r^{(i)\sharp} \otimes f_r^{(i)})^{\vee})\ T((f_s^{(i)\sharp} \otimes f_s^{(i)})^{\vee}))^{1/2};\ T \in \mathcal{M}\} \\
&\leqslant \sup\{(\sum_{i=1}^{M} T((f_r^{(i)\sharp} \otimes f_r^{(i)})^{\vee}))^{1/2};\ T \in \mathcal{M}\} \\
&\quad \sup\{(\sum_{i=1}^{M} T((f_s^{(i)\sharp} \otimes f_s^{(i)})^{\vee}))^{1/2};\ T \in \mathcal{M}\} \\
&= (p_{\mathcal{M}}(\check{a}_{rr})\, p_{\mathcal{M}}(\check{a}_{ss}))^{1/2}
\end{aligned}$$

are implied. Thus

$$p_{\mathcal{M}}((\sum_{\substack{r+s=m\\ r\neq s}}\sum_{i=1}^{M} f_r^{(i)\#}\otimes f_s^{(i)})^\vee)\leqslant \sum_{\substack{r+s=m\\ r\neq s}} p_{\mathcal{M}}(\check{a}_{rs})$$
$$\leqslant \sum_{\substack{r+s=m\\ r\neq s}} (p_{\mathcal{M}}(\check{a}_{rr})\, p_{\mathcal{M}}(\check{a}_{ss}))^{1/2}$$

follow for each $m\in \boldsymbol{N}^*$. Hence (B_i) is satisfied, where $(\omega_n)_{n=0}^\infty$, $\omega_n=1$. The validity of (B_{ii}) with $d_n=1$ ($n=0, 1, 2, \cdots$) is a consequence of

$$p_{\mathcal{M}}((\sum_{i=1}^{M} f_n^{(i)\#}\otimes f_n^{(i)})^\vee) = \sup\{\sum_{i=1}^{M} T((f_n^{(i)\#}\otimes f_n^{(i)})^\vee);\ T\in\mathcal{M}\}$$
$$\overset{(**)}{\leqslant} \sup\{\sum_{i=1}^{M'} T((f_n^{(i)\#}\otimes f_n^{(i)}));\ T\in\mathcal{M}\} = p_{\mathcal{M}}((\sum_{i=1}^{M'} f_n^{(i)\#}\otimes f_n^{(i)})^\vee),$$

where $\check{f}_n^{(i)}\in F$, $i=1, 2, \cdots, M'$, $M\leqslant M'$, $n=0, 1, 2, \cdots$.
((+) follows from Lemma 2.1c). (*) is a consequence of $\sum_{i=1}^{M}(a_i b_i)^{1/2}\leqslant (\sum_{i,j=1}^{M} a_i b_j)^{1/2}$, where $a_i\geqslant 0$, $b_i\geqslant 0$ ($i=1, 2, \cdots, M$). (**) is yielded by the $\{F, \#\}$-positivity of $T\in\mathcal{M}$.)

An immediate consequence of Theorem 4.2 and Lemma 4.3 is the following.

Corollary 4.4. *Let us be given an alg-# cone $\{F, \#\}$ in a topological tensor-algebra $E_\otimes[\tau]$ such that the assumptions of Lemma* 4.3 *are satisfied. Further, let every system of semi-norms $\mathfrak{P}(\tau)$, which defines τ, satisfy the following: For each $p\in\mathfrak{P}(\tau)$ and each sequence $(r_n)\in \boldsymbol{R}_+^{N^*}$, the semi-norm $f\to\sum_{n=0}^{\infty} r_{2n}\, p(\check{f}_{2n})$, $f\in E_\otimes$, is also τ-continuous. Then, $\{F, \#\}$ is τ-normal if and only if there is a system of semi-norms $\tilde{\mathfrak{P}}$ that defines τ and satisfies* (B_i), (B_{ii}).

Remark. The further assumption of Corolary 4.4 is satisfied for the topologies ε_P, π_P, ι_P, ε_∞, π_∞, $\varepsilon_\otimes$, $\pi_\otimes$, $\iota_\otimes$, which are given in (4).

Condition (B_{iii}) is discussed by the following lemma and the remark to it. It is shown that, in the case of a filtrated alg-# cone $\{F, \#\}$, there are l.c. topologies τ on $E_\otimes$ such that $\{F, \#\}$ is τ-normal, and however, (B_{iii}) is not satisfied.

Lemma 4.5. *Let us be given a filtrated alg-# cone in a topological tensor-algebra $E_\otimes[\tau]$. Further, let* (B_{ii}) *be satisfied. Then, $\{F, \#\}$ is τ-normal.*

Proof. Using Theorem 2.3a) and Lemma 2.1f), there is an $n\in\boldsymbol{N}^*$ such that

$$\{F, \sharp\} \subset \check{E}_{2n} \,. \tag{35}$$

Take any τ-continuous semi-norm p, and consider $U=\{g\in E_{\otimes}; p(g)<1\}$. If $h\in[U]$, then there are $u^{(i)}\in U$, $k^{(i)}\in\{F, \sharp\}$ $(i=1, 2)$ such that $h=u^{(1)}+k^{(1)}=u^{(2)}-k^{(2)}$. Hence,

$$p(h)\leqslant p(u^{(1)})+p(k^{(1)})\leqslant 1+d_{2n}\,p(k^{(1)}+k^{(2)})$$
$$= 1+d_{2n}\,p(u^{(2)}-u^{(1)})<1+2d_{2n}$$

are implied. Thus, $[U]\subset(1+2d_{2n})U$. This proves the lemma under consideration.

Remarks. a) The filtratedness of $\{F, \sharp\}$ implies that (B_i) is satisfied.

b) If $\sharp$ is continuous on $E_1[t_1]$ and fulfils in addition (25), then every filtrated alg-$\sharp$ cone is normal with respect to every l.c. topology τ which satisfies $\varepsilon_P<\tau<\varepsilon_{\otimes}$.

Proof. Using the continuity of $\sharp$, there is a system of semi-norms $\mathfrak{P}(t_1)$ that defines t_1 and satisfies $p_1(g_1^{\sharp})=p_1(g_1)$ for all $p_1\in\mathfrak{P}(t_1)$, $g_1\in E_1$. Note that

$$\tau_{|E_{2n}} = \varepsilon_{2n} \,,$$

where n is taken from (35). Using now that

$$\{p_{2n} = p\otimes_{\varepsilon}\cdots\otimes_{\varepsilon}p \quad (2n \text{ factors});\ p\in\mathfrak{P}(t_1)\} \tag{36}$$

defines ε_{2n}, an analogous consideration as in (*) of the proof of Example 3.4c) yields that (B_{ii}) is satisfied. Lemma 4.5 implies finally the assertion to be shown.

c) Obviously, there are l.c. topologies τ on $E_{\otimes}$ such that the assumptions of b) are satisfied, and however, (B_{iii}) does not apply. As a concrete example take

$$\mathfrak{P}(\tau) = \{f\rightarrow\sum_{n=0}^{\infty}p_n(f_n);\ p_1\in\mathfrak{P}(t_1)\},$$
$$p_n = p_1\otimes_{\varepsilon}\cdots\otimes_{\varepsilon}p_1 \quad (n \text{ factors}),\ n=1, 2, \cdots, p_0(f_0)=|f_0| \,.$$

In the case of involutive cones let us construct some examples of normal topologies. For $E_{\otimes}^{+}$, normal topologies are also considered in [10], [11] [22], [28], [33].

Example 4.6. Let us be given an l.c. space $E_1[t_1]$, a set of semi-norms $\mathfrak{P}(t_1)=\{p^{(\alpha)};\alpha\in A_1\}$ which defines t_1 (A_1 denotes a directed set of indices), and an involutive cone $\{F, \#\}$ in $E_{\otimes}$. Further, let

$$p^{(\alpha)}(f_1) = p^{(\alpha)}(f_1^{\#})$$

for all $\alpha \in A_1$, $f_1 \in E_1$. For each sequence $(\alpha^i)_{i=1}^{\infty} \in (A_1)^N$ with

$$p^{(\alpha^1)}(f_1) \leqslant p^{(\alpha^2)}(f_1) \leqslant \cdots, \tag{37}$$

$f_1 \in E_1$, let us consider the semi-norm $f_n \to \varepsilon_{(\alpha)}(f_n)$, $f_n \in E_n$ $(n=1, 2, \cdots)$, where

$$\varepsilon_{(\alpha)}(f_n) = \begin{cases} p_1 \otimes_{\varepsilon} \cdots p_s \otimes_{\varepsilon} p_s \otimes_{\varepsilon} \cdots \otimes_{\varepsilon} p_1(f_n) & \text{for } n=2s \\ p_1 \otimes_{\varepsilon} \cdots \otimes_{\varepsilon} p_s \otimes_{\varepsilon} p_{s+1} \otimes_{\varepsilon} p_s \otimes_{\varepsilon} \cdots \otimes_{\varepsilon} p_1(f_n) & \text{for } n=2s+1. \end{cases}$$

For each sequence $(\gamma_n) \in \boldsymbol{R}_+^{N^*}$, let us define the semi-norm

$$\varepsilon_{(\gamma)(\alpha)}(f) = \sum_{n=0}^{\infty} \gamma_n \varepsilon_{(\alpha)}(f_n),$$

$$f \in E_{\otimes}, \quad \varepsilon_{(\alpha)}(f_0) = |f_0| .$$

If $\Gamma \subset \boldsymbol{R}_+^{N^*}$ satisfies Definition 1.1i), and $B \subset (A_1)^N$ fulfils that for each $\alpha \in A_1$, $k \in \boldsymbol{N}$ there is some $(\beta^i)_{i=1}^{\infty} \in B$ with $\beta^k > \alpha$, then let us introduce a graded topology $\varepsilon_{\Gamma, B}$ on $E_{\otimes}$ by

$$\{\varepsilon_{(\gamma)(\alpha)}; (\gamma_n) \in \Gamma, (\alpha^i) \in B\},$$

see Definition 1.1. Notice also that

$$\varepsilon_{\Gamma, B|E_n} = \varepsilon_n$$

$(n=0, 1, 2, \cdots)$, $\varepsilon_1 = t_1$, ε_0 denotes the Euclidean toplogy on E_0.

Let us say that Γ *satisfies condition* (N), if for each $(\gamma_n) \in \Gamma$ the following are fulfilled:

- $(\mathrm{N_i})$ $1 = \gamma_0 \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \cdots$,
- $(\mathrm{N_{ii}})$ $\gamma_{2m} \geqslant \max\{(\gamma_{m+i})^2/\gamma_{2i};\ i=0, 1, \cdots, m-1\}$, $m=1, 2, \cdots$,
- $(\mathrm{N_{iii}})$ there are two constants $c>0$, $1<\lambda\leqslant 2$ and two sequences $(\gamma_n') \in \Gamma$, $(k_n)_{n=0}^{\infty} \in \mathfrak{A}_{(\omega, \beta)}$, $\beta_n = 2+n^{\lambda+1}$, $\omega_n = 1$, such that $\gamma_n k_n \leqslant c\gamma_n'$, $n=0, 1, 2, \cdots$.

Lemma. *If Γ satisfies* (N), *then $\{F, \#\}$ is $\varepsilon_{\Gamma, B}$-normal.*

Proof. For each $(\gamma_n) \in \Gamma$, $(\alpha^i) \in B$, the following estimations are valid:

$$\varepsilon_{(\gamma)(\alpha)}\Big(\Big(\sum_{\substack{r+s=n \\ r \neq s}} \sum_{j=1}^{M} f_r^{(j)\#} \otimes f_s^{(j)}\Big)^{\vee}\Big)$$

$$\leqslant \sum_{\substack{r+s=n\\ r\neq s}} r_n \varepsilon_{(\alpha)}((\sum_{j=1}^{M} f_r^{(j)\#} \otimes f_s^{(j)})^\vee)$$

$$\overset{(*)}{\leqslant} \sum_{\substack{r+s=n\\ r\neq s}} r_n(\varepsilon_{(\alpha)}((\sum_{j=1}^{M} f_r^{(j)\#} \otimes f_r^{(j)})^\vee)\, \varepsilon_{(\alpha)}((\sum_{j=1}^{M} f_s^{(j)\#} \otimes f_s^{(j)})^\vee))^{1/2}$$

$$\overset{(+)}{\leqslant} \sum_{\substack{r+s=n\\ r\neq s}} (\varepsilon_{(\gamma)(\alpha)}((\sum_{j=1}^{M} f_r^{(j)\#} \otimes f_r^{(j)})^\vee)\, \varepsilon_{(\gamma)(\alpha)}((\sum_{j=1}^{M} f_s^{(j)\#} \otimes f_s^{(j)})^\vee))^{1/2},$$

$f^{(j)} \in F$, $j=1, 2, \cdots, M$. Hence $\varepsilon_{(\gamma)(\alpha)}$ satisfies (A) with $\omega_n=1$ $(n=0, 1, 2, \cdots)$. Thus $(B_{\rm i})$ applies. Further, for each $(\alpha^i) \in B$, $M' \geqslant M$ $(M, M' \in N)$ it follows

$$\varepsilon_{(\alpha)}(\sum_{j=1}^{M} f_n^{(j)\#} \otimes f_n^{(j)})$$

$$= \sup\{|\sum_{j=1}^{M} T^{(1)} \otimes \cdots \otimes T^{(2n)}(f_n^{(j)\#} \otimes f_n^{(j)})|;\ T^{(i)}, T^{(2n-i+1)} \in U^0_{\alpha^i},$$

$$i = 1, 2, \cdots, n\}$$

$$\overset{(**)}{=} \sup\{\sum_{j=1}^{M} |T^{(1)} \otimes \cdots \otimes T^{(n)}(\check{f}_n^{(j)})|^2;\ T^{(i)} \in U^0_{\alpha^i},\quad i = 1, 2, \cdots, n\}$$

$$\leqslant \sup\{\sum_{j=1}^{M'} |T^{(1)} \otimes \cdots \otimes T^{(n)}(\check{f}_n^{(j)})|^2;\ T^{(i)} \in U^0_{\alpha^i},\quad i = 1, 2, \cdots, n\}$$

$$= \varepsilon_{(\alpha)}(\sum_{j=1}^{M'} f_n^{(j)\#} \otimes f_n^{(j)}),$$

where $\check{f}_n^{(j)} \in F$, $U^0_{\alpha^i} = \{T \in E';\ |T(f)| \leqslant p^{(\alpha^i)}(f)$ for all $f \in E_1\}$. Hence, $(B_{\rm ii})$ applies, where $d_n=1$ $(n=0, 1, 2, \cdots)$. Finally, $(N_{\rm iii})$ yields $(B_{\rm iii})$. Using Theorem 4.2, the lemma under consideration is shown.
((*) follows from (37) and an analogous consideration as given in Example 3.4c). (+) is a consequence of $r_m \leqslant \min\{(r_{2r} r_{2s})^{1/2};\ r+s=m\}$, which is implied by $(N_{\rm ii})$. (**) follows analogously to equation (*) of Example 3.4c).)

Remarks. a) Setting $B_\infty = \{(\alpha^i)_{i=1} \in (A_1)^N;\ \alpha^1 = \alpha^2 = \cdots\}$, it follows that $\varepsilon_{\mathbf{R}_+^{N*}, B_\infty} = \varepsilon_\infty$. Further, there is a Γ which satisfies (N) and fulfils $\varepsilon_{\Gamma, B_\infty} = \varepsilon_\infty$. Thus, $\{F, \#\}$ is ε_∞-normal. The ε_∞-normality of $\mathcal{S}_\otimes^+$ was first shown in [22].

b) Let the antilinear bijection $\#$ be continuous on $E_1[t_1]$ and satisfy (25). Arguing as above, it follows that every alg-$\#$ cone $\{F, \#\}$ is ε_∞-normal.

c) If the σ-topologies exist on E_n $(n=2, 3, \cdots)$, then all the considerations of the present example apply also to σ-topologies.

Because $\varepsilon_\otimes$ is well-adapted to the structure of $E_\otimes$ considered as an l.c.

space, it is of interest to investigate whether or not a given alg-# cone is $\varepsilon_\otimes$-normal. In the case of $\mathcal{S}_\otimes^+$, this question was answered by G. Lassner in [22]. The following considers $E_\otimes^+$.

Theorem 4.7. *Let us be given an l.c. space $E_1[t_1]$ such that t_1 is metrizable and the involution $f_1 \to f_1^*$, $f_1 \in E_1$, is t_1-continuous. Then, the following are equivalent.*

i) *$E_\otimes^+$ is $\varepsilon_\otimes$-normal,*

ii) *$E_1[t_1]$ is normable.*

Proof. ii)$\Rightarrow$i): Assuming ii), $\varepsilon_\otimes = \varepsilon_\infty$ follows from [12]. Then, i) is a consequence of Remark a) to Example 4.6.

i)$\Rightarrow$ii): The proof uses the following two assertions.

(I) *If T denotes an $\varepsilon_\otimes$-continuous and positive linear functional on $E_\otimes$, then the mapping*

$$A_T : E_\otimes[\varepsilon_\otimes] \times E_\otimes[\varepsilon_\otimes] \to \boldsymbol{C}$$

is jointly continuous, where $A_T(f, g) = T(f^* g)$, $f, g \in E_\otimes$.

(II) *If $E_1[t_1]$ is metrizable but non-normable, then there is an $\varepsilon_\otimes$-continuous linear functional S on $E_\otimes$ such that A_S is not jointly continuous.*

Assume now that i) is satisfied, and that ii) does not apply. Take a linear functional S from (II). Using i), there are $\varepsilon_\otimes$-continuous and positive linear functionals $T^{(j)}$, $j=1, \cdots, 4$, such that

$$S = T^{(1)} - T^{(2)} + i(T^{(3)} - T^{(4)}) .$$

Noticing that there is a system of semi-norms $\mathfrak{P}(\varepsilon_\otimes)$ which defines $\varepsilon_\otimes$ and satisfies $q(f^*) = q(f)$ for all $q \in \mathfrak{P}(\varepsilon_\otimes)$, $f \in E_\otimes$, it follows from (I) that there is a $q' \in \mathfrak{P}(\varepsilon_\otimes)$ such that

$$|T^{(j)}(f^* g)| \leqslant q'(f^*)\, q'(g) = q'(f)\, q'(g) ,$$

$f, g \in E_\otimes$, $j=1, \cdots, 4$. Thus, $S(f^* g) \leqslant 4\, q'(f)\, q'(g)$. But this is a contradiction to (II).

Proof of (I). Recalling Example 3.4a) and Theorem 3.5b), it follows

$$|T(f^* f)| \leqslant \sum_{n=0}^{\infty} (1 + A_{n,\lambda})\, T_{2n}(f_n^* \otimes f_n) ,$$

$1 < \lambda < \infty$, $f \in E_\otimes$. Setting $r_n = 1 + A_{n,\lambda}$ and using

$$|T_{2n}(f_n^* \otimes f_n)| \leqslant p_{2n}(f_n^* \otimes f_n) = (p_n(f_n))^2,$$

where p_n is taken from (35), $f_n \in E_n$, it is implied that

$$|T(f^* f)| \leqslant (\sum_{n=0}^{\infty} \sqrt{r_n}\, p_n(f_n))^2. \tag{37}$$

Using that the right-hand side of (37) is a $\varepsilon_{\otimes}$-continuous seminorm, Lemma 2.1c) implies (I).

Proof of (II). The proof follows straightforwardly from the Theorem of Kolmogorov and a Hahn-Banach argument, see [14].

Remarks **4.8.** a) If (B_i) of Definition 4.1 is replaced by

(B_i') p satisfies condition (A′) with respect to the convex hull (31) and a sequence (ω_n),

then all the considerations of the present chapter apply also to the convex hull (31) of finitely many alg-♯ cones.

b) Examples 3.7 and 4.6 imply that $\mathcal{S}_{\otimes}^{+} + \{\mathcal{F}, \sharp\}$ (convex hull of the cone of positivity and the cone of reflection positivity) is $\varepsilon_{\Gamma,B}$-normal, if Γ satisfies condition (N). Especially, $\mathcal{S}_{\otimes}^{+} + \{\mathcal{F}, \sharp\}$ is ε_{∞}-normal.

Acknowledgements

This work was mainly done during the author's stay at the RIMS of Kyoto University. The author wants to express his gratitude to the Japanese Government for providing a scholarship and he also wishes to thank the members of RIMS, in particular Professor H. Araki, for their very kind hospitality and for many stimulating and helpful discussions on the subject of this paper.

References

[1] Alcántara, J., *The choice of tensor-product topologies in the definition of tensor-algebras*, Lecture given at the Congress on Algeraic QFT and rings of unbounded operators, Göttingen, 1982.

[2] Alcántara, J. and Dubin, D.A., I*-algebras and their applications, *Publ. RIMS, Kyoto Univ.*, **17**, (1981), 179–199.

[3] Antoine, J.-P. and Ôta, S., Unbounded GNS representations of a *-algebra in a Krein space, *Letters in Math. Phys.*, **18** (1989), 267–274.

[4] Bongaarts, P.J.M., Maxwell's equations in axiomatic quantum field theory. I. Field tensor and potentials, *J. Math. Phys.*, **18** (1977), 1510–1516; II. Covariant and non-covariant gauges, *J. Math. Phys.*, **23** (1982), 1881–1898.

[5] Borchers, H.J., On the structure of the algebra of field operators, *Nuovo Cimento*, **24** (1962), 214–236.

[6] Borchers, H.J., Algebraic aspects of Wightman field theory, in: *Statistical Machanics and Field Theory*, Sen & Weil (Eds.), Halsted Press, New York (1972).

[7] Dubin, D.A. and Hennings, M.A., Regular tensor algebras, *Publ. RIMS, Kyoto Univ.*, **25** (1989), 971–999.

[8] Gradstein, I.S. and Ryshik, I.M., *Tables, Integrals, Sums, Sequences, and Derivations* (in Russian), Moscow, 1962.

[9] Hegerfeldt, G.C., From Euclidean to relativistic fields and the notion of Markov fields, *Commun. Math. Phys.*, **35** (1974), 155–171.

[10] Hofmann, G., *Die Testfunktionenalgebra und ihre Anwendung in der axiomatischen Quantenfeldtheorie*, Thesis A, Leipzig, 1975.

[11] ————, Ein hinreichendes Kriterium für die Normalität von Kegeln in Tensoralgebren und einige Anwendungen, *Wiss. Z. KMU Leipzig*, **27** (1978), 261–269.

[12] ————, Topologien auf Tensoralgebren, *Wiss. Z. KMU Leipzig*, **33** (1984), 16–24.

[13] ————, On the topological structure of tensor-algebras and the closure of the cone of positive elements, in: *Operator Algebras and Applications*, H. Araki (Edt.), Sûrikaiseki Kenkyûsho Kôkyûroku, Kyoto (1985), 26–47.

[14] ————, *Zur Halbordnungsstruktur topologischer Tensoralgebren*, Thesis B, Univ. Leipzig, 1989.

[15] ————, On the cone of α-positivity for QFT with indefinite metric, to appear.

[16] Ito, K.R., Canonical linear transformation on Fock space with an indefinite metric, *Publ. RIMS, Kyoto Univ.*, **14** (1978), 503–556.

[17] Jakóbczyk, L., Borchers algebra formulation of an indefinite inner product quantum filed theory, *J. Math. Phys.*, **25** (1984), 617–622.

[18] Jakóbczyk, L. and Strocchi, F., Krein structures for Wightman and Schwinger functions, *J. Math. Phys.*, **29** (1988), 1231–1235.

[19] Köthe, G., *Topological vector spaces, I & II*. Springer-Verlag, Berlin, New York, Heidelberg, 1984, 1979.

[20] Lassner, G., On the structure of the test fucntion algebra, *Dubna preprint*, JINR E2-5254 (1970).

[21] ————, Topological algebras of operators, *Rep. Math. Phys.*, **3** (1972), 279–293.

[22] ————, Über die Realisierbarkeit topologischer Tensoralgebren, *Math. Nachr.*, **62** (1974), 89–101.

[23] Lassner, G. and Uhlmann, A., On positive functionals on algebras of test functions for quantum fields, *Commun. Math. Phys.*, **7** (1968), 152–159.

[24] Neumark, M.A., *Normierte Algebren*, Akademie-Verl., Berlin, 1963.

[25] Osterwalder, K. and Schrader, R., Axioms for Euclidean Green's functions, I & II, *Commun. Math. Phys.*, **31** (1973), 83–112 and **42** (1975), 281–305.

[26] Schaefer, H.H., *Topological vector spaces*, Springer-Verlag, Berlin, 1971.

[27] Schmüdgen, K., The order structure of topological *-algebras, *Rep. Math. Phys.*, **7** (1975), 215–227.

[28] ————, Graded and filtrated topological *-algebras, I. Graded normal topologies. *Rep. Math. Phys.*, **18** (1980), 211–229.

[29] Simon, B., *The $P(\varphi)_2$ Euclidean (quantum) field theory*, Princeton University Press, 1974.

[30] Uhlmann. A., Über die Definition der Quantenfelder nach Wightman und Haag, *Wiss. Z. KMU Leipzig*, **11** (1962), 213–217.

[31] Wyss, W., *On Wightman's thoery of quantized fields*, Lectures in theoretical physics, University of Colorado, Boulder 1968, Gordon and Breach, 1969.

[32] ————, The field algebra and its positive linear functionals, *Commun. Math. Phys.*, **27** (1972), 223–234.

[33] Yngvason, J., On the algebra of test functions for field operators, *Commun. Math.*

Phys., **34** (1973), 315–333.

[34] Yngvason, J., Translationally invariant states and the spectrum ideal in the algebra of test functions for quantum fields, *Commun. Math. Phys.*, **81** (1981), 401–418.

[35] ———, On the locality ideal in the algebra of test functions for quantum fields, *Publ. RIMS, Kyoto Univ.*, **20** (1984), 1063–1081.