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A Continuity Principle for the Bergman Kernel Function

By

Klas DIEDERICH* and Takeo OHSAWA**

§0. Statement of the Results

Let $D \subset \mathbf{C}^n$ be a bounded domain with C^∞-smooth pseudoconvex boundary, and let $p \in \partial D$ be any point. By a two-sided bumping family of D at p we mean a family of smoothly bounded pseudoconvex domains $\{D_t\}_{-1 \le t \le 1}$ satisfying the following properties:

1) $D_0 = D$,
2) $D_{t_1} \subset D_{t_2}$ if $t_1 < t_2$,
3) $\{\partial D_t\}_{-1 \le t \le 1}$ is a C^∞-family of real hypersurfaces in $\mathbf{C}^n$,
4) for any neighborhood U of p in $\mathbf{C}^n$ there exists a $t_0 < 0$ such that $D_{t_0} \backslash D_{-t_0} \subset U$.

Remark. A two-sided bumping family of D at p exists, of course, if ∂D is strictly pseudoconvex at p. Recently, it was shown by Cho [Ch], that such a family also exists, if ∂D is of finite type at p.

By a peak function at p we mean a continuous function f on $\bar{D}$ which is holomorphic on D and satisfies $f(p)=1$ and $|f(z)|<1$ on $\bar{D} \backslash \{p\}$. By $K_D(z, w)$ we denote the Bergman kernel function of D and we put $K_D(z) = K_D(z, z)$. Finally, we write ds_D^2 for the Bergman metric of D.

Our goal is to prove continuity results in the parameter t of a bumping family for the Bergman kernel, the Bergman metric and related functions. Namely, we will show the following theorems:

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* Fachbereich 7 Mathematik, Bergische Universität GHS Wuppertal, Gauss Str. 20, D-5600 Wuppertal 1, Germany.
** Department of Mathematics, Nagoya University, 464-01 Nagoya, Japan.

Theorem 1. *Let $D \subset C^n$ be a bounded domain with C^∞-smooth pseudoconvex boundary and let $\{D_t\}_{-1\leq t\leq 1}$ be a two-sided bumping family of D at some point $p \in \partial D$ where ∂D is strictly pseudoconvex. Then there is for any $\varepsilon > 0$ and any neighborhood U of p a number $t_0 \in (0, 1)$ such that*

$$|K_D(z)\, K_{D_t}^{-1}(z) - 1| < \varepsilon$$

for all $t \in (-t_0, t_0)$ and $z \in D \backslash U$.

This theorem and the analogous result for the Bergman metric will be consequences of the continuity principle for so-called *maximizing functions.* We define:

Definition 1. Let D be a bounded domain in C^n and $w \in D$ an arbitrary point. We denote $N_0 = N \cup \{0\}$, fix an $\alpha \in N_0^n$ and put

$$H^2_{(\alpha)}(D, w) = \left\{g \in H^2(D) \colon \left(\frac{\partial}{\partial z}\right)^\beta g(w) = 0 \ \forall\ \beta \in N_0^n \text{ with } |\beta| \leq |\alpha|, \beta \neq \alpha\right\}.$$

By $B_D^{(\alpha)}(z, w) \in H^2_{(\alpha)}(D, w)$ we denote the (unique) function satisfying

$$B_D^{(\alpha)}(w, w) = \max\left\{\left|\left(\frac{\partial}{\partial z}\right)^\alpha g(w)\right| \text{for } g \in H^2_{(\alpha)}(D, w) \text{ with } \|g\|_D = 1\right\}$$

and we write $B_D^{(\alpha)}(w) = B_D^{(\alpha)}(w, w)$.

We will show

Theorem 2. *Let D, $p \in \partial D$ be as in Theorem 1 and fix $\alpha \in N_0^n$. Then there exists for any $\varepsilon > 0$ and any neighborhood U of p a number $t_\alpha \in (0, 1)$ such that*

$$\left|\frac{B_D^{(\alpha)}(w)}{B_{D_t}^{(\alpha)}(w)} - 1\right| < \varepsilon$$

for all $t \in (-t_\alpha, t_\alpha)$ and $w \in D \backslash U$.

As an immediate consequence one obtains

Corollay. *Under the assumptions of Theorem 1 there exists for any $\varepsilon > 0$ and any neighborhood U of p a $t' \in (0, 1)$ such that*

$$(1-\varepsilon)\, ds_D^2 \leq ds_{D_t}^2 \leq (1+\varepsilon)\, ds_D^2$$

on $D \backslash U$ for all $t \in (-t', t')$.

§1. The Maximizing Functions

From now on, unless explicitly stated otherwise, we always suppose, that D

and $p\in\partial D$ satisfy the hypothesis of Theorem 1. Furthermore, we fix an $\alpha\in\boldsymbol{N}_0^n$ and let $B_z(w)=B(w, z)=B_D^{(\alpha)}(w, z)$ be the corresponding maximizing function as defined in Section 0. The first crucial tool for the proof of the theorems is the following

Lemma 1. *For any* $\varepsilon>0$ *and any neighborhood* U *of* p *there exists a* $t_1\in(-1, 0)$ *such that*

$$||B_z||^2_{D\setminus D_t}<\varepsilon$$

for all t *such that* $t_1<t<0$ *and for all* $z\in D\setminus U$.

Proof. We may assume $\varepsilon<1$. Let f be a peak function of D at p. Since $\sup_{D\setminus U}|f|<1$ there exists an integer m such that $\sup_{D\setminus U}|f|^m<\varepsilon/4$. Therefore one can find a $t_1\in(-1, 0)$ such that

$$\sup_{D\setminus D_t}|1-f^m|<\frac{\varepsilon}{4}$$

for all t such that $t_1<t<0$.
Let $z\in D\setminus U$ be any point and put

$$\psi_z=(1-f^m(z))^{-1}(1-f^m)\,B_z\,.$$

Then we have

$$\left(\frac{\partial}{\partial w}\right)^\beta\psi_z(z)=\left(\frac{\partial}{\partial w}\right)^\beta B_z(z)\,,$$

for all $\beta\in\boldsymbol{N}_0^n$ with $|\beta|\leq|\alpha|$. Therefore $||\psi_z||_D\geq 1$.
On the other hand, one has

$$\begin{aligned}||\psi_z||_D^2&=\left(1-\frac{\varepsilon}{4}\right)^{-2}||(1-f^m)\,B_z||_D^2\\&<\left(1-\frac{\varepsilon}{4}\right)^{-2}||B_z||^2_{D_t}+\frac{\varepsilon^2}{16}\left(1-\frac{\varepsilon}{4}\right)^{-2}||B_z||^2_{D\setminus D_t}\\&=\left(1-\frac{\varepsilon}{4}\right)^{-2}-\left(1-\frac{\varepsilon^2}{16}\right)\left(1-\frac{\varepsilon}{4}\right)^{-2}||B_z||^2_{D\setminus D_t}.\end{aligned}$$

Hence we get

$$||B_z||^2_{D\setminus D_t}<\left\{1-\left(1-\frac{\varepsilon}{4}\right)^2\right\}\left(1-\frac{\varepsilon^2}{16}\right)^{-1}<\varepsilon\,.$$

This finishes the proof.

§2. An Approximation Lemma

The second technical lemma needed for the proof of the theorems deals with uniform approximation on bumping families. For this we fix such a family $\{D_t\}_{-1\leq t\leq 1}$ of D at p. We can choose a corresponding C^∞-family $\{\rho_t\}$ of defining functions, i.e. a C^∞ function $\rho_t(z)=\rho(t,z)$ on $[-1,1]\times\mathbb{C}^n$ such that $D_t=\{z\in\mathbb{C}^n: \rho_t(z)<0\}$ and $d\rho_t\neq 0$ on ∂D_t and such that the functions $\rho_t(z)$ are strictly plurisubharmonic in a neighborhood of $\overline{D_1\backslash D_{-1}}$. With this we put $r_t=2\rho_0-\rho_t$ and $U_t=\{z\in D: r_t(z)<0\}$.

We show:

Lemma. *There exists a constant C such that for any $\varphi\in H^2(D)$ and $t\in[0,1]$ one can find $\varphi_t\in H^2(D_t)$ satisfying*

$$\|\varphi-\varphi_t\|_D\leq C\|\varphi\|_{U_t}$$

and

$$\|\varphi_t\|_{D_t}\leq\|\varphi\|_D+C\|\varphi\|_{U_t}.$$

Proof. We put $h_t=\rho_t^{-1}r_t$. Then

(1)
$$\begin{aligned}\partial h_t &= -h_t\,\rho_t^{-1}\,\partial\rho_t+\rho_t^{-1}\,\partial r_t\\ &= -(h_t+1)\,\rho_t^{-1}\,\partial\rho_t+2\rho_t^{-1}\,\partial\rho_0\,.\end{aligned}$$

Note that there exist constants C_0 and C_ε (for any $\varepsilon>0$) such that

(2)
$$\partial\rho_0\bar\partial\rho_0\leq C_0\partial\rho_t\bar\partial\rho_t+C_\varepsilon|\rho_t|^{2-\varepsilon}ds_e^2 \text{ on } D$$

where ds_e^2 denotes the eulidean metric.
Let ds_t^2 be the metric on D_t defined by

$$ds_t^2 = ds_e^2+c_0\,\partial\bar\partial\log(-\rho_t)\,,$$

where $\partial\bar\partial\log(-\rho_t)$ stands for the complex Hessian of $\log(-\rho_t)$, and c_0 is a sufficiently small positive constant, so that ds_t^2 is a metric for all t. Let $|\cdot|_t$ denote the pointwise length with respect to ds_t^2. Then, combining the strict plurisubharmonicity of ρ_t with (1) and (2), we obtain an estimate

(3)
$$|\partial h_t|_t^2\leq -2c_0(h_t+1)^2+C'_\varepsilon|\rho_0|^{2-\varepsilon}\rho_t^{-1} \text{ on } D$$

for all $\varepsilon>0$. Here C'_ε may depend on ε.
Note that $U_t=\{z\in D_t: -1<h_t(z)<0\}$. Hence

$$|\rho_t| = |h_t^{-1} r_t| \geq |r_t| \geq 2|\rho_0| - |\rho_t| \text{ on } U_t .$$

Therefore, one has on U_t

$$|\partial h_t|_t^2 \leq -2c_0 + C'_\varepsilon |\rho_0|^{1-\varepsilon} .$$

In particular, we obtain on U_t for some constant C_1

$$|\partial h_t|_t^2 \leq C_1 . \tag{4}$$

Let $\chi: \boldsymbol{R} \to [0, 1]$ be any C^∞-function satisfying $\chi = 0$ on $(-\infty, -1]$ and $\chi = 1$ on $[0, \infty)$. Given any function $\varphi \in H^2(D)$ and $t \in [0, 1]$ we put

$$u_t = \bar{\partial} (\chi(h_t) \varphi) \wedge dz_1 \wedge \cdots \wedge dz_n .$$

By (4) there exists a constant C_2 independent of t such that

$$||u_t||_t \leq C_2 \, ||\varphi||_{U_t} ,$$

where $||\cdot||_t$ denotes the L^2-norm with respect to ds_t^2. Since the potential function $||z||^2 - c_0 \log(-\rho_t)$ of ds_t^2 has a bounded gradient with respect to ds_t^2, with the bound C_3 being independent of t, in virtue of the L^2-vanishing theorem of Donnelly-Fefferman [D-F] there exists an $(n, 0)$-form v_t on D_t satisfying

$$\bar{\partial} v_t = u_t \quad \text{and} \quad ||v_t||_{D_t} \leq C_3 \, ||u_t||_t ,$$

where we note that the L^2-norm of v_t does not depend on the choice of Hermitian metrics since v_t is of type (n, 0). Let us now define $\varphi_t \in H^2(D_t)$ by

$$\varphi_t \, dz_1 \wedge \cdots \wedge dz_n = \chi(h_t) \, \varphi \wedge dz_1 \wedge \cdots \wedge dz_n - v_t .$$

Then we obtain

$$||\varphi - \varphi_t||_D \leq (1 + C_2 \, C_3) \, ||\varphi||_{U_t}$$

and

$$||\varphi_t||_{D_t} \leq ||\varphi||_D + C_2 \, C_3 \, ||\varphi||_{U_t} .$$

Thus the constant $C = 1 + C_2 \, C_3$ satisfies the requirement.

§3. The Proof of the Continuity Principle

In order to avoid hiding the essentials behind technical details, we will prove here in detail Theorem 2 for the case $\alpha = 0$. Theorem 1 is an immediate consequence of this. At the end we will then indicate, how the proof has to be modified in order to give Theorem 2 for general α.

Let D, $\{D_t\}_{-1\le t\le 1}$, $p\in\partial D$ and the neighborhood U of p be given so as to satisfy the conditions of Theorem 1, take $z\in D\backslash U$ and let $B_z=B_D^{(0)}(\cdot, z)$ be the corresponding maximizing function. Choose a smooth family of defining functions ρ_t for the bumping family as in Section 2. By applying the approximation lemma we obtain functions $B_{z,t}\in H^2(D_t)$ for $0<t\ll 1$ such that

$$||B_{z,t}-B_z||_D^2\le C\,||B_z||_{U_t}^2 \quad \text{and} \quad ||B_{z,t}||_D^2\le 1+C\,||B_z||_{U_t}^2 .$$

Here U_t is as before and C is a constant independent of t. Hence we obtain

$$|B_{z,t}(z)-B_z(z)|^2\le C\,||B_z||_{U_t}^2\,K_D(z) \tag{5}$$

and

$$K_{D_t}(z)\ge(1+C\,||B_z||_{U_t}^2)^{-1}|B_{z,t}(z)|^2 . \tag{6}$$

Combining (5) and (6) we obtain

$$K_{D_t}(z)\ge(1+C\,||B_z||_{U_t}^2)^{-1}\,(1-C\,||B_z||_{U_t}^2)\,K_D(z) .$$

Since $K_{D_t}(z)\le K_D(z)$ we have

$$\lim_{t\to+0} K_D(z)\,K_{D_t}^{-1}(z)=1$$

by Lemma 1. On the other hand it follows directly from Lemma 1, that

$$\lim_{t\to-0} K_D(z)\,K_{D_t}^{-1}(z)=1$$

Thus the proof of Theorem 2 for $\alpha=0$ and of Theorem 1 is finished.

The proof of Theorem 2 for arbitrary α is completely similar except that we must have approximating functions $B_{z,t}^{(\alpha)}\in H_{(\alpha)}^2(D_t, z)$ for $B_D^{(\alpha)}(\cdot, z)$. In order to realize this additional restriction one only has to replace the use of Donnelly-Fefferman's vanishing theorem by that with weight functions of the form $N \log|z-w|^2$ $(N\gg 1)$. For such a modification of Donnelly-Fefferman's vanishing theorem the reader is referred to Ohsawa-Takegoshi [O-T].

In order to see, that the Corollary on the Bergman metric follows from Theorem 2, we only have to recall, that

$$ds_D^2(X, X)=\sup\{|Xg(z)|^2\,K_D^{-1}(z)\colon ||g||_D=1 \quad \textit{and} \quad g(z)=0\}$$

for any $z\in D$ and any tangent vector X at z.

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