Publ. RIMS, Kyoto Univ.
28 (1992), 587–593

A Note on Coherent States Related to Weighted Shifts

By

Włodzimierz MLAK* and Jan STOCHEL**

§1. Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Denote by S a forward isometric shift such that $Se_n = e_{n+1}$ for $n \geq 0$ and by D a diagonal operator determined by the equality $De_n = w_n e_n$ for $n \geq 0$. Assume that $w_n > 0$ for $n \geq 0$. The operator $W := SD$ is called a *weighted shift* (*with weights* w_n). Notice that W is an invertible and irreducible operator such that $\mathfrak{D}(W) = \mathfrak{D}(D)$ ($\mathfrak{D}(A)$ stands for the domain of an operator A). Since S is bounded and D is self-adjoint, we have

$$W^* = DS^*. \tag{1.1}$$

Using (1.1) one can describe the adjoint W^* of the operator W as follows

$$\mathfrak{D}(W^*) = \{f \in H: \sum_{n=0}^{\infty} |(f, e_{n+1})|^2 w_n^2 < +\infty\} \tag{1.2}$$

and

$$W^* f = \sum_{n=0}^{\infty} (f, e_{n+1}) w_n e_n, \qquad f \in \mathfrak{D}(W^*). \tag{1.3}$$

Thus

$$W^* e_0 = 0, \ W^* e_n = w_{n-1} e_{n-1}, \qquad n > 0. \tag{1.4}$$

The celebrated example of a weighted shift is the quantum creation operator a^+ which acts as follows: $a^+ e_n = \sqrt{n+1}\, e_{n+1}$ for $n \geq 0$. The basic functional model of a^+ belongs to Bargmann [1]. Let us enter into details. Denote by μ the Borel measure on $\boldsymbol{C}^1$ defined by $d\mu(z) = \frac{1}{\Pi} e^{-|z|^2}\, dm(z)$, where m is the planar Lebesgue measure. It is proved in [1] that the set of all entire functions belonging to $L^2(\mu)$ forms a closed linear subspace of $L^2(\mu)$. We denote this subspace by B^2. The main result of [1] states that:

Communicated by H. Araki, May 31, 1991.
1991 Mathematics Subject Classifications: 47B37, 47B20, 81R30
* Instytut Matematyczny PAN, św. Tomasza 30, 31–027 Kraków, Poland
** Instytut Matematyki UJ, Reymonta 4, 30–059 Kraków, Poland

(1.5) a^+ is unitarily equivalent to the operator M of multiplication by the independent variable z in B^2

and

(1.6) $a := (a^+)^*$ is unitarily equivalent to the differential operator $\frac{d}{dz}$ acting in B^2 with $\mathfrak{D}\left(\frac{d}{dz}\right) = \{f \in B^2: \frac{d}{dz} f \in B^2\}$.

Notice that the operator of multiplication by the independent variable z, defined in $L^2(\mu)$, is a normal extension of M. Thus W and consequently a^+ are unbounded subnormal operators.

Recall that $M^* = \frac{d}{dz}$, i.e. $\mathfrak{D}(M^*) = \{f \in B^2: \frac{d}{dz} f \in B^2\}$ and $M^* f(z) = \frac{d}{dz} f(z)$ for $f \in \mathfrak{D}(M^*)$. Define for $u \in \boldsymbol{C}^1$ and $z \in \boldsymbol{C}^1$ the function $h_u(z) = e^{uz}$. It is plain that $h_u \in B^2$ and, moreover, $\frac{d}{dz} h_u(z) = u h_u(z)$ for all $z \in \boldsymbol{C}^1$. Thus $M^* h_u = u h_u$, which means that for every $u \in \boldsymbol{C}^1$, the function h_u is an eigenfunction of M^* with the eigenvalue u. Hence the point spectrum of M^* is equal to $\boldsymbol{C}^1$. Consequently the point spectrum of the annihilation operator a coincides with $\boldsymbol{C}^1$.

The crucial point of our investigations is that the sequence $\{e_n\}_{n=0}^{\infty} \subset B^2$ defined by $e_n(z) = \frac{z^n}{\sqrt{n!}}$ $(n \geq 0)$, is an orthonormal basis of B^2 such that $(Me_n)(z) = z e_n(z) = \sqrt{n+1}\ e_{n+1}(z)$ for $n \geq 0$. Since $h_u(z) = \sum_{n=0}^{\infty} \frac{u^n}{\sqrt{n!}} \cdot \frac{z^n}{\sqrt{n!}} = \sum_{n=0}^{\infty} \frac{u^n}{\sqrt{n!}} e_n(z)$ for all $z \in \boldsymbol{C}$, one can show that for any $u \in \boldsymbol{C}^1$, the series $\sum_{n=0}^{\infty} \frac{u^n}{\sqrt{n!}} e_n$ converges in B^2 to h_u. Setting $w_n = \sqrt{n+1}$ for $n \geq 0$, we can write

$$h_u = e_0 + \sum_{n=1}^{\infty} \frac{u^n}{w_0 \cdot \ldots \cdot w_{n-1}} e_n, \qquad u \in \boldsymbol{C}^1. \tag{1.7}$$

In particular we have $\|h_u\|^2 = e^{|u|^2}$ for $u \in \boldsymbol{C}^1$. Thus the function $e^{-|u|^2/2} h_u \in B^2$ is a normalized eigenfunction of the annihilation operator a. It is called a *coherent normalized* quantum state of the electromagnetic field; such states are used in the quantum optics (see [3] and [5]).

§2. Let W be a weighted shift with positive weights w_n, i.e. $We_n = w_n e_{n+1}$ and $w_n > 0$ for $n \geq 0$. The equality (1.7) suggests that the proper candidate for a coherent state related to W would be the orthogonal series

$$h(z) = e_0 + \sum_{n=1}^{\infty} \frac{z^n}{w_0 \cdot \ldots \cdot w_{n-1}} e_n. \tag{2.1}$$

In order to have a nonempty region of convergence of the series defining $h(z)$, we assume, following the Cauchy-Hadamard theorem, that

$$r(W) = \liminf_{n\to\infty}(w_0 \cdot \ldots \cdot w_{n-1})^{1/n} > 0. \tag{2.2}$$

Denote by $S(W)$ the open disc $\{z \in \boldsymbol{C}^1 : |z| < r(W)\}$. Call $r(W)$ and $S(W)$ the *coherence radius* of W and the *coherence spectrum* of W, respectively.

Let z be an arbitrary element of $S(W)$. Take a real number r_0 such that $|z| < r_0 < r(W)$. Then there exists $n_0 > 0$ and q such that $0 < q < 1$ and

$$\frac{r_0}{(w_0 \cdot \ldots \cdot w_{n-1})^{1/n}} \leq q, \qquad n \geq n_0. \tag{2.3}$$

Therefore

$$\frac{|z|^{2n}}{(w_0 \cdot \ldots \cdot w_{n-1})^2} \leq q^{2n}, \qquad n \geq n_0. \tag{2.4}$$

This in turn implies that the orthogonal series defining $h(z)$ is convergent in H. It follows from (2.1) that

$$(h(z), e_0) = 1 \tag{2.5}$$

and

$$(h(z), e_{n+1}) = \frac{z^{n+1}}{w_0 \cdot \ldots \cdot w_n}, \qquad n \geq 0. \tag{2.6}$$

Thus

$$|(h(z), e_{n+1})|^2 w_n^2 = \frac{|z|^{2(n+1)}}{(w_0 \cdot \ldots \cdot w_{n-1})^2} \qquad n \geq 1. \tag{2.7}$$

One can deduce from (2.4) and (2.7) that

$$\sum_{n=0}^{\infty} |(h(z), e_{n+1})|^2 w_n^2 < +\infty,$$

which shows, by (1.2), that

$$h(z) \in \mathfrak{D}(W^*).$$

Consequently, by (1.3) and (2.6), we get

$$W^* h(z) = z e_0 + \sum_{n=1}^{\infty} \frac{z^{n+1}}{w_0 \cdot \ldots \cdot w_{n-1}} e_n = z h(z).$$

Arguing similarly to the above, one can show that for any $z \in S(W)$, the dimension of the kernel of $zI - W^*$ is equal to 1 (see [6] for the bounded case). Summing up we have proved the following

Theorem 2.1. *If W is a weighted shift with positive weights and $r(W) > 0$, then*

$$h(z) \in \mathfrak{D}(W^*), \qquad z \in S(W), \tag{2.8}$$

$$W^* h(z) = z h(z), \qquad z \in S(W) \tag{2.9}$$

and

$$\dim \operatorname{Ker}(zI - W^*) = 1. \qquad z \in S(W). \tag{2.10}$$

In the sequel we call $h(z)$ the *coherent state* of W at $z \in S(W)$.

Consider again the creation operator $W = a^+$, i.e. $We_n = \sqrt{n+1}\, e_{n+1}$ for $n \geq 0$. Then one can easy check that $r(W) = +\infty$ and $S(W) = C^1$. Moreover, the corresponding coherence state $h(z)$ is expressed by the formula

$$h(z) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} e_n, \qquad z \in C^1 \tag{2.11}$$

and $ah(z) = zh(z)$ for $z \in C^1$, i.e. $h(z)$ is an eigenfunction of the annihilation operator a. Notice that this result has been obtained without appealing to the Bargmann analytic model of a^+ (see Section 1). However this model has inspired all we have presented in Section 2.

3. This section deals with subnormal weighted shifts. Let W be a weighted shift within the space H. Assume that W is a *subnormal* operator, i.e. there is a superspace K of H and a normal operator M within K such that $W \subset M$ (M will be called a *normal extension* of W). It is easy to see that W has the following properties

$$P\mathfrak{D}(M) \subset \mathfrak{D}(W^*) \tag{3.1}$$

and

$$W^* Pf = PM^* f, \qquad f \in \mathfrak{D}(M), \tag{3.2}$$

where P is the orthogonal projection of K onto H. In particular we have $\|W^* e_{n+1}\| \leq \|M^* e_{n+1}\| = \|Me_{n+1}\| = \|We_{n+1}\| = w_{n+1}$ for $n \geq 0$. But $\|W^* e_{n+1}\| = w_n$, so $w_n \leq w_{n+1}$ for $n \geq 0$. This in turn implies that

$$r(W) = \lim_{n\to\infty} w_n. \tag{3.3}$$

If W is unbounded, then, in virtue of (3.3), we have

$$r(W) = +\infty \text{ and } S(W) = C^1. \tag{3.4}$$

On the other hand, if W is bounded, then

$$r(W) = \lim_{n\to\infty} w_n = \|D\| = \|W\|. \tag{3.5}$$

Let M be a normal extension of the weighted shift W with positive weights w_n. Let E be the spectral measure of M. Denote by K_0 the closed linear span of the vectors $E(\sigma) e_0$, where σ runs through the plane Borel sets. Since

$$e_n = (w_0 \cdot \ldots \cdot w_{n-1})^{-1} W^n e_0 = (w_0 \cdot \ldots \cdot w_{n-1})^{-1} M^n e_0, \qquad n \geq 1,$$

we get $e_n \in K_0$ for $n \geq 0$. By the spectral theorem, the part of M in K_0 is canonically isomorphic to the operator of multiplication by the independent variable z in the space $L^2(\mu)$ with $\mu(\cdot) = (E(\cdot)e_0, e_0)$. The unitary isomorphism is uniquely determined by the correspondence $E(\sigma)e_0 \to \chi_\sigma$ (χ_σ stands for the characteristic function of the set σ). Denote by $e_n(\cdot)$ the image (via the canonical isomorphism) of the vector e_n in $L^2(\mu)$, $n \geq 0$. Since $Me_0 = We_0 = w_0 e_1$ and $(Me_0)(z) = z$, we get $e_1(z) = \frac{z}{w_0}$. Similarly, the equality $Me_1 = w_1 e_2$ implies $ze_1(z) = w_1 e_2(z)$, so $e_2(z) = \frac{z^2}{w_0 w_1}$. Applying the induction procedure, we come to the conclusion

$$e_n(z) = \frac{z^n}{w_0 \cdot \ldots \cdot w_{n-1}}, \qquad n \geq 1. \tag{3.6}$$

Thus we have described in an explicit form the orthonormal basis of the picture $B^2(\mu)$ of the space H in $L^2(\mu)$. It is plain — due to $W \subset M$ — that $(\widehat{W}e_n(\cdot))(z) = ze_n(z) = w_n e_{n+1}(z)$ for $n \geq 0$, where $\widehat{W}$ is the picture of W in $B^2(\mu)$. Hence $\widehat{W}$ is the functional model of W. Notice that in the above we have exploited *merely* the simplest form of the spectral theorem. For general theory of functional models for unbounded cyclic operators we refer the reader to [7].

Let us assume that the subnormal weighted shift W satisfies the following condition:

$$\mu(\boldsymbol{C}^1 \setminus S(W)) = 0. \tag{3.7}$$

Given $f \in H$, we define the function $\hat{f}: S(W) \to \boldsymbol{C}$ by $\hat{f}(z) = (f, h(\bar{z}))$ for $z \in S(W)$. It follows from (2.1) and (3.6) that

$$\hat{f}(z) = \sum_{n=0}^{\infty} (f, e_n) e_n(z), \qquad z \in S(W), f \in H. \tag{3.8}$$

Notice that $\hat{f}$ is analytic in $S(W)$. Since $\|f\|^2 = \sum_{n=0}^{\infty} |(f, e_n)|^2$, one can show, using (3.7), (3.8) and the Riesz-Fischer theorem that $\hat{f} \in B^2(\mu)$. Moreover, by the Parseval equality, we have

$$\int_{S(W)} |\hat{f}(z)|^2 d\mu(z) = \|f\|^2, \qquad f \in H. \tag{3.9}$$

An application of the polarization formula gives us

$$(f, g) = \int_{S(W)} \hat{f}(z) \overline{\hat{g}(z)}\, d\mu(z), \qquad f, g \in H. \tag{3.10}$$

Summing up we have proved the following theorem.

Theorem 3.1. *Let W be a subnormal weighted shift with positive weights. If $\mu(C^1 \setminus S(W)) = 0$, then the map $f \to \hat{f}$ is a unitary isomorphism of H onto $B^2(\mu)$.*

Let us consider the Bergmann space A^2. Then the operator W defined on A^2 by

$$(Wf)(z) = zf(z), \qquad |z| < 1, f \in A^2,$$

is a subnormal weighted shift with weights $w_n = \sqrt{\frac{n+1}{n+2}}$. In this particular case $S(W) = \{z \in C^1 \colon |z| < 1\}$ and $\mu = m =$ the planar Lebesgue measure.

§4. Let W be a subnormal weighted shift with corresponding space $B^2(\mu)$. Similarly to Section 3 we assume that $\mu(C^1 \setminus S(W)) = 0$. It follows from Theorem 3.1 that the map $f \to \hat{f}$ sends H onto the whole $B^2(\mu)$. Given $z \in S(W)$, we denote by P_z the orthogonal projection of H onto the one-dimensional space spanned by the vector $h(\bar{z})$. An elementary computation shows that

$$(4.1) \qquad P_z f = \frac{\hat{f}(z)}{\|h(\bar{z})\|^2} h(\bar{z}), \qquad f \in H,\ z \in S(W).$$

(Notice that (2.1) implies $\|h(\bar{z})\|^2 = 1 + \sum_{n=1}^{\infty} \frac{|z|^{2n}}{(w_0 \cdot \ldots \cdot w_{n-1})^2} > 0$ for $z \in S(W)$). It follows from (4.1) that

$$(4.2) \qquad (P_z f, g) = (P_z f, P_z g) = \hat{f}(z)\overline{\hat{g}(z)} \frac{1}{\|h(\bar{z})\|^2}, \qquad f, g \in H,\ z \in S(W).$$

Using (3.10) and (4.1), we get

$$(f, g) = \int_{S(W)} \hat{f}(z)\overline{\hat{g}(z)}\, d\mu(z) = \int_{S(W)} (P_z f, g)\, \|h(\bar{z})\|^2 d\mu(z), \qquad f, g \in H.$$

This can be rewritten in terms of the weak integral as follows

$$(4.3) \qquad f = \int_{S(W)} P_z f \|h(\bar{z})\|^2 d\mu(z), \qquad f \in H.$$

Substituting (4.1) into (4.3) we get

$$(4.4) \qquad f = \int_{S(W)} \hat{f}(z) h(\bar{z})\, d\mu(z), \qquad f \in H.$$

This in turn implies that

$$(4.5) \qquad Af = \int_{S(W)} \hat{f}(z) A h(\bar{z})\, d\mu(z), \qquad f \in H,$$

for any bounded linear operator A within H.

Summing up we have proved the following theorem.

Theorem 4.1. *Let W be a subnormal weighted shift with positive weights. Assume that $\mu(C^1 \setminus S(W)) = 0$. Then*

$$I = \int_{S(W)} P_z \|h(\bar{z})\|^2 d\mu(z), \qquad (\textit{weak integral}), \tag{4.6}$$

where I stands for the identity operator on H. Moreover, if A is a bounded linear operator on H, then the equality (4.5) holds.

Let us consider again the quantum creation operator a^+. It is well-known (cf. [1]) that in that case $B^2(\mu)$ coincides with the Bargmann space B^2 and $d\mu(z) = \frac{1}{\Pi} e^{-|z|^2} \, dm(z)$ (m is the planar Lebesgue measure). Moreover, by (1.7), we have $\|h(z)\|^2 = e^{|z|^2}$ for $z \in C^1$. Since $S(a^+) = C^1$, the integral formula (4.6) turns into

$$\frac{1}{\Pi} \int_C P_z dm(z) = I. \tag{4.7}$$

This is the celebrated Glauber-Klauder basic formula of quantum optics (cf. [3]).

References

[1] Bargmann, V., On a Hilbert space of analytic functions and associated integral transform, *Comm. Pure Appl. Math.*, **19** (1961), 187–214.

[2] Berezin, F.A. and Shubin, M.A., *Schrödinger equation*, (Moscow, 1983).

[3] Klauder, J.R. and Sudarshan, E.C.G., *Fundamentals of quantum optics*, (New York, Amsterdam, 1968).

[4] Mlak, W., Notes on quantum circular operators, Part I, Preprint 303, Institute of Mathematics, Polish Academy of Sciences (1984).

[5] Perina, J., *Quantum statistics of linear and nonlinear optical phenomena*, (Reidel P. Comp., Boston, Lancaster, 1984).

[6] Shields, A.L., Weighted shifts and analytic function theory, *Math. Surveys*, **13**, American Mathematical Society, (Providence, Rhode Island, 1974), 49–128.

[7] Stochel, J., and Szafraniec, F.H., On normal extensions of unbounded operators. III. Spectral properties, *Publ. RIMS, Kyoto Univ.*, **25** (1989), 105–139.

[8] Volkin, H.C., Phase operators, *J. Math. Phys.*, **14** (1973), 1965–1976.