Publ. RIMS, Kyoto Univ.
28 (1992), 679–707

Noether's Inequality for Non-complete Algebraic Surfaces of General Type, II

By

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Abstract

Let V be a nonsingular projective surface of Kodaira dimension $\kappa(V) \geq 0$. Let D be a reduced, effective, nonzero divisor on V with only simple normal crossings. In the present article, a pair (V, D) is said to be a minimal logarithmic surface of general type, if, by definition, $K_V + D$ is a numerically effective divisor of self intersection number $(K_V + D)^2 > 0$ and if $K_V + D$ has positive intersection with every exceptional curve of the first kind on V. Here K_V is the canonical divisor of V. In the case, on the one hand, Sakai [8; Theorem 7.6] proved a Miyaoka—Yau type inequality $(\bar{c}_1^2) := (K_V + D)^2 \leq 3\bar{c}_2 := 3c_2(V) - 3e(D)$. On the other hand, we can easily obtain $(\bar{c}_1^2) \geq \frac{1}{15}\bar{c}_2 - \frac{8}{5}$ by making use of [8; Theorem 5.5]. In the present article, we shall prove that $(\bar{c}_1^2) \geq \frac{1}{9}\bar{c}_2 - 2$ provided that the rational map $\Phi_{|K_V+D|}$ defined by the complete linear system $|K_V + D|$ has a surface as the image of V. Moreover, if the equality holds, then the logarithmic geometric genus $\bar{p}_g := h^0(V, K_V + D) = \frac{1}{2}(\bar{c}_1^2) + 2 = 3$, D is an elliptic curve and V is the canonical resolution in the sense of Horikawa associated with a double covering $h: Y \to \mathbf{P}^2$. In addition, the branch locus B of h is a reduced curve of degree eight and the singular locus Sing B consists of points of multiplicity ≤ 3 except for at most one "simple quadruple point".

Introduction

This is a succession of the previous paper [9]. We work over an algebraically closed field k of characteristic zero. Let V be a nonsingular projective surface defined over k. If V is a minimal surface of general type, we have the following inequality due to M. Noether:

$$p_g(V) \leq \frac{1}{2}c_1(V)^2 + 2\,.$$

This inequality, together with the Noether formula $12\chi(\mathcal{O}_V) = c_1(V)^2 + c_2(V)$,

Communicated by K. Saito, May 18, 1990.
1991 Mathematics Subject Classification: 14J29
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implies the following inequality:

$$c_1(V)^2 \geq \frac{1}{5}c_2(V) - \frac{36}{5}.$$

The first (resp. second) inequality is called, in the present article, the first (resp. second) Noether inequality. A surface V is said to lie on the first (resp. second) Noether line if the first (resp. second) Noether inequality becomes an equality. The surfaces lying on Noether lines were studied by Horikawa [4] and [5].

In [9], we extended the first Noether inequality to the case of a minimal logarithmic surface of general type (see the Definition below). In the present article, we shall give a second Noether inequality for logarithmic surfaces of general type and classify those surfaces lying on the first or second Noether line. It is not, however, straightforward to derive a second Noether inequality from the first one as in the complete case, since we can not make avail of a formula corresponding to the Noether formula.

Let V be a nonsingular projective surface and let D be a reduced, effective divisor on V with only simple normal crossings. Such a pair (V, D) is called *a logarithmic surface* (*log surface*, for short). We denote by K_V the canonical divisor of V.

Definition. A log surface (V, D) is *of general type* if the logarithmic Kodaira dimension $\bar{\kappa}(V-D) := \kappa(V, K_V + D) = 2$. A log surface (V, D) of general type is *minimal* if $K_V + D$ is numerically effective (nef, for short) and if $K_V + D$ has positive intersection with every (-1)-curve on V (i.e., there are no redundant (-1)-curves on (V, D) in the sense of Sakai [11]).

In the case where $\bar{\kappa}(V-D) \geq 0$ and where $K_V + D$ has positive intersection with every (-1)-curve on V, the divisor $K_V + D$ is nef if and only if D is semi-stable in the sense of Sakai [8]. In the case $K_V + D$ is nef, $K_V + D$ is of general type if and only if $(K_V + D)^2 > 0$. Semi-stable curves on surfaces were studied by [8]. The definition of minimal log surface of general type given in [9] is more general and include the case where D is a $\mathbf{Q}$-divisor.

Set $\bar{p}_g := \dim H^0(V, K_V + D)$, $(\bar{c}_1^2) := (K_V + D)^2$ and $\bar{c}_2 := c_2(V) - e(D)$, where $e(D)$ is the Euler number of D. The following Theorem TZ is a part of [9; Theorem 2.10]. The cases where $K_V + D$ is ample and where $K_V + D$ is numerically effective and big were treated by Fujita [2] and Sakai [8], respectively.

Theorem TZ. *Let (V, D) be a log surface of general type such that $K_V + D$ is nef, $\bar{p}_g \geq 3$ and $|K_V + D|$ is not composed with a pencil. Write $|K_V + D| = |C| + G$, where $|C|$ and G are respectively the movable part and the fixed part. Then, replacing C by a general member of $|C|$ if necessary, we have the following assertions*:

(1) $\bar{p}_g \le (\bar{c}_1^2) + 2$.

(2) *Suppose the Kodaira dimension* $\kappa(V) \ge 0$. *Then* $\bar{p}_g \le \frac{1}{2}(\bar{c}_1^2) + 2$. *If the equality holds then we have*:

(2-1) $\mathrm{Bs}\,|K_V + D| = \phi$, *hence* $C \sim K_V + D$,

(2-2) $\frac{1}{2}(C^2) + 1 \le g(C) = p_a(C) \le (C^2) + 1$,

(2-3) $1 + p_g(V) \le h^1(V, C) + \chi(\mathcal{O}_V) = h^0(V, -D) + g(C) + 1 - \frac{1}{2}(C^2)$ *and*

(2-4) *the morphism* $\Phi_{|C|}\colon V \to \mathbf{P}^N$ $(N := \bar{p}_g - 1)$ *is either a birational morphism onto a surface of degree* $2(N-1)$, *or a morphism of degree two onto a normal rational surface* $\overline{W}$ *of degree* $N-1$.

Concerning Theorem TZ, (2), we consider in §1 a log surface (V, D) of general type satisfying the following condition:

$$(*)\quad K_V + D \text{ is nef}, \quad \kappa(V) \ge 0, \quad \bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + 2 \quad \text{and} \quad \deg \Phi_{|K_V+D|} = 2\,.$$

For such a pair (V, D), let C be a general member of $|K_V + D|$, let $\overline{W} := \Phi_{|C|}(V)$ and let $\eta\colon W \to \overline{W}$ be a minimal resolution. Denote by $\tilde{\Phi}$ the rational map $\eta^{-1} \circ \Phi_{|C|}\colon V \to W$. Let $h\colon Y \to W$ be the normalization of W in the function field $k(V)$ of V. If $g(C) = \frac{1}{2}(C^2) + 1$ or $(C^2) + 1$, the structure of the log surface (V, D) is explicitly described in Lemmas 1.3 and 1.4. We summarize in the following Theorem A a part of the results in Lemmas 1.4 and 1.2 which is to be used later.

Theorem A. *Let* (V, D) *be a log surface of general type satisfying the above condition* (*). *Assume that* $g(C) = (C^2) + 1$. *Then the following assertions hold.*

(1) $\tilde{\Phi}$ *is a morphism.*

(2) $\Phi_{|C|}$ *contracts* D *to points.*

(3) *If* D *is not contracted to points by* $\tilde{\Phi}$ *then* $\overline{W}$ *is a cone of degree* $N-1$ *in* $\mathbf{P}^N$.

(4) *Suppose* D *is contracted to points by* $\tilde{\Phi}$. *Let* B *be the branch locus of* $\tilde{\Phi}$. *Then* $B \sim 2F$ *where* $F := H - K_W$ *for a general member* H *of* $|\eta^*\mathcal{O}_{\overline{W}}(1)|$, *and* $h\colon Y \to W$ *is the double covering defined by a relation* $\mathcal{O}(B) \cong \mathcal{O}(F)^{\otimes 2}$.

From §2, we assume always that (V, D) is a minimal log surface of general type. Then V is a minimal resolution of Y in the case of Theorem A, (4) (cf. Lemma 1.4). In §2, we consider the process of canonical resolution Y^* of Y in the sense of Horikawa [4; p.48] and employ the notations B_i, E_i,

m_i, P_i used in the process and set before Definition 2.1. By making use of Horikawa's computation [4; Lemma 6] of invariants $\chi(\mathcal{O}_{Y^*})$ and $(K_{Y^*}^2)$ in terms of singularities of B, we prove:

Theorem B. *Let (V, D) be a minimal log surface of general type treated in Theorem* A, (4). *Assume $D \neq 0$ and $H^1(V, K_V + D) = 0$. Then the following assertions hold.* (*See* Lemma 2.8 *for the details.*)

(1) *We have $q(V) = 0$, $\bar{p}_g = p_g(V) + 1$, and $(K_V^2) \geq (K_{Y^*}^2) = 2(\bar{p}_g - 3) \geq 0$. Moreover, $m_b = 4$, 5 for some b and $m_i = 2$, 3 whenever $i \neq b$.*

(2) *Suppose $V = Y^*$. Then P_b is a simple quadruple point of B_b (see* Definition 2.4). *All possible "singularity types of B_b at P_b" and the corresponding configurations of $D(= T)$ are given in the Table* 1 *attached at the end of Proposition* 2.5. *In particular, $(D^2) = -2$.*

To be precise, by "the singularity type of B_b at P_b" in Theorem B, (2), we mean the singularity type of $E_{b+1} + B_{b+1}$ near the intersection of E_{b+1} and B_{b+1}. D is classified to the types I_2, $I_{3,3}$, $I_{4,2^t}$, $I_{3,3,2^t}$ and $I_{3,2^c,3,2^d}$. Each type of T or equivalently each type of $E_{b+1} + B_{b+1}$ in the Table 1 is realizable (cf. Proposition 2.6 and its Remark).

After these preparations, we shall look for an inequality which is to be called a second Noether inequality in the case of log surfaces. Sakai [8; Theorem 5.5] is crucial in the proof of the following Theorem C.

Theorem C. *Let (V, D) be a minimal log surface of general type. Assume that $\kappa(V) \geq 0$ and $D \neq 0$. Write $D = \sum_{i=1}^n D_i$ with irreducible components D_i's and set $r := \sum_{i<j}(D_i, D_j)$. Then we have*

$$(K_V, 4K_V + 3D) \geq 0 \qquad \textit{and} \qquad \bar{p}_g \geq \frac{1}{12}\bar{c}_2 - \frac{1}{4}(\bar{c}_1^2)\,.$$

Theorem C, together with an inequality $\bar{p}_g \leq (\bar{c}_1^2) + 2$ (cf. [2; Corollary 1.10] and [8; Theorems 6.1 and 6.5]), implies the following:

Corollary. *With the same hypotheses as in Theorem* C, *we have*:

$$(\bar{c}_1^2) \geq \frac{1}{15}\bar{c}_2 - \frac{8}{5}\,.$$

The above result, together with the inequality $(\bar{c}_1^2) \leq 3\bar{c}_2$ proved by Sakai [8; Theorem 7.6], gives an effective restriction on the region of non-complete algebraic surfaces $V - D$ of general type to exist.

In order to find an inequality of the form $(\bar{c}_1^2) \geq \gamma\bar{c}_2 + \delta$ as above with two rational numbers γ (>0) and δ, it is not sufficient to assume only $K_V + D$ is nef and big and it is necessary to assume (V, D) is minimal as defined above.

Indeed, suppose that there are rational numbers γ (>0) and δ such that the inequality $(\bar{c}_1^2) \geq \gamma \bar{c}_2 + \delta$ holds for every log surface (V, D) with property that $K_V + D$ is nef and big. Let $\sigma: W \to V$ be the blowing-up of m distinct nonsingular points $x_1, \dots, x_m$ of D and let $B = \sigma'(D)$ be the proper transform of D. Then $K_W + B \sim \sigma^*(K_V + D)$ and $(K_W + B)^2 = (K_V + D)^2$. Hence $K_W + B$ is nef and big. Note that $c_2(W) = c_2(V) + m$ and $e(B) = e(D)$. Then we must have $(K_V + D)^2 = (K_W + B)^2 \geq \gamma(c_2(W) - e(B)) + \delta = \gamma(c_2(V) - e(D)) + \delta + \gamma m$. This leads to a contradiction by taking $m \to +\infty$. Therefore, we have to assume that $K_V + D$ has positive intersection with every (-1)-curve on V.

More precisely, we prove:

Theorem D. *With the same hypotheses as in Theorem* C, *assume further that* $\bar{p}_g \geq 3$ *and* $|K_V + D|$ *is not composed with a pencil. Then the following two assertions hold.*

(1) $(\bar{c}_1^2) \geq \frac{1}{9}\bar{c}_2 - 2$.

(2) *Suppose* $(\bar{c}_1^2) < \frac{1}{9}\bar{c}_2 - 1$. *Then* $(\bar{c}_1^2) = \frac{1}{9}\bar{c}_2 - 2 + \frac{\alpha - 6}{9}$ *for some* $\alpha \in \{6, 7, \dots, 14\}$, *and we have either* $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + \frac{3}{2}$ *with* $\alpha \geq 9$ *or* $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + 2$.

(3) *If the second case in* (2) *with* $\alpha \leq 11$ *takes place, then either*

(3-1) $\bar{p}_g = 3$, $(\bar{c}_1^2) = 2$, $\alpha = 6 + e(D)$, $(K_V^2) = 0$ *and* $\kappa(V) = 1$, or

(3-2) $\bar{p}_g = 3$, $(\bar{c}_1^2) = 2$, $\alpha = 7 + e(D)$, $(K_V^2) = 1$ *and* $\kappa(V) = 2$.

Remark. More properties of the log surface (V, D) fitting the first case (resp. second case) of (2) are given in Lemma 3.6 (resp. 3.7). A log surface (V, D) fitting the case (3-1) (resp. (3-2)) is a log surface treated in Theorem B where $Y^* = V$ (resp. Y^* is the blowing-up of the minimal surface V with a nonsingular point of D as the center) and we refer to Proposition 2.6 and its Remark for the existence of such log surfaces.

We shall fix the following terminology and notations. Let V be a nonsingular projective surface. If E is a nonsingular rational curve on V with $(E^2) = -n$, we call E a $(-n)$-curve. A divisor H on V is called *numerically effective* (*nef*, for short) if $(H, R) \geq 0$ for every curve R. A nef divisor H is called *big* if $(H^2) > 0$.

K_V: canonical divisor of V
$\kappa(V)$: Kodaira dimension of V
$h^i(V, H) := \dim H^i(V, H) = \dim H^i(V, \mathcal{O}_V(H))$
$q(V) := \dim H^1(V, \mathcal{O}_V)$ the irregularity of V
$p_g(V) := \dim H^0(V, K_V)$ the geometric genus of V

$\bar{p}_g(V) := \dim H^0(V, K_V + D)$
$c_i(V)$: the i-th chern class of V
$(\bar{c}_1^2) := (K_V + D)^2$
$e(D)$: Euler number of a reduced, effective divisor D
$\bar{c}_2 := c_2(V) - e(D)$
$\mathcal{O}_{\overline{W}}(1)$: the sheaf of hyperplane sections if $\overline{W}$ is embedded into a projective space in some known way
$p_a(A)$: arithmetic genus of an irreducible curve A
$g(A)$: geometric genus of an irreducible curve A
$|H|$ or $|\mathcal{O}(H)|$: complete linear system defined by H
$\Phi_{|H|}$: rational map $V \to \mathbb{P}^{\dim |H|}$ defined by $|H|$
$f_*(H)$: direct image by a morphism f with finite degree
$f^*(H)$: total transform by a morphism f
$f'(H)$: proper transform by a birational morphism f
$H_1 \sim H_2$: linear equivalence
$H_1 \equiv H_2$: numerical equivalence
Σ_n, M_n: Hirzebruch surface of degree $n \geq 0$ and a minimal section M_n on Σ_n satisfying $(M_n^2) = -n$.

The author would like to thank Professor M. Miyanishi for constant encouragement during the preparation of the article and thank Professor S. Tsunoda for valuable discussion. He would also like to thank the referee for valuable suggestions that make the article more readable.

§1. The Case where $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + 2$

In the present section, we shall consider log surfaces (V, D) of general type satisfying the condition (cf. Theorem TZ in the Introduction):

$$(*) \quad K_V + D \text{ is nef, } \kappa(V) \geq 0,\ \bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + 2 \text{ and } \Phi_{|C|}: V \to \mathbb{P}^N\ (N := \bar{p}_g - 1)$$

is a morphism of degree two onto a normal, rational surface $\overline{W}$ of degree $N - 1$, where C is a general member of $|K_V + D|$.

Actually, the above surfaces $\overline{W}$ were completely classified (cf. Nagata [7; Theorem 7]). Note that $(\bar{c}_1^2) = (C^2) = 2(N-1)$. Let $\eta: W \to \overline{W}$ be a minimal resolution. We put $\eta = \mathrm{id}$ if W is nonsingular. Let $\overline{H}$ be a general member of $|\mathcal{O}_{\overline{W}}(1)|$ and set $H = \eta^*\overline{H}$. Then $C \sim \Phi_{|C|}^*(\overline{H})$. Suppose $\overline{W} \neq \mathbb{P}^2$. Then $W = \Sigma_e$, a Hirzebruch surface of degree e. We let $\pi: \Sigma_e \to \mathbb{P}^1$ be a $\mathbb{P}^1$-fibration and L a general fiber of π. If $\overline{W}$ is nonsingular then $0 \leq N - e - 3 \equiv 0 \pmod 2$ and $H \sim M_e + \frac{1}{2}(N + e - 1)L$.

Suppose $\overline{W}$ is singular. Then $e = N - 1 \geq 2$ and η is just the contraction of the minimal section M_{N-1} of Σ_{N-1}. Moreover, $H \sim M_{N-1} + (N-1)L$. We set $\bar{L} := \eta(L)$. Then $\bar{L}$ is a line of $\mathbf{P}^N$ contained in the cone $\overline{W}$ and we have $\overline{H} \sim (N-1)\bar{L}$. We have also $Cl(\overline{W}) = \mathbf{Z}[\bar{L}]$, $\mathrm{Pic}(\overline{W}) = \mathbf{Z}[\overline{H}]$ and $(H^2) = (\overline{H}^2) = N - 1$.

Let us begin with the following:

Lemma 1.1. *Assume the above condition (*) and assume further that $\overline{W}$ is singular. Let $\bar{h}$: $\overline{Y} \to \overline{W}$ be the normalization of $\overline{W}$ in the function field $k(V)$ of V. Set $\alpha = 4(1 + (C^2) - g(C))/(C^2)$. Then the following assertions hold.*

(1) $0 \leq \alpha \leq 2$.

(2) *We have $\mathcal{O}(K_{\overline{Y}})^{\otimes 2(N-1)} \cong \bar{h}^*\mathcal{O}_{\overline{W}}(1)^{\otimes (N-1)(2-\alpha)}$. Hence $\overline{Y}$ has only $\mathbf{Q}$-Gorenstein singularities. Moreover, $K_{\overline{Y}}$ is ample if $\alpha < 2$, i.e., if $g(C) > \frac{1}{2}(C^2) + 1$.*

Proof. (1) follows from the result $\frac{1}{2}(C^2) + 1 \leq g(C) \leq (C^2) + 1$ (cf. Theorem TZ, (2)).

(2) Since $\overline{W}$ is singular, $W = \Sigma_{N-1}$, $Cl(\overline{W}) = \mathbf{Z}[\bar{L}]$, $\mathrm{Pic}(\overline{W}) = \mathbf{Z}[\overline{H}]$ and $(\overline{H}^2) = N - 1$. Set $\varphi = \Phi_{|C|}$. We can prove that $\varphi^*\varphi_*(N-1)K_V = 2(N-1)K_V + \Theta$ with an integral exceptional divisor Θ of φ. Then we have $(\overline{H}, \varphi_* K_V) = \frac{1}{2}(C, \varphi^*\varphi_* K_V) = (C, K_V) = (2-\alpha)(C^2)/2 = (2-\alpha)(N-1)$. Hence $\varphi_*(N-1)K_V \sim (N-1)(2-\alpha)\overline{H}$ and $2(N-1)K_V + \Theta \sim (N-1)(2-\alpha)C$. Let $\bar{g}$: $V \to \overline{Y}$ be a birational morphism satisfying $\bar{h} \circ \bar{g} = \varphi$. Then Θ is also an exceptional divisor of $\bar{g}$ and we have $2(N-1)K_{\overline{Y}} \sim (N-1)(2-\alpha)\bar{g}_*(C) \sim (N-1)(2-\alpha)\bar{h}^*(\overline{H}) \in |\bar{h}^*\mathcal{O}_{\overline{W}}(1)^{\otimes (N-1)(2-\alpha)}|$. Then the rest of assertion (2) follows if one notes that $\overline{H}$ is an ample Cartier divisor and $\bar{h}$ is a finite morphism.

Let $\tilde{\Phi}$ be the rational map $\eta^{-1} \circ \Phi_{|C|}$: $V \to W$. If $\tilde{\Phi}$ is not a morphism then $\overline{W}$ is singular. More precisely, we have:

Lemma 1.2. *Assume the above condition (*) and assume further that $\tilde{\Phi}$: $V \to W$ is not a morphism. Then the following assertions hold.*

(1) *We have $\bar{p}_g = \frac{1}{2}(C^2) + 2 = 4$ and $g(C) = 4$. Hence $W = \Sigma_2$ and $\overline{W}$ is a quadric cone in $\mathbf{P}^3$.*

(2) *There is a nonsingular curve Δ of genus two such that $C \sim 2\Delta$.*

Proof. The idea in [5; Lemma 1.5] is used in the proof and we omit the proof.

Let $u: S \to T$ be a morphism of degree two between nonsingular projective surfaces. Write $K_S \sim u^*(K_T) + R_u$ with the ramification divisor R_u. Denote by B_u the branch locus $u_*(R_u)$ of u. Then B_u is a reduced divisor. Let $u_1: S \to S_1$ be a composite of blowing-downs of (-1)-curves, which are exceptional curves of u, such that the morphism $u_2: S_1 \to T$ induced by u has no (-1)-curves as exceptional curves. The ramification divisor R_{u_2} (resp. the branch locus B_{u_2}) of u_2 is equal to $u_{1*}(R_u)$ (resp. B_u). Moreover, $u_2^*(B_{u_2}) - 2R_{u_2}$ is an effective exceptional divisor of u_2 (cf. [4; Lemma 3]).

Assume the condition (*) at the beginning of § 1 and assume further that the rational map $\tilde{\Phi} = \eta^{-1} \circ \Phi_{|C|}: V \to W$ is a morphism. Lemma 1.2 shows that this is the case provided $g(C) = \frac{1}{2}(C^2) + 1$ or $g(C) = (C^2) + 1$. These cases will be elucidated in Lemmas 1.3 and 1.4, respectively. Let $\tau: V \to X$ be a composite of blowing-downs of (-1)-curves which are contracted by $\tilde{\Phi}$, such that the morphism $f: X \to W$ induced by $\tilde{\Phi}$ has no (-1)-curves contractible by f. So, $f \circ \tau = \tilde{\Phi}$. Write $K_X \sim f^*(K_W) + R$ with the ramification divisor R. Denote by B the branch locus $f_*(R)$. Then B is a reduced divisor and $f^*(B) - 2R$ is an effective exceptional divisor of f.

Let H be a general member of $|\eta^*\mathcal{O}_{\overline{W}}(1)|$. Then $C \sim \tilde{\Phi}^*(H)$. Set $\tilde{C} = \tau_*(C)$. Then $\tilde{C} \sim f^*(H)$ and $C = \tau^*(\tilde{C})$. Set $\tilde{D} = \tau_*(D)$. Since $C \sim K_V + D$, we have $\tilde{C} \sim K_X + \tilde{D}$. Hence $R \sim \tilde{C} - \tilde{D} - f^*(K_W)$, $B \sim 2H - f_*(\tilde{D}) - 2K_W$ and $f^*(B) - 2R \sim 2\tilde{D} - f^*f_*(\tilde{D})$.

Let $h: Y \to W$ (resp. $\bar{h}: \overline{Y} \to \overline{W}$) be the normalization of W (resp. $\overline{W}$) in the function field $k(V)$ and let $g: X \to Y$ be a birational morphism such that $h \circ g = f$. There is also a birational morphism $\eta_y: Y \to \overline{Y}$ such that $\bar{h} \circ \eta_y = \eta \circ h$.

Lemma 1.3. *Assume the condition (*) at the beginning of § 1 and* $g(C) = \frac{1}{2}(C^2) + 1$. *Let X' be a nonsingular minimal model of V. Then the following assertions hold.*

(1) *We have $2K_{X'} \sim 0$ and X' is a minimal resolution of $\overline{Y}$.*

(2) *Suppose X is a minimal surface. Then X is a* K3*-surface and $h: Y \to W$ is the double covering defined by a relation $\mathcal{O}(B) \cong \mathcal{O}(-K_W)^{\otimes 2}$.*

(3) *Suppose X is not a minimal surface. Then $\overline{W}$ is singular and there exists only one (-1)-curve on X.*

Proof. We refer to the proof of Lemma 1.4 below.

Lemma 1.4. *Assume the condition (*) at the beginning of § 1 and that $g(C) = (C^2) + 1$ or equivalently that D is contracted by the morphism $\Phi_{|C|}$.*

(1) *Suppose $\tilde{\Phi}_*(D) \neq 0$. Then $\overline{W}$ is singular, $W = \Sigma_{N-1}$ and $\tilde{\Phi}_*(D) = M_{N-1}$ or $2M_{N-1}$. Moreover, we have $\mathcal{O}(2K_{\overline{Y}}) \cong \bar{h}^*\mathcal{O}_{\overline{W}}(2)$.*

(2) *Suppose* $\tilde{\Phi}_*(D) = 0$. *Then the following assertions hold.*

(2-1) *Let* B *be the branch locus of* $\tilde{\Phi}: V \to W$. *Then* $B \sim 2F$ *where* $F = H - K_W$, $f^*F \sim R + \tilde{D}$ *and* $h: Y \to W$ *is the double covering defined by a relation* $\mathcal{O}(B) \cong \mathcal{O}(F)^{\otimes 2}$.

(2-2) $g: X \to Y$ *is a minimal resolution.*

(2-3) *We have* $\mathcal{O}(K_Y) \cong h^*\eta^*\mathcal{O}_{\overline{W}}(1)$. *Hence* $\mathcal{O}(K_{\overline{Y}}) \cong \bar{h}^*\mathcal{O}_{\overline{W}}(1)$ *and* $g^*(K_Y) \sim K_X + \tilde{D}$.

Proof. The condition $g(C) = (C^2) + 1$ is equivalent to $(C, D) = 0$. The latter condition is equivalent to a condition that D is contracted by $\Phi_{|C|} = \eta \circ \tilde{\Phi}$. We have also $(\tilde{C}, \tilde{D}) = (C, D) = 0$. Hence $\tilde{D}$ is contracted by $\eta \circ f$ because $\mathcal{O}(\tilde{C}) \cong f^*\eta^*\mathcal{O}_{\overline{W}}(1)$.

(1) Assume $\tilde{\Phi}_*(D) = f_*(\tilde{D}) \neq 0$. Then $\overline{W}$ is singular and $f_*(\tilde{D})$ is a multiple of the minimal section M_{N-1} of W. Since $\tilde{D}$ is a reduced divisor and since $\deg f = 2$, we see that $f_*(\tilde{D}) = M_{N-1}$ or $2M_{N-1}$. Then, as in Lemma 1.1, we can prove that $\Phi_{|C|}^*\Phi_{|C|*}(K_V) \sim 2K_V + \Theta$, $\Phi_{|C|*}(K_V) \in |\mathcal{O}_{\overline{W}}(2)|$ and $2K_{\overline{Y}} \in |\bar{h}^*\mathcal{O}_{\overline{W}}(2)|$.

(2) Assume $f_*(\tilde{D}) = 0$. Then $B \sim 2(H - K_W)$ and $f^*(H - K_W) \sim R + \tilde{D}$. Thus, we can apply [4; Lemma 4]. Let $\tilde{h}: \tilde{Y} \to W$ be the double covering defined by a relation $\mathcal{O}(B) \cong \mathcal{O}(H - K_W)^{\otimes 2}$. Then $\tilde{Y}$ is normal and X is the minimal resolution of $\tilde{Y}$. Hence $\tilde{Y} \cong Y$. We identify $\tilde{Y}$ (resp. $\tilde{h}$) with Y (resp. h). (2-1) and (2-2) are proved.

Note that $K_Y \sim h^*(K_W + (H - K_W)) \in |h^*\eta^*\mathcal{O}_{\overline{W}}(1)|$ and $g^*(K_Y) \sim g^*h^*(H) = f^*(H) \sim \tilde{C} \sim K_X + \tilde{D}$. Since $\eta \circ h$ is factorized by $\bar{h}$, we have $\mathcal{O}(K_{\overline{Y}}) \cong \bar{h}^*\mathcal{O}_{\overline{W}}(1)$. This proves (2-3).

We end this section by proving Theorem A in the Introduction. Suppose D is not contracted to points by $\tilde{\Phi}$. Then $\overline{W}$ is a singular surface. So, $\overline{W}$ is a cone of degree $N - 1$ in $\mathbf{P}^N$ (see the argument at the beginning of § 1). Theorem A, (3) is then proved. The assertions (1), (2) and (4) of Theorem A are proved in Lemmas 1.2 and 1.4.

§ 2. Canonical Resolutions

In the present section, we shall consider those double coverings which appeared in Theorem A, (4). We shall fix the following notations.

Let $\overline{W}$ be a surface of degree $N - 1$ in $\mathbf{P}^N$, which is not contained in any hyperplane. Then $\overline{W}$ is normal and rational. Let $\eta: W \to \overline{W}$ be a minimal resolution. Let H be a general member of $|\eta^*\mathcal{O}_{\overline{W}}(1)|$.

Suppose $\overline{W} \neq \mathbf{P}^2$. Then $W = \Sigma_e$ and we let $\pi: W \to \mathbf{P}^1$ be a $\mathbf{P}^1$-fibration and L a general fiber. We know that either $\overline{W}$ is a cone and $H \sim M_e + eL$ with $e = N - 1$, or $\eta = \mathrm{id}$ and $H \sim M_e + \frac{1}{2}(N + e - 1)L$ with $0 \leq N - e - 3 \equiv 0 \pmod 2$.

Set $F := H - K_W$. Let B be a reduced, effective divisor such that $B \sim 2F$. Set $B_0 := B$, $F_0 := F$ and $W_0 := W$. Suppose P_0 is a singular point of B with multiplicity m_0. Let $\sigma_1: W_1 \to W_0$ be the blowing-up of the point P_0 and set $E_1 := \sigma_1^{-1}(P_0)$, $B_1 := \sigma_1^* B_0 - 2[m_0/2]E_1$ and $F_1 := \sigma_1^* F_0 - [m_0/2]E_1$. Here $[m_0/2]$ is the largest integer satisfying $m_0/2 \geq [m_0/2]$. Then we have again $B_1 \sim 2F_1$. If P_1 is a singular point of B_1 with multiplicity m_1, we let $\sigma_2: W_2 \to W_1$ be the blowing-up of P_1 and set $E_2 := \sigma_2^{-1}(P_1)$, $B_2 := \sigma_2^* B_1 - 2[m_1/2]E_2$ and $F_2 := \sigma_2^* F_1 - [m_1/2]E_2$. Continue this process. Then at the n-th step for some $n \geq 0$, we obtain B_n and F_n such that $B_n \sim 2F_n$ and B_n is nonsingular. (B_n is not necessarily irreducible.) Set $\tilde{B} := B_n$ and $\tilde{F} := F_n$.

Let $\gamma_i: Y_i \to W_i$ $(0 \leq i \leq n)$ be a double covering defined by the relation $\mathcal{O}(B_i) \cong \mathcal{O}(F_i)^{\otimes 2}$ and a nonzero global section corresponding to B_i. Then we have a birational morphism $\tilde{\sigma}_i: Y_i \to Y_{i-1}$ satisfying $\gamma_{i-1} \circ \tilde{\sigma}_i = \sigma_i \circ \gamma_i$. Set $Y := Y_0$, $Y^* := Y_n$, $W^* = W_n$, $\gamma := \gamma_0$, $\tilde{\gamma} := \gamma_n$, $\sigma := \sigma_1 \ldots \sigma_n: W^* \to W$ and $\tilde{\sigma} := \tilde{\sigma}_1 \ldots \tilde{\sigma}_n: Y^* \to Y$. Since $\tilde{B}$ is nonsingular, Y^* is nonsingular and $\tilde{\sigma}$ is a resolution which is not necessarily minimal. According to Horikawa [4; p. 48], we make the following:

Definition 2.1. $\tilde{\sigma}: Y^* \to Y$ is called *the canonical resolution of Y associated with the double covering $\gamma: Y \to W$.*

In the following Lemmas 2.2 and 2.3 and Propositions 2.5 and 2.6, we shall use the above assumptions and notations: $F = H - K_W$, $B \sim 2F$, etc.

Lemma 2.2. *Set* $T := \tilde{\gamma}^*(\sum_{i=0}^{n-1}([m_i/2] - 1)(\sigma_{i+2} \ldots \sigma_n)^* E_{i+1})$. *Then we have*:

(1)

$$K_{Y^*} + T \sim \tilde{\gamma}^* \sigma^* H \in |\tilde{\gamma}^* \sigma^* \eta^* \mathcal{O}_{\overline{W}}(1)|,$$

$$(T^2) = -2 \sum_{i=0}^{n-1} ([m_i/2] - 1)^2, \quad (K_{Y^*}^2) = 2(H^2) + (T^2) = 2(N-1) + (T^2) \quad \text{and}$$

$$\chi(\mathcal{O}_{Y^*}) = N + 2 - \frac{1}{2} \sum_{i=0}^{n-1} [m_i/2]([m_i/2] - 1).$$

(2) *We have* $h^0(Y^*, K_{Y^*} + T) = N + 1 = \frac{1}{2}(K_{Y^*} + T)^2 + 2$. *The morphism* $\Phi_{|K_{Y^*}+T|}$ *is a composite of the morphism* $\eta \circ \sigma \circ \tilde{\gamma}$ *and the closed embedding* $\overline{W} \to \mathbb{P}^N$. *By the abuse of notation, we shall write* $\Phi_{|K_{Y^*}+T|} = \eta \circ \sigma \circ \tilde{\gamma}$.

(3) $K_{Y^*} + T$ *is nef and big.* T *is contracted by the morphism* $\sigma \circ \tilde{\gamma}$: $Y^* \to W$ *and hence* $(K_{Y^*} + T, T) = 0$.

Proof. By the abuse of notation, we denote by the same letter E_i the total transform on W^* of the (-1)-curve E_i. Then $T = \tilde{\gamma}^*(\sum_{i=0}^{n-1}([m_i/2] - 1)E_{i+1})$, and $(E_i, E_j) = 0$ (resp. -1) if $i \neq j$ (resp. if $i = j$). Hence $(T^2) = -2\sum_{i=0}^{n-1}([m_i/2] - 1)^2$. By the construction of the canonical resolution, we have:

$$\tilde{B} = \sigma^*B - 2\sum_{i=0}^{n-1} [m_i/2]E_{i+1}\,,$$

$$\tilde{F} = \sigma^*F - \sum_{i=0}^{n-1} [m_i/2]E_{i+1}\,,$$

$$K_{W^*} \sim \sigma^*K_W + \sum_{i=1}^{n} E_i \qquad \text{and} \qquad K_{Y^*} \sim \tilde{\gamma}^*(K_{W^*} + \tilde{F})\,.$$

Noting that $F = H - K_W$ with $H \in |\eta^*\mathcal{O}_{\overline{W}}(1)|$, we deduce easily the first assertion in (1).

Applying [4; Lemma 6], we obtain $(K_{Y^*}^2) = 2(K_W + F)^2 - 2\sum_{i=0}^{n-1}([m_i/2] - 1)^2$ and $\chi(\mathcal{O}_{Y^*}) = \frac{1}{2}(F, K_W + F) + 2\chi(\mathcal{O}_W) - \frac{1}{2}\sum_{i=0}^{n-1}[m_i/2]([m_i/2] - 1)$. Then the last two assertions in (1) follow from the fact that $F = H - K_W$ and the following:

Claim (1). $(F, H) = 2N$ and $(H^2) = N - 1$.

Since $\overline{W}$ has degree $N - 1$ in $\mathbf{P}^N$, we have $(H^2) = N - 1$. If $N = 2$, then $W = \overline{W} = \mathbf{P}^2$ and the Claim (1) is clear.

Suppose $N \geq 3$ and W is nonsingular. Then $W = \overline{W} = \Sigma_e$, $H \sim M_e + \frac{1}{2}(N + e - 1)L$ and $K_W \sim -2M_e - (e + 2)L$. Hence $(F, H) = (H^2) - (K_W, H) = 2N$.

Suppose $\overline{W}$ is singular. Then $W = \Sigma_{N-1}$ and $H \sim M_{N-1} + (N - 1)L$. We have also $(F, H) = 2N$. The Claim (1) is proved.

Note that $(\sigma^*H - \tilde{F}, \sigma^*H) = (H^2) - (F, H) = -N - 1 < 0$. Hence $H^0(W^*, \sigma^*H - \tilde{F}) = 0$. Note also that $\tilde{\gamma}_*\mathcal{O}_{Y^*} \cong \mathcal{O}_{W^*} \oplus \mathcal{O}(-\tilde{F})$. Using the projection formula, we then obtain $H^0(Y^*, \tilde{\gamma}^*\sigma^*H) \cong H^0(W^*, \sigma^*H) \cong H^0(\overline{W}, \mathcal{O}_{\overline{W}}(1))$. Therefore, $h^0(Y^*, K_{Y^*} + T) = h^0(\overline{W}, \mathcal{O}_{\overline{W}}(1))$ and $\Phi_{|K_{Y^*}+T|} = \Phi_{|\mathcal{O}_{\overline{W}}(1)|} \circ \eta \circ \sigma \circ \tilde{\gamma}$. We can also easily show that $\Phi_{|\mathcal{O}_{\overline{W}}(1)|}$ is a closed embedding of $\overline{W}$ into $\mathbf{P}^N$. Then (2) follows because the fact $(H^2) = N - 1$ implies the second equality in (2).

Since H is nef and big, so is $K_{Y^*} + T$. Note that $\sigma_*\tilde{\gamma}_*T = 2\sigma_*\sum_{i=0}^{n-1}([m_i/2] - 1)E_{i+1} = 0$. Hence $\Phi_{|K_{Y^*}+T|*}T = 0$ and $(K_{Y^*} + T, T) = 0$. The assertion (3) is proved.

We shall give a sufficient condition for $\kappa(Y^*) \geq 0$.

Lemma 2.3. *Assume that* $m_b = 4$ *or* 5 *for some* $b \geq 0$, *and* $m_i = 2$ *or* 3 *for every* $i \neq b$. *Then* $T = \tilde{\gamma}^*(\sigma_{b+2} \cdots \sigma_n)^*E_{b+1}$ *and the following assertions hold.*

(1) *$(T^2) = -2$ and $(K_{Y^*}^2) = 2(N-2) \geq 0$.*

(2) *$p_g(Y^*) = N$, $q(Y^*) = 0$, $\kappa(Y^*) \geq 1$ and $H^1(Y^*, K_{Y^*} + T) = 0$.*

(3) *Suppose $\kappa(Y^*) = 1$. Then $N = 2$, $W = \overline{W} = \mathbb{P}^2$ and $\gamma: Y \to \mathbb{P}^2$ is a double covering ramified over a curve B of degree eight. Moreover, Y^* is a minimal surface and hence a minimal resolution of Y.*

Proof. (1) follows from Lemma 2.2. By the projection formula and by the fact that $p_g(W^*) = 0$ and $F = H - K_W$, we have:

$$\begin{aligned} H^0(Y^*, K_{Y^*}) &\cong H^0(W^*, K_{W^*} + \tilde{F}) + H^0(W^*, K_{W^*}) \\ &= H^0(W^*, \sigma^* H - (\sigma_{b+2} \ldots \sigma_n)^* E_{b+1}) \\ &\cong H^0(W_{b+1}, (\sigma_1 \ldots \sigma_{b+1})^* H - E_{b+1}) \,. \end{aligned}$$

Since $|(\sigma_1 \ldots \sigma_b)^* H|$ is base point free, we have $p_g(Y^*) = h^0(W_b, (\sigma_1 \ldots \sigma_b)^* H) - 1 = h^0(\overline{W}, \mathcal{O}_{\overline{W}}(1)) - 1 = N$ (see the proof of Lemma 2.2). By Lemma 2.2, we have $\chi(\mathcal{O}_{Y^*}) = N + 1$. Hence $q(Y^*) = 0$. Since $p_g(Y^*) = N \geq 2$, we have $\kappa(Y^*) \geq 1$. The assertion (3) follows from the result $(K_{Y^*}^2) = 2(N-2) \geq 0$.

In order to finish the proof of (2), it remains to verify $H^1(Y^*, K_{Y^*} + T) = 0$. Let A be a general member of $|K_{Y^*} + T|$. By Lemma 2.2, (2), $\Phi_{|A|}$ is a morphism onto a surface. Hence A is a nonsingular irreducible curve with $g(A) = 1 + \frac{1}{2}(A, A + K_{Y^*}) = 1 + (A^2)$ because $(A, T) = 0$ by Lemma 2.2, (3). Note that $h^0(Y^*, A) = N + 1 = 2 + \frac{1}{2}(A^2)$ by Lemma 2.2. Using the fact $q(Y^*) = 0$ and considering the cohomologies of the following exact sequence

$$0 \to \mathcal{O}_{Y^*} \to \mathcal{O}_{Y^*}(A) \to \mathcal{O}_A(A) \to 0 \,, \tag{2.3}$$

we can obtain $h^0(Y^*, A) = 1 + h^0(A, A_{|A})$ and the following:

$$h^1(Y^*, A) + \chi(\mathcal{O}_{Y^*}) = h^0(Y^*, K_{Y^*} - A) + g(A) + 1 - \frac{1}{2}(A^2) \,.$$

Since $h^0(Y^*, K_{Y^*} - A) = h^0(Y^*, -T) = 0$ and since $\chi(\mathcal{O}_{Y^*}) = 1 + p_g(Y^*) = 2 + \frac{1}{2}(A^2)$, we obtain $H^1(Y^*, A) = 0$. This proves (2).

Definition 2.4. Let T be a reduced, effective divisor on a nonsingular surface W and let P be a singular point of T with multiplicity four. Let $\sigma: W' \to W$ be the blowing-up of the point P and set $E := \sigma^{-1}(P)$. Then P is said to be *a simple quadruple point of* T if the following two conditions are satisfied.

(1) Each point Q of $E \cap \sigma'(T)$ is a smooth or double point of $\sigma'(T)$.

(2) If $Q \in E \cap \sigma'(T)$ is a double point of $\sigma'(T)$, then either at least one tangent of $\sigma'(T)$ at Q is different from that of E or the point Q of $\sigma'(T)$ is an ordinary cusp with E as its tangent.

We shall consider when pairs (Y^*, T) treated in Lemma 2.3 satisfy the hypotheses of Theorem A. In view of Lemma 2.2, it is equivalent to asking when T is a reduced, effective divisor with only simple normal crossings. A sufficient condition will be found in the following Proposition 2.5. More precisely, we shall give a relationship between the configuration of T and that of B_b near a quadruple point.

Proposition 2.5. *Assume that* $m_b = 4$ *or* 5 *for some* $b \geq 0$ *and* $m_i = 2$ *or* 3 *for every* $i \neq b$. *Set* $T := \tilde{\gamma}^*(\sigma_{b+2} \cdots \sigma_n)^* E_{b+1}$. *Then we have*:

(1) *T is a reduced divisor if and only if P_b is a simple quadruple point of B_b. If this is the case, then $B_{b+1} = \sigma'_{b+1} B_b$ and T is connected.*

(2) *Suppose R is reduced. Then all possible configurations of $E_{b+1} + B_{b+1}$ near the intersection points $E_{b+1} \cap B_{b+1}$ and the corresponding configurations of T are given in the Table 1 below.*

In particular, T has only simple normal crossings if and only if T is one of the types I_2, $I_{3,3}$, $I_{4,2^t}$, $I_{3,3,2^t}$ and $I_{3,2^c,3,2^d}$. If this is the case, then T is an elliptic curve or a loop of nonsingular rational curves.

We shall use the following notations in the Table 1. Set $\Gamma := \tilde{\gamma}^*(\sigma_{b+2} \cdots \sigma_n)' E_{b+1}$ and $\Delta := T - \Gamma$. We give a decomposition into irreducible components: $\Gamma = \sum_{i=1}^{s} \Gamma_i$, $\Delta = \sum_{j=1}^{t} \Delta_j$. Then s $= 1$ or 2, and Δ is void or consists of one or two chains of (-2)-curves. More precisely, one of the following five cases occurs:

Case (2-1). $T = \Gamma$, T is an irreducible curve with $p_a(T) = 1$ and $(T^2) = -2$, and T is one of the types I_2, I'_2 and I''_2.

Case (2-2). $T = \Gamma$, T consists of exactly two (-3)-curves Γ_1 and Γ_2 with $(\Gamma_1, \Gamma_2) = 2$, and T is one of the types $I_{3,3}$ and $I'_{3,3}$.

Case (2-3). Γ is a (-4)-curve, Δ is a chain and T is one of the types $I_{4,2^t}$, $I'_{4,2}$ $(t = 1)$ and $I'_{4,2^2}$ $(t = 2)$.

Case (2-4). Γ is a chain with two (-3)-curves, Δ is a chain and T is one of the types $I_{3,3,2^t}$ and $I'_{3,3,2}$ $(t = 1)$.

Case (2-5). Γ consists of two disjoint (-3)-curves, Δ consists of two disjoint chains and T is of type $I_{3,2^c,3,2^d}$ with $t = c + d$.

In the Table 1, though B_{b+1} is drawn as if it is reducible, it might be irreducible. The intersections of irreducible components of T and the intersection of E_{b+1} with B_{b+1} are easily obtained by the above description, the configurations in the Table 1 and the fact that $(K_{Y^*} + T, T) = 0$, $(T^2) = -2$ and $(E_{b+1}, B_{b+1}) = 4$. A component of $E_{b+1} + B_{b+1}$, marked by a symbol $*$ on it, is the (-1)-curve E_{b+1}. By a bracketed number (α), between two local irreducible components of $E_{b+1} + B_{b+1}$ at a point P, we mean that two local components meet each other with order of contact α at the point P. By a pair $(2, \alpha)$ of integers, which is written over a cusp Q of a component of B_{b+1}, we mean that the cusp Q is of type $(2, \alpha)$. Self intersection numbers of components of T are also given.

Table 1

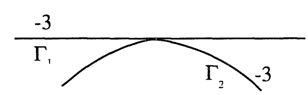
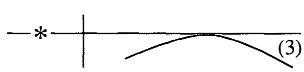
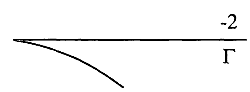
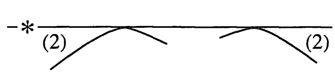
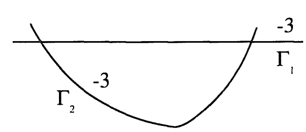
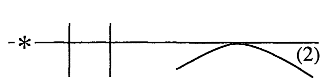
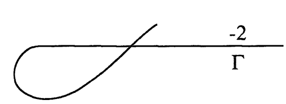
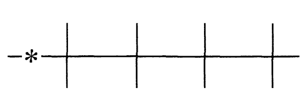
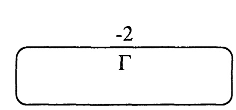
No	$E_{b+1}+B_{b+1}$	T	Type of T
1			$I'_{3,3}$
2			I''_2
3			$I_{3,3}$
4			I'_2
5			I_2

Table 1

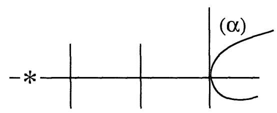
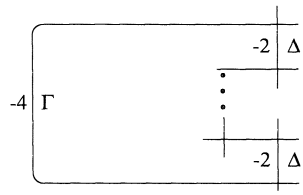
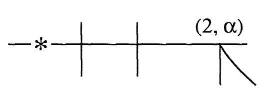
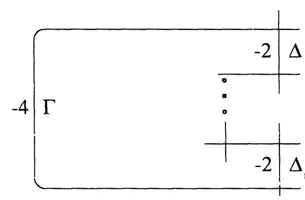
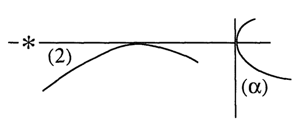
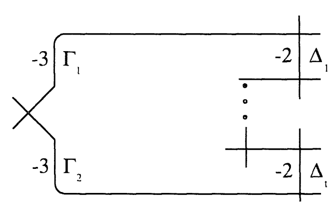
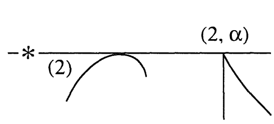
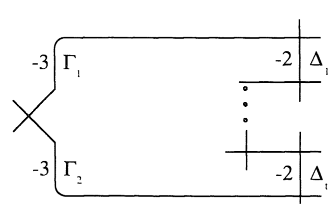
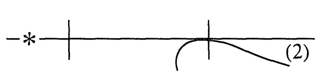
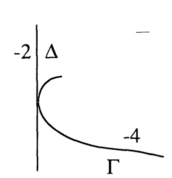
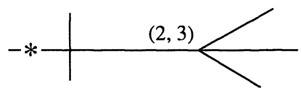
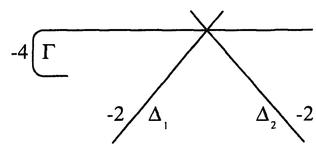
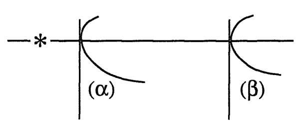
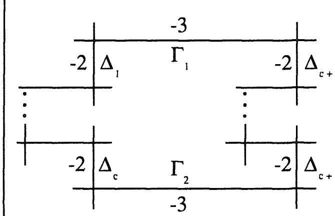
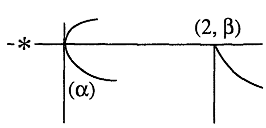
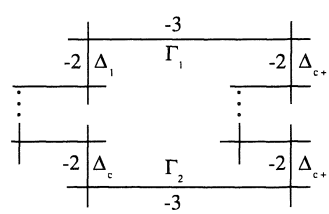
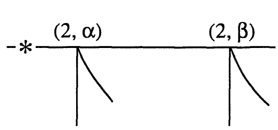
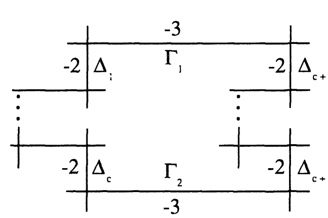
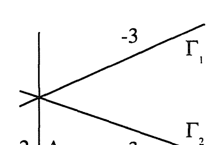
No	$E_{b+1} + B_{b+1}$	T	Type of T
6			$I_{4,2^t}$ $(t = 2\alpha - 1 \geq 1)$
7			$I_{4,2^t}$ $(t = \alpha - 1 \geq 2)$
8			$I_{3,3,2^t}$ $(t = 2\alpha - 1 \geq 1)$
9			$I_{3,3,2^t}$ $(t = \alpha - 1 \geq 2)$
10			$I'_{4,2}$

Table 1

No	$E_{b+1} + B_{b+1}$	T	Type of T
11			$I'_{4,2^2}$
12			$I_{3,2^c,2^d}$ (c = 2α - 1 ≥ 1) (d = 2β - 1 ≥ 1)
13			$I_{3,2^c,3,2^d}$ (c = 2α - 1 ≥ 1) (d = β - 1 ≥ 2)
14			$I_{3,2^c,3,2^d}$ (c = α - 1 ≥ 2) (d = β - 1 ≥ 2)
15			$I'_{3,3,2}$

Proof of Proposition 2.5. By Lemma 2.3, (1) and by Lemma 2.2, (3), we see that $(T^2) = -2$ and $(K_{Y^*} + T, T_i) = 0$ for every component T_i of T. Note that T is reduced if and only if $(\sigma_{b+2} \dots \sigma_n)^* E_{b+1}$ is reduced and contains no components of $\tilde{B}$. If $m_b = 5$, then $B_{b+1} = \sigma_{b+1}^* B_b - 2[m_b/2] E_{b+1} = \sigma'_{b+1} B_b + E_{b+1}$, $\tilde{B} \geq (\sigma_{b+2} \dots \sigma_n)' B_{b+1} \geq (\sigma_{b+2} \dots \sigma_n)' E_{b+1}$ and $(\sigma_{b+2} \dots \sigma_n)^* E_{b+1} \geq (\sigma_{b+2} \dots \sigma_n)' E_{b+1}$. This implies that T is not reduced. So, in order to prove Proposition 2.5, we may (and shall) assume $m_b = 4$. Then $B_{b+1} = \sigma'_{b+1} B_b$, $(E_{b+1}, B_{b+1}) = 4$ and $(\sigma_{b+2} \dots \sigma_n)' E_{b+1}$ is not contained in $\tilde{B}$ by the definition.

Claim (1). $(\sigma_{b+2} \dots \sigma_n)^* E_{b+1}$ has no common components with $\tilde{B}$ if and only if each point of $E_{b+1} \cap B_{b+1}$ is a smooth or double point of B_{b+1}.

Suppose $Q \in E_{b+1} \cap B_{b+1}$ is a singular point of B_{b+1} with multiplicity ≥ 3. Changing the order of blowing-ups $\sigma_i's$ $(i \geq b + 2)$ if necessary, we may assume that $Q = P_{b+1}$. Then the multiplicity of B_{b+1} at P_{b+1} is equal to m_{b+1} by the definition of $m_i's$. Hence $m_{b+1} = 3$ by the assumptions. Then $B_{b+2} = \sigma'_{b+2} B_{b+1} + E_{b+2}$, where $E_{b+2} = \sigma_{b+2}^{-1}(P_{b+1}) \leq \sigma_{b+2}^* E_{b+1}$. So, $\tilde{B} \geq (\sigma_{b+3} \dots \sigma_n)' E_{b+2}$ and $(\sigma_{b+2} \dots \sigma_n)^* E_{b+1} \geq (\sigma_{b+3} \dots \sigma_n)' E_{b+2}$, which implies that there is a common component in $\tilde{B}$ and $(\sigma_{b+2} \dots \sigma_n)^* E_{b+1}$.

Suppose $P_{b+1} \in E_{b+1} \cap B_{b+1}$ is a double point of B_{b+1}. Then B_{b+2} is equal to $\sigma'_{b+2} B_{b+1}$ and does not contain the (-1)-curve $E_{b+2} = \sigma_{b+2}^{-1}(P_{b+1})$. Moreover, each point of $E_{b+2} \cap B_{b+2}$ is a smooth or double point of B_{b+2}.

The Claim (1) follows easily from the above arguments. By making use of the Claim (1), we can prove Proposition 2.5.

Remark. Actually, after changing the order of blowing-ups $\sigma_i's$ in the process of the canonical resolution if necessary, we may assume that either $b = 0$, $m_0 = 4$ and $m_i = 2, 3$ $(i \geq 1)$, or $b = 1$, $P_0 = \sigma_1(P_1)$, $m_0 = 3$, $m_1 = 4$ and $m_i = 2, 3$ $(i \geq 2)$.

If the second case above occurs, then $E_2 + B_2 (= E_2 + \sigma'_2 E_1 + \sigma'_2 \sigma'_1 B)$ is given in one of the rows No. 2, No. 4, No. 5, No. 6, No. 7, No. 10 and No. 11 in the Table 1.

The following Proposition 2.6 concerns the existence of surfaces treated in Theorem D, (3), which is stated in the Introduction. We retain the hypotheses and notations before Definition 2.1. We assume furthermore that $N = 2$ and $W = \overline{W} = \mathbf{P}^2$. In this case, $F \in |\mathcal{O}_{\mathbf{P}^2}(4)|$, B is a reduced curve of degree eight in $\mathbf{P}^2$, $\gamma\colon Y \to \mathbf{P}^2$ is a double covering defined by a relation $\mathcal{O}(B) \cong \mathcal{O}(F)^{\otimes 2}$, and $\tilde{\sigma}\colon Y^* \to Y$ is the canonical resolution associated with the double covering γ.

Proposition 2.6. *Assume that* $N = 2$, $W = \overline{W} = \mathbf{P}^2$, $m_b = 4$ *for some* $b \geq 0$ *and* $m_i = 2$ *or* 3 *for every* $i \neq b$ *and assume* P_b *is a simple quadruple point of*

B_b. *Assume further that* $T = \tilde{\gamma}^*(\sigma_{b+2} \dots \sigma_n)^* E_{b+1}$ *has simple normal crossings and* $e(T) \le 7$. *Then either* (Y^*, T) *fits the Case* (3-1) *of Theorem* D, *or* Y^* *contains only one* (-1)*-curve* E *and* $(X, \delta(T))$ *fits the Case* (3-2) *of Theorem* D *where* δ: $Y^* \to X$ *is the blowing-down of* E *to a nonsingular point of* $\delta(T)$.

Proof. By Lemma 2.2, the pair (Y^*, T) satisfies that $\bar{p}_g = \frac{1}{2}(K_{Y^*} + T)^2 + 2 = 3$, $K_{Y^*} + T$ is a nef and big divisor and $\Phi_{|K_{Y^*}+T|}$: $Y^* \to \mathbb{P}^2$ is a surjective morphism of degree two. Moreover, $c_2(Y^*) = 12\chi(\mathcal{O}_{Y^*}) - (K_{Y^*}^2) = 12(1 - 0 + 2) - 0$ (cf. Lemma 2.3).

If $\kappa(Y^*) = 1$, then Y^* is a minimal surface (cf. Lemma 2.3) and hence (Y^*, T) is a minimal log surface of general type. Thus, (Y^*, T) satisfies all the hypotheses in Theorem D and fits the case (3-1) there. Indeed, we have $(\bar{c}_1^2) = \frac{1}{9}\bar{c}_2 - 2 + \frac{\alpha - 6}{9}$ where $\alpha = 6 + e(T)$.

Suppose $\kappa(Y^*) \ne 1$. Then $\kappa(Y^*) = 2$ (cf. Lemma 2.3). Let δ: $Y^* \to X$ be a birational morphism onto a nonsingular minimal model. Then we have:

(2.6) $$1 \le (\delta^* K_X^2) \le (\delta^* K_X, K_{Y^*} + T) - 1 \le (K_{Y^*} + T)^2 - 1 = 1$$

because $K_{Y^*} + T$ is 1-connected and $K_{Y^*} + T - \delta^* K_X \ge T > 0$. Thus, every inequality in (2.6) becomes an equality. So, $(K_X^2) = 1$ and δ is the blowing-down of a single (-1)-curve E on Y^* because $(K_{Y^*}^2) = 0$. Moreover, $0 = (K_{Y^*} + T - \delta^* K_X, K_{Y^*} + T) = (E, K_{Y^*} + T)$ (cf. Lemma 2.2, (3)). Hence $(E, T) = 1$ and $Q := \delta(E)$ is a nonsingular point of $\delta(T)$ (cf. Table 1). Furthermore, $\delta^*(K_X + \delta(T)) \sim K_{Y^*} + T$ and $\Phi_{|K_{Y^*}+T|} = \Phi_{|K_X+\delta(T)|} \circ \delta$. Thus, $(X, \delta(T))$ satisfies all the hypotheses in Theorem D and fits the Case (3-2) there. Indeed, we have $c_2(X) (= c_2(Y^*) - 1) = 35$ and $(\bar{c}_1^2) = \frac{1}{9}\bar{c}_2 - 2 + \frac{\alpha - 6}{9}$ where $\alpha = 7 + e(\delta(T))$.

Remark. (1) For every $1 \le n \le 15$, we can actually construct a pair (Y^*, T) (or equivalently a reduced curve B of degree eight in $\mathbb{P}^2$) such that T is given in n-th row of the Table 1.

(2) The second case in Proposition 2.6 occurs if and only if the second case in the Remark to Proposition 2.5 occurs. If this is the case, then $\bar{E} := \tilde{\gamma}(E)$ is the proper transform on $\tilde{W}$ of the (-2)-curve $\sigma_2' E_1$ on W_2. Hence $\bar{E}$ is also a (-2)-curve and contained in the branch locus $\tilde{B}$ of the double covering $\tilde{\gamma}$: $Y^* \to W^*$.

We shall apply the above arguments on (Y^*, T) to the log surfaces in Theorem A. For these surfaces, the morphism $\Phi_{|K_V+D|}$ is a composite of a morphism $\tilde{\Phi}$: $V \to W$ and the resolution η: $W \to \overline{W}$. We shall assume furthermore that $\tilde{\Phi}_* D = 0$ and (V, D) is a minimal log surface of general type. Then, with notations of Lemma 1.4, $V = X$ and $g: V \to Y$ is a minimal resolution. We shall also use the notations set before Definition 2.1. Let $\tilde{\sigma}: Y^* \to Y$ be the canonical resolution of Y associated with the double covering $\gamma := h: Y \to W$, the latter being defined in Theorem A. Then there is a birational morphism $\delta: Y^* \to V$ satisfying $\tilde{\Phi} \circ \delta = \sigma \circ \tilde{\gamma} = \gamma \circ \tilde{\sigma}$.

Lemma 2.7. *Let (V, D) be a minimal log surface of general type satisfying the hypotheses of Theorem* A *and the condition $\tilde{\Phi}_* D = 0$. With the above notations and the notations in Lemma* 2.2, *we have $\delta^* D = R_\delta + T$, where R_δ is the ramification divisor of δ. In particular, $D = \delta_* T$ and $(T^2) \le (D^2) \le 0$.*

Proof. Note that $K_V + D \sim \tilde{\Phi}^* H$ with $H \in |\eta^* \mathcal{O}_{\overline{W}}(1)|$. Hence $\delta^*(K_V + D) \sim \tilde{\gamma}^* \sigma^* H \sim K_{Y^*} + T \sim \delta^* K_V + R_\delta + T$ (cf. Lemma 2.2). So, $\delta^* D \sim R_\delta + T$. Thus, two divisors are equal because D and hence $\delta^* D$ have negative definite intersection matrix by the condition $\tilde{\Phi}_* D = 0$.

We end this section with the following result which will imply Theorem B in the Introduction.

Lemma 2.8. *Assume the hypotheses of Theorem* B. *Then we have*:

(1) *We have $q(V) = 0$, $\bar{p}_g = p_g(V) + 1$ and $(K_V^2) \ge (K_{Y^*}^2) = 2(\bar{p}_g - 3) \ge 0$. Moreover, we have $m_b = 4$ or 5 for some $b \ge 0$ and $m_i = 2$ or 3 for every $i \ne b$.*

(2) *Suppose $V = Y^*$, i.e., $\delta = \mathrm{id}$ and $D = T$. Then P_b is a simple quadruple point of B_b. Moreover, the divisor D has one of the types I_2, $I_{3,3}$, $I_{4,2^t}$, $I_{3,3,2^t}$ and $I_{3,2^c,3,2^d}$ given in the Table* 1 *of Proposition* 2.5. *In particular, $(D^2) = -2$.*

Proof. Let C be a general member of $|K_V + D|$. Then we have $g(C) = (C^2) + 1$. By Theorem TZ, we have $1 + p_g(V) \le h^1(V, K_V + D) + \chi(\mathcal{O}_V) = h^0(V, -D) + g(C) + 1 - \frac{1}{2}(C^2) = 2 + \frac{1}{2}(C^2) = \bar{p}_g$. The condition $H^1(V, K_V + D) = 0$ implies that $q(V) = 0$ and $1 + p_g(V) = \chi(\mathcal{O}_V) = \bar{p}_g$. We shall apply Lemma 2.2. In the present case, we have $N = \bar{p}_g - 1$. Then we obtain $\chi(\mathcal{O}_V) = \chi(\mathcal{O}_{Y^*}) = \bar{p}_g + 1 - \frac{1}{2}\sum_{i=0}^{n-1} [m_i/2]([m_i/2] - 1)$. Hence $\sum_{i=0}^{n-1} [m_i/2]([m_i/2] - 1) = 2$. So, $m_b = 4$ or 5 for some $b \ge 0$ and $m_i = 2$ or 3 for every $i \ne b$. Since V is a minimal resolution of Y, we have $(K_V^2) \ge (K_{Y^*}^2) = 2(\bar{p}_g - 3) \ge 0$ by Lemma 2.3, (1).

Suppose $V = Y^*$. Then $D = T$ and T is a reduced, effective divisor with only simple normal crossings. Then (2) follows from Proposition 2.5.

Remark. If $\kappa(V) = 1$, then $V = Y^*$, $\bar{p}_g = 3$ and V is a minimal surface. Indeed, $\kappa(V) = 1$ implies that $(K_V^2) \le 0$ and hence $(K_V^2) = (K_{Y^*}^2) = 2(\bar{p}_g - 3) = 0$ by the proof above. Then follows the assertion.

§ 3. The Inequality $(\bar{c}_1^2) \geq \frac{1}{9}\bar{c}_2 - 2$

In the present section, we shall consider minimal log surfaces (V, D) of general type. Sakai [8; Theorem 7.6] proved:

Theorem 3.1. *Let (V, D) be a minimal log surface of general type. Then the inequality $(\bar{c}_1^2) \leq 3\bar{c}_2$ holds true.*

We are going to find an inequality of the form $(\bar{c}_1^2) \geq \gamma\bar{c}_2 + \delta$ for two rational numbers γ (>0) and δ. This, together with the inequality in Theorem 3.1, gives an effective restriction on the region of log surfaces (V, D) of general type to exist.

In the proof of Theorem C, we shall use the following notations. Let $\varepsilon(-1)$ be the largest reduced, effective divisor whose support is a union of connected components of D satisfying either one of the following two conditions:

(1) Δ_i is an elliptic curve with $(\Delta_i^2) = -1$.

(2) Δ_i is a loop of nonsingular rational curves Δ_{ij}'s with $(\Delta_{i1}^2) = -3$ and $(\Delta_{ij}^2) = -2$ for every $j \geq 2$.

Now we can prove Theorem C by making use of Sakai [8; Theorem 5.5]. Note that $(\bar{c}_1^2) - (K_V^2) = (D, 2K_V + D)$. Note also that $\bar{c}_2 - c_2(V) = -e(D) = r - \sum_{i=1}^n e(D_i) = r + \sum_{i=1}^n (2g(D_i) - 2) = (D, K_V + D) - r$. The assumption that $D \neq 0$ implies that $H^2(V, K_V + D) \cong H^0(V, -D) = 0$. Then, by the Riemann-Roch theorem, we have:

$$\bar{p}_g = h^1(V, K_V + D) + \frac{1}{2}(D, K_V + D) + \frac{1}{12}\{(K_V^2) + c_2(V)\}$$

$$= h^1(V, K_V + D) + \frac{1}{12}\{(K_V^2) + \bar{c}_2 + r + 5(D, K_V + D)\}$$

$$= \frac{1}{12}\bar{c}_2 - \frac{1}{4}(\bar{c}_1^2) + h^1(V, K_V + D) + \frac{1}{12}\{r + 8(D, K_V + D) + (K_V, 4K_V + 3D)\}.$$

In order to finish the proof of Theorem C, we have only to show that $(K_V, 4K_V + 3D) \geq 0$. Sakai [8; Theorem 5.5] proved that $|4K_V + 3D| \neq \phi$ and $\mathrm{Bs}|4K_V + 3D| \subseteq \mathrm{Supp}\,\varepsilon(-1)$. Writing $|4K_V + 3D| = |L_4| + G_4$, where $|L_4|$ and G_4 are respectively the movable part and the fixed part, we have $\mathrm{Supp}\,G_4 \subseteq \varepsilon(-1)$. By the definition of $\varepsilon(-1)$, we have $(K_V, E) = 0$ or 1 for every component E of $\varepsilon(-1)$. Thus, we obtain $(K_V, G_4) \geq 0$ and $(K_V, 4K_V + 3D) = (K_V, L_4) + (K_V, G_4) \geq 0$ because $\kappa(V) \geq 0$ by the assumption. Theorem C is proved.

We now consider the case where $|K_V + D|$ is not composed with a pencil. Our goal is Theorem D. We shall use the following notations.

Write $|K_V + D| = |C| + G$ with the movable part $|C|$ and the fixed part G. We choose C to be a sufficiently general member of $|C|$. If x_1 is a base point of $|C|$, let m_1 be the multiplicity of the curve C at x_1 and let $f_1\colon V_1 \to V$ be the blowing-up of the point x_1. If x_2 is a base point of $|f_1'(C)|$, let m_2 be the multiplicity of the curve $f_1'(C)$ at x_2 and let $f_2\colon V_2 \to V_1$ be the blowing-up of the point x_2. Continue this process. Then at the b-th step for some $b \geq 0$, the linear system $|f'(C)|$ is base point free. Here we set $V' = V_b$ and $f = f_1 \dots f_b\colon V' \to V$. Then $\bar{p}_g = h^0(V', f'(C))$ and $(f'(C))^2 = (C^2) - \sum_{i=1}^{b} m_i^2$. Since general members of $|f'(C)|$ are nonsingular, the geometric genus $g(C)$ is equal to $p_a(f'C)$. Note that $\Phi_{|f'(C)|}\colon V' \to \mathbf{P}^N$ $(N := \bar{p}_g - 1)$ is a morphism onto a surface $\overline{W}$.

In Lemmas 3.6 and 3.7 below, we shall use the following analogue of Beauville [1; Lemma 5.5] and its Remark and Corollary.

Proposition 3.4. *Let (V, D) be a log surface of general type such that $K_V + D$ is nef and $\Phi_{|K_V+D|}$ is a rational map onto a surface $\overline{W}$ in $\mathbf{P}^N$ $(N := \bar{p}_g - 1)$. Let H be a general hyperplane of $\overline{W}$ and set $h = h^0(H, \mathcal{O}_H(H))$, $\tilde{h} = h^0(f'C, \mathcal{O}_{f'C}(f'C))$, $g = g(H)$ (the geometric genus) and $d = \deg \overline{W} = (H^2)$. Then $\bar{p}_g \leq h + 1 \leq \tilde{h} + 1$ and $d \cdot \deg \Phi_{|K_V+D|} = (f'C)^2 \leq (C^2) \leq (K_V + D)^2$.*

Suppose furthermore $\bar{p}_g \geq \frac{1}{3}(7 + d \cdot \deg \Phi_{|K_V+D|})$. Then one of the following cases occurs.

(1) $\deg \Phi_{|K_V+D|} = 2$ *and $\overline{W}$ is birational to a ruled surface.*

(2) $\deg \Phi_{|K_V+D|} = 1$, $d = g - 1 \geq 5$ *and* $\bar{p}_g = h + 1 = \tilde{h} + 1 = \frac{1}{3}(7 + d)$. *Moreover, $\mathcal{O}_{f'C}(2f'C)$ is isomorphic to the canonical line bundle of $f'C$.*

(3) $\deg \Phi_{|K_V+D|} = 1$, $g \leq d \leq 2g - 2$ *and* $\tilde{h} \leq \frac{1}{2}d + 1$.

(4) $\deg \Phi_{|K_V+D|} = 1$, $d \geq 2g - 1$ *and* $\tilde{h} = d + 1 - g$. *Moreover,* $\kappa(V) = -\infty$.

Proof. Set $Z := f'(C)$. With the above notations, $\Phi_{|Z|}\colon V' \to \mathbf{P}^N$ $(N = \bar{p}_g - 1)$ is a morphism onto a surface $\overline{W}$. Hence $d \cdot \deg \Phi_{|Z|} = (Z)^2 \leq (C^2) \leq (C, K_V + D) \leq (K_V + D)^2$. Considering the following exact sequence:

$$0 \to \mathcal{O}_{\overline{W}} \to \mathcal{O}_{\overline{W}}(H) \to \mathcal{O}_H(H) \to 0\,,$$

we obtain $\bar{p}_g = h^0(V', Z) = h^0(\overline{W}, H) \leq 1 + h$. Let φ be the restriction map of the morphism $\Phi_{|Z|}$ to a general member $f'(C)$ of $|f'(C)|$. Then $H = \varphi(Z)$ is a general member of $|H|$. Note that $\varphi^*\mathcal{O}_H(H) \cong \mathcal{O}_Z(Z)$ and hence $h \leq \tilde{h}$.

Assume further $\bar{p}_g \geq \frac{1}{3}(7 + d \cdot \deg \Phi_{|Z|})$. Hence $3\bar{p}_g - 7 \geq d \cdot \deg \Phi_{|Z|} \geq (\bar{p}_g - 2) \deg \Phi_{|Z|}$ because $d \geq N - 1$. So, $\deg \Phi_{|Z|} = 1$ or 2. In the case

$\deg \Phi_{|Z|} = 2$, we assert that $\overline{W}$ is birational to a ruled surface. Indeed, if the assertion is false, then $d = \deg \overline{W} \geq 2(N-1)$ (cf. Beauville [1; Lemma 1.4]) and hence $3\bar{p}_g - 7 \geq 4(\bar{p}_g - 2)$. This is a contradiction.

We now consider the case where $\deg \Phi_{|Z|} = 1$. Then $\varphi: f'(C) \to H$ is a birational morphism. So, we have $g(C) = g(H) = g$, $(Z)^2 = (H^2) = d$ and $g(Z) - 1 - \deg Z_{|Z} = g - 1 - d$. We have the following three cases.

Case $1 \leq d \leq g-1$. We shall show that the Case (2) occurs. Applying Beauville [1, Lemma 5.1], we obtain $\tilde{h} \leq \frac{1}{3}(4 + \deg Z_{|Z}) = \frac{1}{3}(4+d)$. So, $\bar{p}_g \leq h + 1 \leq \tilde{h} + 1 \leq \frac{1}{3}(7+d)$. By the assumption, we must have $\bar{p}_g = h + 1 = \tilde{h} + 1$ and $\tilde{h} = \frac{1}{3}(4+d)$. By Beauville [1; Remark 5.2], we obtain $d = g-1$ and $\mathcal{O}_Z(2Z) \cong \omega_Z$.

Note that $\Phi_{|H|}$ and hence $\Phi_{|\mathcal{O}_H(H)|}$ gives rise to an isomorphism of H onto its image. So, if $h = 2$ then $H \cong \mathbb{P}^1$ and $d = 1$. This contradicts the equality $h = \frac{1}{3}(4+d)$. Therefore, we must have $h \geq 3$ and $d \geq 5$. Thus, all the assertions in the Case (2) are verified.

Case $g \leq d \leq 2g-2$. Applying the Clifford index theorem (cf. Martens [6; §2.31]), we have $\tilde{h} \leq \frac{1}{2}d + 1$. Hence the Case (3) occurs.

Case $d \geq 2g-1$, i.e., $(K_{V'}, Z) \leq -1$ in view of the genus formula. Then $\kappa(V) = -\infty$. Applying the Riemann-Roch theorem, we obtain $\tilde{h} = d + 1 - g$. Hence the Case (4) occurs.

Remark. (1) Suppose $\deg \Phi_{|K_V+D|} = 1$. Then we have $2(g-1-d) = 2(p_a(Z) - 1 - (Z)^2) = (Z, K_{V'} - Z) = (Z, f^*K_V + \sum_{i=1}^b E_i - f^*C + \sum_{i=1}^b m_i E_i) = (C, K_V - C) + \sum_{i=1}^b m_i(m_i+1) \geq (C, K_V - C)$. Here $E_i := (f_{i+1} \cdots f_b)^* f_i^{-1}(x_i)$ is the total transform on V' of the (-1)-curve $f^{-1}(x_i)$ on V_i.

(2) If the Case (3) or (4) occurs, then $g - 1 - d \leq -1$ and hence $(C, K_V - C) \leq -2$. Indeed, in the Case (4), we have $g - 1 - d \leq \mathrm{Min}\,\{g-2, -g\}$.

Corollary 3.5. *Let (V, D) be a log surface of general type such that $K_V + D$ is nef, $\kappa(V) \geq 0$, $\dim \Phi_{|K_V+D|}(V) = 2$, $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + 2$ and $(D, K_V + D) = 0$. Then $\Phi_{|K_V+D|}$ is a morphism of degree two onto a normal, rational surface of degree $N-1$ $(N := \bar{p}_g - 1)$ in $\mathbb{P}^N$.*

Proof. In view of Theorem TZ in the Introduction, we have only to show that $\deg \Phi_{|K_V+D|} = 2$. Note that $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + 2 \geq \frac{1}{3}(7 + \bar{c}_1^2) \geq \frac{1}{3}(7 + d \cdot \deg \Phi_{|K_V+D|})$. So, the hypotheses of Proposition 3.4 are satisfied. Hence the Case (1), (2),

(3) or (4) in Proposition 3.4 occurs. Since $C \sim K_V + D$ and $(C, K_V - C) = (K_V + D, -D) = 0$, the Case (3) or (4) is impossible by the Remark, (2) above. If the Case (2) occurs, then we have $\frac{1}{2}(\bar{c}_1^2) + 2 = \bar{p}_g = \frac{1}{3}(7 + d) \leq \frac{1}{3}(7 + \bar{c}_1^2)$. So, $(\bar{c}_1^2) = 2$, $\bar{p}_g = 3$ and $\Phi_{|K_V+D|}: V \to \mathbf{P}^2$ is a birational morphism. This contradicts the assumption $\kappa(V) \geq 0$. Thus, the Case (1) occurs and $\deg \Phi_{|K_V+D|} = 2$.

Theorem D in the Introduction will consist of the subsequent two lemmas. We consider first the case where $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + \frac{3}{2}$. Set $N = \bar{p}_g - 1$.

Lemma 3.6. *Assume the same hypotheses as in Theorem* D *and assume that* $\bar{p}_g \leq \frac{1}{2}(\bar{c}_1^2) + \frac{3}{2}$. *Assume further that* $\bar{p}_g = \frac{1}{12}\bar{c}_2 - \frac{1}{4}(\bar{c}_1^2) + \frac{\beta}{12}$ *with an integer* $\beta \leq 11$. *Then the following three assertions hold.*

(1) $H^1(V, K_V + D) = 0$, $(K_V^2) = \frac{1}{4}(\beta - e(D) + 3D^2)$ *and every connected component of* D *is an elliptic curve or a loop of* $\mathbf{P}^1$*'s.*

(2) $\beta \geq \beta - e(D) \geq 2\left(\bar{p}_g - \frac{3}{2}\right) \geq 3.$

(3) *Suppose* $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + \frac{3}{2}$. *Then* D *is connected and* C, *replaced by a general member of* $|C|$ *if necessary, is a nonsingular irreducible curve. Moreover, one of the following cases takes place*:

(3a) $\operatorname{Bs}|C| = \phi$, $G = D = \varepsilon(-1)$ *and* V *is a minimal surface of general type satisfying* $q(V) = 1$ *and* $p_g(V) = \frac{1}{2}(K_V^2) + 2$. *Moreover,* $\Phi_{|K_V|}$ *is a morphism of degree two onto a normal, rational surface of degree* $N - 1$ *in* $\mathbf{P}^N$.

(3b) *We have* $q(V) = 0$, $\bar{p}_g = p_g(V) + 1$ *and* $\operatorname{Bs}|K_V + D| = \phi$. *Moreover, either* $\bar{p}_g = 3$ *and* $\Phi_{|K_V+D|}: V \to \mathbf{P}^2$ *is a morphism of degree* 3, *or* $\bar{p}_g = 4$ *and* $\Phi_{|K_V+D|}$ *is a birational morphism onto a quintic surface in* $\mathbf{P}^3$.

(3c) $G = 0$, $q(V) = 0$, $\bar{p}_g = p_g(V) + 1$ *and* $|C|$ *contains a base point* P. *Moreover, let* $f: V' \to V$ *be the blowing-up of the point* P *then* $|f'(C)|$ *is base point free. Finally,* $\Phi_{|f'(C)|}$ *is a morphism of degree two onto a normal, rational surface of degree* $N - 1$ *in* $\mathbf{P}^N$.

Proof. Replace C by a general member of $|C|$. Then C is a reduced, irreducible curve because we have $\dim \Phi_{|C|}(V) = 2$ by the assumption in Theorem D. Write $\bar{p}_g = \frac{1}{12}\bar{c}_2 - \frac{1}{4}(\bar{c}_1^2) + \frac{\beta}{12}$ with an integer $\beta \leq 11$. By the proof of Theorem C, we have $H^1(V, K_V + D) = 0$ and $\beta = r + 8(D, K_V + D) + (K_V, 4K_V + 3D) \geq r + 8(D, K_V + D) \geq 8(D, K_V + D) \geq 0$. Note that $\frac{1}{2}(D, K_V + D) =$

$\chi(\mathcal{O}(K_V + D)) - \chi(\mathcal{O}_V) \in \mathbb{Z}$. So, we must have

$$(D, K_V + D) = 0\,, \qquad \beta = r + (K_V, 4K_V + 3D)\,.$$

Then follow (1) and $(\bar{c}_1^2) = \frac{1}{4}(\beta - e(D) - D^2)$ because $r = e(D)$ in our case.

Suppose $\mathrm{Bs}|3K_V + 2D|$ is not contained in $\mathrm{Supp}\,\varepsilon(-1)$. Then $(\bar{c}_1^2) = 1$ or 2 by Sakai [8; Theorem 5.5]. This leads to $\bar{p}_g \leq \frac{1}{2}(\bar{c}_1^2) + \frac{3}{2} < 3$, which contradicts the hypothesis. Therefore, $|3K_V + 2D| \neq \phi$ and $\mathrm{Bs}|3K_V + 2D| \subseteq \mathrm{Supp}\,\varepsilon(-1)$. As in Theorem C, we can prove that $(K_V, 3K_V + 2D) \geq 0$. So, $\frac{3}{4}(\beta - r + 3D^2) \geq 2(D^2)$ and hence $\beta - r \geq -\frac{1}{3}(D^2)$. On the other hand, we have $\bar{p}_g \leq \frac{1}{8}(\beta - r - D^2) + \frac{3}{2}$ and hence $-(D^2) \geq 8\left(\bar{p}_g - \frac{3}{2}\right) - (\beta - r)$. Then the assertion (2) follows.

We assume furthermore that $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + \frac{3}{2}$ and shall verify the assertion (3). We use the same notations as those before Proposition 3.4 and set $Z := f'(C)$. Note that $\Phi_{|Z|}\colon V' \to \mathbb{P}^N$ is a morphism onto a surface $\overline{W}$.

Set $L := Z_{|Z}$. Then $0 < \deg L \leq 2g(Z) - 2$ because $\kappa(V') \geq 0$. Hence the Clifford index $c(L) := \deg L + 2(1 - h^0(Z, L)) \geq 0$ (cf. Martens [6; §2.31]). Namely, $h^0(Z, Z_{|Z}) \leq \frac{1}{2}(Z^2) + 1$. Consider the cohomologies of the following exact sequence:

(3.6) $$0 \to \mathcal{O}_{V'} \to \mathcal{O}_{V'}(Z) \to \mathcal{O}_Z(Z) \to 0\,.$$

We obtain $\frac{1}{2}(\bar{c}_1^2) + \frac{3}{2} = \bar{p}_g \leq 1 + h^0(Z, Z_{|Z}) \leq 2 + \frac{1}{2}(Z^2) \leq 2 + \frac{1}{2}(C^2) \leq 2 + \frac{1}{2}(K_V + D)^2$. Hence $h^0(V, C) = \bar{p}_g = 1 + h^0(Z, Z_{|Z})$ and one of the following three cases occurs. In order to prove the assertion (3), we shall consider these cases separately.

Case (1) $(K_V + D)^2 - 1 = (C^2) = (Z^2)$ and $h^0(Z, Z_{|Z}) = 1 + \frac{1}{2}(Z^2)$. We shall show that the Case (3a) occurs. The condition $(C^2) = (Z^2)$ implies that $\mathrm{Bs}|C| = \phi$ and $V = V'$. In particular, general members of $|C|$ are non-singular. The condition $1 = (K_V + D)^2 - (C^2) = (C, G) + (G, K_V + D)$ implies that $G \neq 0$, $(C, G) = 1$ and $(G, K_V + D) = 0$ because $K_V + D = C + G$ is a nef and big divisor and is hence 1-connected. Note also that $(G^2) = (G, K_V + D - C) = -1$. Hence G is a reduced divisor and a connected component of $\varepsilon(-1)$ (cf. [8; Lemma 4.12]). In particular, G is a connected component of D.

We claim that $h^1(V, C) = h^0(V, K_V - C) = 1$. Indeed, considering the cohomologies of the following exact sequence:

$$0 \to \mathcal{O}(C) \to \mathcal{O}(K_V + D) \to \mathcal{O}_G(K_V + D) \to 0\,,$$

we obtain that $H^1(V, C) \cong H^0(G, \mathcal{O}_G(K_V + D))$ and $H^0(V, K_V - C) \cong H^2(V, C) \cong H^1(G, \mathcal{O}_G(K_V + D))$ because $h^0(V, C) = h^0(V, K_V + D) = \bar{p}_g$, $H^1(V, K_V + D) = 0$ and $H^2(V, K_V + D) \cong H^0(V, -D) = 0$. Note that the dualizing sheaf ω_G of G is isomorphic to $\mathcal{O}_G$ and note that $\mathcal{O}_G(K_V + D) = \mathcal{O}_G(K_V + G) \cong \omega_G$. Then the claim follows.

On the other hand, we have $K_V - C \sim G - D \leq 0$. Hence we must have $D = G$ and $C \sim K_V$. Then $q(V) = 1$. This fits the Case (3a). Indeed, since $p_g(V) = \frac{1}{2}(K_V^2) + 2 > \frac{1}{3}(7 + K_V^2)$ in the case $p_g(V) \geq 4$ and since V is not a rational surface, $\Phi_{|K_V|}\colon V \to \mathbf{P}^N$ is a morphism of degree two onto a normal, rational surface of degree $N - 1$ in $\mathbf{P}^N$ (cf. Beauville [1; Theorem 5.5]).

Case (2) $(K_V + D)^2 = (C^2) = (Z^2)$ and $h^0(Z, Z_{|Z}) = \frac{1}{2} + \frac{1}{2}(Z^2)$. We shall show that the Case (3b) occurs. As in the previous Case (1), we can prove that $(C, G) = 0$, $G = 0$, $C \sim K_V + D$, $\mathrm{Bs}|C| = \phi$, $V = V'$ and that general members of $|C|$ are nonsingular curves with genus $g(C) = 1 + \frac{1}{2}(C, C + K_V) = 1 + (C^2)$. Considering the cohomologies of the exact sequence (3.6), we obtain:

$$\begin{aligned} 1 + p_g(V) \leq h^1(V, C) + \chi(\mathcal{O}_V) &= h^0(V, K_V - C) + g(C) - (C^2) + h^0(C, C_{|C}) \\ &= \frac{3}{2} + \frac{1}{2}(C^2) = \bar{p}_g\,. \end{aligned}$$

Since $H^1(V, K_V + D) = 0$, we must have $q(V) = 0$ and $\bar{p}_g = 1 + p_g(V)$.

Claim (1). D is connected.

Considering the cohomologies of the following exact sequence:

$$0 \to \mathcal{O}(K_V) \to \mathcal{O}(K_V + D) \to \mathcal{O}_D(K_V + D) \to 0\,,$$

we obtain an exact sequence

$$0 \to H^0(V, K_V) \to H^0(V, K_V + D) \to H^0(D, (K_V + D)_{|D}) \to 0$$

because $q(V) = 0$. Note that $\mathcal{O}_D(K_V + D)$ is isomorphic to the dualizing sheaf of D, which is trivial. Hence we have $h^0(D, \mathcal{O}_D) = \bar{p}_g - p_g(V) = 1$. So, D is connected.

To show that the Case (3b) occurs, it remains to verify the last assertion in the Case (3b). Note that $\deg \Phi_{|K_V+D|} = (K_V + D)^2/\deg \overline{W} = (2\bar{p}_g - 3)/\deg \overline{W}$ which is an odd integer. If $\bar{p}_g = 3$, then $\deg \Phi_{|K_V+D|} = 3$.

Suppose $\bar{p}_g \geq 4$. Then we have $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + \frac{3}{2} \geq \frac{1}{3}(7 + \bar{c}_1^2) = \frac{1}{3}(7 + \deg \overline{W} \cdot \deg \Phi_{|K_V+D|})$. Hence the hypotheses of Proposition 3.4 are satisfied. So, $\deg \Phi_{|K_V+D|} = 1$ and the Case (2), (3) or (4) in Proposition 3.4 occurs. Since $(C, K_V - C) = (K_V + D, -D) = 0$, only the Case (2) in Proposition 3.4 is possible by the Remark, (2) to Proposition 3.4. Hence we have $\frac{1}{2}(\bar{c}_1^2) + \frac{3}{2} = \bar{p}_g = \frac{1}{3}(7 + \deg \overline{W}) = \frac{1}{3}(7 + \bar{c}_1^2)$. This implies $(\bar{c}_1^2) = 5$ and $\bar{p}_g = 4$.

Case (3) $(K_V + D)^2 = (C^2) = (Z^2) + 1$ and $h^0(Z, Z_{|Z}) = 1 + \frac{1}{2}(Z^2)$. As in the previous case, we can show that the Case (3c) occurs.

In the following lemma, we shall consider the case where $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + 2$. Set $N := \bar{p}_g - 1$.

Lemma 3.7. *Assume the same hypotheses as in Theorem* D. *Assume further that* $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + 2 = \frac{1}{12}\bar{c}_2 - \frac{1}{4}(\bar{c}_1^2) + \frac{\alpha}{12}$ *for an integer* $\alpha \leq 11$. *Then we have*:

(1) $\bar{p}_g = 3$, *D is an elliptic curve or a loop of* $\mathbf{P}^1$*'s, the hypotheses of Theorem* B *are satisfied and V is a minimal surface. Hence we have a birational morphism* $\delta\colon Y^* \to V$ (cf. Lemma 2.7).

(2) *One of the following two cases occurs.*

(2a) $\kappa(V) = 1$, $V = Y^*$, $(D^2) = -2$ *and* $\alpha = 6 + e(D) \geq 6$.

(2b) $\kappa(V) = 2$, $(K_V^2) = 1$, $(D^2) = -1$, $\alpha = 7 + e(D) \geq 7$, *δ is the blowing-down of a unique* (-1)*-curve on Y^* to a nonsingular point of D and* $T = \delta'(D)$.

Proof. By Theorem TZ in the Introduction, $|K_V + D|$ is base point free and every general member C of $|K_V + D|$ is a nonsingular irreducible curve of genus ≥ 2. As in Lemma 3.6, we can prove that $H^1(V, C) = 0$, $(D, K_V + D) = 0$, $\alpha = e(D) + (K_V, 4K_V + 3D) \geq e(D) \geq 0$, $(K_V^2) = \frac{1}{4}(\alpha - e(D) + 3D^2)$ and $(\bar{c}_1^2) = \frac{1}{4}(\alpha - e(D) - D^2)$. Note also that $g(C) = 1 + (C^2)$. By Corollary 3.5, $\Phi_{|C|}\colon V \to \mathbf{P}^N$ is a morphism of degree two onto a normal, rational surface $\overline{W}$ of degree $N - 1$. Then the hypotheses of Theorem A are satisfied. In particular, $\Phi_{|C|}$ is the composite morphism of a morphism $\tilde{\Phi}\colon V \to W$ and the minimal resolution $\eta\colon W \to \overline{W}$. Thus, to finish the proof of the assertion (1), we have only to verify that $\bar{p}_g = 3$, V is a minimal surface and the claim below. Indeed, as in Lemma 2.8, Theorem TZ implies that $q(V) = 0$ and $\bar{p}_g = p_g(V) + 1$. So, we can prove that D is connected as in Claim (1) of Lemma 3.6.

Claim. D is contractible by $\tilde{\Phi}$.

Suppose, on the contrary, that $\tilde{\Phi}_* D \neq 0$. Then $\overline{W}$ is a cone in $\mathbf{P}^N$ and $W = \Sigma_{N-1}$ (cf. Theorem A). In particular, $\bar{p}_g = N + 1 \geq 4$. Since $\deg \Phi_{|C|} = 2$ and since D is an elliptic curve or a loop of $\mathbf{P}^1$'s, we have $\tilde{\Phi}_* D = 2M_{N-1}$ with the minimal section M_{N-1} of Σ_{N-1} because the condition $(D, K_V + D) = 0$ implies that $\eta_* \tilde{\Phi}_* D = \Phi_{|C|*} D = 0$.

In the case where D is an elliptic curve, we have $\tilde{\Phi}^* M_{N-1} = D + \Theta$ with an effective, integral, exceptional divisor Θ of $\tilde{\Phi}$.

In the case where D is a loop of $\mathbf{P}^1$'s, we write $D = \sum_{i=1}^n D_i$, where D_i's are irreducible components and $(D_i, D_{i+1}) = (D_n, D_1) = 1$ $(1 \leq i \leq n-1)$. If $\tilde{\Phi}_* D_q = 2M_{N-1}$ for some q, then $\tilde{\Phi}(D - D_q)$ is a single point P and P is a singular point of $\tilde{\Phi}(D_q)$. This contradicts that M_{N-1} is nonsingular. So, we may assume that $\tilde{\Phi}_*(D_1) = \tilde{\Phi}_*(D_p) = M_{N-1}$ and $\tilde{\Phi}_*(D - D_1 - D_p) = 0$ for some $p > 1$. Then there is an effective, integral, $\tilde{\Phi}$-exceptional divisor Θ and there are nonnegative integers δ_i such that $\delta_1 = \delta_p = 1$, $\tilde{\Phi}^* M_{N-1} = \sum_{i=1}^n \delta_i D_i + \Theta$ and D and Θ have no common components.

In both cases, note that Θ is also contracted by $\Phi_{|C|}$, i.e., $(\Theta, K_V + D) = 0$. Note also that Θ has negative intersection matrix and contains no (-1)-curves because (V, D) is minimal. Hence Θ consists of (-2)-curves and is disjoint from D. Thus we have $0 = (\Theta, \tilde{\Phi}^* M_{N-1}) = (\Theta^2)$. Hence $\Theta = 0$, and $\tilde{\Phi}^* M_{N-1} = D$ or $\sum_{i=1}^n \delta_i D_i$ in the case where D is an elliptic curve or a loop of $\mathbf{P}^1$'s, respectively.

We assert that $\tilde{\Phi}^* M_{N-1} = D$ in both cases. Suppose this assertion is verified. Then we have $(D^2) = -2(N-1) = -2(\bar{p}_g - 2) = -\frac{1}{4}(\alpha - e(D) - D^2)$ and $\alpha \geq \alpha - e(D) = -3(D^2) = 6(\bar{p}_g - 2) \geq 12$. This contradicts the hypothesis. So, the claim is true.

It remains to show that $\delta_i = 1$ for every i when D is a loop. By the minimality of (V, D) and by $(D_i, K_V + D) = 0$, we have $a_i := -(D_i^2) \geq 2$ for every i. The fact $0 = (\tilde{\Phi}^* M_{N-1}, D_i) = (\sum_{i=1}^n \delta_i D_i, D_i)$ $(i \neq 1, p)$ implies that $\delta_i \geq 1$ because D is connected and $\delta_1 = \delta_p = 1 > 0$, and implies that $\delta_3 (= a_2\delta_2 - \delta_1 \geq 2\delta_2 - \delta_1) \geq \delta_2, \ldots, \delta_p \geq \delta_{p-1} \geq \cdots \geq \delta_2, \delta_1 \geq \delta_n \geq \delta_{n-1} \geq \cdots \geq \delta_{p+1}$. So, $\delta_i = 1$ for $1 \leq i \leq n$. This proves the claim.

By virtue of the above claim, the hypotheses of Theorem B are satisfied. Hence we have $\frac{1}{4}(\alpha - e(D) + 3D^2) = (K_V^2) \geq (K_{Y^*}^2) = 2(\bar{p}_g - 3)$. So, $11 \geq \alpha \geq \alpha - e(D) \geq 8(\bar{p}_g - 3) - 3(D^2) \geq 3 + 8(\bar{p}_g - 3)$. If $\bar{p}_g \geq 4$, then we must have $\bar{p}_g = 4$, $(D^2) = -1$ and $(K_V^2) = (K_{Y^*}^2)$, i.e., $V = Y^*$. This contradicts the result $(D^2) = -2$ in Theorem B. Therefore, $\bar{p}_g = 3$. If $\kappa(V) = 1$, then Lemma 3.7 follows from Lemma 2.8 and its Remark.

Suppose $\kappa(V) \neq 1$. Then $\kappa(V) = 2$ because $p_g(V) = 2$ (cf. Theorem B). Let $\tau: V \to X$ be a birational morphism onto a nonsingular minimal model. By Lemma 2.2 and Theorem B, (1), $(K_{Y^*} + T)^2 = 2$ and $(K_{Y^*}^2) = 0$. By the proof in Proposition 2.6, we have $(K_X^2) = 1$ and $\tau \circ \delta$ is the blowing-down of a single (-1)-curve E on Y^* to a nonsingular point Q of $\tau_* \delta_* T = \tau_* D$ (cf. Lemma 2.7). So, $V = X$ or Y^*. Since (V, D) is minimal and since $(E, K_{Y^*} + T) = 0$ (cf. Proposition 2.6), we have $V = X$. Note that $(T^2) = -2$ (cf. Lemma 2.3). So, (V, D) fits the Case (2b) (cf. Lemma 2.7) and Lemma 3.7 is proved.

Now we can prove Theorem D in the Introduction. Suppose $\bar{p}_g \leq \frac{1}{2}(\bar{c}_1^2) + 1$. Then $\bar{p}_g \geq \frac{1}{12}\bar{c}_2 - \frac{1}{4}(\bar{c}_1^2) + \frac{1}{4}$ by Lemma 3.6. Hence $(\bar{c}_1^2) \geq \frac{1}{9}\bar{c}_2 - 1$. So, to prove Theorem D, we may assume that $\bar{p}_g \geq \frac{1}{2}(\bar{c}_1^2) + \frac{3}{2}$. Note also that $\bar{p}_g \leq \frac{1}{2}(\bar{c}_1^2) + 2$ (cf. Theorem TZ). Suppose $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + \frac{3}{2}$. Then, by Lemma 3.6, we have $\bar{p}_g = \frac{1}{12}\bar{c}_2 - \frac{1}{4}(\bar{c}_1^2) + \frac{\alpha - 6}{12}$ with an integer $\alpha \geq 9$. Hence $(\bar{c}_1^2) = \frac{1}{9}\bar{c}_2 - 2 + \frac{\alpha - 6}{9}$. Suppose $\bar{p}_g = \frac{1}{2}(\bar{c}_1^2) + 2$. Then, by Lemma 3.7, we have $\bar{p}_g = \frac{1}{12}\bar{c}_2 - \frac{1}{4}(\bar{c}_1^2) + \frac{\alpha}{12}$ with an integer $\alpha \geq 6$. Hence $(\bar{c}_1^2) = \frac{1}{9}\bar{c}_2 - 2 + \frac{\alpha - 6}{9}$. The assertions in (3-1) and (3-2) then follow from Lemma 3.7. Theorem D is thus proved.

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