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A Class of Extremal Positive Maps in 3 × 3 Matrix Algebras

Dedicated to Professor Jun Tomiyama on the sixtieth anniversary of his birthday

By

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Abstract

We shall provide a large class of extremal positive maps in $M_3(\mathbf{C})$ which are neither 2-positive nor 2-copositive and study the algebraic structure of the set of all positive linear maps in $M_3(\mathbf{C})$.

§1. Introduction

Let $M_n(\mathbf{C})$ be the $n \times n$ matrix algebras and $P(M_n)$ be the set of all positive linear maps in $M_n(\mathbf{C})$. One of the basic problems about the structure of the set $P(M_n)$ is whether the set $P(M_n)$ can be decomposed as the algebraic sum of simplier classes in $P(M_n)$. Tow convex classes were candidates, that is, the class of completely positive maps and the class of completely copositive maps. With these classes the program was successful at least for $M_2(\mathbf{C})$ [13]. That this is not the case for higher dimensional algebras was shown by Choi [3] at first by an example of indecomposable maps in $M_3(\mathbf{C})$. Recently, Kye [6], Tanahasi and Tomiyama [12], and the author [8, 9] have studied strong positive indecomposable maps in $M_n(\mathbf{C})$ such that they can not be decomposed into a sum of a 2-positive map and a 2-copositive map. Another approach to the set $P(M_n)$ is to study extremal points in $P(M_n)$. In [11], Størmer investigated the extremal unital positive maps in C^*-algebras and completely characterized the class of extremal unital positive maps in $M_2(\mathbf{C})$. This is, however, no algebraic formula which enables one to construct general extremal positive maps, even in $M_3(\mathbf{C})$. It is, therefore, of interest to tackle the class

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of extremal atomic maps, that is, extremal positive maps which are neither 2-positive nor 2-copositive.

In the present note, we shall provide a large class of extremal atomic maps in $M_3(\mathbf{C})$ and try to determine the algebraic structure of the set $P(M_3)$.

For nonnegative real numbers, c_1, c_2, c_3 with $c_1 \times c_2 \times c_3 = 1$, we define the linear map $\Theta[2; c_1, c_2, c_3]$ in $M_3(\mathbf{C})$ by

$$\Theta[2; c_1, c_2, c_3]([x_{i,j}]) = \begin{bmatrix} x_{1,1} + c_1 x_{3,3} & -x_{1,2} & -x_{1,3} \\ -x_{2,1} & x_{2,2} + c_2 x_{1,1} & -x_{2,3} \\ -x_{3,1} & -x_{3,2} & x_{3,3} + c_3 x_{2,2} \end{bmatrix}$$

for each $[x_{i,j}] \in M_3(\mathbf{C})$. Then, in [6] we know that $\Theta[2; c_1, c_2, c_3]$ is atomic map and in particular $\Theta[2; 1, 1, 1]$ is extremal [5].

Our main result is following:

Theorem. *For nonnegative real numbers c_1, c_2, c_3 with $c_1 \times c_2 \times c_3 = 1$, $\Theta[2; c_1, c_2, c_3]$ are extremal.*

This is the affirmative answer to the question described in [6].

To prove Theorem, we use Choi and Lam's method [5]. For $\phi \in P(M_n)$ we have an hermitian biquadratic form B_ϕ on the set of complex variable $\lambda = (\lambda_1, \dots, \lambda_n)$, $\mu = (\mu_1, \dots, \mu_n)$ of $\mathbf{C}^n$, defined by $B_\phi(\lambda, \mu) = (\phi(\mu^*\mu)\lambda^* | \lambda^*)$. If ϕ is decomposable, there exist bilinear forms $g_p(\lambda, \mu) = \sum \beta^p_{i,j}\lambda_i\mu_j$ and dual bilinear forms $h_p(\lambda, \mu) = \sum \gamma^p_{i,j}\bar{\lambda}_i\mu_j$ such that $B_\phi = \sum (\overline{g_p}g_p + \overline{h_p}h_p)$, where $\beta^p_{i,j}$ and $\gamma^p_{i,j} \in \mathbf{C}$. The converse is also true. Considering the set of real variable $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ of $\mathbf{R}^n$, the study of decomposability is related with Hilbert's classical problem of whether a positive semi-definite real forms (=psd forms) must be the sum of all sequare of other (real) polynomials. Let $P_{n,m}$ be the set of all psd in n variable of degree m. A form $F \in P_{n,m}$ is said to be extremal if $F = F_1 + F_2$, $F_i \in P_{n,m}$, should imply $F_i = \lambda_i F$, where λ_i is nonnegative real number with $\lambda_1 + \lambda_2 = 1$. If we write $\varepsilon(P_{n,m})$ to denote the set of all extremal forms in $P_{n,m}$, an elementary result in the theory of convex bodies shows that $\varepsilon(P_{n,m})$ spans $P_{n,m}$. We stress that if $\phi \in P(M_n)$ maps $M_n(\mathbf{R})$ into itself, $B_\phi \in \varepsilon(P_{2n,4})$ implies that ϕ is extremal in $P(M_n)$. Therefore, to get Theorem, it suffices to prove $B_\Theta[2; c_1, c_2, c_3] \in \varepsilon(P_{6,4})$.

We show in §2 $B_\Theta[2; c_1, c_2, c_3] \in \varepsilon(P_{6,4})$.

In §3, we study the algebraic structure of the set $P(M_3)$.

§2. Extremal Biquadratic Forms

Let $P_{n,m}$ be the set all psd forms in n variables of degree m and $\varepsilon(P_{n,m})$ be the set of all extremal psd forms in $P_{n,m}$. For nonnegative real number c_1, c_2, c_3 with $c_1 \times c_2 \times c_3 = 1$, we define a biquadratic form $B_\Theta[2; c_1, c_2, c_3]$

by

$$B_\Theta[2; c_1, c_2, c_3]\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

$$= [y_1, y_2, y_3]\Theta[2; c_1, c_2, c_3] \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{bmatrix} [y_1, y_2, y_3]^t$$

$$= x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 - 2(x_1x_2y_1y_2 + x_1x_3y_1y_3 + x_2x_3y_2y_3) + c_1x_3^2y_1^2 + c_2x_1^2y_2^2 + c_3x_2^2y_3^2 ,$$

where $[x_1, x_2, x_3]$, $[y_1, y_2, y_3] \in \mathbf{R}^3$ and t means the transpose map in $M_3(\mathbf{C})$. Although we know $B_\Theta[2; c_1, c_2, c_3]$ is psd [6], we give the proof of it for the completeness.

Lemma 2.1 ([3], [5]). *$B_\Theta[2; c_1, c_2, c_3]$ is psd.*

Proof. By the arithmetic-geometric inequality and $c_1 \times c_2 \times c_3 = 1$, we have

$$c_1x_3^2y_1^2 + c_2x_1^2y_2^2 + c_3x_2^2y_3^2 \geq 3(x_1y_1x_2y_2x_3y_3)^{2/3} .$$

Using this, we get

$$\begin{aligned} B_\Theta[2; c_1, c_2, c_3]\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} &= x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 - 2(x_1x_2y_1y_2 + x_1x_3y_1y_3 + x_2x_3y_2y_3) \\ &\quad + c_1x_3^2y_1^2 + c_2x_1^2y_2^2 + c_3x_2^2y_3^2 \\ &\geq x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 - 2(x_1x_2y_1y_2 + x_1x_3y_1y_3 + x_2x_3y_2y_3) \\ &\quad + 3(x_1y_1x_2y_2x_3y_3)^{2/3} . \end{aligned}$$

We put $x_1y_1 = a$, $x_2y_2 = b$, $x_3y_3 = c$, then we have only to show

$$a^2 + b^2 + c^2 - 2(ab + bc + ac) + 3(abc)^{2/3} \geq 0$$

for $a, b, c \geq 0$. By symmetry, we may assume that c is smallest. Using the arithmetic-geometric inequality again, we have

$$\begin{aligned} a^2 + b^2 + c^2 - 2(ab+bc+ac) + 3(abc)^{2/3} &\geq a^2 + b^2 - 2(ab+bc+ac) + 4c\sqrt{ab} \\ &= (a-b)^2 + 4c\sqrt{ab} - 2(a+b)c \\ &= (a-b)^2 - 2c(\sqrt{a} - \sqrt{b})^2 \\ &= (\sqrt{a} - \sqrt{b})^2[(\sqrt{a} + \sqrt{b})^2 - 2c] \\ &\geq 0 . \end{aligned}$$

□

For the proof of the extremality of $B_\Theta[2; c_1, c_2, c_3]$, according to the Choi and Lam's program we construct a psd form $Q[c_1, c_2, c_3]$ in $P_{4,4}$. To get $Q[c_1, c_2, c_3]$, we plug in;

$$
\begin{aligned}
B_\Theta[2; c_1, c_2, c_3]\begin{pmatrix} x & w & z \\ y & z & w \end{pmatrix}
&= x^2y^2 + c_1y^2z^2 + c_2x^2z^2 + c_3w^4 - 4xyzw \\
&= Q[c_1, c_2, c_3](x, y, z, w).
\end{aligned}
$$

Note that their extremality of $B_\Theta[2; 1, 1, 1]$ and $Q[1, 1, 1]$ were proved by Choi and Lam [5].

Let $\ni(Q)$ be the set of real zero of $Q[c_1, c_2, c_3]$ which may be the viewed as a projective set on $\mathbb{P}^3$. We see easily, then, that $\ni(Q)$ consists of the following 7 points; $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and

$$
\left(\sqrt{c_1}, \sqrt{c_2}, 1, \sqrt{\frac{1}{c_3}}\right), \quad \left(-\sqrt{c_1}, \sqrt{c_2}, -1, \sqrt{\frac{1}{c_3}}\right), \quad \left(\sqrt{c_1}, -\sqrt{c_2}, -1, \sqrt{\frac{1}{c_3}}\right),
$$

$$
\left(-\sqrt{c_1}, -\sqrt{c_2}, 1, \sqrt{\frac{1}{c_3}}\right).
$$

It is easily to see that $Q[c_1, c_2, c_3]$ are not a sum of squares of polinomials. Moreover, we have the following result;

Proposition 2.2. $Q[c_1, c_2, c_3] \in \epsilon(P_{4,4})$.

Proof. Let $K \in P_{4,4}$ and $Q[c_1, c_2, c_3] \geq K \geq 0$. To begin with, there are 35 possible monominal terms in K. It is obvious that K cannot contain x^4, y^4, and z^4. Since K and $Q[c_1, c_2, c_3] - K$ are psd, we know K cannot contain x^3y, x^3z, x^3w, y^3x, y^3z, y^3w, z^3x, z^3y, z^3w, x^2w^2, y^2w^2, and z^2w^2. Thus

$$
K(0, y, 0, w) = \alpha yw^3 + \beta w^4 \geq 0.
$$

This gives $\alpha = 0$. Therefore, we can eliminate xw^3, yw^3, and zw^3 from K. Hence, there are 17 possible monominal terms in K and K can be written as follows;

$$
\begin{aligned}
K(x, y, z, w) = x^2(p_1y^2 + p_2yz + p_3yw + p_4z^2 + p_5zw) \\
+ x(p_6y^2z + p_7y^2w + p_8yz^2 + p_9yw^2 + p_{10}yzw + p_{11}z^2w + p_{12}zw^2) \\
+ p_{13}y^2z^2 + p_{14}y^2zw + p_{15}yz^2w + p_{16}yzw^2 + p_{17}w^4.
\end{aligned}
$$

To try to eliminate more monominal terms from K, we use Reznic's idea [10]. Let $P(x_1, \ldots, x_n) = \sum a_i x^{\gamma_i}$ be real form with degree $2m$, where $a_i \neq 0$ and γ_i's are distinct n-tuples on $\mathbb{R}^n$. The cage of P, $C(P)$, is the convex hull of the γ_i's, viewed as vectors in $\mathbb{R}^n$ lying in the hyperplane $u_1 + \cdots + u_n = 2m$.

Reznic showed that if both of f and g are psd forms, then $C(f+g) \supseteq C(f)$ [10, Theorem 1]. Using this result,

$$C(Q) = \{(2\alpha_1 + 2\alpha_3 + \alpha_5, 2\alpha_1 + 2\alpha_2 + \alpha_5, 2\alpha_2 + 2\alpha_3 + \alpha_5, \alpha_5 + 4\alpha_4); a_i \geq 0, \sum \alpha_i \leq 1\}$$
$$\supseteq C(K).$$

From this observation, K can be written as follows,

$$K(x, y, z, w) = q_1 x^2 y^2 + q_2 x^2 yz + q_3 x^2 z^2 + q_4 xy^2 z + q_5 xyz^2 + q_6 xyw^2 + q_7 xyzw + q_8 xzw^2 + q_9 y^2 z^2 + q_{10} yzw^2 + q_{11} w^4 .$$

Since K is psd, all partial derivations of K vanish on $\ni(Q)$. From a partial derivation of K with respect to w, we get

$$\frac{\partial}{\partial w} K\left(\sqrt{c_1}, \sqrt{c_2}, 1, \sqrt{\frac{1}{c_3}}\right) = 2q_6 \sqrt{\frac{c_1 c_2}{c_3}} + q_7 \sqrt{c_1 c_2} + 2q_8 \sqrt{\frac{c_1}{c_3}} + 2q_{10} \sqrt{\frac{c_2}{c_3}} + 4q_{11} \left(\sqrt{\frac{1}{c_3}}\right)^3 = 0,$$

$$\frac{\partial}{\partial w} K\left(-\sqrt{c_1}, \sqrt{c_2}, -1, \sqrt{\frac{1}{c_3}}\right) = -2q_6 \sqrt{\frac{c_1 c_2}{c_3}} + q_7 \sqrt{c_1 c_2} + 2q_8 \sqrt{\frac{c_1}{c_3}} - 2q_{10} \sqrt{\frac{c_2}{c_3}} + 4q_{11} \left(\sqrt{\frac{1}{c_3}}\right)^3 = 0,$$

$$\frac{\partial}{\partial w} K\left(\sqrt{c_1}, -\sqrt{c_2}, -1, \sqrt{\frac{1}{c_3}}\right) = -2q_6 \sqrt{\frac{c_1 c_2}{c_3}} + q_7 \sqrt{c_1 c_2} - 2q_8 \sqrt{\frac{c_1}{c_3}} + 2q_{10} \sqrt{\frac{c_2}{c_3}} + 4q_{11} \left(\sqrt{\frac{1}{c_3}}\right)^3 = 0,$$

$$\frac{\partial}{\partial w} K\left(-\sqrt{c_1}, -\sqrt{c_2}, 1, \sqrt{\frac{1}{c_3}}\right) = 2q_6 \sqrt{\frac{c_1 c_2}{c_3}} + q_7 \sqrt{c_1 c_2} - 2q_8 \sqrt{\frac{c_1}{c_3}} - 2q_{10} \sqrt{\frac{c_2}{c_3}} + 4q_{11} \left(\sqrt{\frac{1}{c_3}}\right)^3 = 0,$$

so we obtain

$$\text{(1)} \qquad \begin{cases} q_6 = q_8 = q_{10} = 0 \\ q_7\sqrt{c_1c_2} + 4q_{11}\left(\sqrt{\dfrac{1}{c_3}}\right)^3 = 0 . \end{cases}$$

Similarly, from the other partial derivations of K we obtain

$$\text{(2)} \qquad \begin{cases} q_2 = q_4 = q_5 = 0 \\ 2q_1\sqrt{c_1}c_2 + 2q_3\sqrt{c_1} + q_7\sqrt{\dfrac{c_2}{c_3}} = 0 , \end{cases}$$

$$\text{(3)} \qquad 2q_1c_1\sqrt{c_2} + q_7\sqrt{\frac{c_1}{c_3}} + 2q_9\sqrt{c_2} = 0 ,$$

$$\text{(4)} \qquad 2q_3c_1 + q_7\sqrt{\frac{c_1c_2}{c_3}} + 2q_9c_2 = 0 .$$

From (3) $\times \sqrt{c_2}$ − (4), we get $q_3 = c_2q_1$, thus from (2),

$$\text{(5)} \qquad q_7 = -4q_1 .$$

From (1), (3), and (5), we get $q_9 = c_1q_1$ and $q_{11} = c_3q_1$.

Therefore, we obtain $K(x, y, z, w) = q_1Q[c_1, c_2, c_3](x, y, z, w)$ and the proof is completed. □

Remark. In [3], Choi and Lam gave another explicit example of non-square psd extremal ternary quatric form $S(x, y, z)$, that is,

$$S(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2 .$$

As in the case of $Q[c_1, c_2, c_3]$ we expect the extremality of $S[c_1, c_2, c_3]$;

$$S[c_1, c_2, c_3](x, y, z) = c_1x^4y^2 + c_2y^4z^2 + c_3z^4x^2 - 3x^2y^2z^2 ,$$

where c_1, c_2, and c_3 are nonnegative real numbers with $c_1 \times c_2 \times c_3 = 1$. We know, however, that $S[c_1, c_2, c_3]$ is not extremal in $P_{3,6}$ except for $c_1 = c_2 = c_3 = 1$ from [10, Theorem 2]. □

Suppose $F\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$ is a biquadratic form such that

$$B_\Theta[2; c_1, c_2, c_3] \geq F \geq 0 .$$

Since

$$B_\Theta[2; c_1, c_2, c_3]\begin{pmatrix} x & w & z \\ y & z & w \end{pmatrix} = x^2y^2 + c_1y^2z^2 + c_2x^2z^2 + c_3w^4 - 4xyzw ,$$

$$B_\Theta[2; c_1, c_2, c_3]\begin{pmatrix} w & z & x \\ z & w & y \end{pmatrix} = x^2y^2 + c_1x^2z^2 + c_2w^4 + c_3y^2z^2 - 4xyzw\,,$$

$$B_\Theta[2; c_1, c_2, c_3]\begin{pmatrix} z & x & w \\ w & y & z \end{pmatrix} = x^2y^2 + c_1w^4 + c_2y^2z^2 + c_3x^2z^2 - 4xyzw\,,$$

from the previous proposition we get

$$F\begin{pmatrix} x & w & z \\ y & z & w \end{pmatrix} = \lambda_1 B_\Theta[2; c_1, c_2, c_3]\begin{pmatrix} x & w & z \\ y & z & w \end{pmatrix},$$

$$F\begin{pmatrix} w & z & x \\ z & w & y \end{pmatrix} = \lambda_2 B_\Theta[2; c_1, c_2, c_3]\begin{pmatrix} w & z & x \\ z & w & y \end{pmatrix},$$

$$F\begin{pmatrix} z & x & w \\ w & y & z \end{pmatrix} = \lambda_3 B_\Theta[2; c_1, c_2, c_3]\begin{pmatrix} z & x & w \\ w & y & z \end{pmatrix}.$$

By comparing the coefficient of x^2y^2, y^2z^2, and z^2x^2, we get $\lambda_1 = \lambda_2 = \lambda_3$. As in the same argument in [5, Theorem 4.4], we obtain the main result in this section.

Theorem 2.3.

$$B_\Theta[2; c_1, c_2, c_3] \in \epsilon(P_{6,4})\,.$$

§ 3. The Algebraic Structure of $P(M_3)$

Let $P(M_n)$ be the set of all positive linear maps in $M_n(\mathbf{C})$. For each $k = 1, 2, \ldots$, a map $\varphi \in P(M_n)$ is said to be k-positive (respectively, k-copositive) if the k-multiplicity map $\varphi(k)$ (respectively, the k-comultiplicity map $\varphi^c(k)$);

$$\varphi(k);\ [a_{i,j}] \in M_k(M_n(\mathbf{C})) \mapsto [\varphi(a_{i,j})]_{i,j=1}^k$$

$$(\text{respectively, } \varphi^c(k);\ [a_{i,j}] \in M_k(M_n(\mathbf{C})) \mapsto [\varphi(a_{j,i})]_{i,j=1}^k)\,,$$

is positive. If φ is k-positive for every k, then φ is said to be completely positive. It is, however, known that every n-positive map in $M_n(\mathbf{C})$ is completely positive and the class of completely positive maps is equal to

$$\left\{\sum_i V_i(\ )V_i^*;\ V_i \in M_n(\mathbf{C})\right\}.$$

Completely copositive maps are defined in a similar way and the saturation of copositivity in $M_n(\mathbf{C})$ also occur. In particular, the class of completely copositive maps is equal to

$$\left\{\sum_j W_j(\ )^t W_j^*;\ W_j \in M_n(\mathbf{C})\right\},$$

where t means the transpose map in $M_n(\mathbf{C})$. A map $\varphi \in P(M_n)$ is said to be decomposable if φ can be a sum of a completely positive map and a completely copositive map. As a new candidate for the previous basic problem, Tanahasi and Tomiyama [12] have introduced the following concept;

Definition. A map $\varphi \in P(M_n)$ is said to be *atomic* if φ can not be decomposed into a sum of a 2-positive map and a 2-copositive map.

Note that the class of atomic maps is not a positive cone [8].

A map $\varphi \in P(M_n)$ is said to be extremal if $\varphi = \varphi_1 + \varphi_2$, $\varphi_i \in P(M_n)$, should imply $\varphi_i = \lambda_i \varphi$, where λ_1, λ_2 are nonnegative real numbers, $\lambda_1 + \lambda_2 = 1$.

Since $\Theta[2; c_1, c_2, c_3](M_3(\mathbf{R})) \subset M_3(\mathbf{R})$, from Theorem 2.3 and [6, Theorem 3.2] we obtain the main result;

Theorem 3.1. *For negative number c_1, c_2, c_3 with $c_1 \times c_2 \times c_3 = 1$, $\Theta[2; c_1, c_2, c_3]$ are extremal atomic maps.*

For the completeness, we give the elementary proof of the atomic property of $\Theta[2; c_1, c_2, c_3]$.

Proof. At first, we give a proof of the positivity of $\Theta[2; c_1, c_2, c_3]$ as in the argument of [4, Appendix B].

Since $B_\Theta[2; c_1, c_2, c_3]$ is a psd form by Lemma 2.1, for every rank one positive semidefinite complex matrix $[\overline{\alpha_i}\alpha_j]$,

$$\Theta[2; c_1, c_2, c_3] \begin{bmatrix} \overline{\alpha_1}\alpha_2 & \overline{\alpha_1}\alpha_2 & \overline{\alpha_1}\alpha_3 \\ \overline{\alpha_2}\alpha_1 & \overline{\alpha_2}\alpha_2 & \overline{\alpha_2}\alpha_3 \\ \overline{\alpha_3}\alpha_1 & \overline{\alpha_3}\alpha_2 & \overline{\alpha_3}\alpha_3 \end{bmatrix}$$

$$= \begin{bmatrix} \overline{\lambda_1} & 0 & 0 \\ 0 & \overline{\lambda_2} & 0 \\ 0 & 0 & \overline{\lambda_3} \end{bmatrix} \Theta[2; c_1, c_2, c_3] \begin{bmatrix} |\alpha_1|^2 & |\alpha_1\alpha_2| & |\alpha_1\alpha_3| \\ |\alpha_2\alpha_1| & |\alpha_2|^2 & |\alpha_2\alpha_3| \\ |\alpha_3\alpha_1| & |\alpha_3\alpha_2| & |\alpha_3|^2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

turns out to be positive semidefinite too, where $\lambda_i (i = 1, 2, 3)$ are complex numbers of modulus 1 with $\alpha_i = \lambda_i |\alpha_i|$. Hence by linearity, $\Theta[2; c_1, c_2, c_3](X) \geq 0$ for every positive semidefinite complex matrix X.

Let $\xi = [0, 1, 1, 1, 1, 0] \in \mathbf{R}^6$ and $x = \xi^t \xi$. We have, then,

$$(\Theta[2; c_1, c_2, c_3](2)(x)\eta^t | \eta^t) = -1 ,$$

where $\eta = [0, 1, 1, 1, 0, -1] \in \mathbf{R}^6$. Hence we know $\Theta[2; c_1, c_2, c_3]$ is not 2-positive.

Let $\{e_{i,j}\}_{1 \leq i,j \leq n}$ be canonical matrix units for $M_3(\mathbf{C})$. It is easily seen

$$\Theta[2; c_1, c_2, c_3](2) \begin{pmatrix} e_{1,1} & e_{2,1} \\ e_{1,2} & e_{2,2} \end{pmatrix} \not\geq 0 ,$$

and $\Theta[2; c_1, c_2, c_3]$ is not 2-copositive.

Since $\Theta[2; c_1, c_2, c_3]$ is extremal, $\Theta[2; c_1, c_2, c_3]$ is atomic. □

Thanks to the previous Theorem, to clear the structure of the set $P(M_3)$ it is important to analysis the existence of extremal 2-positive maps which is not 3-positive, that is, completely positive. In [2], Choi gave examples of $(n-1)$-positive maps $\varphi_n \in P(M_n)$ $(n \geq 3)$ which are not n-positive;

$$\varphi_n(X) = (n-1)\mathrm{trace}(X)1_n - X\,,$$

where trace$(\cdot)$ means the canonical trace in $M_n(\mathbf{C})$. According to Ando's note, φ_n is purely $(n-1)$-positive; If ψ_1 is $(n-1)$-positive and ψ_2 n-positive in $P(M_n)$ satisfying $\varphi_n = \psi_1 + \psi_2$, then $\psi_2 = 0$. In particular, φ_3 is a decomposable map [1]. But using the concept of atom, we get the another aspect of φ_n. Indeed,

$$\begin{aligned}\varphi_n(X) &= (n-1)\varepsilon(X) + \varepsilon(SXS^*) - X + \psi_2(X)\\ &= \psi_1(X) + \psi_2(X) \quad X \in M_n(\mathbf{C})\,,\end{aligned}$$

where ε is a canonical projection of $M_n(\mathbf{C})$ to the diagonal part, S is the rotation matrix in $M_n(\mathbf{C})$ such that $S = [\delta_{i,j+1}]$. From [9, Theorem], ψ_1 is atomic and obviously ψ_2 is completely positive.

On the other hand, in the study of contractive projections on C^*-algebras we know an arbitrary $\left[\frac{n}{2}\right]$-positive contractive projection in $P(M_n)$ automatically becomes a completely positive map [7, Theorem 3.1], where [ ] means Gaussian symbol. Therefore, we can pose the following problem;

Problem 3.2. *Let $n \geq 3$. Is an arbitrary extremal $(n-1)$-positive map in $P(M_n)$ completely positive?*

If this problem is true, we can completely determine the algebraic structure of $P(M_3)$, that is,

Problem 3.3. *For any $\varphi \in P(M_3)$, can φ be written as a positive linear sum of decomposable maps and atomic maps?*

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