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Clebsch-Gordan Coefficients for $\mathcal{U}_q(\mathfrak{su}(1, 1))$ and $\mathcal{U}_q(\mathfrak{sl}(2))$, and Linearization Formula of Matrix Elements

By

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Abstract

The tensor product of two representations of the discrete series and the limit of the discrete series of $\mathcal{U}_q(\mathfrak{su}(1, 1))$ is decomposed into the direct sum of irreducible components of $\mathcal{U}_q(\mathfrak{sl}(1, 1))$, and the Clebsch-Gordan coefficients with respect to this decomposition are computed in two ways. In some cases, the tensor product of an irreducible unitary representation of $\mathcal{U}_q(\mathfrak{su}(2))$ and a representation of the discrete series of $\mathcal{U}_q(\mathfrak{su}(1, 1))$ is decomposed into the direct sum of irreducible components of $\mathcal{U}_q(\mathfrak{sl}(2))$, and the Clebsch-Gordan coefficients with respect to this decomposition are calculated, too. Making use of these coefficients, the linearization formula of the matrix elements is obtained.

§0. Introduction

The real form $\mathcal{U}_q(\mathfrak{su}(2))$ of the quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2))$ has been studied in Mathematical Physics. In particular, the finite dimensional unitary representations of $\mathcal{U}_q(\mathfrak{su}(2))$ have been considered so far. Jimbo [4] constructed the finite dimensional irreducible unitary representations of $\mathcal{U}_q(\mathfrak{su}(2))$, and proved that the tensor product of two irreducible unitary representations of $\mathcal{U}_q(\mathfrak{su}(2))$ is decomposed into the direct sum of irreducible unitary representations of $\mathcal{U}_q(\mathfrak{su}(2))$. Further, Kirillov and Reshetikhin [5] calculated the Clebsch-Gordan coefficients for $\mathcal{U}_q(\mathfrak{su}(2))$ with respect to the above decomposition. Ruegg [9] generalized the $\mathfrak{su}(2)$-invariants theory to $\mathcal{U}_q(\mathfrak{su}(2))$, which van der Waerden has used. Making use of this method, he also calculated the Clebsch-Gordan coefficients for $\mathcal{U}_q(\mathfrak{su}(2))$. Masuda et al. [7, 8] studied the quantum group $SU_q(2)$, and expressed the matrix elements

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associated with the irreducible unitary representations of $\mathcal{U}_q(\mathfrak{su}(2))$ by the basic hypergeometric series ${}_2\phi_1\left[\begin{matrix} a_1, a_2 \\ b_1 \end{matrix}; q, z\right]$.

On the other hand, the study of the real form $\mathcal{U}_q(\mathfrak{su}(1,1))$ of $\mathcal{U}_q(\mathfrak{sl}(2))$ has just started. Masuda et al. [6] constructed the series of the infinite dimensional irreducible unitary representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$ (cf. [10]), and proved that the matrix elements associated with the irreducible unitary representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$ are also expressed by the basic hypergeometric series ${}_2\phi_1$.

In this paper, we prove that the tensor product of two representations which belong to discrete series and the limit of the discrete series of $\mathcal{U}_q(\mathfrak{su}(1,1))$ is decomposed into the direct sum of irreducible components of $\mathcal{U}_q(\mathfrak{su}(1,1))$, and compute the Clebsch-Gordan coefficients for $\mathcal{U}_q(\mathfrak{su}(1,1))$ with respect to this decomposition in two ways. Moreover we prove that, in some cases, the tensor product of an irreducible unitary representation of $\mathcal{U}_q(\mathfrak{su}(2))$ and a representation of the discrete series of $\mathcal{U}_q(\mathfrak{su}(1,1))$ is decomposed into the direct sum of irreducible components of $\mathcal{U}_q(\mathfrak{sl}(2))$, and compute the Clebsch-Gordan coefficients for $\mathcal{U}_q(\mathfrak{sl}(2))$ with respect to this decomposition. Making use of these coefficients, we obtain the linearization formula of the matrix elements associated with the discrete series of $\mathcal{U}_q(\mathfrak{su}(1,1))$.

The plan of this paper is as follows. First, in Section 1, we define the real form $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{U}_q(\mathfrak{su}(1,1))$ of the quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2))$, and introduce their irreducible unitary representations, $W_l \left(l \in \mathbb{N} \cup \mathbb{N} + \frac{1}{2}\right)$ of $\mathcal{U}_q(\mathfrak{su}(2))$ and $V_l \left(l \in \left\{-\frac{1}{2}, 0, \frac{1}{2}, 1, \dots\right\}\right)$ of $\mathcal{U}_q(\mathfrak{su}(1,1))$. We call $V_l \left(l \in \mathbb{N} \cup \mathbb{N} + \frac{1}{2}\right)$ discrete series, and call $V_{-1/2}$ the limit of the discrete series. In Section 2 we study the decomposition of the tensor product $V_{l_1} \otimes V_{l_2}$ of $\mathcal{U}_q(\mathfrak{su}(1,1))$. The result is as follows:

Theorem 2.1. *$V_{l_1} \otimes V_{l_2} \simeq \bigoplus_{l \in L_1(l_1, l_2)} V_l$ as unitary representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$, where $L_1(l_1, l_2) = \left\{l \in \mathbb{N} \cup \mathbb{N} + \frac{1}{2} \middle| l \geq l_1 + l_2 + 1, l - l_1 - l_2 \in \mathbb{N}\right\}$.*

In Section 3, we calculate the Clebsch-Gordan coefficients for $\mathcal{U}_q(\mathfrak{su}(1,1))$ with respect to the decomposition in Theorem 2.1. For $l \in L_1(l_1, l_2)$ we define $I_l = \{l+1, l+2, \dots\}$, and let $\{\xi^l_m | m \in I_l\}$ (resp. $\{\xi^{l_1}_{m_1} | m_1 \in I_{l_1}\}$, $\{\xi^{l_2}_{m_2} | m_2 \in I_{l_2}\}$) be an orthonormal basis of V_l (resp. V_{l_1}, V_{l_2}). From Theorem 2.1, ξ^l_m is denoted as

$$\xi^l_m = \sum_{m_1 \in I_{l_1}, m_2 \in I_{l_2}} \begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix} \xi^{l_1}_{m_1} \otimes \xi^{l_2}_{m_2},$$

where all $\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix} \in \mathbb{C}$ but finite are zero. Then we compute the Clebsch-

Gordan coefficients $\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}$ by solving the recurrence relation. In Section 4, we generalize the $\mathfrak{su}(2)$-invariants theory to $\mathcal{U}_q(\mathfrak{su}(1, 1))$, and calculate the Clebsch-Gordan coefficients making use of this method. Let V be a representation of $\mathcal{U}_q(\mathfrak{sl}(2))$. The vector $I \in V$ is $\mathcal{U}_q(\mathfrak{sl}(2))$ invariant if $kI=I$, $eI=fI=0$. Using that the dimension of the subspace of the $\mathcal{U}_q(\mathfrak{sl}(2))$ invariant vectors of some representation V is less than or equal to 1, we obtain another expression of the Clebsch-Gordan coefficients. At the end of this section we express the Clebsch-Gordan coefficients for $\mathcal{U}_q(\mathfrak{su}(1, 1))$ by the basic hypergeometric series ${}_3\phi_2\left[\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}; q, z\right]$. In Section 5, we first consider the decomposition of the tensor product $W_{l_1} \otimes V_{l_2}$ of $\mathcal{U}_q(\mathfrak{sl}(2))$, where $l_1 - l_2 < 1$ and $l_2 \in \mathbb{N} \cup \mathbb{N} + \frac{1}{2}$. The result is

Theorem 5.1. *$W_{l_1} \otimes V_{l_2} \simeq \bigoplus_{l \in L_2(l_1, l_2)} V_l$ as representations of $\mathcal{U}_q(\mathfrak{sl}(2))$, where*

$$L_2(l_1, l_2) = \left\{ l \in \left\{ -\frac{1}{2}, 0, \frac{1}{2}, 1, \dots \right\} \middle| l + l_1 - l_2 \in \mathbb{N},\ -l_1 + l_2 \le l \le l_1 + l_2 \right\}.$$

Next we calculate the Clebsch-Gordan coefficients for $\mathcal{U}_q(\mathfrak{sl}(2))$ with respect to the decomposition in Theorem 5.1. Let $\{\tilde{\xi}^l_m | m \in I_l\}$ be an orthogonal basis of V_l and $\{\tilde{x}^{l_1}_{m_1} | m_1 \in J_{l_1}\}$ be an orthonormal basis of W_{l_1}. By Theorem 5.1, $\tilde{\xi}^l_m$ is denoted as

$$\tilde{\xi}^l_m = \sum_{m_1 \in J_{l_1}, m_2 \in I_{l_2}} \left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right] \tilde{x}^{l_1}_{m_1} \otimes \tilde{\xi}^{l_2}_{m_2},$$

where all $\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right] \in \mathbb{C}$ but finite are zero. We call these coefficients $\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right]$ the Clebsch-Gordan coefficients for $\mathcal{U}_q(\mathfrak{sl}(2))$. Then we calculate the Clebsch-Gordan coefficients $\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right]$ in the same way as Section 4. At the end of this section we express the Clebsch-Gordan coefficients for $\mathcal{U}_q(\mathfrak{sl}(2))$ by the basic hypergeometric series ${}_3\phi_2$. Finally, in Section 6, we introduce the matrix elements $\tilde{p}^{(l)}_{ij}$ associated with the representation W_l $\left(l \in \mathbb{N} \cup \mathbb{N} + \frac{1}{2}\right)$ and $\tilde{w}^{(l)}_{ij}$ associated with V_l $\left(l \in \left\{-\frac{1}{2}, 0, \frac{1}{2}, 1, \dots\right\}\right)$, and prove the linearization formula of the matrix elements $\tilde{w}^{(l)}_{ij}$. In particular, we show the three-term recurrence relation of the matrix elements $\tilde{w}^{(l)}_{ij}$.

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§ 1. $\mathcal{U}_q(\mathfrak{su}(2))$, $\mathcal{U}_q(\mathfrak{su}(1,1))$ and Their Irreducible Unitary Representations

The quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2))$ is the algebra over $\mathbb{C}$ with a complex parameter q ($q \neq 0, \pm 1$) generated by $k^{\pm 1}$, e, f with the following relations [2, 4]:

$$kk^{-1} = k^{-1}k = 1\,, \quad kek^{-1} = qe\,, \quad kfk^{-1} = q^{-1}f, \quad [e, f] = \frac{k^2 - k^{-2}}{q - q^{-1}}\,.$$

This algebra has a Hopf algebra structure. The coproduct $\Delta\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2)) \otimes \mathcal{U}_q(\mathfrak{sl}(2))$ is defined on the generators as

$$\Delta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1}\,, \quad \Delta(e) = e \otimes k + k^{-1} \otimes e\,, \quad \Delta(f) = f \otimes k + k^{-1} \otimes f\,.$$

The counit $\varepsilon\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathbb{C}$ is defined by

$$\varepsilon(k^{\pm 1}) = 1\,, \qquad \varepsilon(e) = \varepsilon(f) = 0\,.$$

The antipode $S\colon \mathcal{U}_q(\mathfrak{sl}(2)) \to \mathcal{U}_q(\mathfrak{sl}(2))$ is defined by

$$S(k) = k^{-1}\,, \qquad S(e) = -qe\,, \qquad S(f) = -q^{-1}f\,.$$

In the sequel we assume that $q^m \neq 1$ for any integer m.

A $*$ structure $A \ni a \mapsto a^* \in A$ of a Hopf algebra $(A, \Delta, \varepsilon, S)$ over $\mathbb{C}$ is a morphism satisfying the following conditions [6]:

$*$ is a conjugate linear, anti-automorphism of A such that

$$*^2 = \mathrm{id}\,.$$

$$\Delta \circ * = (* \otimes *) \circ \Delta\,.$$

$$\varepsilon(a^*) = \overline{\varepsilon(a)}\,.$$

$$(* \circ S)^2 = \mathrm{id}\,.$$

We regard a pair of A and $*$ structure as a real form of A. Then, for $-1 < q < 1$ ($q \neq 0$), we define $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{U}_q(\mathfrak{su}(1,1))$ as real forms of $\mathcal{U}_q(\mathfrak{sl}(2))$ with the following $*$ structures:

$$\mathcal{U}_q(\mathfrak{su}(2))\colon k^* = k\,, \qquad e^* = f\,, \qquad f^* = e\,.$$

$$\mathcal{U}_q(\mathfrak{su}(1,1))\colon k^* = k\,, \qquad e^* = -f\,, \qquad f^* = -e\,.$$

Let V be a representation of $\mathcal{U}_q(\mathfrak{sl}(2))$. For any $\alpha \in \mathbb{C}\backslash\{0\}$, we set $V(\alpha) = \{v \in V | kv = \alpha v\}$. Whenever $V(\alpha) \neq \{0\}$, we call it a weight space of V and call α a weight of V. We additionally assume that V has an Hermitian inner product $\langle \cdot, \cdot \rangle$. If this inner product satisfies the condition

$$\langle av, w \rangle = \langle v, a^*w \rangle \qquad \text{for } a \in \mathcal{U}_q(\mathfrak{sl}(2)),\ v,\ w \in V,$$

we call V a unitary representation of the real form of $\mathcal{U}_q(\mathfrak{sl}(2))$.

Let $0<q<1$. For $l \in \mathbb{N}+\frac{1}{2}$ we introduce the finite dimensional irreducible unitary representations $W_l=\bigoplus_{j \in J_l} \mathbb{C} x_j^l$ of $\mathscr{U}_q(\mathfrak{su}(2))$ [4, 5], where $J_l= \{-l,-l+1, \ldots, l\}$. The action of $\mathscr{U}_q(\mathfrak{su}(2))$ is given on the generators as

$$\begin{cases} k x_j^l=q^j x_j^l, \\ e x_j^l=q^{-(l-j-1)}[l-j] x_{j+1}^l, \\ f x_j^l=q^{-(l+j-1)}[l+j] x_{j-1}^l, \end{cases}$$

where $[m]=\dfrac{1-q^{2m}}{1-q^2}$ for $m \in \mathbb{Z}$. The Hermitian inner product on W_l is given on the basis by

$$\langle x_j^l, x_j^l\rangle=\delta_{ij} d_j^l$$

with $d_j^l=q^{(l-j)(l+j)} \dfrac{[l-j]![l+j]!}{[2l]!}$, where $[m]!=\prod_{i=1}^m[i]$ for $m \in \mathbb{N} \backslash\{0\}$ and $[0]!=1$.

Next, for $l \in\left\{-\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots\right\}$ we introduce the infinite dimensional irreducible unitary representations $V_l=\bigoplus_{j \in I_l} \mathbb{C} \xi_j^l$ of $\mathscr{U}_q(\mathfrak{su}(1,1))$ [6], where $I_l= \{l+1, l+2, \ldots\}$. The action of $\mathscr{U}_q(\mathfrak{su}(1,1))$ is as follows.

$$\begin{cases} k \xi_j^l=q^{-j} \xi_j^l, \\ e \xi_j^l=-q^{(5/2)+l-2j}[j-l-1] \xi_{j-1}^l, \\ f \xi_j^l=q^{(1/2)-l}[j+l+1] \xi_{j+1}^l . \end{cases}$$

The Hermitian inner product is given by

$$\langle\xi_i^l, \xi_j^l\rangle=\delta_{ij} c_j^l,$$

where $c_j^l=q^{-(j-l)(j-l-1)} \dfrac{[j-l-1]![2l+1]!}{[j+l]!}$. We call V_l $\left(l \in \mathbb{N} \cup \mathbb{N}+\frac{1}{2}\right)$ discrete series, and call $V_{-1/2}$ the limit of the discrete series.

Remark. Masuda et al. [6] have constructed all series of irreducible unitary representations of $\mathscr{U}_q(\mathfrak{su}(1,1))$ $(0<q<1)$ (cf. [10]).

Let $l \in \mathbb{C}$ and I_l be a subset of $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$. We define a representation $V_l=\bigoplus_{j \in I_l} \mathbb{C} \xi_j^l$ of $\mathscr{U}_q(\mathfrak{sl}(2))$ with the action

$$\begin{cases} k\xi_j^l = q^{-j}\xi_j^l\,, \\ e\xi_j^l = -q^{(5/2)+l-2j}\dfrac{1-q^{2(j-l-1)}}{1-q^2}\xi_{j-1}^l\,, \\ f\xi_j^l = q^{(1/2)-l}\dfrac{1-q^{2(j+l+1)}}{1-q^2}\xi_{j+1}^l\,. \end{cases}$$

V_l is an irreducible unitary representation of $\mathscr{U}_q(\mathfrak{su}(1,1))$ in the following cases.

The case of $I_l \subset \mathbb{Z}$:

(1) $l \in \mathbb{N}$, and $I_l = \{l+1, l+2, \dots\}$ or $I_l = \{-l-1, -l-2, \dots\}$,

(2) $l = -\dfrac{1}{2} + \sqrt{-1}\lambda \left(0 \le \lambda \le \dfrac{\pi}{2h}\right)$ and $I_l = \mathbb{Z}$,

(3) $l = -\dfrac{1}{2} + \dfrac{\sqrt{-1}\pi}{2h} + s\ (s > 0)$ and $I_l = \mathbb{Z}$,

(4) $-\dfrac{1}{2} < l < 0$ and $I_l = \mathbb{Z}$.

The case of $I_l \subset \mathbb{Z} + \dfrac{1}{2}$:

(1) $l \in \mathbb{N} + \dfrac{1}{2}$, and $I_l = \{l+1, l+2, \dots\}$ or $I_l = \{-l-1, -l-2, \dots\}$,

(2) $l = -\dfrac{1}{2} + \sqrt{-1}\lambda \left(0 \le \lambda \le \dfrac{\pi}{2h}\right)$ and $I_l = \mathbb{Z} + \dfrac{1}{2}$,

(3) $l = -\dfrac{1}{2} + \dfrac{\sqrt{-1}\pi}{2h} + s\ (s > 0)$ and $I_l = \mathbb{Z} + \dfrac{1}{2}$,

(1′) $l = -\dfrac{1}{2}$, and $I_l = \left\{\dfrac{1}{2}, \dfrac{3}{2}, \dots\right\}$ or $I_l = \left\{-\dfrac{1}{2}, -\dfrac{3}{2}, \dots\right\}$,

where $q = e^{-h}$. For each family, an Hermitian inner product on V_l is defined by

$$\langle \xi_i^l, \xi_j^l \rangle = \delta_{ij} c_j^l$$

with

$$\frac{c_{j+1}^l}{c_j^l} = \begin{cases} -\dfrac{1-q^{2(l-j)}}{1-q^{2(l+j-1)}} & \text{for the family (1), (4), (1′)}, \\ q^{-2j-1} & \text{for the family (2)}, \\ q^{2s-2j-1}\dfrac{1+q^{-2s+2j+1}}{1+q^{2s+2j+1}} & \text{for the family (3)}. \end{cases}$$

Further, any irreducible unitary representation of $\mathcal{U}_q(\mathfrak{su}(1,1))$ is isomorphic to one of the above families.

Let V_1 and V_2 be representations of $\mathcal{U}_q(\mathfrak{sl}(2))$. The tensor product of the representations V_1 and V_2 is a representation of $\mathcal{U}_q(\mathfrak{sl}(2))$ on the vector space $V_1 \otimes V_2$ with the action

$$a \cdot \zeta = \Delta(a)\zeta \qquad \text{for } a \in \mathcal{U}_q(\mathfrak{sl}(2))\,, \quad \zeta \in V_1 \otimes V_2\,.$$

We denote this representation by $V_1 \otimes V_2$. If V_1 and V_2 are unitary representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$, then the tensor product $V_1 \otimes V_2$ becomes a unitary representation of $\mathcal{U}_q(\mathfrak{su}(1,1))$ with the Hermitian inner product

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \xi, \xi' \rangle \langle \eta, \eta' \rangle \qquad \text{for } \xi, \xi' \in V_1,\ \eta, \eta' \in V_2\,.$$

§2. Decomposition of Tensor Product of Two Representations of Discrete Series and Limit of Discrete Series of $\mathcal{U}_q(\mathfrak{su}(1,1))$

In the sequel $0 < q < 1$. To the end of Section 4, we fix $l_1, l_2 \in \left\{-\frac{1}{2}, 0, \frac{1}{2}, 1, \dots\right\}$. We have the decomposition of $V_{l_1} \otimes V_{l_2}$ into the direct sum of irreducible components of $\mathcal{U}_q(\mathfrak{su}(1,1))$.

Theorem 2.1. $V_{l_1} \otimes V_{l_2} \simeq \bigoplus_{l \in L_1(l_1, l_2)} V_l$ *as unitary representations of* $\mathcal{U}_q(\mathfrak{su}(1,1))$, *where* $L_1(l_1, l_2) = \left\{ l \in \mathbb{N} \cup \mathbb{N} + \frac{1}{2} \,\middle|\, l \geq l_1 + l_2 + 1,\ l - l_1 - l_2 \in \mathbb{N} \right\}$.

To prove this theorem, we first construct the basis $\{\zeta^l_m \mid l \in L_1(l_1, l_2),\ m \in I_l\}$ of $\bigoplus_{l \in L_1(l_1, l_2)} V_l$ in $V_{l_1} \otimes V_{l_2}$. For $l \in L_1(l_1, l_2)$ we define ζ^l_m $(m \in I_l)$ as follows.

$$\zeta^l_{l+1} = \sum_{m_1 = l_1 + 1}^{l - l_2} a_{m_1} \xi^{l_1}_{m_1} \otimes \xi^{l_2}_{l+1-m_1}\,,$$

where

$$a_{m_1} = (-1)^{m_1 - l_1 - 1} q^{(m_1 - l_1 - 1)(2m_1 - l + l_1 + l_2)} \begin{bmatrix} l - l_1 - l_2 - 1 \\ m_1 - l_1 - 1 \end{bmatrix}$$

and

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \dfrac{[n]!}{[m]![n-m]!} & \text{if } 0 \leq m \leq n\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For $m \geq l + 2$

$$\zeta^l_m = \left\{ \prod_{j = l+1}^{m-1} (q^{(1/2) - l}[j + l + 1])^{-1} \right\} f^{m-l-1} \zeta^l_{l+1}\,.$$

Lemma 2.2. *The vectors ζ^l_m are non-zero for all $l \in L_1(l_1, l_2)$ and $m \in I_l$.*

Proof. By definition, it is trivial that $\zeta^l_{l+1} \neq 0$. Making use of the formula

$$(2.1)\quad e^{m-l-1}f^{m-l-1} - \left(\prod_{j=1}^{m-l-1} \frac{q^j - q^{-j}}{q - q^{-1}}\right)\left(\prod_{j=1}^{m-l-1} \frac{q^{1-j}k^2 - q^{-(1-j)}k^{-2}}{q - q^{-1}}\right) \in \mathcal{U}_q(\mathfrak{sl}(2))\cdot e$$

for $m \geq l + 2$, we have $e^{m-l-1}f^{m-l-1}\zeta^l_{l+1} \neq 0$. This implies the result. □

We note that ζ^l_m satisfies

$$\begin{cases} k\zeta^l_m = q^{-m}\zeta^l_m\,, \\ e\zeta^l_m = -q^{(5/2)+l-2m}[m-l-1]\zeta^l_{m-1}\,, \\ f\zeta^l_m = q^{(1/2)-l}[m+l+1]\zeta^l_{m+1}\,, \end{cases}$$

and consequently, $\sum_{m \in I_l} \mathbb{C}\zeta^l_m$ is a representation of $\mathcal{U}_q(\mathfrak{sl}(2))$.

Proposition 2.3. *The set of vectors $\{\zeta^l_m | l \in L_1(l_1, l_2),\ m \in I_l\}$ is linearly independent over $\mathbb{C}$.*

Proof. Suppose that

$$\sum_{\substack{l \in L_1(l_1, l_2) \\ m \in I_l}} \alpha^l_m \zeta^l_m = 0\,,$$

where all $\alpha^l_m \in \mathbb{C}$ but finite are zero. Then, for any $m \geq l_1 + l_2 + 2$ we have

$$\sum_{l_1+l_2+1 \leq l \leq m-1} \alpha^l_m \zeta^l_m = 0\,,$$

because

$$V_{l_1} \otimes V_{l_2} = \bigoplus_{j \geq l_1+l_2+2} (V_{l_1} \otimes V_{l_2})(q^{-j})\,.$$

For the proof, it suffices to show the following.

If $\displaystyle\sum_{l_1+l_2+1 \leq l \leq m-1} \alpha^l_m \zeta^l_m = 0$, then $\alpha^l_m = 0$ for $l_1 + l_2 + 1 \leq {}^{\forall}l \leq m - 1$.

We prove this by the induction on $m \geq l_1 + l_2 + 2$. It is trivial in the case of $m = l_1 + l_2 + 2$. Applying e to both sides of $\sum_{l_1+l_2+1 \leq l \leq m} \alpha^l_{m+1}\zeta^l_{m+1} = 0$, we obtain

$$\sum_{l_1+l_2+1 \leq l \leq m-1} \alpha^l_{m+1}(-1)q^{(1/2)+l-2m}[m-l]\zeta^l_m = 0\,.$$

The induction hypothesis leads us to $\alpha^l_{m+1} = 0$ for $l_1 + l_2 + 1 \leq l \leq m - 1$, and hence $\alpha^l_{m+1} = 0$ for $l_1 + l_2 + 1 \leq l \leq m$. □

Moreover we use the following lemma.

Lemma 2.4. (i) $\langle \zeta_m^l, \zeta_{m'}^{l'} \rangle = \delta_{ll'}\delta_{mm'} c_m^l \langle \zeta_{l+1}^l, \zeta_{l+1}^l \rangle$.
(ii) $\bigoplus_{m \in I_l} \mathbb{C}\zeta_m^l \simeq V_l$ *as unitary representations of* $\mathscr{U}_q(su(1, 1))$.

Proof. (i) We first note that the formula (2.1) shows that

$$\langle \zeta_m^l, \zeta_{m'}^{l'} \rangle = 0 \qquad \text{unless } m - l = m' - l'. \tag{2.2}$$

Suppose $m - l = m' - l'$ and $l \neq l'$, then $m \neq m'$. This means that the weight of ζ_m^l is not equal to that of $\zeta_{m'}^{l'}$, and, as a result, $\langle \zeta_m^l, \zeta_{m'}^{l'} \rangle = 0$. From (2.1) and (2.2), one can show the result.

(ii) From (i) one can show this easily, then we omit the proof. □

Now we prove Theorem 2.1. For $n \geq l_1 + l_2 + 2$ $(n - l_1 - l_2 \in \mathbb{N})$

$$dim(V_{l_1} \otimes V_{l_2})(q^{-n}) = dim\left(\bigoplus_{\substack{l \in L_1(l_1, l_2) \\ m \in I_l}} \mathbb{C}\zeta_m^l\right)(q^{-n}),$$

and consequently, $V_{l_1} \otimes V_{l_2} = \bigoplus_{\substack{l \in L_1(l_1, l_2) \\ m \in I_l}} \mathbb{C}\zeta_m^l$. Making use of Lemma 2.4, we obtain Theorem 2.1. □

§3. Clebsch-Gordan Coefficients for $\mathscr{U}_q(su(1, 1))$

In addition to l_1 and l_2, we fix $l \in L_1(l_1, l_2)$ to the end of Section 4. Let $\tilde{\xi}_m^l = (c_m^l \langle \zeta_{l+1}^l, \zeta_{l+1}^l \rangle)^{-(1/2)} \zeta_m^l$ (resp. $\tilde{\xi}_{m_1}^{l_1} = (c_{m_1}^{l_1})^{-(1/2)} \xi_{m_1}^{l_1}$, $\tilde{\xi}_{m_2}^{l_2} = (c_{m_2}^{l_2})^{-1/2} \xi_{m_2}^{l_2}$). The set $\{\tilde{\xi}_m^l | m \in I_l\}$ (resp. $\{\tilde{\xi}_{m_1}^{l_1} | m_1 \in I_{l_1}\}$, $\{\tilde{\xi}_{m_2}^{l_2} m_2 \in I_{l_2}\}$) is an orthonormal basis of V_l (resp. V_{l_1}, V_{l_2}). By Theorem 2.1, $\tilde{\xi}_m^l$ is denoted as

$$\tilde{\xi}_m^l = \sum_{m_1 \in I_{l_1}, m_2 \in I_{l_2}} \begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix} \tilde{\xi}_{m_1}^{l_1} \otimes \tilde{\xi}_{m_2}^{l_2}, \tag{3.1}$$

where all $\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix} \in \mathbb{C}$ but finite are zero. We call these coefficients $\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}$ the Clebsch-Gordan coefficients for $\mathscr{U}_q(su(1, 1))$.

The next proposition is the key to obtain these coefficients.

Proposition 3.1. *We have*

$$f^{m-l-1}\zeta_{l+1}^l = \sum_{m_1 \in I_{l_1}, m_2 \in I_{l_2}} r\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix} \xi_{m_1}^{l_1} \otimes \xi_{m_2}^{l_2}, \tag{3.2}$$

where

$$r\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}$$
$$= \delta_{m,m_1+m_2}(-1)^{m_1-l_1-1} q^{(1/2-l_2+m_1)(m-l-1)} q^{(m_1-l_1-1)(2m_1-l+l_1+l_2)}$$

$$\times \sum_{k\geq 0}(-1)^k q^{3k^2+k(2l-4m_1-2m+3)}\frac{[m-l-1]![m_1+l_1]!}{[k]![m-l-1-k]![m_1+l_1-k]!}$$

$$\times\frac{[m_2+l_2]![l-l_1-l_2-1]!}{[m_1-l_1-1-k]![(m_2+l_2)-(m-l-1)+k]![(m_2-l_2-1)-(m-l-1)+k]!}.$$

The sum over k is taken such that none of the factorials could have a negative integer.

Proof. The weight of $f^{m-l-1}\zeta^l_{l+1}$ is q^{-m}, and, as a result,

$$r\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m\end{bmatrix}=0 \qquad \text{unless } m_1+m_2=m\,.$$

Applying f to both sides of (3.2), we obtain the recurrence relation of $r\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m-m_1 & m\end{bmatrix}$:

$$\text{(3.3)}\quad r\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m-m_1+1 & m+1\end{bmatrix}$$

$$= q^{-(1/2)-l_1+m_1-m}[m_1+l_1]r\begin{bmatrix} l_1 & l_2 & l \\ m_1-1 & m-m_1+1 & m\end{bmatrix}$$

$$+ q^{(1/2)-l_2+m_1}[m-m_1+l_2+1]r\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m-m_1 & m\end{bmatrix}.$$

Putting $a(m_1,m)=\dfrac{q^{-(1/2+m_1-l_2)(m-l-1)}}{[m_1+l_1]![m-m_1+l_2]!}r\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m-m_1 & m\end{bmatrix}$, (3.3) turns out to be

$$a(m_1,m+1)=q^{l-l_1+l_2-2m}a(m_1-1,m)+a(m_1,m)\,.$$

Solving this recurrence relation under the condition

$$a(m_1,l+1)=(-1)^{m_1-l_1-1}q^{(m_1-l_1-1)(2m_1-l+l_1+l_2)}$$

$$\times\frac{[l-l_1-l_2-1]!}{[m_1-l_1-1]![l-l_2-m_1]![m_1+l_1]![l+l_2-m_1+1]!},$$

we get

$$a(m_1,m)=(-1)^{m_1-l_1-1}q^{(m_1-l_1-1)(2m_1-l+l_1+l_2)}$$

$$\times\sum_{k=0}^{m-l-1}(-1)^k q^{3k^2+k(2l-4m_1-2m+3)}\frac{1}{[l+l_1+l_2+1]!}$$

$$\times\begin{bmatrix} m-l-1 \\ k\end{bmatrix}\begin{bmatrix} l+l_1+l_2+1 \\ m_1+l_1-k\end{bmatrix}\begin{bmatrix} l-l_1-l_2-1 \\ m_1-l_1-1-k\end{bmatrix}$$

Hence Proposition 3.1 is proved. □

Moreover we use the following lemma.

Lemma 3.2.

(i) $\langle f^{m-l-1}\zeta^l_{l+1}, f^{m-l-1}\zeta^l_{l+1}\rangle$

$$= q^{-(m+l-1)(m-l-1)}\frac{[m-l-1]![m+l]!}{[2l+1]!}\langle \zeta^l_{l+1}, \zeta^l_{l+1}\rangle .$$

(ii) $\displaystyle\sum_{k=0}^{h} q^{2k(k+m-h)}\begin{bmatrix} n \\ k\end{bmatrix}\begin{bmatrix} m \\ h-k\end{bmatrix} = \begin{bmatrix} m+n \\ h\end{bmatrix}.$

(iii) $\langle \zeta^l_{l+1}, \zeta^l_{l+1}\rangle = q^{-(l-l_1-l_2-1)(l-l_1-l_2)}$

$$\times \frac{[2l]![l-l_1-l_2-1]![2l_1+1]![2l_2+1]!}{[l-l_1+l_2]![l+l_1-l_2]![l+l_1+l_2+1]!}.$$

Proof. We omit the proof of (ii) (cf. [1]). One can easily prove (i) by (2.1) and (iii) by (ii), and then we also omit the proof. □

By the definition of $r\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m\end{bmatrix}$

$$\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m\end{bmatrix} = r\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m\end{bmatrix}(c^{l_1}_{m_1})^{1/2}(c^{l_2}_{m_2})^{1/2}\langle f^{m-l-1}\zeta^l_{l+1}, f^{m-l-1}\zeta^l_{l+1}\rangle^{-1/2},$$

and hence we obtain

Theorem 3.3. *The Clebsch-Gordan coefficient is expressed as*

(3.4)

$$\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m\end{bmatrix}$$

$$= \delta_{m,m_1+m_2}(-1)^{m_1-l_1-1}q^{2m_1(m-l-1)+(m_1+l_1)(m_1-l_1-1)}$$

$$\times\left(\frac{[l-l_1+l_2]![l+l_1-l_2]![l-l_1-l_2-1]![l+l_1+l_2+1]![2l+1]}{[m-l-1]![m+l]![m_1-l_1-1]![m_1+l_1]![m_2-l_2-1]![m_2+l_2]!}\right)^{1/2}$$

$$\times \sum_{k\geq 0}(-1)^k q^{3k^2+k(2l-4m_1-2m+3)}$$

$$\times \frac{[m-l-1]![m_1+l_1]![m_1-l_1-1]![m_2+l_2]![m_2-l_2-1]!}{[k]![m-l-1-k]![m_1+l_1-k]![m_1-l_1-1-k]![(m_2+l_2)-(m-l-1)+k]!}$$

$$\times \frac{1}{[(m_2-l_2-1)-(m-l-1)+k]!},$$

where the sum over k is taken such that none of the factorials could have a negative integer.

§4. Another Way to Calculate Clebsch-Gordan Coefficients

We first define $\mathscr{U}_q(\mathfrak{sl}(2))$ invariant vectors.

Definition 4.1. Let V be a representation of $\mathscr{U}_q(\mathfrak{sl}(2))$. $I \in V$ is a $\mathscr{U}_q(\mathfrak{sl}(2))$ *invariant vector* if I satisfies the condition

$$kI = I, \qquad eI = fI = 0.$$

Let V be a set of complex valued functions which are defined on the Cartesian product $I_{l_1} \times I_{l_2} \times I_l$. V is a representation of $\mathscr{U}_q(\mathfrak{sl}(2))$ with the following action on the generators: For $F \in V$,

$$(k^{\pm 1}F)(m_1, m_2, m) = q^{\pm(-m_1-m_2+m)}F(m_1, m_2, m),$$

$$\begin{aligned}(eF)(m_1, m_2, m) = &-q^{(1/2)-m_1-m_2+m}([m_1+l_1+1][m_1-l_1])^{1/2}F(m_1+1, m_2, m)\\ &-q^{(1/2)+m_1-m_2+m}([m_2+l_2+1][m_2-l_2])^{1/2}F(m_1, m_2+1, m)\\ &-q^{(3/2)+m_1+m_2-m}([m-l-1][m+l])^{1/2}F(m_1, m_2, m-1),\end{aligned}$$

$$\begin{aligned}(fF)(m_1, m_2, m) = &\,q^{(3/2)-m_1-m_2+m}([m_1-l_1-1][m_1+l_1])^{1/2}F(m_1-1, m_2, m)\\ &+q^{(3/2)+m_1-m_2+m}([m_2-l_2-1][m_2+l_2])^{1/2}F(m_1, m_2-1, m)\\ &+q^{(1/2)+m_1+m_2-m}([m+l+1][m-l])^{1/2}F(m_1, m_2, m+1).\end{aligned}$$

The following proposition plays an important role in this section.

Proposition 4.2. *The dimension of the subspace of $\mathscr{U}_q(\mathfrak{sl}(2))$ invariant vectors of V is less than or equal to* 1.

Proof. Let I_1 and I_2 be $\mathscr{U}_q(\mathfrak{sl}(2))$ invariant vectors such that $I_1 \neq 0$ and $I_2 \neq 0$. To prove this proposition, it suffices to show that there exists $\alpha \in \mathbb{C}$ such that $I_1 = \alpha I_2$. By the condition $kI_i = I_i$ $(i = 1, 2)$

$$I_i(m_1, m_2, m) = 0 \qquad \text{unless } m_1 + m_2 = m.$$

We set $\tilde{I}_i(m_1, m) = I_i(m_1, m - m_1, m)$. The condition $eI_i = fI_i = 0$ turns out to be

$$\begin{aligned}(4.1)\qquad &q^{3/2}([m_1+l_1+1][m_1-l_1])^{1/2}\tilde{I}_i(m_1+1, m)\\ &+q^{(3/2)+2m_1}([m-m_1+l_2][m-m_1-l_2-1])^{1/2}\tilde{I}_i(m_1, m)\\ &+q^{1/2}([m-l-1][m+l])^{1/2}\tilde{I}_i(m_1, m-1) = 0,\end{aligned}$$

$$\begin{aligned}(4.2)\qquad &q^{1/2}([m_1-l_1-1][m_1+l_1])^{1/2}\tilde{I}_i(m_1-1, m)\\ &+q^{(1/2)+2m_1}([m-m_1-l_2][m-m_1+l_2+1])^{1/2}\tilde{I}_i(m_1, m)\\ &+q^{3/2}([m+l+1][m-l])^{1/2}\tilde{I}_i(m_1, m+1) = 0.\end{aligned}$$

Since $I_i \neq 0$, $\tilde{I}_i(l_1+1, l+1) \neq 0$ by (4.1) and (4.2). Then we put $\alpha = \dfrac{\tilde{I}_1(l_1+1, l+1)}{\tilde{I}_2(l_1+1, l+1)}$. Making use of (4.1) and (4.2), one can show $\tilde{I}_1(m_1, m) = \alpha\tilde{I}_2(m_1, m)$ by induction on m_1 and m. □

We define two vectors $I, J \in V$:

$$I(m_1, m_2, m) = \delta_{m, m_1+m_2}\left(\sum_{(\nu,\alpha,\beta)\in S(m_1,m)} I_{\nu,\alpha,\beta}\right) r_1(m_1, m_2, m),$$

$$J(m_1, m_2, m) = (-1)^{m-l-1} q^{-m}\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix},$$

where

$$I_{\nu,\alpha,\beta} = (-1)^{\nu+\alpha+\beta} q^{-(1/2)(l+l_1+l_2+3)-\nu-(\alpha^2+\beta^2)-2(l+1)(\alpha+\beta)} q^{-2(l_1-l_2)(\alpha-\beta)+\alpha(l-l_1-l_2-1-\nu)}$$
$$\times\, q^{(l-l_1-l_2-1-\nu)(-l+l_1-l_2-1-\beta)+\alpha(-l+l_1-l_2-1-\beta)}$$
$$\times \begin{bmatrix} l-l_1-l_2-1 \\ \nu \end{bmatrix}\begin{bmatrix} l+l_1-l_2+\alpha \\ \alpha \end{bmatrix}\begin{bmatrix} l-l_1+l_2+\beta \\ \beta \end{bmatrix},$$

$$S(m_1, m) = \{(\nu, \alpha, \beta) \in \mathbb{N}^3 \mid \alpha+\beta = m-l-1,\ \alpha+\nu = m_1-l_1-1\},$$

$$r_1(m_1, m_2, m) = r_1(l_1, m_1) r_1(l_2, m_2) r_1(l, m),$$

and

$$r_1(l, m) = (-1)^{m-l-1} q^{1/2(m-l-1)(m+3l+2)}\left(\frac{[m-l-1]![2l+1]!}{[m+l]!}\right)^{1/2}.$$

Proposition 4.3. *The vectors I and J are $\mathcal{U}_q(s\ell(2))$ invariant.*

For the proof, we use the following lemma.

Lemma 4.4.

(i) $$-q^{(3/2)-m}([m+l][m-l-1])^{1/2}\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m-1 \end{bmatrix}$$
$$= -q^{(1/2)-m_1-m_2}([m_1+l_1+1][m_1-l_1])^{1/2}\begin{bmatrix} l_1 & l_2 & l \\ m_1+1 & m_2 & m \end{bmatrix}$$
$$-q^{(1/2)+m_1-m_2}([m_2+l_2+1][m_2-l_2])^{1/2}\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2+1 & m \end{bmatrix}.$$

(ii) $q^{(1/2)-m}([m-l][m+l+1])^{1/2}\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m+1 \end{bmatrix}$

$$= q^{(3/2)-m_1-m_2}([m_1-l_1-1][m_1+l_1])^{1/2}\begin{bmatrix} l_1 & l_2 & l \\ m_1-1 & m_2 & m \end{bmatrix}$$

$$+ q^{(3/2)+m_1-m_2}([m_2-l_2-1][m_2+l_2])^{1/2}\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2-1 & m \end{bmatrix}.$$

Proof. Applying e and f to both sides of (3.1), we have the above formulas. □

Proof of Proposition 4.3. It is trivial that $kI = I$ and $kJ = J$. By Lemma 4.4 (i),

$$(eJ)(m_1, m_2, m)$$

$$= (-1)^{m-l}q^{(1/2)-m_1-m_2}([m_1+l_1+1][m_1-l_1])^{1/2}\begin{bmatrix} l_1 & l_2 & l \\ m_1+1 & m_2 & m \end{bmatrix}$$

$$+ (-1)^{m-l}q^{(1/2)+m_1-m_2}([m_2+l_2+1][m_2-l_2])^{1/2}\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2+1 & m \end{bmatrix}$$

$$+ (-1)^{m-l-1}q^{(5/2)+m_1+m_2-2m}([m-l-1][m+l])^{1/2}\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m-1 \end{bmatrix}$$

$$= 0.$$

In the same way as above, one can prove $fJ = 0$ by Lemma 4.4 (ii), and then we omit it.

Next we show $eI = fI = 0$. Making use of the formula

$$r_1(l, m) = -q^{m+l}\left(\frac{[m-l-1]}{[m+l]}\right)^{1/2} r_1(l, m-1),$$

we have

$$(eI)(m_1, m_2, m)$$

$$= \delta_{m,m_1+m_2+1} r_1(m_1, m_2, m)\left\{q^{(5/2)+m_1+l_1}[m_1-l_1]\left(\sum_{\substack{(\nu,\alpha,\beta)\\ \in S(m_1+1,m)}} I_{\nu,\alpha,\beta}\right)\right.$$

$$+ q^{(3/2)+m_1+m+l_2}[m-m_1-l_2-1]\left(\sum_{\substack{(\nu,\alpha,\beta)\\ \in S(m_1,m)}} I_{\nu,\alpha,\beta}\right)$$

$$\left.+ q^{(1/2)-m-l}[m+l]\left(\sum_{\substack{(\nu,\alpha,\beta)\\ \in S(m_1,m-1)}} I_{\nu,\alpha,\beta}\right)\right\}.$$

We calculate the each term in the above equation as follows.

$$q^{(5/2)+m_1+l_1}[m_1 - l_1]\Bigg(\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1+1,m)}} I_{v,\alpha,\beta}\Bigg)$$

$$= \sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1+1,m)}} q^{(5/2)+v+\alpha+2l_1}([\alpha] + q^{2\alpha}[v])I_{v,\alpha,\beta}$$

$$= \sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m-1)}} q^{(7/2)+v+\alpha+2l_1}[\alpha + 1]I_{v,\alpha+1,\beta} + \sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m)}} q^{(7/2)+v+3\alpha+2l_1}[v + 1]I_{v+1,\alpha,\beta}\,.$$

$$q^{(3/2)+m_1+m+l_2}[m - m_1 - l_2 - 1] \sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m)}} I_{v,\alpha,\beta}$$

$$= \sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m)}} q^{(7/2)+v+2\alpha+\beta+l+l_1+l_2}([l - l_1 - l_2 - 1 - v] + q^{2(l-l_1-l_2-1-v)}[\beta])I_{v,\alpha,\beta}$$

$$= \sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m)}} q^{(7/2)+v+2\alpha+\beta+l+l_1+l_2}[l - l_1 - l_2 - 1 - v]I_{v,\alpha,\beta}$$

$$+ \sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m-1)}} q^{(5/2)-v+2\alpha+\beta+3l-l_1-l_2}[\beta + 1]I_{v,\alpha,\beta+1}\,.$$

$$q^{(1/2)-m-l}[m + l]\Bigg(\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m-1)}} I_{v,\alpha,\beta}\Bigg)$$

$$= \sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m-1)}} q^{-(3/2)-\alpha-\beta-2l}$$

$$\times ([l + l_1 - l_2 + 1 + \alpha] + q^{2(l+l_1-l_2+1+\alpha)}[l - l_1 + l_2 + 1 + \beta])I_{v,\alpha,\beta}$$

$$= \sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m-1)}} q^{-(3/2)-\alpha-\beta-2l}[l + l_1 - l_2 + 1 + \alpha]I_{v,\alpha,\beta}$$

$$+ \sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m-1)}} q^{(1/2)+\alpha-\beta+2l_1-2l_2}[l - l_1 + l_2 + 1 + \beta]I_{v,\alpha,\beta}\,.$$

Thus we get

$$(eI)(m_1, m_2, m)$$

$$= \delta_{m,m_1+m_2+1}r_1(m_1, m_2, m)\Bigg\{\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m-1)}} (q^{(7/2)+v+\alpha+2l_1}[\alpha+1]I_{v,\alpha+1,\beta}$$

$$+ q^{-(3/2)-\alpha-\beta-2l}[l+l_1-l_2+1+\alpha]I_{v,\alpha,\beta}) + \sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m)}} (q^{(7/2)+v+3\alpha+2l_1}[v+1]I_{v+1,\alpha,\beta}$$

$$+q^{(7/2)+v+2\alpha+\beta+l+l_1+l_2}[l-l_1-l_2-1-v]I_{v,\alpha,\beta})$$

$$+\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m-1)}}(q^{(5/2)-v+2\alpha+\beta+3l-l_1-l_2}[\beta+1]I_{v,\alpha,\beta+1}$$

$$+q^{(1/2)+\alpha-\beta+2l_1-2l_2}[l-l_1+l_2+1+\beta]I_{v,\alpha,\beta})\Big\}.$$

By straightforward computation, we can show that the each term in the above equation is zero. Thus we obtain $eI=0$. Moreover

$$(fI)(m_1,m_2,m)$$
$$=\delta_{m+1,m_1+m_2}(-1)r_1(m_1,m_2,m)\Big\{q^{(1/2)-m_1-l_1}[m_1+l_1]\Big(\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1-1,m)}}I_{v,\alpha,\beta}\Big)$$
$$+q^{-(1/2)+3m_1-m-l_2}[m_2+l_2]\Big(\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m)}}I_{v,\alpha,\beta}\Big)$$
$$+q^{(5/2)+m+l}[m-l]\Big(\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m+1)}}I_{v,\alpha,\beta}\Big)\Big\}.$$

We calculate under the condition $m+1=m_1+m_2$.

$$q^{(1/2)-m_1-l_1}[m_1+l_1]\Big(\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1-1,m)}}I_{v,\alpha,\beta}\Big)$$
$$=\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1-1,m)}}q^{-(3/2)-v-\alpha-2l_1}([l+l_1-l_2+1+\alpha]-q^{2(2+v+\alpha+2l_1)}$$
$$\times[l-l_1-l_2-1-v])I_{v,\alpha,\beta}$$
$$=\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1-1,m)}}q^{-(3/2)-v-\alpha-2l_1}[l+l_1-l_2+1+\alpha]I_{v,\alpha,\beta}$$
$$-\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1-1,m)}}q^{(5/2)+v+\alpha+2l_1}[l-l_1-l_2-1-v]I_{v,\alpha,\beta}.$$

$$q^{-(1/2)+3m_1-m-l_2}[m_2+l_2]\Big(\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m)}}I_{v,\alpha,\beta}\Big)$$
$$=\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m)}}q^{(3/2)+v+2\alpha-\beta-l+3l_1-l_2}([l-l_1+l_2+1+\beta]-[v])I_{v,\alpha,\beta}$$
$$=\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1,m)}}q^{(3/2)+v+2\alpha-\beta-l+3l_1-l_2}[l-l_1+l_2+1+\beta]I_{v,\alpha,\beta}$$
$$-\sum_{\substack{(v,\alpha,\beta)\\ \in S(m_1-1,m)}}q^{(5/2)+v+2\alpha-\beta-l+3l_1-l_2}[v+1]I_{v+1,\alpha,\beta}.$$

$$q^{(5/2)+m+l}[m-l]\Big(\sum_{\substack{(\nu,\alpha,\beta)\\ \in S(m_1,m+1)}} I_{\nu,\alpha,\beta}\Big)$$

$$= \sum_{\substack{(\nu,\alpha,\beta)\\ \in S(m_1,m+1)}} q^{(5/2)+\alpha+\beta+2l}([\alpha]+q^{2\alpha}[\beta])I_{\nu,\alpha,\beta}$$

$$= \sum_{\substack{(\nu,\alpha,\beta)\\ \in S(m_1-1,m)}} q^{(7/2)+\alpha+\beta+2l}[\alpha+1]I_{\nu,\alpha+1,\beta} + \sum_{\substack{(\nu,\alpha,\beta)\\ \in S(m_1,m)}} q^{(7/2)+3\alpha+\beta+2l}[\beta+1]I_{\nu,\alpha,\beta+1}\,.$$

Therefore we obtain

$$\begin{aligned}
&(fI)(m_1, m_2, m)\\
&= \delta_{m+1,m_1+m_2}(-1)r_1(m_1, m_2, m)\\
&\quad\times\Bigg\{\sum_{\substack{(\nu,\alpha,\beta)\\ \in S(m_1-1,m)}} (q^{-(3/2)-\nu-\alpha-2l_1}[l+l_1-l_2+1+\alpha]I_{\nu,\alpha,\beta}\\
&\quad+ q^{(7/2)+\alpha+\beta+2l}[\alpha+1]I_{\nu,\alpha+1,\beta})\\
&\quad- \sum_{\substack{(\nu,\alpha,\beta)\\ \in S(m_1-1,m)}} (q^{(5/2)+\nu+\alpha+2l_1}[l-l_1-l_2-1-\nu]I_{\nu,\alpha,\beta}\\
&\quad+ q^{(5/2)+\nu+2\alpha-\beta-l+3l_1-l_2}[\nu+1]I_{\nu+1,\alpha,\beta})\\
&\quad+ \sum_{\substack{(\nu,\alpha,\beta)\\ \in S(m_1,m)}} (q^{(3/2)+\nu+2\alpha-\beta-l+3l_1-l_2}[l-l_1+l_2+1+\beta]I_{\nu,\alpha,\beta}\\
&\quad+ q^{(7/2)+3\alpha+\beta+2l}[\beta+1]I_{\nu,\alpha,\beta+1})\Bigg\}\\
&= 0\,.
\end{aligned}$$

Hence the proof of Proposition 4.3 is completed. □

Proposition 4.2 and the value of $\begin{bmatrix} l_1 & l_2 & l \\ l_1+1 & m-l_1-1 & m \end{bmatrix}$ lead us to the following theorem.

Theorem 4.5. *The Clebsch-Gordan coefficient is expressed as*

(4.3)

$$\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}$$

$$= \delta_{m,m_1+m_2} q^{2(l_1+1)\{(m-l-1)-(m_1-l_1-1)\}}$$

$$\times\left(\frac{[m-l-1]![m_1-l_1-1]![m_2-l_2-1]![l-l_1-l_2-1]![l+l_1+l_2+1]![2l+1]}{[m+l]![m_1+l_1]![m_2+l_2]![l+l_1-l_2]![l-l_1+l_2]!}\right)^{1/2}$$

$$\times \sum_{v} (-1)^v q^{v^2+v(4l_1+3)}$$

$$\times \frac{[(m+l)-(m_2+l_2)-1-v]![m_2+l_2+v]!}{[v]![l-l_1-l_2-1-v]![m_1-l_1-1-v]![(m-l-1)-(m_1-l_1-1)+v]!},$$

where the sum over v *is taken such that none of the factorials could have a negative integer.*

There is a simple relation between the Clebsch-Gordan coefficients and the basic hypergeometric series.

Let $0 < q < 1$ and $r \in \mathbb{N}\backslash\{0\}$. We define basic hypergeometric series ${}_r\phi_{r-1}$ by

$${}_r\phi_{r-1}\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_{r-1} \end{matrix}; q, z\right] = \sum_{j=0}^{\infty} \frac{(a_1; q)_j (a_2; q)_j \ldots (a_r; q)_j}{(q; q)_j (b_1; q)_j \ldots (b_{r-1}; q)_j} z^j,$$

where

$$(a; q)_j = \begin{cases} 1, & j = 0, \\ (1-a)(1-aq)\ldots(1-aq^{j-1}), & j = 1, 2, \ldots, \end{cases}$$

and it is assumed that the parameters $b_1, \ldots, b_{r-1}$ are such that the denominator factors in the terms of the series are never zero [3]. Then the Clebsch-Gordan coefficient (3.4) is expressed by the basic hypergeometric series ${}_3\phi_2$ as follows.

(3.4)′ If $(m_2 - l_2 - 1) - (m - l - 1) \geq 0$,

$$\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}$$

$$= \delta_{m,m_1+m_2} (-1)^{m_1-l_1-1} q^{2m_1(m-l-1)+(m_1+l_1)(m_1-l_1-1)}$$

$$\times \left(\frac{[l-l_1+l_2]![l+l_1-l_2]![l-l_1-l_2-1]![l+l_1+l_2+1]![2l+1]}{(m-l-1]![m+l]![m_1-l_1-1]![m_1+l_1]![m_2-l_2-1]![m_2+l_2]!} \right)^{1/2}$$

$$\times \frac{[m_2+l_2]![m_2-l_2-1]!}{[(m_2+l_2)-(m-l-1)]![(m_2-l_2-1)-(m-l-1)]!}$$

$$\times {}_3\phi_2\left[\begin{matrix} q^{-2(m-l-1)}, q^{-2(m_1+l_1)}, q^{-2(m_1-l_1-1)} \\ q^{2\{(m_2+l_2)-(m-l-1)+1\}}, q^{2\{(m_2-l_2-1)-(m-l-1)+1\}} \end{matrix}; q^2, q^2\right].$$

(3.4)″ If $(m_2 - l_2 - 1) - (m - l - 1) \leq 0$,

$$\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}$$

$$= \delta_{m,m_1+m_2} (-1)^{l+l_1+l_2+1} q^{2m_1(m-l-1)+(m_1+l_1)(m_1-l_1-1)}$$

$$\times q^{\{(m-l-1)-(m_2-l_2-1)\}(-3m+m_2-l+3l_2+3)}$$

$$\times\left(\frac{[m-l-1]![m_1-l_1-1]![m_1+l_1]![m_2+l_2]![l-l_1+l_2]![l+l_1+l_2+1]!}{[m+l]![m_2-l_2-1]![l+l_1-l_2]![l-l_1-l_2-1]!}\right)^{1/2}$$

$$\times\frac{[2l+1]^{1/2}}{[(m-l-1)-(m_2-l_2-1)]![2l_2+1]!}$$

$$\times\,{}_3\phi_2\left[\begin{matrix} q^{-2(l+l_1-l_2)},\ q^{-2(l-l_1-l_2-1)},\ q^{-2(m_2-l_2-1)} \\ q^{2\{(m-l-1)-(m_2-l_2-1)+1\}},\ q^{2(2l_2+2)} \end{matrix};q^2,q^2\right].$$

On the other hand, the Clebsch-Gordan coefficient (4.3) is also expressed by the basic hypergeometric series ${}_3\phi_2$ as

(4.3)′ If $(m-l-1)-(m_1-l_1-1)\geq 0$,

$$\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}$$

$$=\delta_{m,m_1+m_2}q^{2(l_1+l)\{(m-l-1)-(m_1-l_1-1)\}}$$

$$\times\left(\frac{[m-l-1]![m_2+l_2]![m_2-l_2-1]![l+l_1+l_2+1]![2l+1]}{[m+l]![m_1+l_1]![m_1-l_1-1]![l-l_1-l_2-1]![l+l_1-l_2]![l-l_1+l_2]!}\right)^{1/2}$$

$$\times\frac{[(m+l)-(m_2+l_2)-1]!}{[(m-l-1)-(m_1-l_1-1)]!}$$

$$\times\,{}_3\phi_2\left[\begin{matrix} q^{-2(m_1-l_1-1)},\ q^{-2(m_2+l_2+1)},\ q^{-2(l-l_1-l_2-1)} \\ q^{2\{(m-l-1)-(m_1-l_1-1)+1\}},\ q^{-2\{(m+l)-(m_2+l_2)-1\}} \end{matrix};q^2,q^2\right].$$

(4.3)″ If $(m-l-1)-(m_1-l_1-1)\leq 0$,

$$\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}$$

$$=\delta_{m,m_1+m_2}q^{\{(m-l-1)-(m_1-l_1-1)\}\{(m-l-1)-(m_1+l_1)\}}$$

$$\times\left(\frac{[m_1-l_1-1]![l-l_1-l_2-1]![l+l_1+l_2+1]![2l+1]}{[m-l-1]![m+l]![m_1+l_1]![m_2-l_2-1]![m_2+l_2]![l+l_1-l_2]!}\right)^{1/2}$$

$$\times\frac{[(m_1+l_1)+(m_2-l_2-1)]!}{[(m_1-l_1-1)-(m-l-1)]!}$$

$$\times\,{}_3\phi_2\left[\begin{matrix} q^{-2(m-l-1)},\ q^{-2(m_2-l_2-1)},\ q^{2(l-l_1+l_2+1)} \\ q^{-2\{(m_1+l_1)+(m_2-l_2-1)\}},\ q^{2\{(m_1-l_1-1)-(m-l-1)+1\}} \end{matrix};q^2,q^2\right].$$

§5. Clebsch-Gordan Coefficients for $\mathcal{U}_q(\mathfrak{sl}(2))$

We fix $l_1, l_2 \in \mathbb{N}\cup\mathbb{N}+\frac{1}{2}$ such that $l_1-l_2<1$. The tensor product $W_{l_1}\otimes V_{l_2}$ is decomposed into the direct sum of irreducible components of $\mathcal{U}_q(\mathfrak{sl}(2))$.

Theorem 5.1. *$W_{l_1} \otimes V_{l_2} \simeq \bigoplus_{l \in L_2(l_1,l_2)} V_l$ as representations of $\mathscr{U}_q(\mathfrak{sl}(2))$, where*

$$L_2(l_1, l_2) = \left\{ l \in \left\{ -\frac{1}{2}, 0, \frac{1}{2}, 1, \dots \right\} \middle| \, l + l_1 - l_2 \in \mathbb{N}, \, -l_1 + l_2 \le l \le l_1 + l_2 \right\}.$$

Proof. We define the set of vectors $\{\zeta_m^l | l \in L_2(l_1, l_2), \, m \in I_l\}$ in $W_{l_1} \otimes V_{l_2}$ as follows.

$$\zeta_{l+1}^l = \sum_{m_1=-l+l_2}^{l_1} b_{m_1}^l x_{m_1}^{l_1} \otimes \xi_{m_1+l+1}^{l_2},$$

$$\zeta_m^l = \left\{ \prod_{j=l+1}^{m-1} (q^{(1/2)-l}[j+l+1])^{-1} \right\} f^{m-l-1} \zeta_{l+1}^l \qquad \text{for } m \ge l+2,$$

where $b_{m_1}^l = q^{-(1/2)(l_1-m_1)(2l+l_1-2l_2+3m_1+2)} \begin{bmatrix} l+l_1-l_2 \\ l_1-m_1 \end{bmatrix}$. Making use of these vectors, this theorem is proved in the same way as Theorem 2.1, and then we omit the proof. □

In addition to l_1 and l_2, we fix $l \in L_2(l_1, l_2)$. Let $\tilde{\xi}_m^l = (c_m^l)^{-1/2} \zeta_m^l$ and $\tilde{x}_{m_1}^{l_1} = (d_{m_1}^{l_1})^{-1/2} x_{m_1}^{l_1}$. The set $\{\tilde{\xi}_m^l | m \in I_l\}$ is an orthogonal basis of V_l and the set $\{\tilde{x}_{m_1}^{l_1} | m_1 \in J_{l_1}\}$ is an orthonormal basis of W_{l_1}. By Theorem 5.1, $\tilde{\xi}_m^l$ is denoted as

$$\tilde{\xi}_m^l = \sum_{m_1 \in J_{l_1}, m_2 \in I_{l_2}} \left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right] \tilde{x}_{m_1}^{l_1} \otimes \tilde{\xi}_{m_2}^{l_2}, \tag{5.1}$$

where all $\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right] \in \mathbb{C}$ but finite are zero. We call these coefficients $\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right]$ the Clebsch-Gordan coefficients for $\mathscr{U}_q(\mathfrak{sl}(2))$.

Let V be a set of complex valued functions which are defined on the Cartesian product $J_{l_1} \times I_{l_2} \times I_l$. V is a representation of $\mathscr{U}_q(\mathfrak{sl}(2))$ with the following action: For $F \in V$,

$$(k^{\pm 1}F)(m_1, m_2, m) = q^{\pm(m_1-m_2+m)} F(m_1, m_2, m),$$

$$\begin{aligned}(eF)(m_1, m_2, m) = {} & q^{-m_2+m-l_1+(1/2)}([l_1-m_1+1][l_1+m_1])^{1/2} F(m_1-1, m_2, m) \\ & - q^{-m_1-m_2+m+(1/2)}([m_2+l_2+1][m_2-l_2])^{1/2} F(m_1, m_2+1, m) \\ & - q^{-m_1+m_2-m+(3/2)}([m-l-1][m+l])^{1/2} F(m_1, m_2, m-1),\end{aligned}$$

$$\begin{aligned}(fF)(m_1, m_2, m) = {} & q^{-m_2+m-l_1+(1/2)}([l_1+m_1+1][l_1-m_1])^{1/2} F(m_1+1, m_2, m) \\ & + q^{-m_1-m_2+m+(3/2)}([m_2-l_2-1][m_2+l_2])^{1/2} F(m_1, m_2-1, m) \\ & + q^{-m_1+m_2-m+(1/2)}([m+l+1][m-l])^{1/2} F(m_1, m_2, m+1).\end{aligned}$$

Proposition 5.2. *The dimension of the space of $\mathscr{U}_q(\mathfrak{sl}(2))$ invariant vectors is less than or equal to* 1.

Proof. We omit the proof, because one can prove this in the same way as Proposition 4.2. □

We define two vectors $I, J \in V$:

$$I(m_1, m_2, m) = \delta_{m_1+m,m_2}\left(\sum_{(\nu,\alpha,\beta)\in T(m_1,m)} I_{\nu,\alpha,\beta}\right) r_2(m_1, m_2, m),$$

$$J(m_1, m_2, m) = (-1)^{m-l-1} q^{-m} \left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{matrix}\right]\right],$$

where

$$\begin{aligned} I_{\nu,\alpha,\beta} = (-1)^{\nu+\beta} & q^{(1/2)(l_1+l_2-l-2\alpha)+(1/2)(l+l_1-l_2-2\nu)-2(l+l_1+l_2+2)\beta} \\ & \times q^{-\beta(\beta-1)-(1/2)(l+l_1+l_2+2+2\beta)+\alpha(l+l_1-l_2-\nu)} \\ & \times q^{(l+l_1-l_2-\nu)(-l-l_1-l_2-2-\beta)+\alpha(-l-l_1-l_2-2-\beta)} \\ & \times \begin{bmatrix} l_1+l_2-l \\ \alpha \end{bmatrix} \begin{bmatrix} l+l_1-l_2 \\ \nu \end{bmatrix} \begin{bmatrix} l+l_1+l_2+1+\beta \\ \beta \end{bmatrix}, \end{aligned}$$

$$T(m_1, m) = \{(\nu, \alpha, \beta) \in \mathbb{N}^3 \mid \alpha + \nu = l_1 - m_1, \alpha + \beta = m - l - 1\},$$

$$r_2(m_1, m_2, m) = r_2(l_1, m_1) r_1(l_2, m_2) r_1(l, m),$$

and

$$r_2(l_1, m_1) = ([l_1 + m_1]![l_1 - m_1]!)^{1/2}.$$

Proposition 5.3. *I and J are $\mathscr{U}_q(\mathfrak{sl}(2))$ invariant.*

For the proof, we use the following.

Lemma 5.4.

$$\begin{aligned} \text{(i)} \quad & -q^{-m+(3/2)}([m+l][m-l-1])^{1/2} \left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m-1 \end{matrix}\right]\right] \\ & = q^{-m_1-m-l_1+(3/2)}([l_1-m_1+1][l_1+m_1])^{1/2} \left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1-1 & m_2 & m \end{matrix}\right]\right] \\ & \quad - q^{-m_1-m_2+(1/2)}([m_2+l_2+1][m_2-l_2])^{1/2} \left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2+1 & m \end{matrix}\right]\right]. \end{aligned}$$

(ii) $q^{-m+(1/2)}([m-l][m+l+1])^{1/2}\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m+1 \end{bmatrix}\right]$

$$= q^{-m_1-m-l_1-(1/2)}([l_1+m_1+1][l_1-m_1])^{1/2}\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1+1 & m_2 & m \end{bmatrix}\right]$$

$$+ q^{-m_1-m_2+(3/2)}([m_2-l_2-1][m_2+l_2])^{1/2}\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2-1 & m \end{bmatrix}\right].$$

Proof. Applying e and f to both sides of (5.1), we have (i) and (ii). □

Proof of Proposition 5.3. It is easy to see that $kI = I$ and $kJ = J$, and, by Lemma 5.4, we get $eJ = fJ = 0$.

We show $eI = fI = 0$. Making use of the formula

$$r_2(l_1, m_1) = \left(\frac{[l_1+m_1]}{[l_1-m_1+1]}\right)^{1/2} r_2(l_1, m_1-1),$$

we obtain

$$\begin{aligned}(eI)&(m_1, m_2, m) \\ &= \delta_{m_1+m, m_2+1} r_2(m_1, m_2, m)\Bigg\{q^{-m_1-l_1+(3/2)}[l_1-m_1+1]\Bigg(\sum_{\substack{(\nu,\alpha,\beta)\\ \in T(m_1-1,m)}} I_{\nu,\alpha,\beta}\Bigg) \\ &\quad + q^{-m_1+m+l_2+(3/2)}[m_1+m-l_2-1]\Bigg(\sum_{\substack{(\nu,\alpha,\beta)\\ \in T(m_1,m)}} I_{\nu,\alpha,\beta}\Bigg) \\ &\quad + q^{(1/2)-m-l}[m+l]\Bigg(\sum_{\substack{(\nu,\alpha,\beta)\\ \in T(m_1,m-1)}} I_{\nu,\alpha,\beta}\Bigg)\Bigg\}.\end{aligned}$$

We continue the calculation of the each term in the above equation.

$$\begin{aligned}q^{-m_1-l_1+(3/2)}&[l_1-m_1+1]\Bigg(\sum_{\substack{(\nu,\alpha,\beta)\\ \in T(m_1-1,m)}} I_{\nu,\alpha,\beta}\Bigg) \\ &= \sum_{\substack{(\nu,\alpha,\beta)\\ \in T(m_1-1,m)}} q^{\alpha+\nu-2l+(1/2)}([\alpha]+q^{2\alpha}[\nu])I_{\nu,\alpha,\beta} \\ &= \sum_{\substack{(\nu,\alpha,\beta)\\ \in T(m_1,m-1)}} q^{\alpha+\nu-2l_1+(3/2)}[\alpha+1]I_{\nu,\alpha+1,\beta} + \sum_{\substack{(\nu,\alpha,\beta)\\ \in T(m_1,m)}} q^{3\alpha+\nu-2l_1+(3/2)}[\nu+1]I_{\nu+1,\alpha,\beta}.\end{aligned}$$

$$\begin{aligned}q^{-m_1+m+l_2+(3/2)}&[m_1+m-l_2-1]\Bigg(\sum_{\substack{(\nu,\alpha,\beta)\\ \in T(m_1,m)}} I_{\nu,\alpha,\beta}\Bigg) \\ &= \sum_{\substack{(\nu,\alpha,\beta)\\ \in T(m_1,m)}} q^{2\alpha+\nu+\beta+l-l_1+l_2+(5/2)}([l+l_1-l_2-\nu]+q^{2(l+l_1-l_2-\nu)}[\beta])I_{\nu,\alpha,\beta}\end{aligned}$$

$$= \sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1,m)}} q^{2\alpha+\nu+\beta+l-l_1+l_2+(5/2)}[l + l_1 - l_2 - \nu] I_{\nu,\alpha,\beta}$$

$$+ \sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1,m-1)}} q^{2\alpha-\nu+\beta+3l+l_1-l_2+(7/2)}[\beta + 1] I_{\nu,\alpha,\beta+1} .$$

$$q^{(1/2)-m-l}[m + l]\left(\sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1,m-1)}} I_{\nu,\alpha,\beta} \right)$$

$$= \sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1,m-1)}} q^{\alpha-\beta-2l_1-2l_2-(3/2)}([l + l_1 + l_2 + 2 + \beta] - [l_1 + l_2 - l - \alpha]) I_{\nu,\alpha,\beta}$$

$$= \sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1,m-1)}} q^{\alpha-\beta-2l_1-2l_2-(3/2)}[l + l_1 + l_2 + 2 + \beta] I_{\nu,\alpha,\beta}$$

$$- \sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1,m-1)}} q^{\alpha-\beta-2l_1-2l_2-(3/2)}[l_1 + l_2 - l - \alpha] I_{\nu,\alpha,\beta} .$$

Thus we get

$$(eI)(m_1, m_2, m)$$

$$= \delta_{m_1+m,m_2+1} r_2(m_1, m_2, m)$$

$$\times \left\{ \sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1,m-1)}} (q^{\alpha+\nu-2l_1+(3/2)}[\alpha + 1] I_{\nu,\alpha+1,\beta} - q^{\alpha-\beta-2l_1-2l_2-(3/2)} \right.$$

$$\times [l_1 + l_2 - l - \alpha] I_{\nu,\alpha,\beta})$$

$$+ \sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1,m)}} (q^{3\alpha+\nu-2l_1+(3/2)}[\nu + 1] I_{\nu+1,\alpha,\beta} + q^{2\alpha+\nu+\beta+l-l_1+l_2+(5/2)}$$

$$\times [l + l_1 - l_2 - \nu] I_{\nu,\alpha,\beta})$$

$$+ \sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1,m-1)}} (q^{2\alpha-\nu+\beta+3l+l_1-l_2+(7/2)}[\beta + 1] I_{\nu,\alpha,\beta+1}$$

$$\left. + q^{\alpha-\beta-2l_1-2l_2-(3/2)}[l + l_1 + l_2 + 2 + \beta] I_{\nu,\alpha,\beta} \right\}$$

$$= 0 .$$

Further

$$(fI)(m_1, m_2, m)$$

$$= \delta_{m_1+m+1,m_2} r_2(m_1, m_2, m) \left\{ q^{-m_1-l_1-(1/2)}[l_1 + m_1 + 1] \left(\sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1+1,m)}} I_{\nu,\alpha,\beta} \right) \right.$$

$$-q^{-3m_1-m-l_2-(1/2)}[m_2+l_2]\Bigg(\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1,m)}} I_{v,\alpha,\beta}\Bigg)-q^{m+l+(5/2)}[m-l]$$

$$\times\Bigg(\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1,m+1)}} I_{v,\alpha,\beta}\Bigg)\Bigg\}.$$

We calculate the each term under the condition $m_1+m+1=m_2$.

$$q^{-m_1-l_1-(1/2)}[l_1+m_1+1]\Bigg(\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1+1,m)}} I_{v,\alpha,\beta}\Bigg)$$

$$=\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1+1,m)}} q^{\alpha+v-2l_1+(1/2)}(q^{2(l+l_1-l_2-v)}[l_1+l_2-l-\alpha]+[l+l_1-l_2-v])I_{v,\alpha,\beta}$$

$$=\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1+1,m)}} q^{\alpha-v+2l-2l_2+(1/2)}[l_1+l_2-l-\alpha]I_{v,\alpha,\beta}$$

$$+\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1+1,m)}} q^{\alpha+v-2l_1+(1/2)}[l+l_1-l_2-v]I_{v,\alpha,\beta}\,.$$

$$q^{-3m_1-m-l_2-(1/2)}[m_2+l_2]\Bigg(\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1,m)}} I_{v,\alpha,\beta}\Bigg)$$

$$=\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1,m)}} q^{2\alpha+v-\beta-l-3l_1-l_2-(3/2)}([l+l_1+l_2+2+\beta]-[v])I_{v,\alpha,\beta}$$

$$=\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1,m)}} q^{2\alpha+v-\beta-l-3l_1-l_2-(3/2)}[l+l_1+l_2+2+\beta]I_{v,\alpha,\beta}$$

$$-\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1+1,m)}} q^{2\alpha+v-\beta-l-3l_1-l_2-(1/2)}[v+1]I_{v+1,\alpha,\beta}\,.$$

$$q^{m+l+(5/2)}[m-l]\Bigg(\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1,m+1)}} I_{v,\alpha,\beta}\Bigg)$$

$$=\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1,m+1)}} q^{\alpha+\beta+2l+(5/2)}([\alpha]+q^{2\alpha}[\beta])I_{v,\alpha,\beta}$$

$$=\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1+1,m)}} q^{\alpha+\beta+2l+(7/2)}[\alpha+1]I_{v,\alpha+1,\beta}+\sum_{\substack{(v,\alpha,\beta)\\ \in T(m_1,m)}} q^{3\alpha+\beta+2l+(7/2)}[\beta+1]I_{v,\alpha,\beta+1}\,.$$

Therefore

$$
\begin{aligned}
&(fI)(m_1, m_2, m) \\
&= \delta_{m_1+m+1,m_2} r_2(m_1, m_2, m) \\
&\quad \times \Bigg\{ \sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1+1,m)}} (q^{\alpha-\nu+2l-2l_2+(1/2)}[l_1 + l_2 - l - \alpha] I_{\nu,\alpha,\beta} \\
&\quad - q^{\alpha+\beta+2l+(7/2)}[\alpha + 1] I_{\nu,\alpha+1,\beta}) \\
&\quad + \sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1+1,m)}} (q^{\alpha+\nu-2l_1+(1/2)}[l + l_1 - l_2 - \nu] I_{\nu,\alpha,\beta} \\
&\quad + q^{2\alpha+\nu-\beta-l-3l_1-l_2-(1/2)}[\nu + 1] I_{\nu+1,\alpha,\beta}) \\
&\quad - \sum_{\substack{(\nu,\alpha,\beta) \\ \in T(m_1,m)}} (q^{2\alpha+\nu-\beta-l-3l_1-l_2-(3/2)}[l + l_1 + l_2 + 2 + \beta] I_{\nu,\alpha,\beta} \\
&\quad + q^{3\alpha+\beta+2l+(7/2)}[\beta + 1] I_{\nu,\alpha,\beta+1}) \Bigg\} \\
&= 0 .
\end{aligned}
$$

□

From Proposition 5.2, we have

Theorem 5.5. *The Clebsch-Gordan coefficient is expressed as*

(5.2)

$$
\begin{aligned}
&\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right] \\
&= \delta_{m_1+m,m_2} q^{(1/2)(m_1-l+l_2)(m_1+l-l_2+1)+2(l_1-m_1)(l+l_1-l_2)-2l_1(m-l-1)} q^{-l_1(m_1+l-l_2+1)} \\
&\quad \times \left(\frac{[l_1+m_1]![l_1-m_1]![m_2-l_2-1]![m-l-1]![2l_2+1]![2l+1]!}{[m_2+l_2]![m+l]![2l_1]!} \right)^{1/2} \\
&\quad \times \sum_{\nu \geq 0} q^{-2\nu(l+2l_1-l_2-m_1-\nu)} \frac{[l+l_1-l_2]![l_1+l_2-l]![m_2+l_2+1+\nu]!}{[\nu]![l+l_1-l_2-\nu]![l_1-m_1-\nu]![(m_2+l_2)-(m+l)+\nu]!} \\
&\quad \times \frac{1}{[(m-l-1)-(l_1-m_1)+\nu]![l+l_1+l_2+2]!},
\end{aligned}
$$

where the sum over ν is taken such that none of the factorials could have a negative integer.

This coefficient is also expressed by the basic hypergeometric series ${}_3\phi_2$:
(5.2)′ If $(m_2 + l_2) - (m + l) \geq 0$ and $(m - l - 1) - (l_1 - m_1) \geq 0$,

$$\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{matrix}\right]\right]$$

$$= \delta_{m_1+m,m_2} q^{(1/2)(m_1-l+l_2)(m_1+l-l_2+1)+2(l_1-m_1)(l+l_1-l_2)-2l_1(m-l-1)} q^{-l_1(m_1+l-l_2+1)}$$

$$\times \left(\frac{[l_1+m_1]![l_1-m_1]![m_2-l_2-1]![m-l-1]![2l_2+1]![2l+1]!}{[m_2+l_2]![m+l]![2l_1]!} \right)^{1/2}$$

$$\times \frac{[l_1+l_2-l]![m_2+l_2+1]!}{[l_1-m_1]![(m_2+l_2)-(m+l)]![(m-l-1)-(l_1-m_1)]![l+l_1+l_2+2]!}$$

$$\times {}_3\phi_2 \left[\begin{matrix} q^{-2(l+l_1-l_2)}, q^{-2(l_1-m_1)}, q^{2(m_2+l_2+2)} \\ q^{2\{(m_2+l_2)-(m+l)+1\}}, q^{2\{(m-l-1)-(l_1-m_1)+1\}} \end{matrix} ; q^2, q^2 \right].$$

(5.2)″ If $(l_1 - m_1) - (m - l - 1) \geq 0$ and $(l_1 + m_1) - (m_2 - l_2 - 1) \geq 0$,

$$\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{matrix}\right]\right]$$

$$= \delta_{m_1+m,m_2} q^{(1/2)(m_1-l+l_2)(m_1+l-l_2+1)+2(l_1-m_1)(l+l_1-l_2)-2l_1(m-l-1)}$$

$$\times q^{-l_1(m_1+l-l_2+1)-2\{(l_1-m_1)-(m-l-1)\}\{(l_1-m_1)+(m_2-l_2-1)\}}$$

$$\times \left(\frac{[l_1+m_1]![l_1-m_1]![2l_2+1]![2l+1]!}{[m_2-l_2-1]![m_2+l_2]![m-l-1]![m+l]![2l_1]!} \right)^{1/2}$$

$$\times \frac{[l+l_1-l_2]![l_1+l_2-l]!}{[(l_1-m_1)-(m-l-1)]![(l_1+m_1)-(m_2-l_2-1)]!}$$

$$\times {}_3\phi_2 \left[\begin{matrix} q^{-2(m-l-1)}, q^{-2(m_2-l_2-1)}, q^{2(l+l_1+l_2+3)} \\ q^{2\{(l_1-m_1)-(m-l-1)+1\}}, q^{2\{(l_1+m_1)-(m_2-l_2-1)+1\}} \end{matrix} ; q^2, q^2 \right].$$

(5.2)‴ If $(m + l) - (m_2 + l_2) \geq 0$ and $(m_2 - l_2 - 1) - (l_1 + m_1) \geq 0$,

$$\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{matrix}\right]\right]$$

$$= \delta_{m_1+m,m_2} q^{(1/2)(m_1-l+l_2)(m_1+l-l_2+1)+2(l_1-m_1)(l+l_1-l_2)-2l_1(m-l-1)}$$

$$\times q^{-l_1(m_1+l-l_2+1)-4l_1\{(m+l)-(m_2+l_2)\}}$$

$$\times \left(\frac{[l_1-m_1]![m_2-l_2-1]![m-l-1]![2l_2+1]![2l+1]!}{[l_1+m_1]![m_2+l_2]![m+l]![2l_1]!} \right)^{1/2}$$

$$\times \frac{[m+l+1]![l+l_1-l_2]!}{[(m+l)-(m_2+l_2)]![(m_2-l_2-1)-(l_1+m_1)]![l+l_1+l_2+2]!}$$

$$\times {}_3\phi_2 \left[\begin{matrix} q^{-2(l_1+m_1)}, q^{-2(l_1+l_2-l)}, q^{2(m+l+2)} \\ q^{2\{(m+l)-(m_2+l_2)+1)}, q^{2\{(m_2-l_2-1)-(l_1+m_1)+1\}} \end{matrix} ; q^2, q^2 \right].$$

§ 6. Linearization Formula of Matrix Elements

Let $\mathscr{A}$ be a full dual space $\mathrm{Hom}_{\mathbb{C}}(\mathscr{U}_q(\mathfrak{sl}(2)), \mathbb{C})$ of $\mathscr{U}_q(\mathfrak{sl}(2))$. We introduce the weak $*$ topology in $\mathscr{A}$ [6]. A sequence $\{\varphi_j\}$ converges to φ in $\mathscr{A}$ if $\varphi_j(a) = \varphi(a)$ $(j \gg 1)$ for any $a \in \mathscr{U}_q(\mathfrak{sl}(2))$. $\mathscr{A}$ is complete with this topology. Moreover we introduce the weak $*$ topology in $\mathrm{Hom}_{\mathbb{C}}(\mathscr{U}_q(\mathfrak{sl}(2))^{\otimes n}, \mathbb{C})$. The algebraic tensor product $\mathscr{A}^{\otimes n}$ is dense in $\mathrm{Hom}_{\mathbb{C}}(\mathscr{U}_q(\mathfrak{sl}(2))^{\otimes n}, \mathbb{C})$, and consequently, one can identify the topological tensor product $\mathscr{A}^{\hat{\otimes} n} = \mathscr{A} \hat{\otimes}_w \mathscr{A} \hat{\otimes}_w \cdots \hat{\otimes}_w \mathscr{A}$ with $\mathrm{Hom}_{\mathbb{C}}(\mathscr{U}_q(\mathfrak{sl}(2))^{\otimes n}, \mathbb{C})$. Then $\mathscr{A}$ is a topological associative algebra with the following multiplication $\mu_{\mathscr{A}}\colon \mathscr{A} \hat{\otimes}_w \mathscr{A} \to \mathscr{A}$.

$$\mu_{\mathscr{A}}(\Phi)(a) = \Phi(\Delta(a)) \qquad \text{for } \Phi \in \mathscr{A} \hat{\otimes}_w \mathscr{A}\,, \qquad a \in \mathscr{U}_q(\mathfrak{sl}(2))\,.$$

We note that the unit $1_{\mathscr{A}}$ is the counit ε.

For $l \in \mathbb{N} \cup \mathbb{N} + \frac{1}{2}$ we define the matrix elements $p_{ij}^{(l)} \in \mathscr{A}$ $(i, j \in J_l)$ associated with W_l (cf. [7, 8]) as

$$a x_j^l = \sum_{i \in J_l} x_i^l p_{ij}^{(l)}(a)\,, \qquad a \in \mathscr{U}_q(\mathfrak{sl}(2))\,.$$

In particular, we define coordinate elements x, u, v, $y \in \mathscr{A}$ as follows.

$$x = p_{1/2,1/2}^{(1/2)}\,, \qquad u = p_{1/2,-1/2}^{(1/2)}\,, \qquad v = p_{-1/2,1/2}^{(1/2)}\,, \qquad y = p_{-1/2,-1/2}^{(1/2)}\,.$$

The elements x, u, v, y satisfy the relations

$$qxu = ux, \qquad qxv = vx\,, \qquad quy = yu\,, \qquad qvy = yv\,, \qquad uv = vu\,,$$

$$xy - q^{-1}uv = yx - quv = 1_{\mathscr{A}}\,.$$

We have a basis of the ring $A(SL_q(2))$ which is generated by the coordinate elements:

$$A(SL_q(2)) = \sum_{0 \le L,M,N}^{\oplus} \mathbb{C} x^L u^M v^N \oplus \sum_{0 < L, 0 \le M,N}^{\oplus} \mathbb{C} u^M v^N y^L\,.$$

Then we define the set of formally analytic elements in $\mathscr{A}$ by

$$A[[SL_q(2)]]$$
$$= \sum x^m v^n \mathbb{C}[[\zeta]] + \sum x^m u^n \mathbb{C}[[\zeta]] + \sum \mathbb{C}[[\zeta]] u^n y^m + \sum \mathbb{C}[[\zeta]] v^n y^m\,,$$

where $\zeta = -q^{-1}uv$. This is a subalgebra of $\mathscr{A}$.

For V_l $\left(l \in \left\{-\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots\right\}\right)$, we have the elements $w_{ij}^{(l)} \in \mathscr{A}(i, j \in I_l)$ determined by

$$a\xi_j^l = \sum_{i \in I_l} \xi_i^l w_{ij}^{(l)}(a)\,, \qquad a \in \mathscr{U}_q(\mathfrak{sl}(2))\,.$$

We regard $w_{ij}^{(l)}$ as the matrix elements associated with V_l [6].

The matrix elements $p_{ij}^{(l)}$ and $w_{ij}^{(l)}$ are expressed by x, u, v, y in $A[[SL_q(2)]]$ [6, 7, 8].

Theorem 6.1. (i) *The matrix elements* $p_{ij}^{(l)}$ $(i, j \in J_l)$ *are*

$$(i+j \geq 0, i \geq j) \qquad p_{ij}^{(l)} = x^{i+j}u^{i-j}q^{(1/2)(i-j)(i+j)+(l-i)(j-i)}\begin{bmatrix} l-j \\ i-j \end{bmatrix} \times {}_2\phi_1\left[\begin{matrix} q^{-2(l-i)}, q^{2(l+i+1)} \\ q^{2(i-j+1)} \end{matrix}; q^2, -quv\right],$$

$$(i+j \geq 0, j \geq i) \qquad p_{ij}^{(l)} = x^{i+j}v^{j-i}q^{(1/2)(i-j)(i+j)+(l-j)(i-j)}\begin{bmatrix} l+j \\ j-i \end{bmatrix} \times {}_2\phi_1\left[\begin{matrix} q^{-2(l-j)}, q^{2(l+j+1)} \\ q^{2(j-i+1)} \end{matrix}; q^2, -quv\right],$$

$$(i+j \leq 0, j \geq i) \qquad p_{ij}^{(l)} = q^{(1/2)(i-j)(i+j)+(i-j)(l+i)}\begin{bmatrix} l+j \\ j-i \end{bmatrix} \times {}_2\phi_1\left[\begin{matrix} q^{-2(l+i)}, q^{2(l-i+1)} \\ q^{2(j-i+1)} \end{matrix}; q^2, -quv\right]v^{j-i}y^{-i-j},$$

$$(i+j \leq 0, i \geq j) \qquad p_{ij}^{(l)} = q^{(1/2)(i-j)(i+j)+(j-i)(l+j)}\begin{bmatrix} l-j \\ i-j \end{bmatrix} \times {}_2\phi_1\left[\begin{matrix} q^{-2(l+j)}, q^{2(l-j+1)} \\ q^{2(i-j+1)} \end{matrix}; q^2, -quv\right]u^{i-j}y^{-i-j}.$$

(ii) *The matrix elements* $w_{ij}^{(l)}$ $(i, j \in I_l)$ *are*

$$(i+j \leq 0, j \leq i) \qquad w_{ij}^{(l)} = q^{(j-i)(l+j)}\frac{(q^{2(l+1-i)}; q^2)_{i-j}}{(q^2; q^2)_{i-j}}x^{-i-j}v^{i-j} \times {}_2\phi_1\left[\begin{matrix} q^{2(l-j+1)}, q^{-2(l+j)} \\ q^{2(i-j+1)} \end{matrix}; q^2, -quv\right],$$

$$(i+j \leq 0, i \leq j) \qquad w_{ij}^{(l)} = q^{(i-j)(l+i)}\frac{(q^{2(l+1+i)}; q^2)_{j-i}}{(q^2; q^2)_{j-i}}x^{-i-j}u^{j-i} \times {}_2\phi_1\left[\begin{matrix} q^{2(l-i+1)}, q^{-2(l+i)} \\ q^{2(j-i+1)} \end{matrix}; q^2, -quv\right],$$

$$(i+j \geq 0, i \leq j) \qquad w_{ij}^{(l)} = q^{(i-j)(l-j)}\frac{(q^{2(l+1+i)}; q^2)_{j-i}}{(q^2; q^2)_{j-i}} \times {}_2\phi_1\left[\begin{matrix} q^{2(l+j+1)}, q^{-2(l-j)} \\ q^{2(j-i+1)} \end{matrix}; q^2, -quv\right]u^{j-i}y^{i+j},$$

$$(i+j \geq 0, j \leq i) \qquad w_{ij}^{(l)} = q^{(j-i)(l-i)} \frac{(q^{2(l+1-i)}; q^2)_{i-j}}{(q^2; q^2)_{i-j}} \times {}_2\phi_1\left[\begin{matrix} q^{2(l+i+1)}, q^{-2(l-i)} \\ q^{2(i-j+1)} \end{matrix}; q^2, -quv\right] v^{i-j} y^{i+j}.$$

We define $\tilde{p}_{ij}^{(l)}$ and $\tilde{w}_{ij}^{(l)}$ as follows:

$$\tilde{p}_{ij}^{(l)} = (d_i^l/d_j^l)^{1/2} p_{ij}^{(l)}, \qquad \tilde{w}_{ij}^{(l)} = (c_i^l/c_j^l)^{1/2} w_{ij}^{(l)}.$$

Since $\xi_{m_1}^{l_1} \otimes \xi_{m_2}^{l_2} = \sum_{\substack{l \in L_1(l_1, l_2) \\ m \in I_l}} \begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix} \xi_m^l$ for $l_1, l_2 \in \left\{-\frac{1}{2}, 0, \frac{1}{2}, 1, \dots\right\}$, we easily see the linearization formula:

$$\tilde{w}_{m_1, m_1'}^{(l_1)} \tilde{w}_{m_2, m_2'}^{(l_2)} = \sum_{l \in L_1(l_1, l_2)} \begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m_1+m_2 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l \\ m_1' & m_2' & m_1'+m_2' \end{bmatrix} \tilde{w}_{m_1+m_2, m_1'+m_2'}^{(l)}.$$

On the other hand, by Theorem 5.1, $\tilde{x}_{m_1}^{l_1} \otimes \tilde{\xi}_{m_2}^{l_2}$ $(l_1 - l_2 < 1)$ is denoted as

$$\tilde{x}_{m_1}^{l_1} \otimes \tilde{\xi}_{m_2}^{l_2} = \sum_{\substack{l \in L_2(l_1, l_2) \\ m \in I_l}} n\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right] \tilde{\xi}_m^l, \tag{6.1}$$

where all $n\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right] \in \mathbb{C}$ but finite are zero.

Proposition 6.2. *If* $l = -\frac{1}{2}$, *then*

$$n\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right] = (-1)^{m_1 - l_1} \left[\begin{bmatrix} l_1 & l_2 & -1/2 \\ m_1 & m_2 & m \end{bmatrix}\right],$$

and if $l \neq -\frac{1}{2}$, *then*

$$n\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right] = (-1)^{m_1 + l - l_2} q^{2(l - l_1 - l_2)(l + l_1 - l_2)} \times \frac{[l - l_1 + l_2]![l + l_1 + l_2 + 1]![2l_1]!}{[2l]![2l_2 + 1]![l + l_1 - l_2]![l_1 + l_2 - l]!} \left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right].$$

For the proof we use the following.

Lemma 6.3. (i) $n\left[\begin{bmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{bmatrix}\right] = 0$ *unless* $m_1 + m = m_2$.

(ii) $$-q^{-m+(3/2)}([m+l](m-l-1])^{1/2}n\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{matrix}\right]\right]$$
$$= q^{-m_2-l_1+1/2}([l_1-m_1][l_1+m_1+1])^{1/2}n\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1+1 & m_2 & m-1 \end{matrix}\right]\right]$$
$$- q^{-m_1-m_2+(3/2)}([m_2+l_2][m_2-l_2-1])^{1/2}n\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2-1 & m-1 \end{matrix}\right]\right].$$

(iii) $$q^{-m+(1/2)}([m-l][m+l+1])^{1/2}n\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{matrix}\right]\right]$$
$$= q^{-m_2-l_1+(1/2)}([l_1+m_1][l_1-m_1+1])^{1/2}n\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1-1 & m_2 & m+1 \end{matrix}\right]\right]$$
$$+ q^{-m_1-m_2+(1/2)}([m_2-l_2][m_2+l_2+1])^{1/2}n\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2+1 & m+1 \end{matrix}\right]\right].$$

(iv) $$\sum_{m_1\in J_{l_1}, m_2\in I_{l_2}}\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{matrix}\right]\right]\cdot n\left[\left[\begin{matrix} l_1 & l_2 & l' \\ m_1 & m_2 & m' \end{matrix}\right]\right] = \delta_{ll'}\delta_{mm'}.$$

(v) $$\sum_{k=0}^{n}(-1)^k\frac{[a-1+k]![c-1]![n]!}{[k]![a-1]![c-1+k]![n-k]!}q^{k(k-2n+1)}$$
$$=\frac{[c-a+n-1]![c-1]!}{[c-a-1]![c+n-1]!}q^{2an},$$

where $c+n-a>0$.

Proof. Applying k, e and f to both sides of (6.1), we obtain (i), (ii) and (iii). The formula (iv) is trivial by the definition of $n\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{matrix}\right]\right]$. We get (v) by the q-Vandermonde sum [3]:

$$_2\phi_1\left[\begin{matrix} q^{2a}, q^{-2n} \\ q^{2c} \end{matrix}; q^2, q^2\right] = \frac{(q^{2(c-a)}; q^2)n}{(q^{2c}; q^2)_n}q^{2an},$$

where $c+n-a>0$. □

Proof of Proposition 6.2. Comparing Lemma 5.4 with Lemma 6.3 (i), (ii) and (iii), we get

$$n\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{matrix}\right]\right] = (-1)^{m_1+l-l_2}\alpha(l, l_1, l_2)\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{matrix}\right]\right],$$

where

$$\alpha(l, l_1, l_2) = (-1)^{l+l_1-l_2}n\left[\left[\begin{matrix} l_1 & l_2 & l \\ l_1 & l_1+l+1 & l+1 \end{matrix}\right]\right]\cdot\left[\left[\begin{matrix} l_1 & l_2 & l \\ l_1 & l_1+l+1 & l+1 \end{matrix}\right]\right]^{-1}.$$

If $l = -\frac{1}{2}$, then $l_2 = l_1 - \frac{1}{2}$. By definition, we have

$$n\left[\left[\begin{matrix} l_1 & l_1 - (1/2) & -(1/2) \\ l_1 & l_1 + (1/2) & 1/2 \end{matrix}\right]\right] = \left[\left[\begin{matrix} l_1 & l_1 - (1/2) & -(1/2) \\ l_1 & l_1 + (1/2) & 1/2 \end{matrix}\right]\right] = 1\,,$$

and, as a result, $\alpha\left(-\frac{1}{2}, l_1, l_1 - \frac{1}{2}\right) = 1$. We assume $l \neq -\frac{1}{2}$, and then $l - l_1 + l_2 \geq 0$. By Lemma 6.3 (iv),

$$\sum_{m_1 = l_2 - l}^{l_1} \left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_1 + l + 1 & l + 1 \end{matrix}\right]\right] \cdot n\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_1 + l + 1 & l + 1 \end{matrix}\right]\right] = 1\,.$$

Thus

$$\alpha(l, l_1, l_2)^{-1} = \sum_{m_1 + l - l_2 = 0}^{l + l_1 - l_2} (-1)^{m_1 + l - l_2} \left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_1 + l + 1 & l + 1 \end{matrix}\right]\right]^2 .$$

Making use of Lemma 6.3 (v), we obtain

$$\sum_{m_1 + l - l_2 = 0}^{l + l_1 - l_2} (-1)^{m_1 + l - l_2} \left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_1 + l + 1 & l + 1 \end{matrix}\right]\right]^2$$

$$= \left\{ \sum_{m_1 + l - l_2 = 0}^{l + l_1 - l_2} (-1)^{m_1 + l - l_2} q^{(m_1 + l - l_2)^2 + (m_1 + l - l_2)} q^{-2(l + l_1 - l_2)(m_1 + l - l_2)} \right.$$

$$\left. \times \frac{[l_1 + m_1]![l + l_1 - l_2]![2l_2 + 1]!}{[l_1 + l_2 - l]![l_1 - m_1]![m_1 + l - l_2]![m_1 + l + l_2 + 1]!} \right\}$$

$$\times q^{-2(l + l_1 - l_2)} \frac{[l + l_1 - l_2]![l_1 + l_2 - l]!}{[2l_1]!}$$

$$= q^{-2(l - l_1 - l_2)(l + l_1 - l_2)} \frac{[2l]![l + l_1 - l_2]![l_1 + l_2 - l]![2l_2 + 1]!}{[l - l_1 + l_2]![l + l_1 + l_2 + 1]![2l_1]!} .$$

Hence we get the result. □

We have the linearization formula.

Theorem 6.4.

$$\tilde{p}^{(l_1)}_{m_1, m_1'} \tilde{w}^{(l_2)}_{m_2, m_2'}$$

$$= \sum_{l \in L_2(l_1, l_2)} \left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m_2 - m_1 \end{matrix}\right]\right] \cdot n\left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1' & m_2' & m_2' - m_1' \end{matrix}\right]\right] \tilde{w}^{(l)}_{m_2 - m_1, m_2' - m_1'} .$$

Proof. For $a \in \mathcal{U}_q(\mathfrak{sl}(2))$, let $\Delta(a) = \sum_i a^i \otimes a_i$. We have

$$\begin{aligned} a(\tilde{x}^{l_1}_{m'_1} \otimes \tilde{\xi}^{l_2}_{m'_2}) &= \sum_i a^i \tilde{x}^{l_1}_{m'_1} \otimes a_i \tilde{\xi}^{l_2}_{m'_2} \\ &= \sum_{m_1, m_2} \tilde{x}^{l_1}_{m_1} \otimes \tilde{\xi}^{l_2}_{m_2} \left(\sum_i \tilde{p}^{(l_1)}_{m_1, m'_1}(a^i) \tilde{w}^{(l_2)}_{m_2, m'_2}(a_i) \right) \\ &= \sum_{m_1, m_2} \tilde{x}^{l_1}_{m_1} \otimes \tilde{\xi}^{l_2}_{m_2} (\tilde{p}^{(l_1)}_{m_1, m'_1} \tilde{w}^{(l_2)}_{m_2, m'_2})(a) . \end{aligned}$$

On the other hand,

$$\begin{aligned} a(\tilde{x}^{l_1}_{m'_1} \otimes \tilde{\xi}^{l_2}_{m'_2}) &= a\left(\sum_{l, m'} n \left[\left[\begin{matrix} l_1 & l_2 & l \\ m'_1 & m'_2 & m' \end{matrix} \right]\right] \tilde{\xi}^{l}_{m'} \right) \qquad \text{(by (6.1))} \\ &= \sum_{l, m} \tilde{\xi}^{l}_{m} \left(\sum_{m'} n \left[\left[\begin{matrix} l_1 & l_2 & l \\ m'_1 & m'_2 & m' \end{matrix} \right]\right] \tilde{w}^{(l)}_{m, m'}(a) \right) \\ &= \sum_{m_1, m_2} \tilde{x}^{l_1}_{m_1} \otimes \tilde{\xi}^{l_2}_{m_2} \\ &\quad \times \left(\sum_{l, m, m'} \left[\left[\begin{matrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{matrix} \right]\right] \cdot n \left[\left[\begin{matrix} l_1 & l_2 & l \\ m'_1 & m'_2 & m' \end{matrix} \right]\right] \tilde{w}^{(l)}_{m, m'}(a) \right) . \end{aligned}$$

Hence we obtain Theorem 6.4. □

Putting $l_1 = 1$ and $m_1 = m'_1 = 0$ in Theorem 6.4, we get the three-term recurrence relation: For $l > 0$

$$\begin{aligned} &(1 + q^{-1}[2]uv)\tilde{w}^{(l)}_{ij} \\ &= -q^{-2(i+j-2l)}([i-l][i+l][j-l][j+l])^{1/2} \frac{[2][i+l+1][j+l+1]}{[2l][2l+1][2l+2]^2} \tilde{w}^{(l-1)}_{ij} \\ &\quad -q^{-3(i+j)+4l+2}(q[i-l-1] + q^{-1}[i+l+2])(q[j-l-1] + q^{-1}[j+l+2]) \\ &\quad \times \frac{[i+l+1][j+l+1]}{[2l][2l+2][2l+3]^2} \tilde{w}^{(l)}_{ij} \\ &\quad - q^{-2(i+j-2l-1)}([i-l-1][i+l+1][j-l-1][j+l+1])^{1/2} \\ &\quad \times \frac{[2][i+l+2][j+l+2]}{[2l+1][2l+2][2l+4]^2} \tilde{w}^{(l+1)}_{ij} . \end{aligned}$$

References

[1] Andrews, G. E., *The theory of partitions*, Encyclopedia of Mathematics and Its Applications, 2, London, Amsterdam, Don Mills: Addison-Wesley, 1976.

[2] Drinfeld, V. G., Quantum groups, in *Proc. Int. Congr. Math.* Berkeley (1986), 798–820, Am. Math. Soc. 1987.

[3] Gasper, G., Rahman, M., *Basic hypergeometric series.* Encyclopedia of Mathematics and Its Applications, **35**, Cambridge, New York, Port Chester: Cambridge University Press 1990.

[4] Jimbo, M., A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation, *Lett. Math. Phys.*, **10** (1985), 63–69.

[5] Kirillov, A. N. and Reshetikhin, N. Yu. Representations of the algebra $U_q(\mathfrak{sl}(2))$, q-orthogonal polynomials and invariants of links, in Kac, V. G. (ed.) *Infinite dimensional Lie algebras and groups,* Proc., CIRM, 1988. Advanced Series in Mathematical Physics, **7**, pp. 285–339, Singapore, New Jersey, London, World Scientific, 1989.

[6] Masuda, T., Mimachi, K., Nakagami, Y., Noumi, M., Saburi, Y. and Ueno, K., Unitary representations of the quantum group $SU_q(1, 1)$ I, II, *Lett. Math. Phys.*, **19** (1990), 187–204.

[7] Masuda, T., Mimachi, K., Nakagami, Y., Noumi, M. and Ueno, K., Representations of quantum groups and a q-analogue of orthogonal polynomials, *C. R. Acad. Sci. Paris*, **307**, Série I (1988), 559–564.

[8] ———, Representations of the quantum group $SU_q(2)$ and the little q-Jacobi polynomials, to appear in *J. Funct. Anal.*

[9] Ruegg, H., A simple derivation of the quantum Clebsch-Gordan coefficients for $SU(2)_q$, *J. Math. Phys.*, **31** (1990), 1085–1087.

[10] Vaksman, L. L. and Korogodsky, L. I., Harmonic analysis on quantum hyperboloids, preprint, 1990, (in Russian).