

Goldman's Type Theorem for Index 3

By

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Abstract

We show a Goldman's type theorem for any inclusions of (not necessarily AFD) factors with the principal graph A_5 . Our main tools are correspondences and sectors.

§1. Introduction and Main Results

In our previous paper [I, Theorem 5.1], we showed that any pair of III_1 AFD factors whose principal graph is the Coxeter graph A_5 is as follows.

$$R_\infty \rtimes S_3 \supset R_\infty \rtimes S_2$$

where R_∞ is the type III_1 AFD factor. This theorem has been known in the case of the type II_1 and III_λ ($0 < \lambda < 1$) AFD factors using classifications of subfactors [Li, O, P]. The aim of the present work is to generalize this theorem for arbitrary factors. Since we will not use classifications of subfactors and group actions, we need no assumption of AFD.

V. Jones' index theory is one of the most important and interesting topics in recent operator algebras, and many connections with other areas of mathematics and mathematical physics are pointed out. As one of his motivations of his definition of index [J], there is Goldman's theorem [G], which says that any inclusion of factors with index 2 is written by the crossed product of a \mathbb{Z}_2 action. One direction of generalizations of this theorem is a characterization of crossed products in terms of indices and higher relative commutants (principal graphs). This was done by M. Pimsner and S. Popa [PP1], A. Ocneanu in the case of II_1 factors, and H. Kosaki [K2] in the case of properly infinite factors. Our approach is considered to be another direction of the generalizations.

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In Section 3, we first prove the main theorem in the case of properly infinite factors using sectors, and next in the case of finite factors using correspondences. Although the latter argument may be valid for general cases, we leave the former because of perspective.

Our basic references are [G.H.J, K1] for the index theory and [I, L2] for the sector theory.

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§2. Preliminaries

In this section we extend the notion of sectors defined by R. Longo in [L2], and show basic facts.

Throughout this paper we assume that von Neumann algebras have separable preduals and Hilbert spaces are separable.

Let M and N be factors. We use the following notations.

$\text{Mor}(N, M)$: the set of unital normal isomorphisms from N into M .

In the case of $N \subset M$,

$E(M, N)$: the set of faithful normal conditional expectations from M to N .

$\rho_1, \rho_2 \in \text{Mor}(N, M)$ are unitary equivalent if and only if there exists a unitary $u \in M$ such that

$$\rho_1 = \text{Ad}(u) \cdot \rho_2.$$

For properly infinite factors M, N , we denote by $\text{Sect}(M, N)$ the quotient of $\text{Mor}(N, M)$ by the above equivalence, and call elements in $\text{Sect}(M, N)$ $M - N$ sectors. If $\rho \in \text{Mor}(N, M)$, ${}_M[\rho]_N$ denotes its class in $\text{Sect}(M, N)$. If $M = N$ $\text{Sect}(M, N)$ coincides with $\text{Sect}(M)$ in [L2, I]. We sometimes associate different sectors with one morphism. For example, let N be a subfactor of M and $\gamma: M \rightarrow N$ the canonical endomorphism [L3]. If we consider γ a map from M to N (resp. from M to M , from N to M , from N to N), we write ${}_N[\gamma]_M$ (resp. ${}_M[\gamma]_M, {}_M[\gamma]_N, {}_N[\gamma]_N$). Sum, product and irreducible decomposition are defined on the sectors in the same way as in [I, §2], except that the product is a map $\text{Sect}(M, N) \times \text{Sect}(N, L) \rightarrow \text{Sect}(M, L)$.

Let L, M, N be properly infinite factors and H be $L - M$ correspondence (For the definition of correspondence see [P2, L2, I]). If $\rho \in \text{Mor}(N, M)$ (resp. $\rho \in \text{Mor}(N, L)$), we define a $L - N$ correspondence H_ρ (resp. $N - M$ correspondence ${}_\rho H$) by

$$x \cdot \xi' \cdot y \equiv x \cdot \xi \cdot \rho(y), \quad x \in L, \quad y \in N, \quad \xi \in H, \quad \xi' \in H_\rho$$

$$\text{(resp. } x \cdot \xi' \cdot y \equiv \rho(x) \cdot \xi \cdot y, \quad x \in N, \quad y \in M, \quad \xi \in H, \quad \xi' \in {}_\rho H)$$

where $\xi = \xi'$ as an element of the Hilbert space. If we consider $L^2(M)$ as $M - M$ correspondence by the natural actions, then the map $\rho \mapsto L^2(M)_\rho$ induces bijection between $\text{Sect}(M, N)$ and the set of equivalence classes of $M - N$ correspondence [L2, Corollary 2.2]. (Note that our convention is different from that in [L2, I], because it is convenient to consider that the left action is standard.) We define a map called conjugation ${}_M[\rho]_N \mapsto {}_N[\bar{\rho}]_M$ using that of the correspondences and the above bijection. For simplicity $\bar{\rho}$ denotes one of representatives of ${}_N[\bar{\rho}]_M$. Let $\gamma: M \rightarrow \rho(N)$ be the canonical endomorphism. Then $\bar{\rho}$ is given as follows [L2, Theorem 3.1].

$$\bar{\rho} = \rho^{-1} \cdot \gamma.$$

Using similar calculation in the proof of [L2, Theorem 3.1], we can show the following relation.

$${}_M(L^2(M)_\rho)_N \cong {}_M({}_{\bar{\rho}}L^2(N))_N. \tag{2.1}$$

The following proposition shows that our convention is natural.

Proposition 2.1. *Let M, N, L be properly infinite factors and ${}_M[\rho_1]_N \in \text{Sect}(M, N)$, ${}_N[\rho_2]_L \in \text{Sect}(N, L)$. Then,*

$${}_M(L^2(M)_{\rho_1})_N \otimes_N (L^2(N)_{\rho_2})_L \cong {}_M(L^2(M)_{\rho_1, \rho_2})_L$$

where \otimes_N means Sauvageot's relative tensor product [S].

Proof. Due to (2.1) and

$${}_N L^2(N)_N \otimes_N L^2(N)_N \cong {}_N L^2(N)_N,$$

we have

$$\begin{aligned} {}_M(L^2(M)_{\rho_1})_N \otimes_N (L^2(N)_{\rho_2})_L &\cong {}_M({}_{\bar{\rho}_1}L^2(N))_N \otimes_N (L^2(N)_{\rho_1})_L \\ &\cong {}_M({}_{\bar{\rho}_1}L^2(N)_{\rho_2})_L \cong {}_M(L^2(M)_{\rho_1, \rho_2})_L. \quad \text{Q.E.D.} \end{aligned}$$

Remark 2.2. If M is a II_1 factor we cannot use sectors. But (2.1) and Proposition 2.1 make sense if we consider only automorphisms. Namely, the following hold.

$${}_M(L^2(M)_\rho)_M \cong {}_M({}_{\rho^{-1}}L^2(M))_M \tag{2.2}$$

$${}_M(L^2(M)_{\rho_1})_M \otimes_M (L^2(M)_{\rho_2})_M \cong {}_M(L^2(M)_{\rho_1, \rho_2})_M \tag{2.3}$$

where $\rho, \rho_1, \rho_2 \in \text{Aut}(M)$ and we use $\bar{\rho} = \rho^{-1}$.

We assume again that M and N are properly infinite factors. Let $\phi: M \rightarrow N$ be a unital normal completely positive map and Ω be a separating and cyclic vector for N . Then there exists a unique sector ${}_N[\rho]_M \in \text{Sect}(N, M)$ and $\xi \in L^2(N)$ satisfying the following [I, subsection 2.2].

$$\langle xJ_N\phi(y^*)J_N\Omega, \Omega \rangle = \langle xJ_N\rho(y^*)J_N\xi, \xi \rangle \quad \text{for } x \in N, y \in M$$

$$\overline{{}_N J_N \rho(M) J_N \xi} = L^2(N)$$

where J_N is the modular conjugation of N . We denote ${}_N[\rho]_M$ by ${}_N[\phi]_M$. If $M \supset N$ and $E \in E(M, N)$, we can compute ${}_N[E]_M$ explicitly as in [L2, Proposition 3.3], [I, subsection 2.2]. Let $\gamma: M \rightarrow N$ be the canonical endomorphism. Then,

$${}_N[E]_M = {}_N[\gamma]_M. \tag{2.4}$$

If we consider E to be a map from M to M , we obtain

$${}_M[E]_M = {}_M[\gamma]_M. \tag{2.5}$$

Now we will move to the index theory. Let ${}_M[\rho]_N \in \text{Sect}(M, N)$. We define the (statistical) dimension $d(\rho)$ by

$$d(\rho) = [M: \rho(N)]_0^{1/2}.$$

In the above equation $[M: \rho(N)]_0$ means Hiai's minimum index [H]. If there is possibility of confusion we will write $d({}_M\rho_N)$. As in [I, subsection 2.3] $d(\rho)$ satisfies the additivity and the multiplicativity [KL], [L4].

In the same way as in [L2, Theorem 4.1], [I, Theorem 2.1], we can show the following proposition.

Proposition 2.3. *Let M, N be a pair of properly infinite factors and ${}_N[\rho_1]_M \in \text{Sect}(N, M)$ ${}_M[\rho_2]_N \in \text{Sect}(M, N)$ with $d(\rho_1), d(\rho_2) < \infty$. If ρ_1 and ρ_2 are irreducible then the following conditions are equivalent:*

- (1) ${}_N[\rho_1\rho_2]_N$ or ${}_M[\rho_2\rho_1]_M$ contains the identity sector.
- (2) ${}_N[\overline{\rho_2}]_M = {}_N[\rho_1]_M$.

If ρ_1 and ρ_2 satisfy the above condition, ${}_N[\rho_1\rho_2]_N$ and ${}_M[\rho_2\rho_1]_M$ contain the identity sector with multiplicity one.

Let $M \supset N$ be a pair of properly infinite factors and $\gamma: M \rightarrow N$ be the canonical endomorphism. Then,

$$M \supset N \supset \gamma(M) \supset \gamma(N) \supset \gamma^2(M) \supset \dots$$

is the tower associated with $M \supset N$. Let $\iota: N \rightarrow M$ be the embedding map. By easy computation we obtain

$${}_N[\overline{\iota}]_M = {}_N[\gamma]_M. \tag{2.6}$$

Then the principal graphs of $M \supset N$ and $N \supset \gamma(M)$ show the fusion rules of the following sequences of sectors [I, §3].

$$\begin{aligned} &{}_M[\iota]_N, \quad {}_M[\iota]_N \cdot {}_N[\gamma]_M = {}_M[\gamma]_M, \quad {}_M[\iota]_N \cdot {}_N[\gamma]_M \cdot {}_M[\iota]_N = {}_M[\gamma]_N \cdots \\ &{}_N[\gamma]_M, \quad {}_N[\gamma]_M \cdot {}_M[\iota]_N = {}_N[\gamma]_N, \quad {}_N[\gamma]_M \cdot {}_M[\iota]_N \cdot {}_N[\gamma]_M = {}_N[\gamma^2]_M \cdots \end{aligned}$$

We call the sectors appearing in the decomposition of the above sectors as descendant sectors.

Before closing this section, we show the following proposition as an application of sectors.

Proposition 2.4. *Let $M \supset N$ be a pair of properly infinite factors with finite depth. If the principal graph of $M \supset N$ has two points whose entries of the Perron-Frobenius eigenvector are minimal and they have different colors, (note that principal graphs are always bipartite), then M is isomorphic to N .*

Proof. Let ${}_M[\rho]_N$ be the descendant sector corresponding to one of the above points which is not the distinguished point. Then the above condition implies $d(\rho) = 1$ as in [I, subsection 2.3]. This means $[M: \rho(N)] = 1$ and so we have the result. Q.E.D.

Remark 2.5. If the principal graph has a graph automorphism which does not preserve the colors of vertices, then the condition of Proposition 2.4 is satisfied. For example, Coxeter graph A_{even} and $D_{\text{odd}}^{(1)}$ satisfy the condition.

§3. The Main Theorem

The main theorem of the present work is as follows.

Theorem 3.1. *Let $M \supset N$ be a pair of factors with the principal graph A_5 . Then there exists L a subfactor of N and an outer action of S_3 on L such that*

$$(M \supset N) \cong (L \rtimes S_3 \supset L \rtimes S_2).$$

Proof. First, we show the theorem in the case of properly infinite factors using sectors. Let $\gamma: M \rightarrow N$ be the canonical endomorphism and $\iota: N \rightarrow M$ the embedding map. The descendant sectors and fusion rules are as follows (see Fig. 1) [I, §5]:

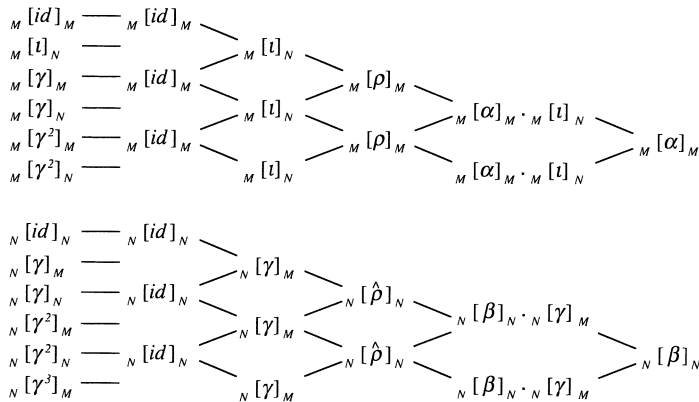


Fig. 1. The diagram of fusion rules of A_5 .

$${}_M[\gamma]_M = {}_M[id]_M \oplus {}_M[\rho]_M, \tag{3.1}$$

$${}_M[\rho]_M \cdot {}_M[l]_N = {}_M[l]_N \oplus {}_M[\alpha]_M \cdot {}_M[l]_N, \tag{3.2}$$

$${}_M[\rho]_M^2 = {}_M[id]_M \oplus {}_M[\alpha]_M \oplus {}_M[\rho]_M, \tag{3.3}$$

$${}_M[\alpha]_M \cdot {}_M[\rho]_M = {}_M[\rho]_M \cdot {}_M[\alpha]_M = {}_M[\rho]_M, \tag{3.4}$$

$$d(\rho) = 2, \quad d(\alpha) = 1. \tag{3.5}$$

$${}_N[\gamma]_N = {}_N[id]_N \oplus {}_N[\hat{\rho}]_N, \tag{3.6}$$

$${}_N[\hat{\rho}]_N \cdot {}_N[\gamma]_M = {}_N[\gamma]_M \oplus {}_N[\beta]_N \cdot {}_N[\gamma]_M, \tag{3.7}$$

$${}_N[\hat{\rho}]_N^2 = {}_N[id]_N \oplus {}_N[\beta]_N \oplus {}_N[\hat{\rho}]_N, \tag{3.8}$$

$${}_N[\beta]_N \cdot {}_N[\hat{\rho}]_N = {}_N[\hat{\rho}]_N \cdot {}_N[\beta]_N = {}_N[\hat{\rho}]_N, \tag{3.9}$$

$$d(\hat{\rho}) = 2, \quad d(\beta) = 1. \tag{3.10}$$

${}_M[\rho]_M, {}_N[\hat{\rho}]_N, {}_M[\alpha]_M, {}_N[\beta]_N$ are self-conjugate and $d(\alpha) = d(\beta) = 1$ implies $\alpha \in \text{Aut}(M), \beta \in \text{Aut}(N)$. We can choose α and β such that $\alpha^2 = id_M, \beta^2 = id_N$ as in [I, Proposition 3.3]. Now we will consider the following inclusions of factors.

$$M \supset N \supset L \equiv N^\beta$$

where N^β is the fixed point algebra of N under β . Let E_1, E_2 be the unique elements in $E(M, N)$ and $E(N, L)$. Due to [L1, Proposition 5.1] there exists an isometry $v \in N$ satisfying the following.

$$\begin{aligned} vy &= \gamma(y)v && \text{for } y \in N \\ E_1(x) &= v^*\gamma(x)v && \text{for } x \in M. \end{aligned}$$

If we define $E \in E(M, N)$ by $E = E_2 \cdot E_1$, we can show the following as in [I, §5].

$${}_M[E]_M = {}_M[\gamma]_M \oplus {}_M[\beta \cdot \gamma]_M.$$

From Fig. 1 and

$${}_N[\gamma^2]_M = {}_N[\gamma]_M \cdot {}_M[l]_N \cdot {}_N[\gamma]_M = \overline{({}_M[l]_N \cdot {}_N[\gamma]_M \cdot {}_M[l]_N)},$$

we can see

$${}_N[\beta \cdot \gamma]_M = \overline{({}_M[\alpha]_M \cdot {}_M[l]_N)} = {}_N[\gamma]_M \cdot {}_M[\alpha]_M$$

where we use (2.6) and self-conjugacy of α . So using (3.1), (3.4) we have

$$\begin{aligned} {}_M[E]_M &= {}_M[\gamma]_M \oplus {}_M[\gamma]_M \cdot {}_M[\alpha]_M \\ &= {}_M[id]_M \oplus {}_M[\alpha]_M \oplus 2{}_M[\rho]_M. \end{aligned} \tag{3.11}$$

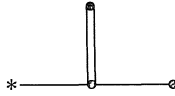


Fig. 2. A part of the principal graph of $M \supset L$

If the depth of $M \supset L$ is 2, we obtain the result due to [I, Corollary 5.4, Proposition 4.1]. Although [I, Proposition 4.2] is also valid for our case, we apply another proof for latter use. Due to (2.5), (3.11) means the principal graph of $M \supset L$ contains the graph in Fig. 2 as a part, and it is easy to see that the Perron-Frobenius eigenvalue of this graph is $\sqrt{6}$. On the other hand, thanks to [KL, L4] we have $[M : L]_0 = [M : N]_0 [N : L]_0 = 6$. So this graph must coincide with the principal graph of $M \supset L$ because of [GHJ, Lemma 1.4.2].

In the case of finite factors, we have to repeat the essentially same argument treating correspondences (bimodules) directly. Before completing the proof, we will prepare a few facts on correspondences. Let

$$N \subset M \subset M_1 \subset M_2 \subset \dots,$$

be the tower associated with $N \subset M$. (For simplicity, we will write ${}_M(M_n)_M$ instead of ${}_M L^2(M_n)_M$.) Then it is well known that M_n is equivalent to

$$\overbrace{M \otimes_N M \otimes_N \cdots \otimes_N M}^{n+1},$$

as an $M - M(N - M, M - N, N - N)$ correspondence [GHJ, Corollary 3.6.5] [J2]. The correspondences appearing in the irreducible decomposition of ${}_M(M_n)_M, {}_N(M_n)_M, {}_M(M_n)_N, {}_N(M_n)_N$, and their restriction-induction graph are as in Fig. 3 (see [O2]). Using the formula in [J1, page 4] and the local index formula in [PP2, Corollary 3.2], we can show the following.

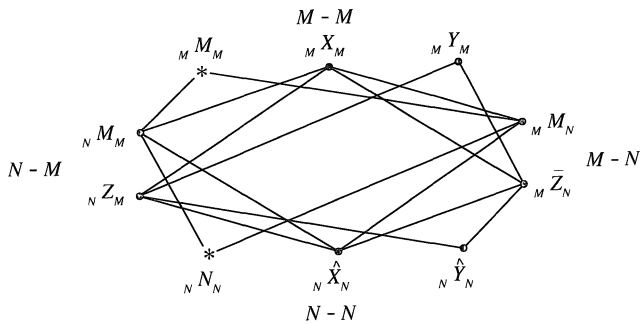


Fig. 3. The restriction-induction graph of A_5

$$\dim_M(X) = \dim(X)_M = 2, \tag{3.12}$$

$$\dim_M(Y) = \dim(Y)_M = 1, \tag{3.13}$$

$$\dim_N(\hat{X}) = \dim(\hat{X})_N = 2, \tag{3.14}$$

$$\dim_N(\hat{Y}) = \dim(\hat{Y})_N = 1. \tag{3.15}$$

Since $\{ {}_M M_M, {}_M X_M, {}_M Y_M \}$ and $\{ {}_N N_N, {}_N \hat{X}_N, {}_N \hat{Y}_N \}$ are closed under conjugation and ${}_M M_M, {}_N N_N$ are self-conjugate, they are all self-conjugate. Due to (3.13), (3.15) and self-conjugacy of ${}_M Y_M, {}_N \hat{Y}_N$, there exist outer automorphisms $\alpha \in \text{Aut}(M), \beta \in \text{Aut}(N)$ satisfying

$$\alpha^2 \in \text{Int}(M) \quad \beta^2 \in \text{Int}(N), \tag{3.16}$$

$${}_M Y_M \cong {}_M({}_\alpha M)_M \cong {}_M(M_\alpha)_M, \tag{3.17}$$

$${}_N \hat{Y}_N \cong {}_N({}_\beta N)_N \cong {}_N(N_\beta)_N, \tag{3.18}$$

where we use (2.2), (2.3). So using the restriction-induction graph, we have

$${}_N Z_M \cong {}_N(M_\alpha)_M \cong {}_N({}_\beta M)_M \cong {}_N({}_\alpha M)_M. \tag{3.19}$$

Lemma 3.2. *For each fixed β, α can be chosen such that*

$$\alpha|_N = \beta.$$

Proof. First we fix α and β . Due to (3.19) the following two $N - M$ actions on $L^2(M)$ are equivalent.

$$x \cdot \xi \cdot y \equiv \beta(x) J y^* J \xi \quad \text{for } x \in N, y \in M, \xi \in L^2(M)$$

$$x \cdot \xi \cdot y \equiv \alpha(x) J y^* J \xi \quad \text{for } x \in N, y \in M, \xi \in L^2(M).$$

where J is the canonical conjugation of M . Then there exists a unitary $w \in \mathbb{B}(L^2(M))$ satisfying

$$w J y^* J = J y^* J w \quad \text{for } y \in M,$$

$$w \beta(x) = \alpha(x) w \quad \text{for } x \in N.$$

From the first equation we obtain $w \in M$. So the second equation shows that $\text{Ad}(w^*) \cdot \alpha$ is the desired automorphism. Q.E.D.

As we computed the fusion rules of sectors, we can show the following.

$${}_M X_M \otimes_M X_M \cong {}_M M_M \oplus {}_M X_M \oplus {}_M(M_\alpha)_M, \tag{3.20}$$

$${}_M({}_\alpha X)_M \cong {}_M(X_\alpha)_M \cong {}_M X_M, \tag{3.21}$$

$${}_N \hat{X}_N \otimes_N \hat{X}_N \cong {}_N N_N \oplus {}_N \hat{X}_N \oplus {}_N(N_\beta)_N, \tag{3.22}$$

$${}_N({}_\beta \hat{X})_N \cong {}_N(\hat{X}_\beta)_N \cong {}_N \hat{X}_N. \tag{3.23}$$

As in the properly infinite case, the following holds.

Lemma 3.3. *We can take β such that $\beta^2 = id_N$.*

Proof. Let $P \equiv \text{Hom}(\widehat{X}_N)$. Then N can be considered a subfactor of P . Due to (3.23), β is implemented by a unitary in P . Thanks to $P \cap N' = \mathbf{C}$ the obstruction of β is trivial [C], and we obtain the result. Q.E.D.

We assume $\beta^2 = id_N$ and $\alpha|_N = \beta$. Note that $\alpha^2 = id_M$ also holds due to (3.16) and $M \cap N' = \mathbf{C}$. Let θ be the unique extension of α to M_1 such that $\theta(e_N) = e_N$ [Kw][Li]. Then, since the principal graph of $M \subset M_1$ is A_5 , [PP3, Proposition 1.7 (iii)] shows that θ is outer. As in the properly infinite case, we will consider the following inclusions of factors.

$$L \equiv N^\beta \subset N \subset M \subset M_1 \subset M_1 \rtimes_{\theta} \mathbf{Z}_2.$$

Let

$$L \subset M \subset \widetilde{M}_1 \subset \widetilde{M}_2 \subset \dots$$

be the tower associated with $L \subset M$.

Lemma 3.4.

- (i) $M \subset M_1 \rtimes_{\theta} \mathbf{Z}_2$ is isomorphic to $M \subset \widetilde{M}_1$
- (ii) $M \subset \widetilde{M}_1$ is an irreducible pair.

Proof. (i): Let J be the canonical conjugation of M and u the canonical implementation of α . By definition, we can see

$$\begin{aligned} \widetilde{M}_1 &= JLJ = J(N' \vee \{u\})J \\ &= JN'J \vee \{u\} = M_1 \vee \{u\}. \end{aligned}$$

Since α preserves N , α commutes with the unique element in $E(M, N)$. Thus u commutes with e_N and so $\text{Ad}(u)|_{M_1} = \theta$ holds. Thanks to [A, II.3] we obtain the result. (ii): If $x + yu \in \widetilde{M}_1 \cap M'$, $x, y \in M_1$. Then $x \in \mathbf{C}$, and y satisfies the following.

$$ya = \alpha(a)y \quad \text{for } a \in M.$$

From $M_1 \cap M' = \mathbf{C}$, y is a multiple of a unitary in M_1 . But this implies $y = 0$ because of [PP3, Proposition 1.7. (ii)]. Q.E.D.

Completion of the proof of Theorem 3.1. Due to Lemma 3.4 the irreducible decomposition of ${}_M(\widetilde{M}_1)_M$ is as follows.

$$\begin{aligned} {}_M(\widetilde{M}_1)_M &\cong {}_M(M_1)_M \oplus {}_M(M_1u)_M \\ &\cong {}_M(M_1)_M \oplus {}_M((M_1)_\alpha)_M \\ &\cong {}_M M_M \oplus {}_M X_M \oplus {}_M(M_\alpha)_M \oplus {}_M(X_\alpha)_M \\ &\cong {}_M M_M \oplus {}_M(M_\alpha)_M \oplus 2_M X_M. \end{aligned}$$

In the above equations we used (3.21). Then as in the properly infinite case, the depth of $L \subset M$ is two. Thanks to [I, Corollary 5.4], the relation between L and M are one of the following. (i) $L \subset M \cong L \subset L \rtimes G$, (ii) $L \subset M \cong M^G \subset M$, where G is a group of order 6. Suppose (ii). Due to the characterization of crossed product [PP1, Corollary 1.1.6], $\dim(\tilde{M}_2 \cap M') = 6$. But $\tilde{M}_2 \cap M'$ is isomorphic to $\text{Hom}({}_M L^2(\tilde{M}_1)_M)$ and hence this is not the case. So (i) holds and G is not commutative i.e. $G \cong S_3$. From $[N:L] = 2$, there exists τ an outer action of \mathbb{Z}_2 on L such that

$$N = L \rtimes_{\tau} \mathbb{Z}_2.$$

Then τ is implemented by a unitary in M . [PP3, Proposition 1.7 (ii)] shows that τ comes from a subgroup action of G up to cocycle conjugacy, and so we obtain the result. Q.E.D.

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