*Publ. RIMS, Kyoto Univ.*
**28** (1992), 845–884

# An $L^2$ Dolbeault Lemma and Its Applications

*Dedicated to Professor Heisuke Hironaka on his 60th birthday*

By

Akira FUJIKI*

## Introduction

Let $X$ be a compact Kähler manifold and $E$ a holomorphic vector bundle on $X$. If $E$ admits a hermitian metric $h$ whose curvature form is sufficiently positive, then the Bochner-Kodaira method leads us to vanishing theorems of relevant sheaf cohomology groups. However, if the metric $h$, and also the Kähler metric $g$ on the base, are given only on some Zariski open subset $U$ of $X$, then what we obtain by the Bochner-Kodaira method is just the corresponding vanishing theorems of $L^2$ cohomology groups defined by using the metrics $h$ and $g$. In order to obtain the vanishing theorem of the sheaf cohomology groups as in the compact case, we need to establish isomorphisms of these two kinds of cohomology groups. This would be done if we establish the corresponding $L^2$ Dolbeault lemma, i.e., if we show that the corresponding $L^2$ Dolbeault complex on $X$ is a resolution of the sheaf of holomorphic sections of $E$.

Such an $L^2$ Dolbeault lemma was first formulated and proved in the context of variations of Hodge structures by Zucker in [31] when $X$ is of dimension one. Later, a generalization to the higher dimensional case was given by Timmerscheid in [28] in a similar context. On the other hand, it seems important to pursue Zucker's method further in the general context as mentioned above.

The first purpose of this paper is then to give a version of such an $L^2$ Dolbeault lemma in arbitrary dimensions, where we assume that $Y := X - U$ is a divisor with only normal crossings in $X$ and that a certain special asymptotic behaviour of the metrics $g$ and $h$ along $Y$ is satisfied (Proposition 2.1). The

Communicated by K. Saito, January 8, 1992.
1991 Mathematics Subject Classification: 32C17
* Department of Mathematics, Faculty of General Education, Kyoto University, Kyoto 606, Japan.

assumption on $g$ is for example satisfied if $g$ is either of Poincaré, ball quotient, or Hilbert modular type along $Y$ (cf. 3.2 for the definition).

We give two applications of the $L^2$ Dolbeault lemma. The first one is as follows. Let $D$ be the unit ball or the polydisc in $\mathbb{C}^n$, and $X$ a smooth toroidal compactification of the quotient $U = D/\Gamma$ of $D$ by an arithmetically defined group $\Gamma$ [1]. Let $\Theta\langle -Y\rangle$ be the sheaf of holomorphic vector fields on $X$ which are tangent to $Y$. Then we show that $H^q(X, \Theta\langle -Y\rangle) = 0$ for any $q < n$ (Theorem 4.1). In particular, the pair $(X, Y)$ is (infinitesimally) rigid under deformations. (See [12][18][25] for the related results).

The second application, which was the original motivation for the present investigation, is concerned with a necessary condition for the existence of a complete Kähler-Einstein metric on a Zariski open subset of a compact complex manifold. We prove that if there exists a complete Kähler-Einstein metric $g$ on $U$ with negative scalar curvature and if $g$ has Poincaré growth along $Y$, then the line bundle $L := K_X \otimes [Y]$ is ample on $X$ (Theorem 5.1). This gives a kind of converse too the existence theorem of R. Kobayashi in [16]. On the other hand, $L$ is not ample for the smooth toroidal compactification $X$ of $D/\Gamma$ as above, though $U$ admits a complete Kähler-Einstein metric of negative scalar curvature induced by the Bergman metric on the ball or the polydisc. By changing the growth condition suitably, we can also formulate and prove an analogue of Theorem 5.1 in relation with these examples (Theorem 5.3). In the case of dimension two there exists a more precise existence theorem for the complete Kähler-Einstein metrics due again to R. Kobayashi [17]. We shall also give a certain converse to this result by a different method (Theorem 6.1).

The paper is arranged as follows. In Section 1 we prove a preliminary lemma which is used to prove the $L^2$ Dolbeault lemma in Section 2. In Section 3 we recall basic facts about $L^2$ cohomology groups and give a result concerning isomorphisms of $L^2$ and sheaf cohomology groups (Proposition 3.3). In the remaining Sections 4, 5 and 6 we give the applications of the Dolbeault lemma as mentioned above.

The author thanks T. Ohsawa for calling his attention to the reference [20] so that he could avoid repeating the known argument.

## § 1. A Preliminary Lemma

1.1. Fix a sufficiently small positive real number $\varepsilon$. Let $d(r)$ and $\mu(r)$ be positive $C^\infty$ functions defined on the half-open interval $(0, \varepsilon]$, and depending also on another $C^\infty$ parameter $s \in S$, where $S$ is a certain $C^\infty$ manifold. We assume the following conditions I and II:

I. We may write $d(r)$ in the form

$$d(r) = a_s r^c (\log 1/r)^q \cdot n(r)$$

where $a_s$ is a positive constant which may depend on $s$, and $c$ and $q$ are real numbers such that

a) for each fixed $s$ we have

$$b_s \leq n(r) \leq c_s$$

for some positive constants $b_s$ and $c_s$ depending possibly on $s$,

b)
$$\lim_{r \to 0} r(\log 1/r)(\log n(r))' = 0$$

uniformly with respect to $s$, where ' denotes the differentiation with respect to $r$, and finally

c) $q \neq -1$ when $c$ is an odd integer.

II. $r^2(\log 1/r)^2 \mu(r)^{-1}$ is bounded uniformly with respect to $s$.

Note that the real numbers $c$ and $q$ above are uniquely determined by the condition a). From I, b) we obtain:

**Lemma 1.1.** *Set* $m(r) = (\log 1/r)^q \cdot n(r) = a_s^{-1} r^{-c} d(r)$. *Then we have*

$$\lim_{r \to 0} r(\log m(r))' = 0 \tag{1}$$

*uniformly with respect to s.*

*Proof.* This follows from the equalities

$$\begin{aligned} r(\log m(r))' &= -q(\log 1/r)^{-1} + r(\log 1/r)(\log n(r))' \cdot (\log 1/r)^{-1} \\ &= (\log 1/r)^{-1}(-q + r(\log 1/r)(\log n(r))') . \end{aligned}$$

*Remark* 1.1. Below, actually we consider only the simple cases where either

$$\text{i)} \quad n(r) \equiv 1 , \qquad \text{or} \qquad \text{ii)} \quad n(r) = e(r)^t , \qquad e(r) = 1 + c_s/\log 1/r ,$$

where $c_s$ is a constant depending on $s$, and $t$ is a real number. (The latter moreover appears only in the Hilbert modular case (cf. 2.1).) In fact, $n(r)$ in ii) satisfies the conditions a) and b) since $1 \leq e(r) \leq 1 + c_s/(\log 1/\varepsilon)$ and

$$r(\log e(r))' = (c_s/(\log 1/r)^2)e(r) \leq 1/(\log 1/r) .$$

Also note that $e(r)$ is an increasing function. Moreover, in the above cases $\mu(r)$ will respectively be of the form

$$\text{i)} \quad \mu(r) = r^2(\log 1/r)^2 , \qquad \text{or} \qquad \text{ii)} \quad \mu(r) = r^2(\log 1/r)^2 e(r)^2$$

so that the condition II is also satisfied.

We distinguish now the following cases:

$$\text{Case 1} \quad 2c+1 \neq d \quad \begin{cases} a) & 2c+1<d \\ b) & 2c+1>d \end{cases}$$

$$\text{Case 2} \quad 2c+1 = d \quad \begin{cases} a) & q>1 \\ b) & q<1\,. \end{cases}$$

**1.2.** Let $\{f_n(r)\}$ be a sequence of $C^\infty$ functions $f_n(r)$ which are defined on $(0, \varepsilon]$, with compact supports, and are parametrized by the integers $n$. They may depend smoothly on the parameter $s \in S$ above. Then we define another sequence $\{u_n(r)\}$ of $C^\infty$ functions $u_n(r)$ on $(0, \varepsilon]$ as follows (cf. [31; p.437]):

In Case 1, a) and Case 2, a)

$$u_n(r) = 2r^n \int_0^r \rho^{-n} \cdot f_{n+1}(\rho)\, d\rho\,,$$

and in Case 1, b) and Case 2, b)

$$u_n(r) = -2r^n \int_r^\varepsilon \rho^{-n} \cdot f_{n+1}(\rho)\, d\rho\,.$$

In this case we shall write

$$u_n(r) = E_n(f_{n+1}(r))\,. \tag{2}$$

We then define weighted $L^2$-norms of $u_n$ and $f_n$ as follows;

$$\|u_n(r)\|^2 = \int_0^\varepsilon |u_n(\rho)|^2\, d(r)\, dr$$

and

$$\|f_n(r)\|^2 = \int_0^\varepsilon |f_n(\rho)|^2 \mu(r)\, d(r)\, dr\,.$$

Then the purpose of this section is to prove the following:

**Lemma 1.2.** *The notations and assumptions being as above, there exists a positive constant $C$ such that for any integer $n$ we have the estimate*

$$\|u_n(r)\|^2 \leq C \|f_{n+1}(r)\|^2\,,$$

*where $C$ is independent of the sequence $\{f_n(r)\}$ and the parameter $s \in S$.*

*Remark* 1.2. If $c$ is not an odd integer, Case 2 does not occur. Then we may replace the condition I, b) (resp. II) by the condition (1) of Lemma 1.1 (resp. II': $r^2\mu(r)^{-1}$ is bounded), as the proof below of the lemma obviously shows.

In what follows we use for simplicity the following notation. Let $A(r)$ and $B(r)$ be $C^\infty$ functions on $(0, \varepsilon]$ which depend possibly on the parameter $s$. Then an inequality $A(r) \lesssim B(r)$ means that there exists a positive constant $C$ such that $A(r) \leq CB(r)$ uniformly with respect to $s$.

**1.3.** *Proof of Lemma* 1.2. (cf. the proof of [31], Proposition 6.4) We treat the four cases separately, and in each case the proof is divided into three steps which are similar to each other. In the first step the argument is independent of the assumptions I and II. Steps 2 and 3 are related respectively to the assumptions I and II.

We take and fix any positive real number $e$ such that for any integer $n$, according as $n$ is in Case 1 a), Case 1 b), Case 2 a), and Case 2 b), we have $2n + c + e + 1 < 0$, $2n + c + e + 1 > 0$, $q - e + 1 > 0$, and $q + e + 1 < 0$, respectively.

Case 1, a): *Step* 1. By the Cauchy-Schwarz inequality we have

$$\begin{aligned} |u_n(r)|^2 &\lesssim r^{2n} \int_0^r \rho^{-2n-e+1} f_{n+1}(\rho)^2 \, d\rho \cdot \int_0^r \rho^{-1+e} \, d\rho \\ &= r^{2n+e} \int_0^r \rho^{-2n-e+1} f_{n+1}(\rho)^2 \, d\rho \, . \end{aligned}$$

Then we have

$$\begin{aligned} \|u_n(r)\|^2 &\lesssim \int_0^\varepsilon r^{2n+e} \left[ \int_0^r \rho^{-2n-e+1} f_{n+1}(\rho)^2 \, d\rho \right] d(r) \, dr \\ &= \int_0^\varepsilon \rho^{-2n-e+1} f_{n+1}(\rho)^2 \left[ \int_\rho^\varepsilon r^{2n+e} \, d(r) \, dr \right] d\rho \, . \end{aligned} \tag{3}$$

*Step* 2. We have

$$\begin{aligned} 0 &\leq -(2n + c + e + 1)/a_s \int_\rho^\varepsilon r^{2n+e} \, d(r) \, dr \\ &= -(2n + c + e + 1) \int_\rho^\varepsilon r^{2n+c+e} m(r) \, dr \\ &= 1/a_s [r^{2n+e+1} \, d(r)]_\varepsilon^\rho + \int_\rho^\varepsilon r^{2n+c+e+1} m'(r) \, dr \, . \end{aligned}$$

Then, since

$$r^{2n+c+e+1} m'(r) / r^{2n+c+e} m(r) = r m'(r)/m(r) \to 0$$

by Lemma 1.1, we get

$$\int_\rho^\varepsilon r^{2n+e} \, d(r) \, dr \lesssim \rho^{2n+e+1} \, d(\rho) \, .$$

*Step* 3. Substituting this into (3) and using the assumption II we get

$$\|u_n(r)\|^2 \preceq \int_0^\varepsilon \rho^2 f_{n+1}(\rho)^2\, d(\rho)\, d\rho$$
$$= \int_0^\varepsilon [f_{n+1}(\rho)^2\mu(\rho)\, d(\rho)][\rho^2\mu(r)^{-1}]\, d\rho \preceq \|f_{n+1}\|^2 .$$

Case 1, b): *Step* 1. In the corresponding argument in a) we replace $-e$ by $+e$, and $\int_0^r$ by $\int_r^\varepsilon$. Then by the same argument we have

$$\|u_n(r)\|^2 \preceq \int_0^\varepsilon \rho^{-2n+e+1} f_{n+1}(\rho)^2 \left[\int_0^\rho r^{2n-e}\, d(r)\, dr\right] d\rho .$$

*Step* 2. Replacing $-e$ by $+e$, and $\int_0^\varepsilon$ by $\int_0^\rho$ in the corresponding argument in a) we get

$$0 \leq (2n + c - e + 1)/a_s \int_0^\rho r^{2n-e}\, d(r)\, dr$$
$$= 1/a_s[r^{2n-e+1}\, d(r)]_0^\rho + \int_0^\rho r^{2n+c-e+1} m'(r)\, dr .$$

Since $r^{2n-e+1}\, d(r)$ tends to 0 when $r$ tends to 0 by the condition I, a), by the same argument using Lemma 1.1 we get

$$\int_0^\rho r^{2n-e}\, d(r)\, dr \preceq \rho^{2n-e+1}\, d(\rho) .$$

*Step* 3. The same argument as in a) works also in this case.

Case 2, a): *Step* 1. As in Case 1 we have

$$|u_n(r)|^2 \preceq r^{2n} \int_0^r \rho^{-2n+1}(\log 1/\rho)^{1+e} f_{n+1}(\rho)^2\, d\rho \cdot \int_0^r \rho^{-1}(\log 1/\rho)^{-1-e}\, d\rho$$
$$= r^{2n}(\log 1/r)^{-e} \int_0^r \rho^{-2n+1} \cdot (\log 1/\rho)^{1+e} f_{n+1}(\rho)^2\, d\rho ,$$

and hence

$$\|u_n(r)\|^2 \preceq \int_0^\varepsilon \rho^{-2n+1}(\log 1/\rho)^{1+e} f_{n+1}(\rho)^2 \cdot \left[\int_\rho^\varepsilon r^{2n}(\log 1/r)^{-e}\, d(r)\, dr\right] d\rho \tag{4}$$

*Step* 2. We have

$$0 \leq (q + 1 - e)/a_s \int_\rho^\varepsilon r^{2n}(\log 1/r)^{-e}\, d(r)\, dr$$
$$= (q + 1 - e) \int_\rho^\varepsilon r^{-1}(\log 1/r)^{q-e} n(r)\, dr$$
$$= 1/a_s[r^{2n+1}(\log 1/r)^{1-e}\, d(r)]_\varepsilon^\rho + \int_\rho^\varepsilon (\log 1/r)^{q+1-e} n'(r)\, dr .$$

Since by the condition I, a) we have

$$(\log 1/r)^{q+1-e}n'(r)/r^{-1}(\log 1/r)^{q-e}n(r) = r(\log 1/r)(\log n(r))' \to 0\,,$$

from this we get

$$\int_\rho^\varepsilon r^{2n}(\log 1/r)^{-e}\, d(r)\, dr \precsim \rho^{2n+1}(\log 1/r)^{1-e}\, d(\rho)\,.$$

*Step* 3. Substituting this in (4) and using the assumption II we have

$$\begin{aligned}\|u_n(r)\|^2 &\precsim \int_0^\varepsilon \rho^2(\log 1/\rho)^2 f_{n+1}(\rho)^2\, d(\rho)\, d\rho \\ &= \int_0^\varepsilon [f_{n+1}(\rho)^2\mu(\rho)\, d(\rho)][\rho^2(\log 1/\rho)^2\mu(\rho)^{-1}]\, d\rho \precsim \|f_{n+1}\|^2\,.\end{aligned}$$

Case 2, b): *Step* 1.

$$\|u_n(r)\|^2 \precsim \int_0^\varepsilon \rho^{-2n+1}(\log 1/\rho)^{1-e} f_{n+1}(\rho)^2\left[\int_0^\rho r^{2n}(\log 1/r)^e\, d(r)\, dr\right] d\rho\,.$$

In fact, if in the corresponding argument in a) we replace $e$ by $-e$, $\int_0^r$ by $\int_r^\varepsilon$, then the argument is quite identical.

*Step* 2. Replacing $-e$ by $+e$, and $\int_\rho^\varepsilon$ by $\int_0^\rho$, in the corresponding argument in a) we have

$$\begin{aligned}0 &\leq -(q+1+e)/a_s \int_0^\rho r^{2n}(\log 1/r)^e\, d(r)\, dr \\ &= 1/a_s[r^{2n+1}(\log 1/r)^{1+e}\, d(r)]_0^\rho + \int_0^\rho (\log 1/r)^{q+1+e} n'(r)\, dr\,.\end{aligned}$$

Then as in the case a) this implies that

$$\int_0^\rho r^{2n}(\log 1/r)^e\, d(r)\, dr \precsim \rho^{2n+1}(\log 1/\rho)^{1-e}\, d(\rho)\,,$$

noting that $r^{2n+1}(\log 1/r)^{1+e}\, d(r)$ tends to 0 as $r$ tends to 0 by the condition I, a).

*Step* 3. The same argument as above works.

It is immediate to see the constants which are implicit in the notations $\precsim$ above are actually independent of $n$. The lemma follows.

## § 2. $\mathbb{L}^2$ Dolbeault Lemma

**2.1.** Let $\mathbb{C}^n = \mathbb{C}^n(z_1, \ldots, z_n)$ be a complex euclidian space for some $n > 0$. For a positive number $\varepsilon$ with $0 < \varepsilon < 1$ we consider the polycylinder $\overline{W} = \overline{W}_\varepsilon$ in $\mathbb{C}^n$;

$$\overline{W} = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n;\, |z_i| < \varepsilon\}\,.$$

Denote by $Y_i$ the hyperplane in $\overline{W}$ defined by $z_i = 0$. Then fix an integer $k$ with $1 \leq k \leq n$, and let $Y = \bigcup_{i=1}^k Y_i$;

$$Y = \{z \in \overline{W};\, z_1 \cdots z_k = 0\}.$$

We put $W = \overline{W} - Y$. In polar coordinates we write $z_i = r_i e^{i\theta_i}$. Let $F$ be a (trivial) holomorphic line bundle defined on $\overline{W}$, with a generating holomorphic section $\sigma$ on $\overline{W}$. Fix a $C^\infty$ hermitian metric $h$ of $F$ over $W$ and denote by $|\sigma|^2$ the square norm of $\sigma$ with respect to $h$. Let $g$ be a Kähler metric on $W$ such that the associated Kähler form $\omega$ is of the following form

$$\omega = \sqrt{-1} \sum_i 1/\mu_i \, dz_i \wedge d\bar{z}_i \,, \tag{5}$$

where $\mu_i = |dz_i|^2$ is a positive function on $W$. Then the volume form $dv$ associated to $\omega$ is written in the form;

$$dv = (\sqrt{-1})^n v \cdot \prod_i dz_i \wedge d\bar{z}_i \,, \qquad v = \prod_{i=1}^n 1/\mu_i \,. \tag{6}$$

We assume that the functions $|\sigma|^2$ and $\mu_i$ (and hence also $v$) depend only on $r_i$, $1 \leq i \leq k$. We set

$$d(r_1, \ldots, r_k) = |\sigma|^2 \cdot v \cdot \prod_{1 \leq i \leq k} r_i \,, \tag{7}$$

and further make the following three assumptions:

A1) The function $d$ is of the form

$$d(r_1, \ldots, r_k) = r_1^{c_1} \ldots r_k^{c_k} (\log 1/r_1)^{b_1} \ldots (\log 1/r_k)^{b_k} L(r_1, \ldots, r_k)^t \,, \tag{8}$$

where

$$L = L(r_1, \ldots, r_k) = \sum_{i=1}^k \log 1/r_i \,,$$

and $c_i$, $b_j$, $t$ are real numbers with $t \geq 0$ such that $q_i := b_i + t \neq -1$ if $c_i$ is an odd integer. We set $a_i = (c_i + 1)/2$ and denote by $[a_i]$ the largest integer which does not exceed $a_i$.

A2) If $1 \leq i \leq k$, then $\mu_i$ is either of the following two forms;

$$\mu_i(r) = r_i^2 (\log 1/r_i)^2 \,, \qquad \text{or} \qquad = r_i^2 L^2 \,. \tag{9}$$

A3) If $k + 1 \leq i \leq n$, then $\mu_i^{-1}$ is bounded (above) on $W$.

Note that A2) implies that the Kähler metric $g$ is (uniformly) complete along $Y$.

In this note we are mainly interested in the following three types of Kähler forms $\omega$.

**Example 2.1.** 1) *Poincaré* (*Growth*) *Type*: We take $\omega = \omega_P$, where

$$\omega_P = \sqrt{-1}\sum_{i=1}^{k}\frac{dz_i \wedge d\bar{z}_i}{(r_i \log 1/r_i)^2} + \sqrt{-1}\sum_{j=k+l}^{n} dz_j \wedge d\bar{z}_j\,;$$

in particular, $\mu_i = r_i^2(\log 1/r_i)^2$, $1 \le i \le k$, and $\mu_j = 1$, $k+1 \le j \le n$, so that A2) and A3) above are satisfied; we have

$$v = \prod_{1 \le i \le k} r_i^{-2}(\log 1/r_i)^{-2}\,.$$

2) *Ball Quotient Type*: We assume that $k = 1$ and take $\omega = \omega_B$, where

$$\omega_B = \sqrt{-1}\frac{dz_1 \wedge d\bar{z}_1}{(r_1 \log 1/r_1)^2} + \sqrt{-1}\sum_{j=2}^{n}\frac{dz_j \wedge d\bar{z}_j}{\log 1/r_1}\,,$$

in particular, $\mu_1 = r_1^2(\log 1/r_1)^2$, $\mu_j = \log 1/r_1$, $k+1 \le j \le n$, so that A2) and A3) are satisfied; also we have

$$v = r_1^{-2}(\log 1/r_1)^{-(n+1)}\,.$$

3) *Hilbert Modular Type*: We take $\omega = \omega_H$, where

$$\omega_H = \sqrt{-1}L^{-2}\left(\sum_{1 \le i \le k} r_i^{-2}\, dz_i \wedge d\bar{z}_i + \sum_{k < j \le n} dz_j \wedge d\bar{z}_j\right);$$

in particular, $\mu_i = r_i^2 L^2$ for $1 \le i \le k$ and $\mu_j = L^{-2}$ for $k+1 \le j \le n$ so that A2) and A3) are again satisfied (cf. Remark 1.1). Also we have

$$v = L^{-2n}\prod_{1 \le i \le k} r_i^{-2}\,.$$

**2.2.** Let $g$ and $h$ be the metrics satisfying A1)–A3) as in 2.1. By using the metrics $g$ and $h$, we may speak as usual of the L$^2$-integrable $F$-valued $(0, q)$-forms on any open subset $V$ of $\overline{W}$ (with measurable coefficients). Then consider the space $L^q = L^q(V, F)$ of $F$-valued $(0, q)$-forms $\varphi$ on $V$ such that both $\varphi$ and $\bar{\partial}\varphi$ are L$^2$-integrable. We can also consider the corresponding sheaves $\mathscr{L}^q(F)$ on $\overline{W}$ which form a complex with respect to the $\bar{\partial}$-operators. Then our L$^2$-Dolbeault lemma is stated as follows.

**Proposition 2.1.** *In the notations of* A1) *the* $L^2$ *Dolbeault complex* $\mathscr{L}^{\bullet}(F)$ *is a resolution of the sheaf* $\mathcal{O}_X(F \otimes [Y_1]^{p_1} \otimes \cdots \otimes [Y_k]^{p_k})$, *where* $p_i = [a_i] - 1$ *if* $c_i$ *is an odd integer and* $q_i > -1$, *and* $p_i = [a_i]$, *otherwise.*

*Remark* 2.1. Two nonnegative functions (or hermitian metrics) $f$ and $g$ defined on $W$ are said to be *equivalent* along $Y$ if for any relatively compact subdomain $V$ of $\overline{W}$ there exists a positive constant $C$ such that $(1/C)g \le f \le Cg$ on $V - Y$. In this case we use the notation $f \sim g$. Since the (local) L$^2$ condition is unchanged if we pass to equivalent metrics or norms, the Dolbeault lemma above still holds if we replace "=" by "$\sim$" in formulas (5) (6) and (8) above.

**Example 2.2.** In each case of Example 2.1 we obtain the following conclusion.

1) *Poincare Type*: If we assume that $|\sigma|^2$ is of the form

$$|\sigma|^2 = r_1^{2a_1} \dots r_k^{2a_k}(\log 1/r_1)^{b_1} \dots (\log 1/r_k)^{b_k}$$

where $b_i \neq -1$ whenever $2a_i$ is an integer, then the function $d$ above (cf. (7)) satisfies A1) with $c_i = 2a_i - 1$ and $q_i = b_i$. Hence, the $\mathrm{L}^2$-Dolbeault complex is a resolution of $\mathcal{O}_X(F \otimes [Y_1]^{p_1} \otimes \cdots \otimes [Y_k]^{p_k})$ in this case.

The result is due to Zucker [31; Prop. 6.4, Prop. 11.5] when $n = 1$. A generalization to the higher dimensional case is found in [28; Prop. D.4 b)].

2) *Ball Quotient Type*: If we assume that $|\sigma|^2$ is of the form

$$|\sigma|^2 = r_1^{2a_1}(\log 1/r_1)^{b_1}$$

where $b_1 \neq n$ if $2a_1$ is an integer, then the function $d$ above satisfies A1) with $c_1 = 2a_1 - 1$ and $q_1 = b_1 - (n + 1)$. Hence, the $\mathrm{L}^2$-Dolbeault complex is a resolution of $\mathcal{O}_X(F \otimes [Y_1]^{p_1})$ in this case.

3) *Hilbert Modular Type*: We assume that $|\sigma|^2$ is of the form

$$|\sigma|^2 = \prod_{1 \leq i \leq k} r_i^{2a_i} \cdot L^u ,$$

where $u \geq 2n$. Then the function $d$ above satisfies A1) with $c_i = 2a_i - 1$, and $q_i = t = u - 2n$. Hence, the $\mathrm{L}^2$-Dolbeault complex is a resolution of $\mathcal{O}_X(F \otimes [Y_1]^{p_1} \otimes \cdots \otimes [Y_k]^{p_k})$ in this case also.

Note that in all the above cases the notations $a_i$ are compatible with those used in the proposition.

**2.3.** *Proof of Proposition* 2.1. A. First we identify $\mathrm{L}^2$-holomorphic sections of $F$ on $W$. Consider any holomorphic section $\beta$ of $F$ of the form $\beta = z_1^{m_1} \dots z_k^{m_k} \cdot \sigma$. Then we have

$$\|\beta\|^2 = C \int_0^\varepsilon \cdots \int_0^\varepsilon \prod_{1 \leq i \leq k} (r_i^{2m_i + c_i}(\log 1/r_i)^{b_i}) \cdot L(r)^t \, dr_1 \dots dr_k$$

for some positive constant $C$. Suppose that $\beta$ is $\mathrm{L}^2$-integrable. Then by the Fubini theorem, the integrand must be integrable with respect to $r_i$ when the other $r_j$, $j \neq i$, are fixed generically. It follows that $h_i := 2m_i + c_i + 1 > 0$, or $h_i = 0$ and $q_i < -1$.

Conversely, if either of these conditions are fulfilled, then in view of the inequality $L \leq \prod_{1 \leq i \leq k} \log 1/r_i$ (for sufficiently small $r_i$) and the assumption $t \geq 0$ we get that $\beta$ is $\mathrm{L}^2$-integrable. From this, one concludes easily that the holomorphic sections of $F$ on $W$ which are $\mathrm{L}^2$-integrable and meromorphic along $Y$ are naturally identified with the sections of the line bundle $F \otimes [Y_1]^{p_1} \otimes \cdots \otimes [Y_k]^{p_k}$.

It remains thus to show that an $L^2$ holomorphic section are necessarily meromorphic along $Y$. By using the Fubini theorem as above, we may reduce the proof to the case where $n = k = 1$. Consider any section $u\sigma$ of $F$, where $u$ is a holomorphic function on $W$. Let $u = \sum_n a_n r^n e^{in\theta}$ be the Fourier series expansion of $u$. Then for any $n$ we have

$$2\pi|a_n|^2 \int_0^\varepsilon r^{2n}\, d(r)\, dr \le \int_W |u\sigma|^2\, dv < +\infty\,,$$

where $r = r_1$ and $\theta = \theta_1$. Hence, if $2n + c < -1$, we must have $a_n = 0$, i.e., $u$ is meromorphic as desired.

B. We next show the exactness of the complex $\mathscr{L}^{\cdot}(F)$ at any positive degree $q$. We proceed in steps.

a) Since the problem is local, by restricting $\overline{W}$ if necessary, we may assume that every object under consideration is defined on the closure $\overline{W}'$ of $\overline{W}$. As in the standard proof of the Dolbeault lemma (cf. [10; p.27]) it suffices to show the following assertion $(*)_i$ for any $i$ with $1 \le i \le n$.

$(*)_i$ Let $f$ be any $\bar\partial$-closed element of $L^q(W, F) = L^q(\overline{W}, F)$ which involves no $d\bar z_j$ for $j < i$. Then for any relatively compact subdomain $\overline{V}$ of $\overline{W}$ there exists an element $g$ of $L^{q-1}(V, F)$ such that the element $f - \bar\partial g$ involves no $d\bar z_j$ with $j \le i$, where $V = \overline{V} \cap W$.

So we fix any $i$ and prove $(*)_i$.

b) Denote by $C^q(W, F)$ the space of $F$-valued $C^\infty(0, q)$-forms on $W$, and $A^q(W, F)$ the subspace of $C^q(W, F)$ consisting of forms which are obtained as restrictions of those defined in a neighborhood of $\overline{W}'$ and whose supports are disjoint from $Y$. We have the direct sum decomposition

$$C^q(W, F) = {}'C^q(W, F) \oplus {}''C^q(W, F)\,,$$

where the elements of ${}'C^q(W, F)$ (resp. ${}''C^q(W, F)$) involve (resp. do not involve) $d\bar z_i$. We denote by $A^q(W, F) = {}'A^q(W, F) \oplus {}''A^q(W, F)$, the similar decomposition for $A^q(W, F)$. We fix any relative compact subdomain $\overline{V}$ of $\overline{W}$ as in $(*)_i$. Let $r_V\colon C^q(W, F) \to C^q(V, F)$ be the natural restriction map. Define a part $bW = bW_i$ of the boundary of $\overline{W}$ by

$$bW = \{(z_1, \dots, z_n) \in \overline{W}';\, |z_i| = \varepsilon\} \cong D \times \cdots \times D \times S \times D \times \cdots \times D\,,$$

where $D = \{z \in \mathbf{C};\, |z| \le \varepsilon\}$, $S = \{z \in D;\, |z| = \varepsilon\}$, and $S$ is on the $i$-th place. Let $C^q(bW, F)$ be the space of $C^\infty$ $q$-forms on the $C^\infty$ manifold with boundary $bW$ which are linear combinations of the forms $d\bar z_J$ and $d\theta_i \wedge d\bar z_{J'}$ with coefficients in $C(bW) := C^0(bW)$, where $J$ and $J'$ do not involve $i$. Then we get the surjective restriction map

$$r\colon A^q(W, F) \to C^q(bW, F)\,,$$

and the corresponding direct sum decomposition $C^q(bW, F) = {}'C^q(bW, F) \oplus {}''C^q(bW, F)$, where for instance the forms in ${}''C^q(bW, F)$ do not involve $d\theta_i$. With respect to the induced Riemannian metric on $bW$ we can speak of $L^2$-norms of the forms in $C^q(bW, F)$ also.

c) In order to prove $(*)_i$ we shall show that for any $q \geq 0$ there exist linear operators

$$G = G_i\colon A^q(W, F) \to {}''C^{q-1}(W, F)\,, \qquad (C^{-1}(W, F) = 0)$$

and

$$P = P_i\colon C^q(bW, F) \to {}''C^q(W, F)$$

with the following three properties:

c1) $G$ and the composite map $r_V P\colon C^q(bW, F) \to {}''C^q(V, F)$ are bounded with respect to the $L^2$-norms.

c2) Write $\bar{\partial} = \sum \bar{\partial}_j$, where $\bar{\partial}_j$ is the $\bar{\partial}$-exterior derivative with respect to $\bar{z}_j$. Then among the resulting operators $A^q(W, F) \to C^q(W, F)$ the following relations hold true:

$$\text{(a)} \quad I = G\bar{\partial}_i + \bar{\partial}_i G + Pr \qquad \text{and} \qquad \text{(b)} \quad 0 = G\bar{\partial}_j + \bar{\partial}_j G\,, \qquad j \neq i\,,$$

where $I$ denotes the natural inclusion $I\colon A^q(W, F) \hookrightarrow C^q(W, F)$.

c3) If an element $\varphi$ of $A^q(W, F)$ does not involve $d\bar{z}_j$ for some $j \neq i$, then the same is true also for $Pr(\varphi)$.

Note that summing up the equations (a) and (b) we have

$$I = G\bar{\partial} + \bar{\partial}G + Pr\,. \tag{10}$$

We then show how the existence of such operators would lead to the proof of $(*)_i$ in the next two steps d) and e).

d) Assume more specifically that for any $\eta \in (0, \varepsilon]$ such operators $G = G_\eta$ and $P = P_\eta$ have already been constructed on $W_\eta \subseteq W_\varepsilon$. (The operators clearly depend on the domain $W_\eta$.) Fix a $\bar{\partial}$-closed element $f$ of $L^q(W, F)$ which involves no $d\bar{z}_j$, $j < i$, as in $(*)_i$. First of all, we note the following fact, which follows readily from the argument in [2; pp. 92, 93] and the completeness of our metric along $Y$.

(A) $A^q(W, F)$ is dense in $L^q(W, F)$ with respect to the norm $\|\varphi\| + \|\bar{\partial}\varphi\|$.

Hence, there exists a sequence $\{f_n\}$ of elements $f_n$ of $A^q(W, F)$, such that $f_n$ and $\bar{\partial}f_n$ converge respectively to $f$ and $\bar{\partial}f$ $(=0)$ in the $L^2$-norm as $n$ tends to $\infty$. Moreover, we may assume that $f_n$ contain no $d\bar{z}_j$, $j < i$ (cf. *loc. cit.*). By the property c1), $G$ and $r_V P$ extend to bounded linear operators of the $L^2$-completions of the corresponding spaces (still denoted by the same letters). In particular, $Gf$ makes sense as an element of $L^{q-1}(W, F)$. Note also that the image of $r_V P$ again consists of forms which involve no $d\bar{z}_i$.

e) By the Fubini's theorem, for almost all $\eta \in (0, \varepsilon]$, $r(f)$ makes sense as an element of $\bar{L}^q(bW_\eta, F)$, the $L^2$-completion of $C^q(bW_\eta, F)$, and moreover, we can assume that $r(f_n)$ converge to $r(f)$ in the $L^2$-norm on $bW_\eta$, after passing to a subsequence if necessary (cf. [3; 3.43]). Then, by replacing $\varepsilon$ by one of such $\eta \in (0, \varepsilon]$ which is sufficiently close to $\varepsilon$ so that $V \subseteq W_\eta$, we may assume that this is true already for $\eta = \varepsilon$. Then applying the formula (10) to $f_n$, and taking the weak limit, we get

$$\bar{\partial}(r_V G(f)) = r_V \bar{\partial} G(f) = r_V(f) - r_V G\bar{\partial}(f) - r_V Pr(f) = r_V(f) - r_V Pr(f),$$

where $\bar{\partial}$ are taken in the sense of currents. This implies that $\bar{\partial}(r_V G(f))$ is actually in $L^q(V, F)$. Moreover, $r_V(f) - \bar{\partial}(r_V G(f))$ $(= r_V Pr(f))$ does not involve any $d\bar{z}_j$ with $j \leq i$ by the remark at the end of d) and the property c3). Thus the element $g := r_V G(f)$ is a desired element having the required properties in $(*)_i$. It remains thus to construct operators $G$ and $P$ as in c) with the properties c1), c2) and c3).

f) We shall reduce the problem of constructing $G$ and $P$ to that of constructing certain other linear maps

$$E\colon A(W) \to C(W) \qquad \text{and} \qquad Q\colon C(bW) \to C(W),$$

where $A(W) = A^0(W, F)$, $C(W) = C^0(W, F)$ and $C(bW) = C^0(bW, F)$; thus $A(W)$ is for instance the space of complex-valued $C^\infty$ functions on $\overline{W}'$ whose supports do not intersect with $Y$. These operators are required to have the two properties f1) and f2) below. For any fixed $z_j$, $j \neq i$, consider elements $a$ of $A(W)$ and $h$ of $C(bW)$ as $C^\infty$ functions on $D = z_1 \times \cdots \times z_{i-1} \times D \times z_{i+1} \times \cdots \times z_n$ and on $S = z_1 \times \cdots \times z_{i-1} \times S \times z_{i+1} \times \cdots \times z_n$, respectively. In general for $\eta > 0$ set $D_\eta = \{z_i \in C; |z_i| \leq \eta\}$ so that $D = D_\varepsilon$. Write $r = r_i$ and $\theta = \theta_i$. On $D$ and $D_\eta$ we shall consider the volume form $d\lambda = d\lambda_i$ defined by

$$dv = d\lambda \wedge (\sqrt{-1})^{n-1} \prod_{j \neq i} dz_j \wedge d\bar{z}_j;$$

similarly on $S$ we consider the volume form $w$ defined by $d\lambda | S = (dr | S) \wedge w$. Then:

f1) We have the following estimates of $L^2$ norms $\| \ \|$, $\| \ \|_\eta$, and $\| \ \|_S$ on $D$, $D_\eta$, and $S$ respectively;

$$\|E(a)\| \leq C \|a d\bar{z}_i\|, \qquad a \in A(W)$$

and

$$\|Qh\|_\eta \leq C_\eta \|h\|_S, \qquad h \in C(bW),$$

where $C$ and $C_\eta$ are positive constants which are independent of $z_j$, $j \neq i$, and $\eta$ is any number in the interval $(0, \varepsilon]$.

f2) Among the resulting operators $A(W) \to C(W)$, the following relations hold:

$$I = {}_i\bar{\partial}E\,, \qquad I = E_i\bar{\partial} + Qr\,, \qquad {}_j\bar{\partial}E = E_j\bar{\partial}\,, \qquad j \neq i\,, \tag{11}$$

where $I\colon A(W) \to C(W)$ is the natural inclusion and ${}_j\bar{\partial}$ is the operator $\partial/\partial\bar{z}_j$ so that $\bar{\partial}_j = {}_j\bar{\partial}\cdot d\bar{z}_j \wedge$.

g) We shall first see how to introduce the operators $G$ and $P$ as in c) in terms of the operators $E$ and $Q$ above (assuming that they have already been constructed). Consider the given generating holomorphic section $\sigma$ of $F$ as giving a fixed trivialization of $F$ on $\overline{W}$. Accordingly, we regard $F$-valued forms naturally as ordinary forms. For simplicity we write $A^q = A^q(W, F)$, $''A^q = {}''A^q(W, F)$ etc. An element $f$ of $''A^q$ is written uniquely in the form

$$f = \sum{}' f_J d\bar{z}_J,$$

where the summation $\sum'$ is taken over all the ordered $q$-tuples $J$ with $i \notin J$.

*Definition of G and P*: We define $G\colon A^q \to {}''C^{q-1}$ by specifying it in the following two cases:

$\alpha$) If $f \in {}'A^q$, write $f = d\bar{z}_i \wedge f_i$ with $f_i = \sum' a_J\, d\bar{z}_J \in {}''A^{q-1}$ uniquely. Then $G(f)$ is by definition a $(q-1)$-form on $W$ obtained by replacing each coefficients $a_J \in A(W)$ of $f_i$ by $E(a_J) \in C(W)$;

$$G(f) = \sum{}' E(a_J) d\bar{z}_J.$$

$\beta$) If $f \in {}''A^q$, by definition $Gf = 0$.

Similarly, we define the operator $P\colon C^q(bW, F) \to {}''C^q$ as follows.

$\alpha$) If $f \in {}'C^q(bW, F)$, then $Pf = 0$ by definition.

$\beta$) If $f \in {}''C^q(bW, F)$, then $Pf$ is the $q$-form obtained from $f = \sum' a_J d\bar{z}_J$, $i \notin J$, $a_J \in C(bW)$, by replacing $a_J$ by $Q(a_J)$;

$$P(f) = \sum{}' Q(a_J)\, d\bar{z}_J.$$

*Verification of the properties* c1)–c3). The property c3) is clear from the above definition of $P$. So it suffices to check c1) and c2).

c1) For $G$: we may assume that $f \in {}'A^q$. Then, in the above notations we have

$$\|Gf\|^2 = \sum_J{}' \|E(a_J)\, d\bar{z}_J\|^2 \le C \sum_J{}' \|a_J\, d\bar{z}_i \wedge d\bar{z}_J\|^2 = C\|f\|^2\,.$$

For $r_V P$: we may assume that $f = \sum' a_J\, dz_J \in {}''C^q(bW, F)$ with $a_J \in C(bW)$. Then:

$$\|r_V P(f)\|^2 = \sum{}' \|r_V P(a_J\, d\bar{z}_J)\|^2 = \sum{}' \|r_V Q(a_J)\, d\bar{z}_J\|^2 \le C_\eta \sum{}' \|a_J\, d\bar{z}_J\|^2_{bW_\eta}\,,$$

where $\eta$ is any number in $(0, \varepsilon]$ such that $V \subseteq W_\eta$.

c2): We check the relations (a) and (b).
(a):
$\alpha$) If $f \in {}'A^q$, by using the first relation of (11) we get

$$(G\bar{\partial}_i + \bar{\partial}_i G)f = \bar{\partial}_i Gf = \sum_i{}' \bar{\partial} E(a_J)\, d\bar{z}_i \wedge d\bar{z}_J$$

$$= \sum{}' a_J\, d\bar{z}_i \wedge d\bar{z}_J = d\bar{z}_i \wedge f_i - 0 = f - Pf\,.$$

$\beta$) If $f \in {}''A^q$, by using the second relation of (11) we get

$$(G\bar{\partial}_i + \bar{\partial}_i G)f = G\bar{\partial}_i f = G\sum{}'({}_i\bar{\partial} a_J)\, d\bar{z}_i \wedge d\bar{z}_J$$

$$= \sum{}' E({}_i\bar{\partial} a_J)\, d\bar{z}_J = \sum{}' a_J\, d\bar{z}_J - \sum{}' Q a_J\, d\bar{z}_J = f - Pf\,.$$

(b) Take any $j$ with $j \neq i$. Then:
$\alpha$) If $f \in {}'A^q$, by the third relation of (11) we get

$$(G\bar{\partial}_j + \bar{\partial}_j G)f = \sum{}' \{G({}_j\bar{\partial} a_J\, d\bar{z}_j \wedge d\bar{z}_i \wedge d\bar{z}_J) +{}_j \bar{\partial} E(a_J)\, d\bar{z}_J\}$$

$$= \sum{}' \{-E({}_j\bar{\partial} a_J)\, d\bar{z}_j \wedge d\bar{z}_J + E({}_j\bar{\partial} a_J)\, d\bar{z}_j \wedge d\bar{z}_J\} = 0\,.$$

$\beta$) If $f \in {}''A^q$,

$$(G\bar{\partial}_j + \bar{\partial}_j G)f = G(\bar{\partial}_j f) = 0\,,$$

since $\bar{\partial}_j f$ is in ${}''A^{q+1}$.

Thus the proof is reduced to the construction of the operators $E$ and $Q$ in f). For this purpose we distinguish the following two cases: Case 1: $1 \leq i \leq k$, and Case 2: $k+1 \leq i \leq n$. (Recall that $Y$ is defined in $\overline{W}$ by the equation $z_1 \cdots z_k = 0$.)

h) We start with Case 1: $1 \leq i \leq k$. Write $r = r_i$ and $\theta = \theta_i$ so that $z_i = re^{\sqrt{-1}\theta}$.

*Definition of E and Q*: *Case of E*: We expand elements $a = a(r, \theta)$ of $A(W)$ in the Fourier series with respect to $\theta$ with parameters $r$ and $z_j$, $j \neq i$:

$$a(r, \theta) = \sum_n a_n(r) e^{in\theta}, \qquad i = \sqrt{-1}\,,$$

where $a_n(r)$ is a $C^\infty$ function on $(0, \varepsilon]$ with compact supports, depending on the parameters $z_j$, $j \neq i$, and $\sum_n$ implies that the summation is over all the integers $n$. (Here and in what follows, we usually suppress $z_j$, $j \neq i$.) Then we define $E(a) \in A(W)$ in the form a Fourier series by

$$E(a)(r, \theta) = \sum_n E_n(a_{n+1}(r)) e^{in\theta}\,,$$

where $E_n(a_{n+1}(r))$ is defined by the formula § 1 (2) before Lemma 1.2. By the rapidly decreasing property of Fourier coefficients it is easy to see that $E(a)$ is in fact a smooth function on $\overline{W}$.

*Case of Q*. Similarly, given any element $h$ of $C(bW)$, expand it in the Fourier series $h = \sum_n h_n e^{in\theta}$, where $h_n$ are independent of $r$, but may depend on other variables $z_j$, and then define $Q(h) \in C(W)$ by

$$Q(h)(r, \theta) = \sum_n{}' h_n (r/\varepsilon)^n e^{in\theta},$$

where $\sum_n'$ denotes taking the summation over all $n$ with $n > -a_i$ (resp. $n \geq -a_i$) if $q_i > -1$ (resp. $q_i < -1$).

*Properties of E and Q*: We shall check the properties f1) and f2) required for $E$ and $Q$. We fix $z_j$ for $j \neq i$.

f1) *Case of E*: We consider the functions $d$ and $\mu = \mu_i$, defined by (7) and (9) respectively, as functions $d = d(r)$ and $\mu = \mu(r)$ of $r = r_i \in (0, \varepsilon]$ depending on the parameter $s = (r_1, \ldots, \hat{r}_i, \ldots, r_k) \in (0, \varepsilon]^{k-1}$. In particular, we may write $d(r)$ uniquely in the form

$$d(r) = a_s r^{c_i} (\log 1/r)^{q_i} \cdot n(r) .$$

Here, $a_s := \prod_{j \neq i} r_j^{c_j} (\log 1/r_j)^{b_j}$ is a positive constant which may depend on $s$, $q_i = b_i + t$ with $q_i \neq -1$ if $c$ is an odd integer, and $n(r) = e(r)^t$, where $e(r) = 1 + c_s/\log 1/r_i$ with $c_s = \sum_{j \neq i} \log 1/r_j$. Then by Remark 1.1, under this convention $d$ satisfies the condition I of 1.1. Similarly, by our assumption A2), $\mu$ satisfies the condition II of 1.1. Now letting $E_n(r) = E_n(a_{n+1}(r))$ we can write $\|E(a)\|^2 = \sum_n \|E_n(r)\|^2$, where

$$\|E_n(r)\|^2 = \sum_n \int_0^\varepsilon |E_n(r)|^2 \, d(r) \, dr .$$

On the other hand, $\|adz\|^2 = \sum_n \|a_n(r)\, dz\|^2$, where

$$\|a_n(r)\, dz\|^2 = \int_0^\varepsilon |a_n(r)|^2 \, d(r) \mu(r) \, dr .$$

Thus by Lemma 1.2 we can conclude that there exists a positive constant $C$ which is independent of the parameter $s$ such that for any $n$ we get $\|E_n(r)\|^2 \leq C\|a_{n+1}(r)\, dz\|^2$. The desired $L^2$ estimate for $E$ follows immediately from this.

*Case of Q*. We have

$$\|h\|_S^2 = d(\varepsilon) \sum_n |h_n|^2 ,$$

and hence

$$\begin{aligned} 1/2\pi \|Q(h)\|_D^2 &= \sum_n{}' \int_0^\varepsilon |h_n|^2 (r/\varepsilon)^{2n} \, d(r) \, dr \\ &= d(\varepsilon) \sum_n{}' |h_n|^2 \int_0^\varepsilon (r/\varepsilon)^{2n+c} [(\log 1/r)/(\log 1/\varepsilon)]^q (n(r)/n(\varepsilon)) \, dr . \end{aligned}$$

Here, the summation is over $n$ with $2n + c > -1$ (resp. $\geq -1$) if $q > -1$ (resp. $< -1$). Moreover $n(r)/n(\varepsilon) \leq 1$ since $t \geq 0$ (cf. Remark 1.1). Thus the integrals in the last term are bounded by a positive constant which can be taken independently of $n$. Hence, we have

$$\|Q(h)\|_D^2 \leq C_1 d(\varepsilon) \sum_n |h_n|^2 = C\|h\|_S^2 ,$$

where $C_1$, and hence $C := C_1 d(\varepsilon)^{-1}$ also, is a constant which is independent of $n$ and of the parameter $s$. Thus if we set $C_\eta = C$, f1) is verified for any $\eta$.

f2) We have to verify the equalities in (11). In view of the relations (cf. [31; p.437])

$$2a_{n+1}(r) = (E_n'(r) - n/rE_n(r)) ,$$

where $'$ denotes the differentiation with respect to $r$, the first and the third equalities are obtained immediately by termwise differentiation and by differentiation with respect to the parameter $z_j$ respectively. We shall prove the second equality. Start with an arbitrary element $u = \sum_n u_n(r)e^{in\theta}$ of $A(W)$. Then we have ${}_i\bar{\partial}u = \sum_n a_n(r)e^{in\theta}$ with

$$2a_{n+1}(r) = r^n[r^{-n}u_n(r)]' .$$

Hence, if we write $E({}_i\bar{\partial}u) = \sum_n v_n(r)e^{in\theta}$, then we have:

$$v_n(r) = 2r^n \int_0^r (1/2)[\rho^{-n}u_n(\rho)]'\, d\rho = u_n(r)$$

in Case 1 of Lemma 1.2, and

$$v_n(r) = -2r^n \int_r^\varepsilon (1/2)[\rho^{-n}u_n(\rho)]'\, d\rho = u_n(r) - (r/\varepsilon)nu_n(\varepsilon)$$

in Case 2 of Lemma 1.2. Hence, we have

$$u = E({}_i\bar{\partial}u) + \sum_n{}'(r/\varepsilon)^n u_n(\varepsilon)e^{in\theta} = E({}_i\bar{\partial}u) + Qr(u)$$

as desired.

i) Next consider Case 2: $k + 1 \leq i \leq n$. We write $z = z_i$ and omit the other variables $z_j$.

*Definitions of $E$ and $Q$*: We set

$$E(a)(z) = 1/2\pi i \int_D \frac{a(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}$$

and

$$Q(a)(z) = 1/2\pi \int_S \frac{a(\zeta)}{\zeta - z} d\theta .$$

*Properties of E and Q*:

f1) Since $k+1 \leq i \leq n$, $d(r_1, \dots, r_k)$ is independent of $r = r_i$. In particular, when $z_j$, $j \neq i$, are all fixed, $d(r)$ is a constant and $d\lambda$ is invariant under translation by $z$.

*Case of E*. Let $\zeta = u + \sqrt{-1}v$. Then we have

$$2\pi^2 |E(a)|^2 \leq \left[\iint_D |a(\zeta)\cdot(\zeta - z)^{-1}|\, du \wedge dv\right]^2 = \left[\iint_{D'} |a(\zeta + z)\cdot\zeta^{-1}|\, du \wedge dv\right]^2$$

$$\leq \int_{D'} |a(\zeta + z)|^2 |\zeta^{-1}|\, du \wedge dv \cdot \int_{D'} |\zeta^{-1}|\, du \wedge dv \qquad \text{(Schwarz)}$$

$$\leq A \int_{D'} |a(\zeta + z)|^2 |\zeta^{-1}|\, du \wedge dv$$

where $D'$ is the translation of $D$ by $z$ and $A := 3\pi \geq \int_{D'} |\zeta^{-1}|\, du \wedge dv$. Hence, we have

$$2\pi^2 \int_D |E(a)|^2\, d\lambda \leq A \int_D \left[\int_{D'} |a(\zeta + z)|^2 |\zeta^{-1}|\, du \wedge dv\right] d\lambda$$

$$= A \int_{D'} \left[\int_D |a(\zeta + z)|^2 |dz|^2 \mu_i(r)^{-1}\, d\lambda\right] |\zeta^{-1}|\, du \wedge dv \quad \text{(Fubini)}$$

$$\leq CA\|a\|^2 \int_D |\zeta^{-1}|\, du \wedge dv \leq CA^2 \|a\|^2$$

by our assumption A3), where $C$ is a positive constant.

*Case of Q*.

$$4\pi^2 \|Q(h)\|_\eta^2 \leq \int_{D_\eta} \left[\int_S |h(\zeta)\cdot(\zeta - z)^{-1}|\, d\theta\right]^2 d\lambda$$

$$\leq \int_{D_\eta} \left[\int_S |h(\zeta)|^2\, d\theta \cdot \int_S d\theta/|\zeta - z|^2\right] d\lambda \leq C_\eta \|h\|_S^2$$

for some positive constant $C_\eta$ which is independent of $s$.

f2) The equalities (11) are well-known (cf. [10; I, D1(6), D2, D(9)]).

## §3. $\mathbb{L}^2$ Cohomology Groups

**3.1.** We first recall some standard $\mathrm{L}^2$ vanishing theorems. Let $X$ be a Kähler manifold of dimension $n$ with a complete Kähler metric $g$. Let $E$ be a holomorphic vector bundle on $X$ of rank $r$ with a hermitian metric $h$. By using the metrics $g$ and $h$, we may speak of the $\mathrm{L}^2$-integrable $E$-valued $(0, q)$-forms on $X$ (with measurable coefficients). Then we consider the space $L^q = L^q(X, E)$ of $E$-valued $(0, q)$-forms $\varphi$ on $X$ such that both $\varphi$ and $\bar{\partial}\varphi$ are $\mathrm{L}^2$-

integrable. $L^{\cdot}$ forms a complex with respect to $\bar{\partial}$, and the associated cohomology groups $H^q(X, E)_{(2)}$ are called the $L^2$ (*Dolbeault*) *cohomology groups with coefficients in E*. For any $q \geq 0$ let $D^q(X, E)$ be the space of $E$-valued $C^\infty$ $(0, q)$-forms with compact supports on $X$. Then with the help of the Kähler metric $g$ the curvature form $F$ of $h$ defines a hermitian endomorphism (denoted by the same letter $F$) of the space $D^q(X, E)$ with its natural inner product defined by $h$ and $g$. Thus $(F\varphi, \varphi)$ gives a hermitian form on $D^q(X, E)$. If we write

$$F = \left(\sum_{1 \leq i, j \leq n} F^a{}_{bi\bar{j}} dz_i \wedge d\bar{z}_j\right)_{1 \leq a, b \leq r}$$

with respect to local coordinates $z_1, \ldots, z_n$ of $X$ and a local trivialization of $E$, then for any $\varphi \in D^q(X, E)$, $F\varphi$ is written in the form

$$(F\varphi)^a_J = \sum (-1)^\alpha F^a{}_{bi\bar{j}_\alpha} \varphi^{bi}{}_{J'_\alpha}, \qquad \varphi = \left(\sum_J \varphi^a_J\right)_{1 \leq a \leq r},$$

where the summation is over all $b$, $i$ and $\alpha$ with $1 \leq b \leq r$, $1 \leq i \leq n$, and $1 \leq \alpha \leq q$; furthermore, $J = (j_1, \ldots, j_q)$ is any $q$-tuple with $j_1 \leq \cdots \leq j_q$ and $J'_\alpha = (j_1 \ldots \hat{j}_\alpha \ldots j_q)$, $\hat{j}_\alpha$ denoting the absense of $j_\alpha$. We say that $(E, h)$ is *Nakano q-positive* (cf. [27; 4.1]) if there exists a positive constant $C$ such that for any $\varphi \in D^q(X, E)$ we have

$$(F\varphi, \varphi) \geq C(\varphi, \varphi). \tag{12}$$

Let $H^j(X, K \otimes E)_{(2)}$ (resp. $H^j(X, E^*)_{(2)}$) be the $L^2$-cohomology groups with coefficients in $K \otimes E$ (resp. the dual $E^*$ of $E$) defined naturally by using the above metrics $g$ and $h$, where $K$ is the canonical bundle of $X$ with the natural hermitian metric induced by $g$. We use the standard $L^2$-vanishing theorem in the following form:

**Lemma 3.1.** *If $(E, h)$ is Nakano q-positive, then the $L^2$-cohomology groups $H^q(X, K \otimes E)_{(2)}$ and $H^{n-q}(X, E^*)_{(2)}$ vanish.*

*Proof.* We have the well-known Bochner-Kodaira inequality (cf. [2], Prop. 15 and the ensuing remark);

$$(F\varphi, \varphi) \leq \|\bar{\partial}\varphi\|^2 + \|\bar{\partial}^*\varphi\|^2, \qquad \varphi \in D^q(X, K \otimes E),$$

where $\bar{\partial}^*$ is the formal adjoint of $\bar{\partial}$. (Here, we note that the usual Ricci term has cancelled out because of the presence of the canonical bundle.) Together with (12) this implies that $E \otimes K$ is $W^{0,q}$-elliptic (or equivalently, $E$ is $W^{n,q}$-elliptic) in the sense of [2; p.89]. It also follows that $E^*$ is $W^{0,n-q}$-elliptic (cf. [2; Lemma 2]). Then the desired vanishing result follows by the argument of Theorem 2 of [2; p.94]. In fact, if, for instance, $\varphi$ is a $\bar{\partial}$-closed element of $L^q(X, K \otimes E)$, then there exists an element $x$ of $L^q(X, K \otimes E)$ for which

$\bar{\partial}^* x$ is also in $L^2$, such that $\varphi = \Box x$ in the weak sense, where $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. Since $\Box dx = d\varphi = 0$, $dx$ is smooth by the regularity theorem, and $d^* dx = 0$ by [2; Prop. 7], so that $\varphi$ is a coboundary; $\varphi = dd^* x$. The proof for $H^{n-q}(X, E^*)_{(2)}$ is the same.

We also record the following well-known fact (cf. e.g. [31; §7]).

**Lemma 3.2.** *Suppose that the $L^2$-cohomology groups $H^q(X, E)_{(2)}$ are finite dimensional for all $q$. Then each $H^q(X, E)_{(2)}$ is naturally isomorphic to the space of $E$-valued harmonic $(0, q)$-forms on $X$.*

**3.2.** Let $X$ be a compact complex manifold, and $Y$ a divisor with only normal crossings in $X$. We set $U = X - Y$. Suppose that we are given a complete Kähler metric $g$ on $U$ with the associated Kähler form $\omega$. By our assumption, for any point $x$ of $Y$ there exists a coordinate neighborhood $\overline{W}_0$ of $x$ in $X$, isomorphic to the polycylinder $\overline{W} = \overline{W}_\varepsilon = \{(z_1, \ldots, z_n); |z_i| < \varepsilon\}$ in $\mathbf{C}^n = \mathbf{C}^n(z_1, \ldots, z_n)$ for some $0 < \varepsilon < 1$, such that $\overline{W} \cap Y = \{z_1 \ldots z_k = 0\}$ for some $1 \leq k \leq n$ (cf. 2.1). (We identify $\overline{W}_0$ and $\overline{W}$.) We call any polycylinder $\overline{W} \cong \overline{W}_0$ as above simply a *polycylinder along $Y$* (*of type $k$*) in what follows. We set $W = \overline{W} \cap U$.

*Definition.* We say that $\omega$ is of *Poincaré* (resp. *ball quotient*, resp. *Hilbert modular*) *type* along $Y$ if for each polycylinder $\overline{W}$ along $Y$ as above the restriction $\omega | W$ is equivalent on $W$ to the Kähler form $\omega_P$ (resp. $\omega_B$, resp. $\omega_H$) in Example 2.1 (cf. Remark 2.1). (In the ball quotient case we assume that $k = 1$.)

More generally, we consider Kähler metrics $g$ such that for each polycylinder $\overline{W}$ along $Y$, the associated form $\omega$ is equivalent on $W$ to a Kähler form $\omega_W$ of the form

$$\omega_W = \sqrt{-1} \sum_i 1/\mu_i \, dz_i \wedge d\bar{z}_i ,$$

where $\mu_i = |dz_i|^2$ are positive functions on $W$. Let $E$ be a holomorphic vector bundle on $X$ with a hermitian metric $h$ on $U$. First we compare the $L^2$-cohomology groups $H^q(U, E)_{(2)}$ with the sheaf cohomology groups $H^q(X, E)$ in general. Let $\mathscr{L}^{\cdot}(E)$ be the corresponding Dolbeault complex of sheaves on $X$ (cf. 2.2). We assume that the Dolbeault lemma holds for $E$, namely that

(A) $\mathscr{L}^{\cdot}(E)$ is a resolution of $\mathcal{O}_X(E)$ on $X$.

If $\mu_i$ are all bounded on $U$, then for any $C^\infty$ function $f$ on $\overline{W}$, its differential $df$ has a bounded norm with respect to $\omega_W$ so that each $\mathscr{L}^q(E)$ is an $\mathscr{A}_X$-module, where $\mathscr{A}_X$ is the sheaf of germs of $C^\infty$ functions on $X$. In particular, it is a fine sheaf. Then as usual, we have natural isomorphisms

$H^q(X, E) \cong H^q(U, E)_{(2)}$ for any $q$. We shall see that the same result is still true under some weaker assumptions. We fix $q \geq 0$ and assume the following condition $(\mathrm{B}) = (\mathrm{B})_q$:

(B) There exist $C^\infty$ positive functions $u$ and $v$ on $U$ satisfying the following conditions:

B1) Any local holomorphic section $s$ of $E$ satisfies $|s|^2 \leq Cu$ locally along $Y$,

B2) For any $\overline{W}$ as above with coordinates $z_i$ we have $|dz_i|^2 \leq C'v$ for any $i$ locally along $Y$,

B3)$_q$ For any $p \leq q$, any $E$-valued $C^\infty$ $p$-form $\varphi$ on $U$ is $\mathrm{L}^2$-integrable, if $|\varphi|^2 \leq C''uv^p$ along $Y$

Here $C$, $C'$ and $C''$ are some positive constants.

For example if $\omega$ is of Poincare type, then $\mu_i$ are all bounded and the sheaves $\mathscr{L}^q(E)$ are all fine. If $\omega$ is of ball quotient (resp. Hilbert modular) type, take $v$ to be any function on $U$ which is equivalent to $\log 1/r_1$ (resp. $L(r)^2$) on each $W$ as above. Then the condition 2) is fulfilled.

**Proposition 3.3.** *Let the notations and assumptions be as above. In particular the conditions* (A) *and* (B) *above are satisfied. Then we have natural isomorphisms* $H^q(X, E) \cong H^q(U, E)_{(2)}$.

*Proof.* Take a sufficiently fine Stein open covering $\mathscr{U} = \{U_\alpha\}$ of $X$. We have then a natural isomorphism $H^q(X, E) \cong H^q(C^\cdot(\mathscr{U}, \mathscr{E}))$, where $(C^\cdot(\mathscr{U}, *), \delta)$ denotes in general a Čech complex, and $\mathscr{E} = \mathscr{O}_X(E)$. Denote by $C^{p,q} = C^q(\mathscr{U}, \mathscr{L}^p)$ the double complex $(C^{p,q}, \bar\partial, \delta)$, where $\mathscr{L}^p = \mathscr{L}^p(E)$. Furthermore, let $'\mathscr{L}^p$ be the subsheaf of $\mathscr{L}^p$ of germs of measurable $E$-valued forms $s$ whose norm square $|s|^2$ is bounded by $Cuv^p$ locally along $Y$ for some constant $C$. We set $'C^{p,q} = C^q(\mathscr{U}, '\mathscr{L}^p)$. Then $('C^{p,\cdot}, \delta)$ also forms a Čech complex. Now let $\rho = \{\rho_\alpha\}$ be a partition of unity subordinate to $\mathscr{U}$. Then by using $\rho$ we define as usual the homotopy operators

$$\gamma = \gamma^{p,q}\colon C^q(\mathscr{U}, j_*\mathscr{L}^p_U) \to C^{q-1}(\mathscr{U}, j_*\mathscr{L}^p_U)\,, \qquad p,\ q \geq 0\,,$$

by

$$\gamma(a)_{\alpha_0\ldots\alpha_{q-1}} = \sum_\alpha \rho_\alpha a_{\alpha\alpha_0\ldots\alpha_{q-1}}\,, \qquad a = (a_{\beta_0\ldots\beta_{q-1}})\,,$$

where $j\colon U \hookrightarrow X$ is the inclusion, and by definition $C^{-1} = \Gamma(X, j_*\mathscr{L}^p_U)$ and $\delta\colon C^{-1} \to C^0$ is the natural inclusion. Then we have $I = \delta\gamma + \gamma\delta$, where $I$ is the identity operator.

*Claim.* If $a$ is a $\bar\partial$-closed element of $'C^{p,q}$, then $\bar\partial\gamma(a)$ is in $'C^{p+1,q-1}$.

*Proof.* Write $a = (a_{\alpha_0\ldots\alpha_q})$ with $\bar\partial a_{\alpha_0\ldots\alpha_q} = 0$. Then $\bar\partial\gamma(a) = \sum_\alpha (\bar\partial\rho_\alpha) \wedge a_{\alpha_0\ldots\hat\alpha_k\ldots\alpha_{q-1}}$. Since $\rho_{\alpha_k}$ is smooth on the whole $U_{\alpha_k}$ and $|dz_j|^2 \leq C'v$ by B2), the claim follows from B3).

Now we start with an arbitrary element $\tilde{\varphi}$ of $H^q(X, E)$ represented by a cocycle $\varphi = (\varphi_{\alpha_0 \dots \alpha_q}) \in C^q(\mathscr{U}, \mathscr{E})$. By Claim above we can define inductively elements $\varphi^j \in {}'C^{j,q-j}$, $0 \leq j \leq q$, with $\delta\varphi^j = 0$ by

$$\varphi^0 = \iota\varphi\,, \qquad \text{and} \qquad \varphi^j = \bar{\partial}\gamma(\varphi^{j-1})\,, \qquad j > 0\,,$$

where $\iota: C^q(\mathscr{U}, \mathscr{E}) \to C^q(\mathscr{U}, {}'\mathscr{L}^0)$ is the natural inclusion (cf. B1)). Then $\varphi^q$ is considered as a $\bar{\partial}$-closed element of $L^q(U, E)$ and we associate to $\tilde{\varphi}$ the class of $\varphi^q$ in $H^q(U, E)_{(2)}$. The independence of the definition from the choice of the representatives $\varphi$, the partition of unity, and the covering $\mathscr{U}$ is checked as usual by using the claim above.

We show the injectivity of the resulting map $u: H^q(X, E) \to H^q(U, E)_{(2)}$. So suppose that we can write $\varphi^q = \bar{\partial}\psi$ for some element $\psi \in L^{q-1}(U, E)$. Then by using the assumption (A) we can find successively elements $\beta^j \in C^{q-2-j,j}$, $0 \leq j \leq q-2$, such that

$$\bar{\partial}\beta^0 = \gamma(\varphi^{q-1}) - \psi\,, \qquad \bar{\partial}\beta^j = \gamma(\varphi^{q-j-1}) - \delta(\beta^{j-1})\,, \qquad 1 \leq j \leq q-2\,,$$

and that $\chi := \gamma(\varphi^0) - \delta(\beta^{q-2})$ belongs to $C^{0,q-1}$ with $\bar{\partial}\chi = 0$ and $\delta\chi = \varphi$. By our assumption (A), $\chi$ is an element of $C^{q-1}(\mathscr{U}, \mathscr{E})$. Thus $\tilde{\varphi}$ vanishes. The injectivity is proved.

The surjectivity is proved similarly by the successive application of the Dolbeault lemma as follows. Given an arbitrary $\bar{\partial}$-closed element $\varphi^q$ in $L^q(U, E)$ we can find elements $\psi^i$ of $C^{q-1-i,i}$, $0 \leq i \leq q-1$, such that $\bar{\partial}\psi^0 = \varphi^q$, and $\bar{\partial}\psi^i = \delta\psi^{i-1}$, $1 \leq i \leq q-1$, and that $\delta\psi^{q-1} = \iota\varphi$ for some element $\varphi$ of $C^q(\mathscr{U}, \mathscr{E})$ (by again using (A)). Then we see immediately that $u(\varphi) =$ the class of $\varphi^q$.

We also note that the next lemma follows immediately from Lemma 3.1.

**Lemma 3.4.** *Let $F$ be a hermitian holomorphic vector bundle on $X$. Let $L$ be a holomorphic line bundle on $X$ with a hermitian metric defined on $U$ such that its curvature form $\omega$ majorates on $U$ some Kähler form $\eta$ defined on the whole $X$, i.e., $\omega - \eta$ is positive definite on $U$. Then there exists a positive integer $n_0$ such that $H^q(U, K \otimes F \otimes L^n)_{(2)} = 0$ for all $n \geq n_0$ and $q > 0$.*

## §4. Certain Infinitesimal Rigidity Theorem

**4.1.** Let $D$ be a bounded symmetric domain in $\mathbf{C}^n$ and $\Gamma$ an arithmetically defined discrete subgroup of the identity component of the group of biholomorphic isometries of $D$ acting properly discontinuously on $D$. We assume that $\Gamma$ is torsion-free so that $U := D/\Gamma$ is a complex manifold. Let $U \hookrightarrow X$ be one of the smooth toroidal compactifications of $U$ [1]; in particular $Y := X - U$ is a divisor with only normal crossings. The Bergman metric on $D$ descends to a complete Kähler metric $g$ on $U$. Denote by $\Theta\langle -Y \rangle$ the sheaf of germs of holomorphic vector fields on $X$ which are tangent to $Y$ at each point of $Y$. $\Theta\langle -Y\rangle$ is a locally free sheaf which is dual to the sheaf $\Omega^1(\log Y)$ of germs of meromorphic 1-forms on $X$ with logarithmic pole along $Y$. In this section we are concerned with the vanishing theorem of the cohomology groups $H^q(X, \Theta\langle -Y\rangle)$ in analogy with the results of Calabi-Vesentini [4] in the compact quotient case; indeed, as an application of our $L^2$ Dolbeault lemma we shall obtain the desired vanishing theorem when $D$ is either the unit ball or the unit polydisc. Namely:

**Theorem 4.1.** *Let $U = D/\Gamma$ and $X$ be as above. Suppose that $D$ is either the unit ball or the unit polydisc in $\mathbf{C}^n$. Suppose further that $\Gamma$ is irreducible (cf. below) when $D$ is the polydisc. Then for any $q<n$ we have $H^q(X, \Theta\langle -Y\rangle) = 0$.*

For $q = 0$, this implies that the group $\mathrm{Aut}(X, Y)$ of biholomorphic automorphisms of $X$ which leave $Y$ invariant is discrete. On the other hand, since $H^1(X, \Theta\langle -Y\rangle)$ is the tangent space of the deformation of the pair $(X, Y)$, we get the following:

**Corollary 4.2.** *The pair $(X, Y)$ is rigid under small and infinitesimal deformations.*

*Remark* 4.1. The fact that $(X, Y)$ is "geometrically rigid", and hence is rigid under small deformations is already known ([12], cf. also [18] [9; IV, § 7]). See also [25] for a related result.

**4.2.** Suppose that $D$ is either the unit ball $B$ in $\mathbf{C}^n$, or the product $H^n$ of $n$ copies of upper half plane $H = \{w \in \mathbf{C};\ v := \mathrm{Im}\ w > 0\}$. When $D = H^n$, suppose further that $\Gamma$ is irreducible in the sense that the projection of $\Gamma$ onto any partial direct factor $(\mathrm{Aut}\ H)^k$, $1 \le k < n$, has a dense image. Let $U \hookrightarrow X^*$ be the Satake compactification of $U$. Then, $X^*$ has only a finite number of isolated singular points, say $p_1, \ldots, p_m$, and $X$ is obtained as a resolution of $X^*$. Let $Y_\nu$ be the connected component of $Y$ corresponding to $p_\nu$. When $D = B$, each $Y_\nu$ is nonsingular and in fact abelian varieties. The following lemma is more or less well-known; see e.g. [12; § 1, b), c)] [32] [17].

**Lemma 4.3.** *The Kähler metric $g$ on $U$ above is of ball quotient type (resp. Hilbert modular type) along $Y$ in the sense of Definition in* 3.2 *if $D = B$ (resp. $H^n$).*

*Proof.* a) We start with the case $D = H^n$. Write $H^n = \{(w_1, \ldots, w_n) \in \mathbf{C}^n;$ $v_i := \mathrm{Im}\ w_i > 0\}$. The Bergman Kähler form $\tilde{\omega}$ on $H^n$ is given up to constants by

$$\tilde{\omega} = \sqrt{-1}/4 \sum_{i=1}^{n} 1/v_i^2\, dw_i \wedge d\overline{w}_i = \sqrt{-1} \sum_{i=1}^{n} \partial\bar{\partial} \log v_i\,.$$

Let $\omega$ be the Kähler form associated to $g$. We have to show that for any polycylinder $\overline{W}$ along $Y$ as in 3.2 the restriction $\omega|W$ is equivalent to the Kähler form $\omega_H$ on $W$ (cf. Example 2.1,3)). We may consider the problem along a component $Y_\nu$ corresponding to the typical cusp "$(i\infty)^n$" since the other cases can be treated similarly (cf. [1; p.42]). We shall follow the exposition in [1; I, §5, III, §1] for the description of $X$ along $E = Y_\nu$. Let $\Gamma_1$ (resp. $\Gamma_2$) be the subgroup of $\Gamma$ consisting of the elements of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right).$$

Then for any $d > 0$ the open subset

$$W_d := \{(w_i) \in H^n; v_1 \dots v_n > d\}$$

is left invariant by $\Gamma_2$ and the natural map $W_d/\Gamma_2 \to U = H^n/\Gamma$ is an open embedding if $d$ is large enough; then the interior $V_d$ of the closure of the image of this map in $X$ forms an open neighborhood of $E$, the structure of which we shall now describe. $\Gamma_1$ is a free abelian group of rank $n$ and it acts on $W_d \subseteq H^n \subseteq \mathbf{C}^n$ by (real) translations. Setting $N = \mathbf{R}^n$, consider $\mathbf{C}^n$ as $\mathbf{C}^n = N + iN$, and $H^n$ as a tube domain $H^n = N + iC$, where $C = \{(v_1, \dots, v_n) \in N; v_i > 0\}$. Now we have

$$W_d' := W_d/\Gamma_1 \subseteq \mathbf{C}^n/\Gamma_1 \cong \mathbf{C}^{*n},$$

and hence, $W_d'$ admits a smooth toroidal embedding $W_d' \subseteq \mathbf{C}^{*n} \subseteq X_{\{\sigma_\alpha\}}$ depending on a certain $\Gamma_2$-invariant polyhedral decomposition $\{\sigma_\alpha\}$ of $(N, C)$. On the other hand, $\Gamma_1$ is a normal subgroup of $\Gamma_2$, and if we denote by $V_d'$ the interior of the closure of $W_d'$ in $X_{\{\sigma_\alpha\}}$, the induced action of $\Gamma_2/\Gamma_1$ extends to a properly discontinuous and fixed point free action on $V_d'$; moreover, $V_d$ is exactly the quotient $V_d'/(\Gamma_2/\Gamma_1)$ which fits in the diagram;

$$V_d'/(\Gamma_2/\Gamma_1) \hookleftarrow W_d/\Gamma_2 \hookrightarrow H^n/\Gamma.$$

Therefore, we have only to consider the growth of the induced Kähler form $\omega'$ on $V_d'$ along $E' := V_d' - W_d'$. $X_{\{\sigma_\alpha\}}$ is covered by coordinate patches $U_\alpha \cong \mathbf{C}^n(z_1^\alpha, \dots, z_n^\alpha)$, one for each $\sigma_\alpha$, in which $E'$ is defined by the equation $z_1^\alpha \dots z_n^\alpha = 0$. We shall describe the growth of $\omega'$ on $U_\alpha$ in terms of the functions $r_i^\alpha := |z_i^\alpha|$. Note that the functions $v_i$ are $\Gamma_1$-invariant, and hence can be thought of as functions on each $U_\alpha$. Then the key observation is contained in the following:

*Claim.* There exist positive real numbers $c_{ij}$, $1 \le i, j \le n$, such that on $U_\alpha$ the following relations hold:

$$v_i = \sum_{j=1}^{n} c_{ij} \log 1/r_j^\alpha, \qquad 1 \le i \le n.$$

*Proof.* For simplicity we write $z_i = z_i^\alpha$ and $r_i = r_i^\alpha$. Recall that the coordinates $z_j$ of $U_\alpha$ is described as follows. There exists a basis $\{e_i\}_{1 \le i \le n}$ of $N$ with $e_i \in \Gamma_1 \cap C \subseteq N$ spanning the polyhedral cone $\sigma_\alpha$, where $N$ is identified with the group of translations of itself. Let $\{e_i^*\}_{1 \le i \le n}$ be the dual basis. Then we get an isomorphism $\mathbf{C}^n/\Gamma_1 \cong \mathbf{C}^{*n} \subseteq \mathbf{C}^n(z_1, \dots, z_n)$ by

$$z_i = \exp(2\pi i e_i^*(w)) , \qquad w = (w_1, \dots, w_n) \in \mathbf{C}^n .$$

Then taking $\mathrm{Im}(1/2\pi i) \log$, we obtain $e_i^*(v) = \dfrac{1}{2\pi} \log 1/r_i$, $1 \le i \le n$, where $w = u + iv$ with $v \in N \subseteq \mathbf{C}^n$ and Im denotes the imaginary part. It follows that

$$v_i = \frac{1}{2\pi} \sum_{j=1}^n e_{ij} \log 1/r_j$$

where $e_{i1}, \dots, e_{in}$ are components of $e_i$ with respect to the standard basis of $\mathbf{R}^n$. Since $e_i \in C$, $c_{ij} := 1/2\pi\ e_{ij}$ are all positive.

Now, using the fact that $v_i$ are pluriharmonic, we compute

$$\partial\bar\partial \log v_i = -v_i^{-2} \lambda_i \wedge \bar\lambda_i$$

where $\lambda_i = \sum_{j=1}^n c_{ij}\, dz_j/z_j$. On the other hand, by Claim we conclude that $v_i$ is equivalent to $L(r_1, \dots, r_n)$ on any relatively compact open subset, say $V$, of $U_\alpha$. Moreover, since $(c_{ij})$ is a constant nonsingular matrix, $\sum_{i=1}^n \lambda_i \wedge \bar\lambda_i$ is equivalent on $V$ to $\sum_{i=1}^n (dz_j/z_j) \wedge (d\bar z_j/\bar z_j)$. Hence, $\omega'$ is equivalent on $V$ to the Kähler form

$$\sqrt{-1} L(r_1, \dots, r_n)^{-2} \sum_{i=1}^n 1/r_i^2\, dz_i \wedge d\bar z_i .$$

The lemma follows from this immediately in the polydisc case.

b) In the ball quotient case one proceeds in the same way. Associated to each singular point of $X^*$ we get an unbounded realization of $D$ in $\mathbf{C}^n(w, u_1, \dots, u_{n-1})$;

$$D = \{\mathrm{Im}\, w > |u_1|^2 + \cdots + |u_{n-1}|^2\} .$$

In this case the group corresponding to $\Gamma_1$ above is cyclic and one of its generators acts on $\mathbf{C}^n$ by translation by a positive real number, say $a$, on the first component. Then we get a canonical toroidal embedding

$$D/\Gamma_1 \subseteq \mathbf{C}^n/\Gamma_1 \cong \mathbf{C}^* \times \mathbf{C}^{n-1} \subseteq \mathbf{C}(z) \times \mathbf{C}^{n-1}(u_1, \dots, u_{n-1}) ,$$

where $z = \exp(2\pi i w/a)$. The Bergman Kähler form on $D$ is given up to constants by $\omega = \sqrt{-1}\partial\bar\partial \log(\mathrm{Im}\, w - \sum_i |u_i|^2)$, which descends to a Kähler form $\omega'$ on $D/\Gamma_1$

$$\omega' = \sqrt{-1}\partial\bar\partial \log\left\{-\log\left[r^{a/2\pi} \exp\left(\sum_i |u_i|^2\right)\right]\right\} .$$

By the same reasoning as in the polydisc case it suffices to check that the growth of $\omega'$ along $\{z=0\}$ is of ball quotient type in the sense of Example 2.2. Since the Hessian of $\log\exp(\sum_i |u_i|^2) = \sum_i |u_i|^2$ is positive definite on $\{z=0\}$, this in fact follows from Lemma 2.3 of [26].

4.3. We denote by $T\langle -Y\rangle$ the vector bundle corresponding to the locally free sheaf $\Theta\langle -Y\rangle$. Though the proof of the theorem is essentially the same in both cases, we first treat the ball quotient case because in the polydisc case a slightly more general result will be given.

*Proof of Theorem* 4.1 *for the case* $D=B$. Let $E=T\langle -Y\rangle$. Let the hermitian metric $h$ on $E|U$ be induced by $g$, identifying $E|U$ with the tangent bundle of $U$. Then the dual $E^*|U$ with the induced hermitian metric has the Nakano $q$-positive curvature form for any $q>0$ (cf. [27, Lemma 4.3 and 6.6A]). Hence by Lemma 3.1 the $L^2$-cohomology group $H^q(U,E)_{(2)}$ vanishes for any $q<n$. Let $\overline{W}$ be any polycilinder along $Y$ with local coordinates $z_1$, $\dots$, $z_n$. Then we have the local decomposition

$$\mathcal{O}(E) \cong \mathcal{O}z_1\partial/\partial z_1 \oplus \mathcal{O}\partial/\partial z_2 \oplus \cdots \oplus \mathcal{O}\partial/\partial z_n\,, \qquad \mathcal{O}=\mathcal{O}_X$$

and the estimates

$$|z_1\partial/\partial z_1|^2 \sim (\log 1/r)^{-2}\,, \qquad |\partial/\partial z_j|^2 \sim (\log 1/r)^{-1}\,, \qquad j>1\,,$$

where $r=r_1$. We can apply the $\mathrm{L}^2$ Dolbeault lemma for the line bundles with generating sections $\sigma = z_1\partial/\partial z_1$ or $\partial/\partial z_j$ (cf. Example 2.2, 2)). It follows that the $\mathrm{L}^2$ Dolbeault complex $\mathscr{L}^{\cdot}(E)$ is a resolution of $\mathcal{O}_X(E)=\Theta\langle -Y\rangle$, i.e., the condition (A) in 3.2 is satisfied. We take any $C^\infty$ function $u$ which is equivalent to $(\log 1/r)^{-1}$ on any $\overline{W}$ as above. Set $v=u^{-1}$. Then the conditions B1) and B2) in 3.2 are clearly satisfied. Moreover, any $C^\infty$ function $\psi$ with $\psi^2 \le (\log 1/r)^{n-1} \sim uv^n$ is $L^2$-integrable with respect to the volume form (6) of 2.1 with $v = r_1^{-2}(\log 1/r_1)^{-(n+1)}$. So the condition $\mathrm{B3})_q$ also is satisfied for any $q\le n$. Hence by Proposition 3.3 we have $H^q(X,E)\cong H^q(U,E)_{(2)}$. Thus the theorem follows in this case.

4.4. Next, we consider the case where $D=H^n$. Let $\tilde{L}_k$ be the pull back of the holomorphic tangent (line) bundle of the $k$-th factor $H_k$ of $H^n$ by the natural projection $H^n\to H_k$. $\tilde{L}_k$ has the natural hermitian metric $\tilde{h}_k$ induced by the Poincaré metric of $H$. The hermitian bundle $(\tilde{L}_k, \tilde{h}_k)$ admits a natural action of $\Gamma$ and it descends to a holomorphic hermitian line bundle $L_k$ on $U=H^n/\Gamma$. For any increasing sequence $1\le i_1<\cdots<i_p\le n$ and an associated sequence of integers $s_{i_1},\dots,s_{i_p}$, denote by $E=E(s_{i_1},\dots,s_{i_p})$ the holomorphic hermitian vector bundle $L_{i_1}^{s_{i1}}\oplus\cdots\oplus L_{i_p}^{s_{ip}}$. Using the hermitian metric on $E$ and $g$ we can define the $\bar\partial$-Laplacian $\square=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial$ as usual, acting on the space of $E$-valued $C^\infty$ forms. We say that an $\mathrm{L}^2$-integrable $C^\infty$ form $\varphi$

is harmonic if $\square\varphi = 0$. Then we have the following theorem of Matsushima-Shimura-Lai-Mok (cf. [20]), which we state in the dual form:

**Lemma 4.4.** *Let $E = E(s_{i_1}, \dots, s_{i_p})$ be as above. Suppose that $s_{i_\alpha} \geq 0$ for any $\alpha$ and $s_{i_\beta} > 0$ for at least one $\beta$. Then there exists no nontrivial $L^2$-integrable $K \otimes E^*$-valued harmonic $(0, q)$-form on $U$ for any $q > 0$.*

*Proof.* Since $\Gamma$ is irreducible, there exists no nontrivial $L^2$-integrable harmonic $E$-valued $(0, k)$-form on $U$ for any $0 \leq k < n$ by [20; Th. 1]. On the other hand, the Hodge $*$-operator and the hermitian metric defines an anti-isomorphism of the space of such forms and the space of $L^2$-integrable $E^*$-valued harmonic $(n, n-k)$-forms, or equivalently (cf. [2; p. 333 (24)]), of $L^2$-integrable $K \otimes E^*$-valued harmonic $(0, n-k)$-forms. The lemma follows.

The line bundles $L_k$ above are special cases of vector bundles considered by Mumford in [24]; in particular it extends naturally to a holomorphic line bundle $\bar{L}_k$ on $X$ (cf. [24; Th. 3.1]). We set $\bar{E}(s_{i_1}, \dots, s_{i_p}) = \bar{L}_{i_1}^{s_{i_1}} \oplus \cdots \oplus \bar{L}_{i_p}^{s_{i_p}}$, which is also the natural extension of $E(s_{i_1}, \dots, s_{i_p})$ in the sense of [24]. In particular, $T\langle -Y\rangle \cong \bar{E}(1, \dots, 1)$ ($n$-tuples of 1) (cf. [24; Prop. 3.4]). Thus, Theorem 4.1 in the polydisc case is a special case of the following:

**Theorem 4.5.** *Let $X$ be a smooth toroidal compactification of the quotient $U = H^n/\Gamma$ as above. Then $H^q(X, \bar{E}(s_{i_1}, \dots, s_{i_p})) = 0$ for any $q < n$ if $s_{i_\alpha} \geq 0$ for any $\alpha$, and $> 0$ for at least one $\alpha$.*

First we note the following:

**Lemma 4.6.** *On any polycylinder $\bar{W}$ along $Y$* (cf. 3.2), *$\bar{L}_i$ admits a generating section $\sigma$ whose squared norm $|\sigma|^2$ is equivalent to $L(r)^{-2}$ on $W$.*

*Proof.* We use the notations in the proof of Lemma 4.3. $\tilde{L}_i|W_d$ descends to a holomorphic line bundle on $W_d'$, and then extends to one on $V_d'$, which we denote by $\bar{L}_i'$. By the construction of $X$ it suffices to show the corresponding statement for $\bar{L}_i'$. Since the typical generating section $\partial/\partial w_i$ of $\tilde{L}_i$ is $\Gamma_1$-invariant, it descends to a generating section $\sigma'$ of $\bar{L}_i'$ on $W_d'$. Its norm square $|\sigma'|^2$ is then given by $1/v_i^2$ on each $U_\alpha$, which is equivalent to $L(r)^{-2}$ there. On the other hand, $\sigma'$ extends to a generating section of $\bar{L}_i'$ by the definition of $\bar{L}_i$ in [24] (cf. 1.3 and 3.1 there). The lemma follows.

*Proof of Theorem* 4.5. By Serre duality it suffices to show that $H^p(X, K_X \otimes \bar{E}^*) = 0$ for $p \geq 1$, where $\bar{E} = \bar{E}(s_{i_1}, \dots, s_{i_p})$. By the above lemma, locally on any polycylinder $\bar{W}$ along $Y$, $K_X \otimes \bar{L}_{i_\alpha}^{s_{i_\alpha}}$ admits a generating section $\tau$ with $|\tau|^2 \sim \prod_{1 \leq i \leq k} r_i^2 \cdot L(r)^{2n+2s_{i_\alpha}}$. Since $a_i = 1$ and $q_i = 2s_i \geq 0$ in the notation of 2.1 and $E$ is a direct sum of such line bundles, by Example 2.2, 3) we conclude that $L^2$-Dolbeault complex for $K_X \otimes \bar{E}^*$ is a resolution of $\mathcal{O}_X(K_X \otimes \bar{E}^*)$, i.e., the condition (A) in 3.2 is satisfied. We take $u$ (resp. $v$) to be any positive $C^\infty$ function which is equivalent to $\prod_{1\le i\le k} r_i^2 \cdot L(r)^{2n+2s}$ (resp. $L(r)^2$) on any polycylinder $\overline{W}$ along $Y$, where $s$ is the maximum of $s_{i_\alpha}$. Then the conditions B1) and B2) in 3.2 are clearly satisfied. Moreover, a $C^\infty$ function $\psi$ with $\psi^2 \le C'' uv^n \sim C'' \prod_{1\le i\le k} r_i^2 \cdot L(r)^{4n+2s}$ is clearly $L^2$-integrable so that the condition B3)$_q$ also is satisfied for any $q$. Thus, by Proposition 3.3 we have natural isomorphisms

$$H^q(X, K_X \otimes \bar{E}^*) \cong H^q(U, K \otimes E^*)_{(2)}\,.$$

In particular, $H^q(U, K \otimes E^*)_{(2)}$ are all finite dimensional. Then, in view of our assumption on $s_{i_\alpha}$, these vanish for all $q < n$ by Lemmas 3.2 and 4.4. The theorem follows.

*Remark* 4.2. By the similar reasoning as above, using Propositions 2.1 and 3.3 we can also show the following: Let $X$ be the compactification of $U = D/\Gamma$ as in Theorem 4.1. If $D = B$ (resp. $H^n$), then there exist natural isomorphisms

$$H^q(U, \Omega_U^p)_{(2)} \cong H^q(X, \Omega^p(\log Y))$$

for $p + q \le n - 2$ (resp. $n - 1$), where $\Omega^0(\log Y) = \mathcal{O}_X$. (Here, one needs the full strength of the condition B3)$_q$.)

## §5. Complete Kähler-Einstein Metrics

5.1. Let $X$ be a Kähler manifold of dimension $n$ with a Kähler metric $g$. The associated Kähler form $\omega$ is written as $\omega = \sqrt{-1}\sum g_{ij}\, d\bar{z}_i \wedge d\bar{z}_j$ for any local coordinates $z_1, \dots, z_n$ on $X$. The Ricci form $\rho$ of $\omega$ is then a real $d$-closed (1, 1)-form on $X$ which is written in the form

$$\rho = \sqrt{-1}\,\bar{\partial}\partial \log \det(g_{ij})$$

with respect to the same coordinates. The metric $g$, or the associated Kähler form $\omega$, is said to be *Kähler-Einstein if*

$$\rho = k\omega \qquad \text{for some real constant} \qquad k = k(g)\,.$$

Here, by replacing $g$ by $(1/|k|)g$ if $k \neq 0$, we can always normalize the constant $k$ so that $k = 1$, 0 or $-1$. In what follows we are concerned only with the case $k = -1$. If $X$ is compact, it is well-known that there exists a Kähler-Einstein metric $g$ on $X$ with $k(g) = -1$ if and only if the canonical bundle $K_X$ of $X$ is ample (cf. [30]).

We are interested in the open case as in the following situation. Let $X$ be a compact connected complex manifold and $U$ a Zariski open subset of $X$. After suitably blowing up the boundary $Y = X - U$ we may assume that $Y$ is a divisor with only normal crossings on $X$ by Hironaka. We ask (necessary) conditions on $(X, Y)$ for $U$ to admit a complete Kähler-Einstein metric $g$ with $k(g) = -1$. We note that such a metric is always unique if one exists ([5; Prop. 5.5]). In the one dimensional case the answer is well-known.

*Example* 5.1. If $n = \dim X = 1$, $Y$ consists of a finite number of points $p_1, \dots, p_d$. Then the following conditions are equivalent: 1) $U := X - Y$ admits a Kähler-Einstein metric. 2) The universal covering of $U$ is isomorphic to the upper half plane $H$. 3) $2g - 2 + d > 0$. 4) $K_X \otimes [Y]$ is ample on $X$, where $[Y]$ is the line bundle defined by the divisor $Y$. In this case the desired metric is (up to constants) induced by the Poincaré metric on $H$.

The implication 4) to 1) above was generalized to the higher dimensional case by R. Kobayashi [16]:

**Theorem $\mathbf{K_1}$** [16]. *Suppose that $X$ is projective and the line bundle $K_X \otimes [Y]$ is ample on $X$. Then there exists a complete Kähler-Einstein metric $g$ on $U$ with $k(g) = -1$.*

If $\dim X > 1$, however, the condition is not necessary. First of all, if we blow up $X$ with center a suitable submanifold of $Y$, then $U$ is unchanged, but $K_{\tilde{X}} \otimes [\tilde{Y}]$ is in general not ample for the resulting pair $(\tilde{X}, \tilde{Y})$. In this respect the condition of the theorem should read: After passing to a suitable bimeromorphic model (inducing the identity of $U$), $L$ is ample on $X$. Namely, one assumes the existence of a good "log canonical model" for $(X, Y)$. More essential examples are provided by the arithmetic quotients of bounded symmetric domains of dimension $>1$.

*Example* 5.2. Let $D$ be a bounded symmetric domain in $C^n$. Let $\Gamma$, $U := D/\Gamma$, and $U \hookrightarrow X$ be as in 4.1. The complete Kähler metric $g$ on $U$ induced by the Bergman metric on $D$ is Kähler-Einstein with $k(g) = -1$. However, $L := K_X \otimes [Y]$ is never ample if $n > 1$. Indeed, if $U \subseteq X^*$ is the Baily-Borel compactification of $U$ and $f: X \to X^*$ is the natural morphism, we may write $L = f^*F$ for some ample line bundle $F$ on $X^*$ (cf. [24; Prop. 3.4(b)]). It is also easy to see that $L$ is never ample on any bimeromorphic model $(X', Y')$ with $U = X' - Y'$.

**5.2.** Thus, in order to get a converse to the above theorem we need to impose certain extra conditions on the metric, which are satisfied by the metric in the above theorem. We contend that such a condition can be given by a suitable growth condition of the metric along $Y$. Namely, observing that the Kähler-Einstein metric in Theorem $K_1$ is of Poincaré type along $Y$ (cf. the proof of [16; Th. 1]), we formulate a converse to Theorem $K_1$ in the following form.

**Theorem 5.1.** *Let $X$ be a compact complex manifold and $U$ a Zariski open subset of $X$ such that $Y := X - U$ is a divisor with only normal crossings. Suppose that there exists a complete Kähler-Einstein metric $g$ on $U$ of Poincaré type along $Y$ with $k(g) = -1$. Then the line bundle $L = K_X \otimes [Y]$ is ample on $U$; in particular, $U$ is quasi-projective and $X$ is Moishezon. If, further, $X$ is either Kähler or algebraic* (*cf.* below), *then $L$ is ample on the whole $X$.*

By "*$L$ is ample on $U$*" we mean that if for all sufficiently large integer $m$, any basis of $H^0(X, L^m)$ defines a birational map of $X$ into a complex projective space which gives a biholomorphic embedding of $U$. On the other hand, we call $X$ *algebraic* if $X$ is a compact complex manifold underlying a smooth complete algebraic variety (defined over $\mathbb{C}$).

The main point of the proof is contained in the following:

**Lemma 5.2.** *Under the assumption of the theorem for any holomorphic vector bundle $E$ on $X$ there exists a positive integer $m_0$ such that $H^q(X, E \otimes L^m) = 0$ for any $q \geq 1$ and $m \geq m_0$.*

*Proof.* For any integer $m \geq 1$ we set

$$L_m = K_X^m \otimes [Y]^{m-1}.$$

Then, since $E \otimes L^m = (E \otimes [Y]) \otimes L_m$, we have only to show the vanishing of $H^q(X, E \otimes L_m)$, $q \geq 1$, for any holomorphic vector bundle $E$ and for any sufficiently large $m$. We prove this by identifying these cohomology groups with the $\mathrm{L}^2$ cohomology groups $H^q(U, E_m)_{(2)}$ with coefficients in $E_m := E \otimes L_m$. More precisely, take any $C^\infty$ hermitian metric $h_E$ on $E$ over $X$ and consider the hermitian metric on $E$ over $U$ obtained by the restriction. On the other hand, on the line bundle $L_m|U \cong K_X^m|U \cong K_U^m$ we put the metric $k_m$ induced by the given Kähler-Einstein metric $g$ on $U$. By using the metrics $g$ and the induced metric $h_m$ on $E_m$ over $U$, we consider the $\mathrm{L}^2$ cohomology groups $H^q(U, E_m)_{(2)}$.

First we claim that the $\mathrm{L}^2$-Dolbeault complex $\mathscr{L}^\cdot(E_m)$ is a resolution of $\mathcal{O}_X(E_m)$. On a polycylinder $\overline{W}$ along $Y$ we take a holomorphic trivialization $E|\overline{W} \cong \overline{W} \times \mathbb{C}^r$; $\mathscr{L}^\cdot(E_m)$ is unchanged on $\overline{W}$ if we replace $h_E$ by the flat metric $h_0$ induced by the standard inner product of $\mathbb{C}^r$ and this trivialization. Then, $E_m|\overline{W}$ is isomorphic to a direct sum of copies of $L_m$ as a hermitian vector bundle. Thus the problem is reduced to the case $E_m = L_m$. There exists then a generating section $\sigma_m$ of $L_m$ with

$$|\sigma_m|^2 = v^{-m}(r_1 \dots r_k)^{-2m+2} = (r_1 \dots r_k)^2(\log 1/r_1 \dots \log 1/r_k)^{2m}.$$

Then, in Example 2.1, 1) we have $a_i = 1$ and $q_i = 2m > -1$; hence $\mathscr{L}^\cdot(L_m)$ is a resolution of $\mathcal{O}_X(L_m)$. Thus our claim is proved.

On the other hand, since the metric is of Poincaré type along $Y$, as we have noted in 3.2 $\mathscr{L}^{\cdot}(E_m)$ is a fine sheaf. Hence we get a natural isomorphism $H^q(X, E_m) \cong H^q(U, E_m)_{(2)}$ for any $q \geq 0$. Since $g$ is Kähler-Einstein, the curvature form of the metric on $L_m|U \cong K_U^m$ is nothing but the multiple $m\omega$ of the original Kähler form $\omega$. Since $\omega$ is of Poincaré type, it majorates some Kähler form on $X$. Thus we can apply Lemma 3.4 to get the lemma.

*Proof of Theorem* 5.1. First, assuming that $X$ is Moishezon we show the last assertion. In this case if $X$ is Kähler, then $X$ is necessarily projective by Moishezon [23]. Hence, we may assume that $X$ is algebraic. Then every coherent analytic sheaf on $X$ admits a resolution of finite length by coherent locally free sheaves (cf. [11; p. 92, Lemma 3.2]). It follows that the conclusion of Lemma 5.2 holds true also for any coherent analytic sheaf $\mathscr{E}$ on $X$, which implies that $L$ is ample on $X$ (cf. [11]; see also the argument below).

It remains to show that in the general case $L$ is ample on $U$. (In that case $X$ would clearly be Moishezon.) We follow the standard argument due to [19]. Let $x$ be any point of $U$ and $\mathscr{m}_x$ the ideal sheaf of $x$ in $X$. We show that there exists an integer $m_0$ such that

$$H^1(X, \mathscr{m}_x L^m) = 0 \qquad \text{for any } m \geq m_0 . \tag{13}$$

Let $f\colon \tilde{X} \to X$ be the blowing up of $X$ at $x$, and set $A = f^{-1}(x)$. Then the assertion is equivalent to:

$$H^1(\tilde{X}, \tilde{L}^m \otimes [A]^{-1}) = 0, \qquad m \geq m_0 , \tag{14}$$

where $\tilde{L}$ is the pull-back of $L$ to $\tilde{X}$ endowed with the induced hermitian metric $\tilde{h}$. As usual, we can find a hermitian metric $h_A$ on $[A]^{-1}$ whose first chern form $\eta$ has support in a neighborhood of $A$ in $\tilde{U} := f^{-1}(U)$ and is positive (definite) in a smaller neighborhood, where chern form $= (\sqrt{-1}/2) \times$ (curvature form). Then there exists a positive integer $m_1$ such that $mf^*\omega - \eta$ is positive on $\tilde{U}$ for any $m \geq m_1$. Then the form $\tilde{\omega}_m := mf^*\omega - \eta$ is the first chern form of the hermitian line bundle $\tilde{L}^m \otimes [A]^{-1}$ and defines a complete Kähler metric on $\tilde{U}$ whose growth at the boundary $\tilde{Y} := f^{-1}(Y) \cong Y$ is of Poincaré type, being the same as that of $\omega$ along $Y$. Then by the same reasoning as in the proof of Lemma 5.2 we can get the vanishing (14) from Lemma 3.1 and Proposition 3.3.

Then by the standard Noetherian argument, if $m_0$ is chosen larger we can assume that (13) is true for all $x \in U$. This then implies that the sections of $H^0(X, L^m)$ have no common zeroes. Similarly, we show that $H^1(\tilde{X}, \mathscr{m}_x \cdot \mathscr{m}_{x'} L^m) = 0$ for any two points $x$, $x'$ of $U$ if $m$ is sufficiently large, which implies as usual that $L$ is ample in the sense of the theorem.

The Kähler-Einstein form determines on $U$ the first chern form of $L$ with its natural metric. Then Theorem 5.1 is considered as giving a criterion of ampleness of a line bundle on $X$ in terms of the positivity of certain chern forms for it on $U$ with suitable growth condition. We hope to treat such situation more generally in [8] by a different method.

**5.3.** We shall slightly generalize the above theorem in the direction of Example 5.2. First we note that in Example 5.2 $L = K_X \otimes [Y]$ is actually very close to being ample. For instance they have the following properties;

(a) $L$ is *semiample*, i.e., $H^0(X, L^m)$ is base point free for all sufficiently large $m$; in fact for a sufficiently large $m$ the rational map $\Phi_m: X \to \mathbf{P}^N$ into some projective space defined by the sections of $L^m$ is a birational morphism which is isomorphic outside $Y$.

(b) $L$ is *nef* and *big*; namely, for any curve $C$ in $X$, $L \cdot C \geq 0$ and $L^n > 0$

(c) $L \cdot C > 0$ for any irreducible curve $C$ which is not contained in $Y$.

(d) $Y$ can be blown down to a subspace of smaller dimension. In particular, when $D$ is the unit ball or the polydisc, $Y$ can be blown down to a finite number of points.

In view of (d) we start with the following situation (cf. also Remark 6.1, 2)). Let $X^*$ be a compact connected normal complex space with only isolated singular points $p_1, \dots, p_m$. Let $f: X \to X^*$ be a resolution of $X^*$ such that $Y_\nu := f^{-1}(p_\nu)$ are divisor with only normal crossings in $X$ and that $f$ is obtained by a blowing up of some ideal sheaf on $X^*$ with support in $\{p_\nu\}$. Let $Y_0$ be a divisor with only normal crossings on $X$ which does not intersect any of $Y_\nu$. We set $Y = Y_0 \cup Y_1 \cup \cdots \cup Y_m$ and $U = X - Y$. We call a Kähler-Einstein metric $g$ on $U$ *admissible* if $g$ is of Poincaré type along $Y_0$, and is either of ball quotient type or of Hilbert modular type along $Y_\nu$, $\nu \geq 1$, (where $Y_\nu$ is assumed nonsingular in the ball quotient case).

**Theorem 5.3.** *The notations being as above, suppose that there exists an admissible Kähler-Einstein metric $g$ with $k(g) = -1$ on $U$. Then $U$ is quasi-projective and $X$ is Moishezon. If, further, $X$ is either Kähler or algebraic, the conditions* (b) *and* (c) *above hold true. Moreover, if every $Y_\nu$ is nonsingular, then even* (a) *holds true; in fact in this case for a sufficiently large $m$ the rational map $\Phi_m$ is a birational morphism which is isomorphic outside the union of $Y_\nu$.*

*Remark* 5.1. In the situation stated before the theorem suppose that $Y_0$ is empty and $Y_\nu$ are all nonsingular with numerically trivial canonical bundles. Then Tsuji [29] has shown that if the conditions (a) and (c) above hold true, then there exists a complete Kähler-Einstein metric $g$ on $U$ assumed with $k(g) = -1$ which are of ball quotient type along $Y$. The above theorem is a kind of converse to his result.

**Lemma 5.4.** *Under the assumption of the theorem we have $H^q(X, K_X^n \otimes [Y]^{n-1}) = 0$ for any $q > 0$ and $n \geq 2$.*

*Proof.* The cohomology groups under consideration are isomorphic to the $L^2$ cohomology groups $H^q(U, K_U^n)_{(2)}$ by Propositions 2.1 and 3.3. On the other hand, the curvature form of the canonical metric on $K_U^n$ induced by the given Kähler-Einstein metric has clearly a (Nakano) positive curvature. Hence the lemma follows from Lemma 3.1.

*Proof of Theorem* 5.3. Let $Y' = \bigcup Y_\nu$. By our assumption there exists an effective divisor $D$ such that its support coincides with $Y'$ and $-D$ is ample in a neighborhood of $Y'$. Hence, we can find a hermitian metric $h_D$ on the line bundles $[D]^{-1}$ whose curvature form $\xi$ has its support contained in a small neighborhood of $D$ and is positive definite in a smaller neighborhood. Then there exists an integer $m_0 > 0$ such that $m\omega + \xi$ is positive on $U$ and majorates some Kähler form defined on $X$ for any $m \geq m_0$. We set $L'_m = L^m \otimes [D]^{-1}$ and put the hermitian metric $g_K^m \otimes h_D$ on $L'_m|U \cong K_U^m \otimes [D]^{-1}|U$, where $g_K$ is the hermitian metric of $K_U$ induced by the given complete Kähler-Einstein metric, whose curvature form is exactly $m\omega + \xi$. By Lemma 3.4 it follows that for any holomorphic vector bundle $E$ on $X$ with a hermitian metric defined on the whole $X$ we have the vanishing of the $L^2$ cohomology group $H^q(U, E \otimes (L'_m)^k)_{(2)}$ if $q > 0$, $m \geq m_0$ and $k$ is sufficiently large.

On the other hand, by using Propositions 2.1 and 3.3 as in the proof of Theorem 5.1, we get natural isomorphisms

$$H^q(U, E \otimes (L'_m)^k)_{(2)} \cong H^q(X, E \otimes L_{mk} \otimes [D]^{-k}) \cong H^q(X, E \otimes [Y]^{-1} \otimes (L'_m)^k)$$

for all $q$, where $L_d = K_X^d \otimes [Y]^{d-1}$ in general. (We may take $v$ to be constant in Proposition 3.3 also in this case.) Again as in the proof of Theorem 5.1, from these we conclude that $L'_m$ is ample for $m \geq m_0$ on $U$ and that $X$ is Moishezon.

Suppose next that $X$ is either Kähler or algebraic. Then as before, $L'_m$ is ample even on the whole $X$, and hence the $\mathbf{Q}$-divisor

$$(1/m)L'_m = K_X + [Y] - 1/m[D]$$

is in the ample cone of the rational Neron severi group of $X$. Taking the limit as $m \to \infty$, we see that $L = K_X \otimes [Y]$ is nef. Moreover, if $B$ is any curve which is not contained in $Y$, we have $L \cdot B > (1/m)D \cdot B \geq 0$ since $D$ has support in $Y$. On the other hand, since $L$ is nef and $\dim H^0(X, L^{mk}) \geq \dim H^0(X, (L'_m)^k) \geq ck^n$ for some positive constant $c$ and for sufficiently large $k$, $L^m$, and hence $L$, is big by a lemma of Sommese (cf. [13; Lemma 3]).

Finally, suppose that $Y_\nu$ are all nonsingular. Since $L$ is nef, $L|Y_\nu$ are also nef on $Y_\nu$. By Kawamata [15] $K_{Y_\nu}$ $(\cong L|Y_\nu)$ are all semiample. On the other hand, $L|Y_0$ is ample since $L^m|Y_0 = L'_m|Y_0$. Thus $L|Y$ is semiample. In view of Lemma 5.4, the long exact sequence associated to the short exact sequence

$$0 \to L_m \to L^m \to L^m|Y \to 0$$

gives rise to a surjection $H^0(X, L^m) \to H^0(Y, L^m_Y)$. It follows that $H^0(X, L^m)$ has no base point on $Y$, while it has one neither on $U$ since $H^0(X, L^m) \supseteq H^0(X, L'_m)$, $L'_m$ is ample and $L^m \cong L'_m$ on $X - Y'$. Hence, $L$ is semiample. The last assertion also is clear from the above descriptions.

In concluding this section we note that the metrics in Example 5.2 have also the following property:

(e) The Kähler form $\omega$ of $g$ extends to a $d$-closed (1, 1)-current $\tilde{\omega}$ on $X$, and moreover, its de Rham class $\gamma$ in $H^2(X, \mathbb{R})$ represents the first chern class $c_1(L)$ of $L$.

In general, if a Kähler form $\omega$ is (at most) of Poincaré growth, then it extends to a $d$-closed (1, 1)-current $\tilde{\omega}$ on $X$ (cf. [24; Prop. 1.2]). However, for the associated class $\gamma$ to coincide with $c_1(L)$ in the Kähler-Einstein case, one needs some extra condition. For instance, this is the case if we assume that the induced metric on the canonical bundle $K_U$ is *good* with respect to the extension $L$ of $K_U$ to $X$ in the sense of Mumford [24; § 1]; moreover, this latter condition is satisfied by the metrics in Example 5.2 [24; Th. 1.4 and Prop. 3.4].

## § 6. Complete Kähler-Einstein Surfaces

**6.1.** In this section we prove another version of the theorems in Section 5 in the case where $\dim X = 2$, in relation with the property (e) discussed at the end of the previous section. (The method of proof, however, is quite independent of the previous arguments; in particular, we make no use of $\mathbb{L}^2$ Dolbeault lemma here.) The result is a kind of converse to that of R. Kobayashi in [17] (cf. also [18]), which we first recall in a form convenient for our purpose. Let $X$ be a projective nonsingular surface and $Y$ a divisor with only normal crossings on $X$. We set $L = K_X \otimes [Y]$ as before.

**Theorem $\mathbb{K}_2$** [17]. *Let the notations be as above. Assume that the conditions* (b) *and* (c) *in* 5.3 *hold true and that the irreducible components $Y_i$ of $Y$ are smooth except possibly for those $Y_i$ with $L \cdot Y_i = 0$. Then there exists a complete Kähler-Einstein metric $g$ with $k(g) = -1$ on $U = X - Y$.*

*Remark* 6.1. 1) We call a nonsingular rational curve on $X$ with self-intersection number $-k$ simply a $(-k)$-curve. Note that the assumption (c) implies that there is no $(-1)$-curve intersecting transersally with $Y$ at exactly one point, and there is no $(-1)$ or $(-2)$-curve which does not intersect with $Y$.

2) Under the above assumptions the situation is essentially the same as that considered in Theorem 5.3 up to chains of nonsingular rational curves which can be contracted to quotient singularities (cf. [17; § 2]). Moreover, the metric obtained in the above theorem has the property (e) above. Indeed,

the metric is good in the sense of [24] (cf. [17; p. 67]) and Theorem 1.4 of [24] applies.

Then the purpose of this section is to prove the converse to the above theorem in the following form:

**Theorem 6.1.** *Let $X$ be a compact connected complex manifold of dimension 2 and $Y$ a divisor with only normal crossings on $X$. Suppose that there exists a complete Kähler metric $g$ on $U := X - Y$ with $k(g) = -1$ and with the property* (e). *Then the conditions* (b) *and* (c) *in* 5.3 *are satisfied, i.e., $L$ is nef and big and $L \cdot C > 0$ for any irreducible curve $C$ which is not contained in $Y$.*

**6.2.** We work in the situation of Theorem 6.1 in what follows. Let $\tilde{\omega}$ and $\gamma$ be as in (e) above, defined with respect to the given Kähler-Einstein metric $g$ on $U$. By the definition, for any $C^\infty$ 2-form $\xi$ on $X$ $\tilde{\omega}(\xi) = \int_U \omega \wedge \xi$. Thus, $\tilde{\omega}$ is a positive current (cf. [21]) and is the simple extension of $\omega$ in the sense of Lelong [21; Chap. 1, § 2]. We also note that $\tilde{\omega}$ is the "minimal" extension of $\omega$ as a positive current. Namely:

**Lemma 6.2.** *Let $\hat{\omega}$ be any extension of $\omega$ to a positive* $(1, 1)$*-current on $X$. Then $\hat{\omega} - \tilde{\omega}$ is a positive current.*

*Proof.* Let $\eta = \hat{\omega} - \tilde{\omega}$. The problem is local. Take any point $x$ of $X$ and take arbitrary local coordinates $z_1$, $z_2$ of $X$ with center $x$ defined in a neighborhood $V$ of $x$. Then it suffices to show that $\eta(\sqrt{-1}f\, dz_1 \wedge d\bar{z}_1)$ is nonnegative for any nonnegative $C^\infty$ function $f$ with compact support in $V$. Take a sequence of $C^\infty$ functions $\alpha_n$ on $V$ with $0 \leq \alpha_n \leq 1$, each of which is equal to the constant 1 in some neighborhood of $V \cap Y$ and converge to the characteristic function $\chi_Y$ of the set $Y$ (cf. [21], p. 12, Remarks). Now we set $\beta = \sqrt{-1}f\, dz_1 \wedge d\bar{z}_1$. Then

$$\eta(\beta) = \eta(\alpha_n\beta) + \eta((1 - \alpha_n)\beta) = \eta(\alpha_n\beta)$$

since $\eta$ has support clearly in $Y$, while $(1 - \alpha_n)\beta$ has support outside $Y$. On the other hand, when $n$ tends to $\infty$, $\tilde{\omega}(\alpha_n\beta)$ converges to zero by the very existence of $\tilde{\omega}$ as a current on $X$. Thus $\eta(\beta) = \lim_{n\to\infty} \eta(\alpha_n\beta) = \lim_{n\to\infty} \hat{\omega}(\alpha_n\beta) \geq 0$ as desired.

Let $\gamma$ be the de Rham class of $\tilde{\omega}$ in $H^2(X, \mathbb{R})$. The main idea of the proof of Theorem 6.1 is contained in:

**Lemma 6.3.** a) *For any irreducible curve $B$ in $X$ which is not contained in $Y$ we have $\gamma \cdot B > 0$.* b) *Let $F$ be an exceptional curve in $X$ which is contractible to a rational singular point, where by convention a smooth point is a special case of rational singular points. Then either* i) *$\gamma \cdot F_i > 0$ for some irreducible component $F_i$ of $F$ or* ii) *$\gamma \cdot F_i = 0$ for all $F_i$.*

*Proof.* a) Let $n: \tilde{B} \to X$ be the normalization of $B$ composed with the inclusion $B \hookrightarrow X$. Since $B \not\subseteq X$, $n^*\tilde{\omega}$ is a $d$-closed positive current on $B$ which represents the class $n^*\gamma$ (cf. e.g. [6; Lemma 2.2]). Thus, its degree, which is equal to $\gamma \cdot B$, is positive.

b) There exist a neighborhood $U$ of $F$ in $X$ and a plurisubharmonic function $u$ on $U - F$ such that

$$\omega|(U - F) = \sqrt{-1}\partial\bar{\partial}u . \tag{15}$$

In fact, $\tilde{\omega}$ defines naturally a class $\hat{\gamma}$ in $H^1(X, \mathscr{P}_X)$, where in general $\mathscr{P}_Z$ denotes the sheaf of germs of pluriharmonic functions on $Z$ (cf. e.g. [7; (2.0)(c)] and [6; p.737(1)]). On the other hand, the restriction map $H^1(U, \mathscr{P}_U) \to H^1(U - F, \mathscr{P}_U)$ is the zero map since $F$ is rationally contractible to a point (cf. the proof of (1) of Prop. 3 of [7]). In particular, the image of $\hat{\gamma}$ in $H^1(U - F, \mathscr{P}_U)$ is zero, which amounts to the equality (15). Let $\pi: U \to \bar{U}$ be the contraction of $F$ to a rational singular point $p$. Consider $u$ as a function on $\bar{U} - \{p\}$. Then $u$ extends to a plurisubharmonic function $\bar{u}$ on $\bar{U}$ as in the proof of Lemma 1 of [6]. We may then pull $\bar{u}$ back to $U$ by $\pi$ to obtain a plurisubharmonic function $\hat{u}$ on $U$ extending $u$ on $U - F$. Let $\hat{\omega} := \sqrt{-1}\partial\bar{\partial}\hat{u}$ be the $d$-closed positive $(1, 1)$-current defined by $\hat{u}$. We compare the two extensions $\tilde{\omega}$ and $\hat{\omega}$ of $\omega$. By Lemma 6.2 $\hat{\omega} - \tilde{\omega}$ is a positive current on $X$ with support in $Y$. Then we may write $\hat{\omega} - \tilde{\omega} = \sum r_i\langle F_i\rangle$ for a unique nonnegative real numbers $r_i$, where $\langle F_i\rangle$ are the integration currents on $F_i$ (cf. the proof of Lemma 2.4 of [6]). Then by taking the cohomology class in $H^2(U, \mathbf{R})$ we obtain

$$\hat{\gamma} - \gamma = 2\pi \sum_i r_i c_1([F_i]) ,$$

where $\hat{\gamma}$ is the class determined by $\hat{\omega}$. However, $\hat{\gamma}$ vanishes because $\hat{\omega}$ admits a global potential on $U$. Hence, $\gamma = -2\pi \sum_i r_i c_1(F_i)$.

Now suppose that $\gamma \cdot F_i \leq 0$ for any $i$. Then we have

$$0 \geq \gamma \cdot F_i = -2\pi \sum_j r_i c_{ij} , \qquad 1 \leq i \leq d ,$$

where $c_{ij} = (F_i \cdot F_j)$. Since $r_i \geq 0$ for all $i$, from this we get $\sum_{1 \leq i,j \leq d} r_i r_j c_{ij} \geq 0$. Since $(c_{ij})$ is negative definite by Grauert, this implies that $r_i = 0$ for all $i$, and therefore $\gamma \cdot F_i = 0$ for all $i$. b) follows.

**6.3.** Let $Y_i$ be the irreducible components of $Y$. We set $\nu_i = L \cdot Y_i$ and $A = \{i; \nu_i \leq 0\}$. Let $Y' = \bigcup_{i \in A} Y_i$ and $Y_\alpha$, $1 \leq \alpha \leq s$, the connected components of $Y'$ which contains at least one $Y_i$ with $\nu_i < 0$.

**Lemma 6.4.** *$Y_\alpha$ consists of a (linear) chain of nonsingular rational curves for any $\alpha$.*

*Proof.* For any irreducible component $Y_i$ of $Y$ set $d_i = (Y - Y_i) \cdot Y_i \geq 0$. Then we have $v_i = (K_X + Y_i) \cdot Y_i + d_i = 2\pi(Y_i) - 2 + d_i$, where $\pi$ denotes the arithmetic genus. Hence, $v_i \leq 0$ if and only if either

a) $Y_i \cong \mathbb{P}^1$ and $d_i \leq 2$ (and in this case $v_i = d_i - 2$), or

b) $\pi(Y_i) = 1$ and $d_i = 0$ (and in this case $v_i = 0$).

Thus, for the irreducible components of $Y_\alpha$ only the case a) occurs. It is easy from this to derive the conclusion of the lemma.

**Lemma 6.5.** *Let $B$ in general be a connected curve in $X$ consisting of a (linear) chain of nonsingular rational curves. Suppose that the self-intersection number $(B_i \cdot B_i) < 0$ for any irreducible component $B_i$ of $B$. Then either of the following is true*: a) *$B$ is exceptional, or* b) *there exist a connected curve $B'$ which is a union of irreducible components of $B$, an open neighborhood $U$ of $B'$ in $X$, and a proper holomorphic map $h\colon U \to D$ of $U$ onto the unit disc $D = \{|z| < 1\}$, with connected fibers such that $h^{-1}(0) = B'$ as a set.*

*Proof.* Suppose that there exists a sequence of bimeromorphic morphisms

$$X \xrightarrow{f_1} X_1 \to \cdots \xrightarrow{f_m} X_m\,, \qquad m \geq 0$$

such that i) if we define $B(k) = f_k \ldots f_1(B)$ for any $k$, $1 \leq k \leq m$, $f_k$ is a blowing down of a $(-1)$-curve contained in $B(k)$, and ii) the self-intersection number of some irreducible component $B(m)_i$ of $B(m)$ are zero. (Note that $B(k)$ always consists of a chain of nonsingular rational curves for each $k$.) Then there exist a neighborhood $U_m$ of $B(m)_i$ and a proper holomorphic map $h_m\colon U_m \to D$ with connected fibers such that $h_m^{-1}(0) = B(m)_i$. Then, if we set $U = (f_m \ldots f_1)^{-1}(U_m)$, the induced map $h := h_m f_m \ldots f_1\colon U \to D$ enjoys the property in b).

So we may assume that there is no sequence of bimeromorphic morphisms as above. This means that if we take any maximal sequence of bimeromorphic morphisms with the property i) above, $B(m)$ consists of nonsingular rational curves with self-intersection number $\leq -2$ on $X_m$ so that its intersection matrix is negative definite. Thus $B(m)$, and hence $B$ itself, can be contractible to a point by a theorem of Grauert; the case a) occurs.

**Lemma 6.6.** *Suppose that $X$ is projective. If $L$ is nef and $L \cdot B > 0$ for any irreducible curve $B$ which is not contained in $Y$, then $L$ is big, i.e., $L^2 > 0$.*

*Proof.* Since $L$ is nef, we have $L^2 \geq 0$. (Recall that $c_1(L)$ is on the boundary of the ample cone.) So, supposing that $L^2 = 0$, we shall derive a contradiction. First we show that $h^0(nL) > 0$ for all sufficiently large $n$. (We write tensor products additively here.) Since $L^2 = 0$, by the Riemann-Roch inequality we get

$$h^0(nL) + h^0(K - nL) \geq (1/2)nL \cdot (nL - K) + \chi(\mathcal{O}_X) = (n/2)L \cdot Y + \chi(\mathcal{O}_X) \geq \chi(\mathcal{O}_X)\,,$$

where $\chi(\mathcal{O}_X)$ is the arithmetic genus of $X$. For any ample divisor $A$ on $X$ and for all sufficiently large $n$ we have $A \cdot (K - nL) < 0$ so that $h^0(K - nL) = 0$. Hence, if the irregularity $q(X)$ of $X$ vanishes, we have $h^0(nL) \geq \chi(\mathcal{O}_X) \geq 1$ as desired. On the other hand, if $q(X) > 0$, the desired conclusion follows from [22; Th. I.2.3]. (In fact, by our assumption, $L \cdot B > 0$ for any irreducible curve $B$ which is not contained in $Y$; hence the condition (2) of Lemma 2.2 in [22] there is not satisfied.) Thus, $h^0(nL) > 0$ for some $n$, i.e., the logarithmic Kodaira dimension $\bar{\kappa}(U)$ of $U = X - Y$ is nonnegative.

Since $L$ is nef, this implies by Kawamata that $L$ is semiample in the sense of 5.3 (a) (cf. [14; §2]). Let $f_n: X \to Z$ be the morphism associated to $H^0(X, L^n)$ for sufficiently large $n$. Then, if $\bar{\kappa}(U) = 0$ or 1, the (irreducible components of) general fiber of $f_n$ contains an irreducible curve $B$ with $B \cdot L = 0$ and $B \not\subseteq Y$. This is a contradiction. But if $\bar{\kappa}(U) = 2$, we have clearly $L^2 > 0$, leading again to a contradiction. Thus the lemma is proved.

*Proof of Theorem* 6.1. First of all, by Lemma 6.3, a) $L \cdot B > 0$ for any irreducible curve $B$ which is not contained in $Y$. Suppose that $L$ is not nef. Then there exist at least one connected component $Y_\alpha$ of $Y'$ as in Lemma 6.4. Then by that lemma $Y_\alpha$ consists of a chain of nonsingular rational curves. If $Y_i \cdot Y_i \geq 0$ for some irreducible component $Y_i$ of $Y_\alpha$, then $Y_i$ is algebraically equivalent to an irreducible curve which is not contained in $Y$. Then by what we have noted above, we have $L \cdot Y_i > 0$, which contradicts the definition of $Y_\alpha$. So we may assume that $Y_i \cdot Y_i < 0$ for any irreducible component $Y_i$. Then by Lemma 6.5 we see that either $Y_\alpha$ is exceptional, in which case it is contracted to a rational singular point, or a certain connected curve $Y'$ in $Y_\alpha$ is algebraically equivalent to an irreducible curve which is not contained in $Y$. The latter case, however, does not occur as we see by the same reasoning as above. Hence $Y_\alpha$ is exceptional. Note that by the definition of $Y_\alpha$ we have $L \cdot Y_i < 0$ for some irreducible component $Y_i$ of $Y_\alpha$. Then by Lemma 6.3, b) we get that $L \cdot Y_j > 0$ for some irreducible component $Y_j$ of $Y_\alpha$. This again contradicts the choice of $Y_\alpha$. Hence, $L$ is nef. Then by Lemma 6.6 $L^2 > 0$ when $X$ is projective.

It remains to check that $X$ is necessarily projective. Indeed, if the algebraic dimension $a(X)$ equals one, then $X$ admits a natural elliptic fibering $f: X \to C$ over a compact Riemann surface $C$ and $K_X \cdot F = 0$ for a general fiber $F$ of $f$. Moreover, since $a(X) = 1$ every component of $Y$ is contained in a fiber of $f$. Hence, $L \cdot F = 0$, contradicting what we have shown above. So suppose that $a(X) = 0$. In this case, every curve $C$ on $X$ is a nonsingular rational curve with negative self-intersection; moreover, $X$ contains no cycle of ratonal curves. (When $X$ is minimal, $K_X$ is trivial, and the assertion is easily checked by using the adjunction formula and the Riemann-Roch theorem. The general case follows from this immediately.) On the other hand, since $L$ is nef, for any component $Y_i$ of $Y$ we have $0 \leq L \cdot Y_i = -2 + (Y - Y_i) \cdot Y_i$, i.e., $(Y - Y_i) \cdot Y_i \geq 2$. Since $Y_i$ is arbitrary, this implies that $Y$ contains a cycle of rational curves. This is a contradiction. Thus $X$ must be projective and the theorem is proved.

## References


[1] Ash, A., Mumford, D., Rappoport, M. and Tai, Y., *Smooth compactification of locally symmetric varieties*, Math. Sci. Press, 1975.

[2] Andreotti, A. and Vesentini, E., Carleman estimates for the Laplace-Beltrami equation on complex manifolds, *Publ. Math. IHES*, **25** (1965), 313–362.

[3] Aubin, T., *Nonlinear analysis on manifolds. Monge-Ampère equations*, Grund. Math. Wiss., 252, Springer, New York Heidelberg Berlin, 1980.

[4] Calabi, E. and Vesentini, E., On compact locally symmetric Kähler manifolds, *Ann. of Math.*, **71** (1960), 472–507.

[5] Cheng, S-Y. and Yau, S-T., On the existence of a complete Kähler metric on noncompact complex manifolds and regularity of Fefferman's equation, *Comm. Pure Appl. Math.*, **33** (1980), 507–544.

[6] Fujiki, A., A theorem on bimeromorphic maps of Kähler manifolds and its applications, *Publ. RIMS, Kyoto Univ.*, **17** (1981), 735–754.

[7] ———, Kählerian normal complex surfaces, *Tohoku Math. J.*, **35** (1983), 101–117.

[8] ———, Weighted line bundles on a quasi-projective manifold, in preparation.

[9] Geer, van der G., *Hilbert modular surfaces*, Ergeb. Math. Grenzgeb., 3. Folge, 16, Berlin Heidelberg New York, 1980.

[10] Gunning, R. and Rossi, H., *Functions of several complex variables*, Prentice Hall, Englewood Cliffs, N.J., 1965.

[11] Hartshorne, R., *Ample subvarieties of algebraic varieties*, Lecture Notes in Math., **156**, Springer, Berlin Heidelberg New York, 1970.

[12] Jost, J. and Yau, S.-T., The strong rigidity of locally symmetric complex manifolds of rank one and finite volume, *Math. Ann.*, **275** (1986), 291–304.

[13] Kawamata, Y., On the classification of noncomplete algebraic varieties, In: *Proc. Copenhagen, Algebraic Geometry*, 1978, (ed. Lønsted), Lecture Notes in Math., **732** Springer, Berlin Heidelberg New York, (1979), 215–232.

[14] ———, A generalization of Kodaira-Ramanujam's vanishing theorem, *Math. Ann.*, **261** (1982), 43–46.

[15] ———, Pluricanonical systems on minimal algebraic varieties, *Invent. Math.*, **79** (1985), 567–588.

[16] Kobayashi, R., Kähler-Einstein metric on an open algebraic manifold, *Osaka J. Math.*, **21** (1984), 399–418.

[17] ———, Einstein-Kähler metrics on open algebraic surfaces of general type, *Tohoku Math. J.*, **37** (1985), 43–77.

[18] ———, Einstein-Kähler $V$-metrics on open Satake $V$-surfaces with isolated singularities, *Math. Ann.*, **272** (1985), 385–398.

[19] Kodaira, K., Kähler varieties of restricted type, *Ann. of Math.*, **60** (1954), 28–48.

[20] Lai, K. F. and Mok, N. L., On a vanishing theorem on irreducible quotients of finite volume of polydiscs, In: *Sem. d'analyse P. Lelong-P. Dolbeault-H. Skoda*, 1983/84, Lecture Notes in Math., **1198**, Springer, Berlin Heidelberg New York, (1986), 163–171.

[21] Lelong, P., *Plurisubharmonic functions and positive differential form*, Gordon and Breach, New York London Paris, 1969.

[22] Miyanishi, M., *Non-complete algebraic surfaces*, Lecture Notes in Math., **857**, Springer, Berlin Heidelberg New York, 1981.

[23] Moishezon, B., On $n$-dimensional compact varieties with $n$ algebraically independent meromorphic functions, *Amer. Math. Soc. Transl.*, **63** (1967), 51–177.

[24] Mumford, D., Hirzebruch's proportionality theorem in the non-compact case, *Invent. Math.*, **42** (1977), 239–272.

[25] Ohsawa, T., On the rigidity of the quotients of bounded symmetric domains, preprint.

[26] Saper, L., $L^2$-cohomology and intersection homology of certain algebraic varieties with isolated singularities, *Invent. Math.*, **82** (1985), 207–255.

[27] Siu, Y. T., Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems, *J. Diff. Geom.*, **17** (1982), 55–138.

[28] Timmerscheidt K., Hodge decomposition for unitary local systems, App. to: Esnault, E. and Vieweg, E., Logarithmic de Rham complexes and vanishing theorems, *Invent. Math.*, **86** (1986), 161–194.

[29] Tsuji, H., A numerical characterization of ball quotients, *Duke Math. J.*, **57** (1988), 537–553.

[30] Yau, S. T., On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I, *Comm. Pure Appl. Math.*, **31** (1978), 339–411.

[31] Zucker, S., Hodge theory with degenerating coefficients: $L^2$ cohomology in the Poincaré metric, *Ann. of Math.*, **109** (1979), 415–476.

[32] ———, The Hodge structures on the intersection homology of varieties with isolated singularities, *Duke Math. J.*, **55** (1987), 603–616.