

Projection Maps for Tensor Products of $gl(r, \mathbf{C})$ -Representations

By

Georgia BENKART*, Daniel BRITTEN** and Frank LEMIRE***

Abstract

We investigate the tensor product $\mathcal{T} = V(\lambda^1) \otimes \cdots \otimes V(\lambda^m)$ of the finite dimensional irreducible $\mathcal{G} = gl(r, \mathbf{C})$ modules labelled by partitions $\lambda^1, \dots, \lambda^m$ of m not necessarily distinct numbers n_1, \dots, n_m respectively. We determine the centralizer algebra $End_{\mathcal{G}}(\mathcal{T})$ and the projection maps of \mathcal{T} onto its irreducible \mathcal{G} -summands and give an explicit construction of the corresponding maximal vectors. In the special case that $n_i = 1$ for $i = 1, \dots, m$, the results reduce to the well-known results of Schur and Weyl.

§ Introduction

The finite dimensional irreducible polynomial representations of the complex general linear Lie group $G = GL(r, \mathbf{C})$, or equivalently of its Lie algebra $\mathcal{G} = gl(r, \mathbf{C})$, are in one-to-one correspondence with the partitions λ having at most r nonzero parts. Let $V(\lambda)$ denote the irreducible \mathcal{G} -module indexed by λ . The natural representation of \mathcal{G} on \mathbf{C}^r corresponds to the representation $V(\{1\})$ labelled by the unique partition of 1. The tensor product of finitely many irreducible polynomial representations is a completely reducible \mathcal{G} -module, and determination of its irreducible summands has long been a problem of interest. Classically, this problem has been tackled by two quite different approaches. In the first, the decomposition of the tensor product $V(\lambda^1) \otimes V(\lambda^2)$ of two arbitrary irreducible \mathcal{G} -modules has been described at the character level by the Littlewood-Richardson rule. In the

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* Department of Mathematics, University of Wisconsin, Madison, Wisconsin USA 53706-1388

** Department of Mathematics, University of Windsor, Windsor, Ontario, CANADA N9P3B4

*** Department of Mathematics, University of Windsor, Windsor, Ontario, CANADA N9P3B4

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second, the M -fold tensor product $T = \otimes^M V(\{1\})$ of the natural representation has been decomposed by showing that the centralizer algebra $End_{\mathcal{G}}(T)$ of \mathcal{G} in the space $End(T)$ of transformations on T is a homomorphic image of the group algebra $\mathbb{C}[S_M]$ of the symmetric group S_M . The Young symmetrizers y_τ give primitive essential idempotents in $\mathbb{C}[S_M]$ which afford a decomposition $\mathbb{C}[S_M] = \bigoplus_{\tau} \mathbb{C}[S_M] y_\tau$ of $\mathbb{C}[S_M]$ into minimal right ideals, and hence they determine the projection maps of $T = \bigoplus_{\tau} y_\tau T$ onto its irreducible summands $y_\tau T$. The maximal vectors for the irreducible summands can be explicitly constructed using the corresponding Young symmetrizers.

In this paper we investigate the tensor product $V(\lambda^1) \otimes \dots \otimes V(\lambda^m)$ of the \mathcal{G} -modules labelled by partitions $\lambda^1, \dots, \lambda^m$ of m not necessarily distinct numbers n_1, \dots, n_m respectively. Although the characters of the irreducible \mathcal{G} -summands of $V(\lambda^1) \otimes \dots \otimes V(\lambda^m)$ are known from the Littlewood-Richardson rule, the explicit submodules corresponding to those characters are not. We identify this tensor product with a summand eT of the M -fold tensor product $T = \otimes^M V(\{1\})$ of the natural representation for $M = n_1 + \dots + n_m$ by using a certain idempotent $e \in \mathbb{C}[S_M]$. It follows then that $eT = \sum_{\tau} e y_\tau T$ where the y_τ 's are the Young symmetrizers in $\mathbb{C}[S_M]$. The submodules $e y_\tau T$ which are nonzero are irreducible, but the sum is no longer direct. (See the example after the statement of Theorem 1.13.) In this paper we describe a distinguished set of labels τ such that the corresponding $e y_\tau$'s are essential idempotents, the sum of the irreducible modules $e y_\tau T$ over this set is direct and gives eT . This enables us to determine the centralizer algebra $End_{\mathcal{G}}(V(\lambda^1) \otimes \dots \otimes V(\lambda^m)) \cong End_{\mathcal{G}}(eT)$ and to explicitly construct the maximal vectors for the irreducible summands occurring in the decomposition of eT . Our methods generalize the classical ones of Schur and Weyl, and indeed our results reduce to their well-known results when $\lambda^1 = \lambda^2 = \dots = \lambda^m = \{1\}$. Encoded into our description of the distinguished labels τ is the Littlewood-Richardson rule.

In the special case that each partition λ^i has only one part, the idempotent e equals the idempotent $|\mathcal{S}|^{-1} \sum_{\sigma \in \mathcal{S}} \sigma$ corresponding to the Young subgroup $\mathcal{S} = S_{I_1} \times \dots \times S_{I_m}$ of S_M where $I_i = \{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_i\}$. Whenever $r \geq M$, the centralizer algebra of \mathcal{G} on $V(\lambda^1) \otimes \dots \otimes V(\lambda^m)$ is the Hecke algebra $e\mathbb{C}[S_M]e$. Thus, we obtain by our investigations a description of primitive idempotents in the Hecke algebra $e\mathbb{C}[S_M]e$. (Compare [C] and

[K], Theorem A.)

§1. Centralizer Algebras

In this section we begin with a few generalities about centralizer algebras and then specialize to study $\mathfrak{gl}(r, \mathbb{C})$ -modules. Initially we suppose that \mathcal{G} is a Lie algebra and T is a \mathcal{G} -module. The *centralizer algebra* $End_{\mathcal{G}}(T)$ of \mathcal{G} on T is the set $\{\phi \in End(T) | \phi(xt) = x\phi(t) \text{ for all } x \in \mathcal{G} \text{ and } t \in T\}$ of linear transformations on T commuting with \mathcal{G} . When T is assumed to be a completely reducible \mathcal{G} -module, the projection maps of T onto its irreducible summands belong to $End_{\mathcal{G}}(T)$, and the \mathcal{G} -submodules of T are the spaces eT where e is an idempotent in $End_{\mathcal{G}}(T)$. Thus, the centralizer algebra is significant for understanding the decomposition of T into irreducible constituents. Similarly, for a submodule eT of T , the centralizer algebra $End_{\mathcal{G}}(eT)$ plays a critical role in analyzing its irreducible summands. As the next result indicates, these centralizer algebras are closely related.

Proposition 1.1. *Let \mathcal{G} be a Lie algebra and T be a completely reducible \mathcal{G} -module. If e is an idempotent in $End_{\mathcal{G}}(T)$, then*

$$End_{\mathcal{G}}(eT) = eEnd_{\mathcal{G}}(T)e|_{eT}.$$

Proof. The elements of $eEnd_{\mathcal{G}}(T)e$ when restricted to eT clearly belong to $End_{\mathcal{G}}(eT)$. To prove the reverse inclusion, let $f \in End_{\mathcal{G}}(eT)$ and let id denote the identity map of T . Extend f to a linear transformation (denoted \hat{f}) on $T = eT \oplus (id - e)T$ by having \hat{f} map $(id - e)T$ to zero. Since $(id - e)T$ is a \mathcal{G} -submodule of T , we have $\hat{f}(xu) = 0 = x\hat{f}(u)$ for all $u \in (id - e)T$ and $x \in \mathcal{G}$. Thus, $\hat{f} \in End_{\mathcal{G}}(T)$. If $u \in eT$, then $\hat{f}(u) = et$ for some $t \in T$. Therefore,

$$(id - e)\hat{f}(u) = (id - e)et = 0.$$

Since $(id - e)\hat{f}$ is zero on $(id - e)T$ as well, $(id - e)\hat{f} = 0$ on T , and $\hat{f} = e\hat{f} \in eEnd_{\mathcal{G}}(T)$. Then on eT , $f = \hat{f}|_{eT} = e\hat{f}|_{eT} = e\hat{f}e|_{eT}$ to show $f \in eEnd_{\mathcal{G}}(T)e|_{eT}$. ■

We suppose now that $\mathcal{G} = \mathfrak{gl}(r, \mathbb{C})$ and $T = \otimes^n V(\{1\})$ of n copies of the natural representation $V(\{1\}) = \mathbb{C}^r$ of \mathcal{G} , which is a completely reducible \mathcal{G} -module. There is an action of the symmetric group S_n on T given by

$$\sigma(u_1 \otimes \cdots \otimes u_n) = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(n)},$$

for $\sigma \in S_n$, which extends to a representation of the group algebra $\mathbb{C}[S_n]$ on T . The action of the group algebra $\mathbb{C}[S_n]$ commutes with that of \mathcal{G} , and thus, the group algebra can be used to decompose T into \mathcal{G} -submodules.

Let $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > \lambda_{\ell+1} = 0\}$ denote a partition of n , and let $\ell(\lambda) = \ell$ be the number of nonzero parts of λ . We write $\lambda \vdash n$ to signify that λ is a partition of n , and use $|\lambda|$ to denote the sum of the parts of λ , which is just n . Associated to λ is its Young frame or Ferrers diagram $\mathcal{F}(\lambda)$ having n boxes with λ_i boxes in the i th row for $i = 1, \dots, \ell$. A *standard tableau of shape* λ is a filling of the boxes in the diagram of λ with the numbers $1, \dots, n$ in such a way that the entries increase from left to right across each row and down each column. Associated to the standard tableau τ are two subgroups of S_n , the *row group* \mathcal{R}_τ of permutations which transform each entry of τ to an entry in the same row, and the *column group* \mathcal{C}_τ of permutations moving each entry of τ to an entry in the same column. These subgroups of S_n enable us to construct the following element in the group algebra $\mathbb{C}[S_n]$:

$$s_\tau = \left(\sum_{\gamma \in \mathcal{C}_\tau} \text{sgn}(\gamma)\gamma \right) \left(\sum_{\rho \in \mathcal{R}_\tau} \rho \right) = \sum_{\substack{\rho \in \mathcal{R}_\tau \\ \gamma \in \mathcal{C}_\tau}} \text{sgn}(\gamma)\gamma\rho.$$

There is some $k \in \mathbb{Z}^+$ such that $s_\tau^2 = ks_\tau$ (see [W], p. 121), so that s_τ is an *essential idempotent*. If $\tilde{s}_\tau = (1/k)s_\tau$, then $\tilde{s}_\tau^2 = \tilde{s}_\tau$, and \tilde{s}_τ is a *Young symmetrizer*. The idempotents \tilde{s}_τ , for τ in the set $\mathcal{ST}(n)$ of all standard tableaux over all partitions λ of n , are primitive idempotents which afford a decomposition of the group algebra $\mathbb{C}[S_n] = \sum_{\tau \in \mathcal{ST}(n)} \tilde{s}_\tau \mathbb{C}[S_n]$ into minimal right ideals. Correspondingly, they determine a decomposition of $T = \otimes^n V(\{1\})$. Thus, $T = \bigoplus_{\tau \in \mathcal{ST}(n)} \tilde{s}_\tau T$ (see [W] and also [BBL]).

The representation of $\mathbb{C}[S_n]$ on T gives a surjective homomorphism $\psi: \mathbb{C}[S_n] \rightarrow \text{End}_{\mathcal{G}}(T)$ with kernel equal to $\bigoplus \mathcal{A}_\lambda$, where \mathcal{A}_λ is the minimal two-sided ideal of $\mathbb{C}[S_n]$ corresponding to the partition λ , and the sum is over all partitions λ of n with $\ell(\lambda) > r$. Hence,

$$(1.2) \quad \text{End}_{\mathcal{G}}(T) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq r}} \mathcal{A}_\lambda.$$

Each \mathcal{A}_λ is the sum of all the minimal right ideals $\tilde{s}_\tau \mathbb{C}[S_n]$ where τ has λ

as its underlying partition. (Compare [Bo] Theorem 4.6.) Thus, when τ has shape λ and $\ell(\lambda) > r$, then $\tilde{s}_\tau T = (0)$. Otherwise, $\tilde{s}_\tau T \cong V(\lambda)$. Therefore when $\ell(\lambda) \leq r$, $V(\lambda)$ occurs as a summand of T with multiplicity equal to the number of standard tableaux of shape λ .

For each partition λ we choose a standard tableau ζ_λ obtained by filling in the diagram of λ with the numbers $1, \dots, n$ in succession beginning with the first row, then the second, and so forth and proceeding from left to right. We let \tilde{s}_λ denote the Young symmetrizer defined by the canonically chosen standard tableau ζ_λ and identify $V(\lambda)$ with $\tilde{s}_\lambda T$.

Example. When $n=2$, there are exactly two partitions of n , $\lambda = \{2\}$ and $\pi = \{1^2\} = \{1 \geq 1\}$, and just two standard tableaux (the canonically chosen one for each partition),

$$\zeta_\lambda = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \quad \text{and} \quad \zeta_\pi = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}.$$

The associated symmetrizers are $\tilde{s}_\lambda = \frac{1}{2}(id + (1\ 2))$ and $\tilde{s}_\pi = \frac{1}{2}(id - (1\ 2))$. Thus, $T = V(\{2\}) \oplus V(\{1^2\})$ is the decomposition of T into irreducible summands. In this case the elements in $V(\{2\})$ are just the symmetric tensors, and those in $V(\{1^2\})$ the anti-symmetric tensors.

Since our eventual aim is the study of the tensor product $V(\lambda^1) \otimes \dots \otimes V(\lambda^m)$, we assume that λ^i is a partition of the integer n_i , and let $M_i = n_1 + \dots + n_i$ for $i=1, \dots, m$. We let ζ^i denote the canonically chosen standard tableau obtained from filling in the diagram of λ^i . However, rather than inserting the integers $1, \dots, n_i$, we use instead the values in $I_i = \{M_{i-1} + 1, \dots, M_i\}$ (where $M_0 = 0$). Let $e_i = \tilde{s}_{\lambda^i}$ denote the associated Young symmetrizer. Then the idempotent e_i belongs to the group algebra of the group S_{I_i} of permutations on I_i , which we identify with a subgroup of S_M for $M = n_1 + \dots + n_m = M_m$. Clearly the idempotents e_i commute, and $e \stackrel{\text{def}}{=} e_1 \cdot e_2 \cdot \dots \cdot e_m$ is an idempotent in $\mathbf{C}[S_M]$. Moreover, if $T = \otimes^M V(\{1\})$, then

$$(1.3) \quad eT \cong V(\lambda^1) \otimes \dots \otimes V(\lambda^m).$$

Throughout this work we identify the tensor product $V(\lambda^1) \otimes \dots \otimes V(\lambda^m)$ with eT . As a consequence of Proposition 1.1 and (1.2) we have:

Proposition 1.4. *For $i=1, \dots, m$ let n_i denote a positive integer and assume*

λ^i is a partition of n_i . Identify $V(\lambda^1) \otimes \cdots \otimes V(\lambda^m)$ with $eT = e \otimes^M V(\{1\})$ as in (1.3). Then $End_{\mathcal{G}}(eT) = e \sum_{\ell(\lambda) \leq r} \lambda \vdash M \mathcal{A}_\lambda e|_{eT} = e \sum_{\ell(\lambda) \leq r} \lambda \vdash M \mathcal{A}_\lambda e$.

Remarks. Observe that if $f = e\phi e$ where $\phi \in \sum_{\ell(\lambda) \leq r} \lambda \vdash M \mathcal{A}_\lambda$ and if $f|_{eT} = 0$, then $f = 0$. Indeed, $e\phi e|_{eT} = 0$ implies $e\phi e = 0$ on T , and hence, $e\phi e \in \sum_{\ell(\mu) > r} \mu \vdash M \mathcal{A}_\mu$.

But then $e\phi e \in \left(\sum_{\ell(\lambda) \leq r} \lambda \vdash M \mathcal{A}_\lambda \right) \cap \left(\sum_{\ell(\mu) > r} \mu \vdash M \mathcal{A}_\mu \right) = (0)$.

One of our main objectives is to describe the projection maps of eT onto its irreducible summands. By Proposition 1.4 these maps are primitive idempotents in $e \sum_{\ell(\lambda) \leq r} \lambda \vdash M \mathcal{A}_\lambda e|_{eT}$. In our description we find it helpful to reverse the order of the row group and column group for every standard tableau τ except for the canonical ones attached to the partitions λ^i . The element

$$y_\tau = \left(\sum_{\rho \in \mathcal{R}_\tau} \rho \right) \left(\sum_{\gamma \in \mathcal{C}_\tau} \text{sgn}(\gamma) \gamma \right) = \sum_{\substack{\rho \in \mathcal{R}_\tau \\ \gamma \in \mathcal{C}_\tau}} \text{sgn}(\gamma) \rho \gamma.$$

is an essential idempotent in $\mathcal{C}[S_M]$, and the essential idempotents y_τ afford a decomposition of $\mathcal{C}[S_M]$ and T just as the s_τ 's and \tilde{s}_τ 's do. If the underlying partition of τ is λ we have $y_\tau T \cong V(\lambda)$ whenever $\ell(\lambda) \leq r$.

The character of the irreducible \mathcal{G} -module $V(\lambda)$ is the Schur function \mathfrak{s}_λ (see [Ma]), and the product of the two Schur functions \mathfrak{s}_ν and \mathfrak{s}_λ is just the character of the tensor product $V(\nu) \otimes V(\lambda)$. The *Littlewood-Richardson rule* (see [J], [S], [T], and [Ma] for proofs) provides an algorithm for decomposing the product of two Schur functions into a sum of Schur functions:

$$\mathfrak{s}_\nu \cdot \mathfrak{s}_\lambda = \sum_{\pi \vdash |\nu| + |\lambda|, \ell(\pi) \leq r} c_{\nu, \lambda}^\pi \mathfrak{s}_\pi,$$

where the coefficients $c_{\nu, \lambda}^\pi$ are the so-called *Littlewood-Richardson coefficients*. Therefore,

$$V(\nu) \otimes V(\lambda) = \sum_{\pi \vdash |\nu| + |\lambda|, \ell(\pi) \leq r} c_{\nu, \lambda}^\pi V(\pi).$$

The Littlewood-Richardson coefficients have a combinatorial description as the number of lattice permutations of shape π/ν and weight λ . If π and ν are partitions with $\pi_i \geq \nu_i$ for all i , then we write $\pi \supseteq \nu$ and say ν is contained in π . The diagram obtained by deleting the boxes of $\mathcal{F}(\nu)$ from the diagram $\mathcal{F}(\pi)$ is the *skew shape* π/ν . The coefficient $c_{\nu, \lambda}^{\pi}$ is zero unless $\pi \supseteq \nu, \lambda$ and $|\pi| = |\nu| + |\lambda|$.

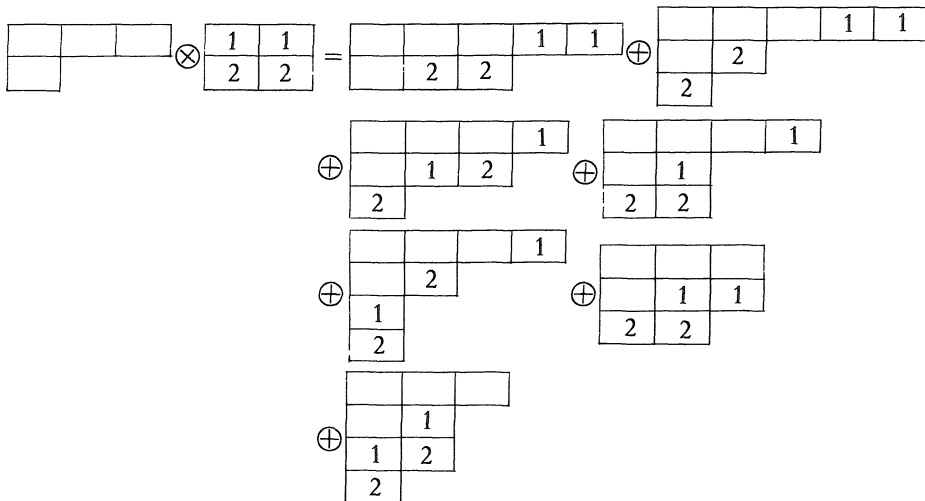
A *lattice permutation* is a word $\omega = b_1 \cdots b_q$ of positive integers such that for each integer $k > 0$ the number of occurrences of k is greater than or equal to the number of occurrences of the integer $k+1$ in $b_1 \cdots b_p$ for $p = 1, \dots, q$. If λ_k denotes the number of occurrences of k in ω , then $\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ is a partition, termed the *weight* of ω . Suppose $\pi \supseteq \nu$ and $|\pi| = |\nu| + |\lambda|$. If $\omega = b_1 \cdots b_q$ is a lattice permutation of weight λ , then one inserts the b_j 's into π/ν by filling in the rows from top to bottom and right to left. For example, if $\nu = \{2, 1^2\}$ and $\pi = \{3^2, 2\}$, then the word $\omega = 1212$ is a lattice permutation of weight $\lambda = \{2^2\}$, which is inserted into π/ν as pictured below:

		1
	1	2
	2	

The coefficient $c_{\nu, \lambda}^{\pi}$ records the number of lattice permutations of weight λ which give semistandard tableaux when inserted into π/ν . By *semistandard* we mean that the entries in the tableau are weakly increasing across each row from left to right and strictly increasing down each column.

There is a procedure for determining the coefficients $c_{\nu, \lambda}^{\pi}$, which we illustrate with the simple example $\nu = \{3, 1\}$, $\lambda = \{2^2\}$. Insert 1's into the first row of the diagram of λ , and 2's into the second. Take the boxes with the 1's from $\mathcal{F}(\lambda)$ and append them one by one to the frame $\mathcal{F}(\nu)$, in such a way that each lies in a different column and the result at each step is the frame of some partition. Next take the boxes with the 2's and adjoin them to the frames just formed. Again, no two of these boxes should lie in the same column and the result at each stage should be the frame of a partition. The 2's should be added in such a way that if the frame is read from top to bottom and right to left, the number of 1's at each step is greater than or equal to the number of 2's; that is, the result of reading the 1's

and 2's in this fashion should be a lattice permutation. Therefore, the algorithm yields the following which we term *the components when the Littlewood-Richardson process is applied to $v \otimes \lambda$* or simply *the components of $v \otimes \lambda$* :



If a component has shape π and if $\ell(\pi) \leq r$, then $V(\pi)$ occurs as a summand of the tensor product. Consequently for $r \geq 4$,

$$\begin{aligned} V(\{3,1\}) \otimes V(\{2^2\}) &= V(\{5,3\}) \oplus V(\{5,2,1\}) \oplus V(\{4,3,1\}) \\ &\quad \oplus V(\{4,2^2\}) \oplus V(\{4,2,1^2\}) \oplus V(\{3^2,2\}) \\ &\quad \oplus V(\{3,2^2,1\}). \end{aligned}$$

In this work we interpret the Littlewood-Richardson algorithm as a process to produce a collection of standard tableaux, which we term Littlewood-Richardson tableaux. To describe these tableaux we use the phrase the *northeast corner* (NE corner) of a position (a,b) or its entry in a tableau to mean all positions (a',b') in the tableau with $a \geq a'$ and $b' \geq b$.

Lemma 1.5. *Let v and λ be two partitions and suppose that θ is a component of $v \otimes \lambda$. If $i+1$ is the entry in row a and column b of θ , then there are at least as many i 's as $(i+1)$'s in the NE corner determined by (a,b) .*

Proof. Assume that the lemma fails at row a and column b of θ and that a is minimal with that property. Then we may suppose that in the NE corner determined by (a,b) there are more $(i+1)$'s than i 's and that the

entry in position (a, b) is $i + 1$. The lattice permutation and semistandard properties of the Littlewood-Richardson procedure insure that i is the (a', b') entry of θ for some $a' < a$ and $b' < b$. The semistandard property also guarantees that the entry in position $(a - 1, b)$ is i . Moreover, if $i + 1$ is in position (a, c) for $c < b$, then i is the entry in position $(a - 1, c)$. Since the $(i + 1)$'s in row a of the NE corner determined by (a, b) are paired with i 's directly above them, there must exist some $i + 1$ in the NE corner determined by (a, b) for which the property fails. Necessarily $i + 1$ occurs in some row higher than row a , but that contradicts the minimality of a . Thus, the lemma must hold. ■

We now describe our procedure for filling in the boxes of a component to yield a standard tableau. Suppose ν and $\lambda = \{\lambda_1 \geq \dots \geq \lambda_\ell > \lambda_{\ell+1} = 0\}$ are partitions. We assume that the boxes of $\mathcal{F}(\nu)$ have been filled with the numbers $1, \dots, |\nu|$ in the canonical way and insert the numbers $|\nu| + 1, \dots, |\nu| + |\lambda|$ into $\mathcal{F}(\lambda)$ from smallest to largest by proceeding from left to right and top to bottom. If θ is a component of $\nu \otimes \lambda$ we substitute the symbol $\alpha_{i,*}$ for each i in θ and refer to the result θ^* . We change the $*$ on each $\alpha_{i,*}$ in θ^* to some j with $1 \leq j \leq \lambda_i$ according to the following steps:

Algorithm 1.6. For $1 \leq j \leq \lambda_1$, let ℓ_j be the length of the j -th column of λ . Set $j = 1$.

(i) Set $i = \ell_j$. Locate the leftmost unlabelled $\alpha_{\ell_j,*}$ and label it $\alpha_{\ell_j,j}$. If $i = 1$, proceed to step (iii).

(ii) Locate the leftmost unlabelled $\alpha_{i-1,*}$ in the northeast corner (NE corner) of $\alpha_{i,j}$ and label it $\alpha_{i-1,j}$. Redefine i to be $i - 1$ and repeat this step until $i = 1$.

(iii) If $j < \lambda_1$, then redefine j to be $j + 1$ and repeat step (i). Otherwise, quit the algorithm.

At the conclusion of this algorithm we replace $\alpha_{i,j}$ with the entry in the i -th row and j -th column of $\mathcal{F}(\lambda)$ which is $|\nu| + \lambda_1 + \dots + \lambda_{i-1} + j$, and then fill each empty box of θ with the corresponding entry in $\mathcal{F}(\nu)$.

We show that this procedure can be accomplished in one and only one way for each component of $\nu \otimes \lambda$. Moreover, the net effect is a standard tableau with underlying partition π where π is the shape of the component. Before addressing those issues, perhaps it is instructive to consider the following example.

Let $\nu = \{3, 2, 1\}$ and $\lambda = \{3^2, 1\}$ so the diagrams of ν and λ are filled according to:

1	2	3
4	5	
6		

7	8	9
10	11	12
13		

We assume that θ is the following component of $\nu \otimes \lambda$:

(1.7)

			1	1
		1	2	
	2			
2	3			

After Algorithm 1.6 has been performed, the component θ has been transformed to:

(1.8)

			$\alpha_{1,2}$	$\alpha_{1,3}$
		$\alpha_{1,1}$	$\alpha_{2,3}$	
	$\alpha_{2,1}$			
$\alpha_{2,2}$	$\alpha_{3,1}$			

and the end product is the standard tableau

1	2	3	8	9
4	5	7	12	
6	10			
11	13			

Lemma 1.9. *Let ν and $\lambda = \{\lambda_1 \geq \dots \geq \lambda_\ell > \lambda_{\ell+1} = 0\}$ be two partitions. Suppose that θ is a component of $\nu \otimes \lambda$ and that each i in θ has been replaced by $\alpha_{i,*}$ to give θ_* . Then Algorithm 1.6 can be successfully performed to change each $\alpha_{i,*}$ in θ_* to $\alpha_{i,j}$ for some j with $1 \leq j \leq \lambda_i$, and at its completion each pair of subscripts (i, j) with $1 \leq i \leq \ell$ and $1 \leq j \leq \lambda_i$ occurs exactly once.*

Proof. Step (i) of Algorithm 1.6 can always be carried out, so that if the algorithm fails to label all $\alpha_{i,*}$, then at some point step (ii) cannot be performed. Assume for $i > 1$ that the algorithm has assigned the indices (i, k) for $1 \leq k < j$ and as well as the indices $(\ell_j, j), (\ell_j - 1, j), \dots, (i, j)$ from the j -th column. Denote this partially labelled version by η . Let (a, b) be the location of $\alpha_{i,j}$ in η . We argue that in the NE corner of η determined by (a, b) there remains at least one unlabelled $\alpha_{i-1,*}$ so that step (ii) can be

performed. Assume to the contrary that all the $\alpha_{i-1,*}$'s in the NE corner of (a, b) have been changed to $\alpha_{i-1,t}$ for some $t < j$. Let $K = \{t | \alpha_{i-1,t} \text{ is in the NE corner of } (a, b) \text{ in } \eta\}$, so that all the elements of K necessarily are less than j . Since θ contains an i in position (a, b) , Lemma 1.5 tells us that θ contains at least one $i-1$ in the NE corner of (a, b) , so that $K \neq \emptyset$. For each $t \in K$, let (c_t, d_t) denote the location of $\alpha_{i,t}$ in η , and let $k \in K$ be such that $d_k \leq d_t$ for all $t \in K$. By the semistandardness of the entries of θ , we have that either (c_k, d_k) is in the NE corner of (a, b) or (a, b) is in the NE corner of (c_k, d_k) . In the first case there are at least $|K| + 1$ i 's in the NE corner of (a, b) in θ , so there must be at least that number of $(i-1)$'s in the NE corner of (a, b) in θ by the lattice permutation property. Only $|K|$ of the $\alpha_{i-1,*}$'s in the NE corner of (a, b) of η have been assigned a second subscript, so one remains, contrary to our assumption. Hence, we may assume that (a, b) is in the NE corner of (c_k, d_k) . Since θ contains at least $|K| + 1$ i 's in the NE corner of (c_k, d_k) , Lemma 1.5 implies that there are at least $|K| + 1$ $(i-1)$'s in the NE corner of (c_k, d_k) in θ . By assumption exactly $|K|$ of those $(i-1)$'s lie in the NE corner of (a, b) . Since k is in K , $\alpha_{i-1,k}$ is in the NE corner of (a, b) . However, by the semistandard property of θ , there exists some $\alpha_{i-1,*}$ in the NE corner of (c_k, d_k) of η further to the left than $\alpha_{i-1,k}$. But this is impossible since the algorithm requires k to be the second subscript in the leftmost $\alpha_{i-1,*}$ in the NE corner of (c_k, d_k) in η . We conclude that Algorithm 1.6 can be performed to label all the $\alpha_{i,*}$ in θ_* . ■

Lemma 1.9 insures that for each component θ of $v \otimes \lambda$ we can carry out the steps in 1.6 in a unique way. This motivates the following definition:

Definition 1.10. *Let v and λ be two partitions, and suppose that θ is a component of $v \otimes \lambda$. Replacing each i in θ by $\alpha_{i,*}$ and then performing the steps in Algorithm 1.6 gives the slant labelling of θ .*

For the component depicted in (1.7) the slant labelling is the one shown in (1.8).

Lemma 1.11. *Let v and $\lambda = \{\lambda_1 \geq \dots \geq \lambda_\ell > \lambda_{\ell+1} = 0\}$ be two partitions, and suppose that θ is a component of $v \otimes \lambda$ that has been given the slant labelling. Replace $\alpha_{i,j}$ in θ by $|v| + \lambda_1 + \dots + \lambda_{i-1} + j$ and fill the empty boxes of θ with the corresponding entries in a standard tableau ζ of shape v having entries in $\{1, \dots, |v|\}$. The result is a standard tableau τ .*

Proof. Suppose that τ has underlying partition π . We need only consider the skew portion of τ of shape π/ν . The Littlewood-Richardson process guarantees that that it is semistandard. Now if i occurs in location (a, b) of θ and i' in position $(a+1, b)$ then $i < i'$. Therefore if τ is not standard, then in the process of creating τ , for some i and some $k < j$, $\alpha_{i,j}$ occurs in position (a, b) of θ_* and $\alpha_{i,k}$ in position $(a, b+1)$. We may assume i is as large as possible with this property. If the length of column k in λ is i , then the length of column j , which is less than or equal to i , is forced to be i as well. But then $\alpha_{i,j}$, which occurs to the left of $\alpha_{i,k}$, should have been labelled $\alpha_{i,k}$ by the algorithm. Thus, we may assume that the length of column k is bigger than i . Then $\alpha_{i+1,k}$ is to the left and below $\alpha_{i,k}$. If $\alpha_{i+1,k}$ is in column b' of θ_* for $b' \leq b$, then the algorithm would have labelled $\alpha_{i,j}$ with k instead of j . Thus, it must be that $\alpha_{i+1,k}$ is in column $b+1$, and by semistandardness it resides in the $(a+1, b+1)$ position. What is in the $(a+1, b)$ location of θ_* ? By semistandardness and the maximality of i it must be $\alpha_{i+1,p}$ for some p with $p < k < j$. But then $\alpha_{i,j}$ should have been labelled $\alpha_{i,p}$ by the algorithm. Consequently, τ is standard. ■

We now want to extend these ideas to the tensor product of arbitrarily many partitions.

Definition 1.12. Let n_1, \dots, n_m denote positive integers (not necessarily distinct) and for $j=1, \dots, m$ fix a partition λ^j of n_j . Let $M_j = n_1 + \dots + n_j$, and set $M = M_m$. A tableau τ of shape π , for π some partition of M , is said to be a $(\lambda^1, \dots, \lambda^m)$ -Littlewood-Richardson tableau (a $(\lambda^1, \dots, \lambda^m)$ -LR tableau for short) provided the following conditions are satisfied:

- (i) For each integer $j=1, \dots, m$, the boxes of τ containing the numbers $1, \dots, M_j$ form a tableau τ^j of shape ν^j where ν^j is a partition of M_j .
- (ii) The tableau τ^1 is the canonically filled tableau of shape λ^1 .
- (iii) For $1 < j \leq m$ assume that θ^j is a component of $\nu^{j-1} \otimes \lambda^j$ of shape ν^j . The tableau τ^j is obtained from the slant labelling of θ^j by substituting $M_{j-1} + \lambda^1_1 + \dots + \lambda^{j-1}_{i-1} + j$ for the entry $\alpha_{i,j}$ and then filling in the empty boxes of θ^j with the entries of τ^{j-1} in the corresponding positions.

Remark. It is apparent from Lemma 1.11 that a $(\lambda^1, \dots, \lambda^m)$ -LR tableau is a standard tableau. By the Littlewood-Richardson rule the number of them is precisely the number of summands when the tensor product $V(\lambda^1) \otimes \dots \otimes V(\lambda^m)$ is decomposed into irreducible \mathcal{G} -representations.

Example. Suppose that $\lambda^1 = \{3, 1\}$, $\lambda^2 = \{2^2\}$ and $\lambda^3 = \{1^3\}$, and $\pi = \{5, 3, 2, 1\}$. Then $M_1 = 4$, $M_2 = 8$, and $M = M_3 = 11$. The tableau

1	2	3	6	9
4	5	10		
7	8			
11				

is a $(\lambda^1, \lambda^2, \lambda^3)$ -LR tableau of shape π .

The ultimate goal of this paper is to establish the following theorem. In its statement the vectors v_i for $i = 1, \dots, r$ denote the standard basis elements for \mathbf{C}^r viewed as a $r \times 1$ matrix, so that v_i has a 1 in the i -th row and zero everywhere else.

Theorem 1.13. For $i = 1, \dots, m$ let n_i denote a positive integer and assume λ^i is a partition of n_i . Identify the $\mathfrak{G} = \mathfrak{gl}(r, \mathbf{C})$ module $V(\lambda^1) \otimes \dots \otimes V(\lambda^m)$ with $eT = e \otimes^M V(\{1\})$ as in (1.3). Let τ be a $(\lambda^1, \dots, \lambda^m)$ -LR tableau of shape π , for π some partition of $M = n_1 + \dots + n_m$. Then:

- (i) e_{y_τ} (and hence $e_{y_\tau}e$) is an essential idempotent in $\mathbf{C}[S_M]$, and thus $e_\tau = \xi_\tau e_{y_\tau} e$ is an idempotent for some scalar ξ_τ .
- (ii) When τ has shape π and $\ell(\pi) \leq r$, then $e_{y_\tau}T = e_\tau T$ is an irreducible \mathfrak{G} -module which is isomorphic to $V(\pi)$. A maximal vector for $e_{y_\tau}T = e_\tau T$ is the vector $e_{y_\tau}\beta_\tau = \xi_\tau^{-1} e_{y_\tau}\beta_\tau$ where $\beta_\tau = u_1 \otimes \dots \otimes u_M$ is the simple tensor constructed according to:

$$u_j = v_i \quad \text{if } j \text{ is in the } i\text{th row of } \tau.$$

- (iii) $eT = \bigoplus \sum_\tau e_{y_\tau}T = \bigoplus \sum_\tau e_\tau T$, where the sum is over all $(\lambda^1, \dots, \lambda^m)$ -LR tableaux τ of shape π for all partitions π of M with $\ell(\pi) \leq r$, is a decomposition of eT into irreducible \mathfrak{G} -representations.

The proof of Theorem 1.13 constitutes the remainder of the paper. The next section develops the necessary ingredients for establishing that e_{y_τ} is an essential idempotent in $\mathbf{C}[S_M]$ and for showing that the vectors $e_{y_\tau}\beta_\tau$ are maximal. The final section is devoted to the proof of part (iii) of Theorem 1.13. We conclude this section with an example to illustrate the main ideas behind Theorem 1.13 and with some remarks.

Example. By the Littlewood-Richardson rule $V(\{2\}) \otimes V(\{1^2\}) \cong V(\{3, 1\}) \oplus V(\{2, 1^2\})$ for all $r \geq 3$. For the tensor product $V(\{2\}) \otimes V(\{1^2\})$

the corresponding idempotent is

$$e = \frac{1}{4}(id + (1\ 2))(id - (3\ 4)).$$

There are three standard tableaux belonging to the partition $\{3,1\}$ of 4:

$$\tau = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array}, \quad \tau' = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array} \quad \text{and} \quad \tau'' = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array},$$

and hence, three corresponding summands $y_\tau T \oplus y_{\tau'} T \oplus y_{\tau''} T$ in $T = \otimes^4 V(\{1\})$. It is easy to verify that $ey_{\tau''} = 0$, but the other two Young symmetrizers survive multiplication by e , and there is a collapsing of the direct sum $y_\tau T \oplus y_{\tau'} T$ into a single irreducible module when e is applied. Our method selects τ as the distinguished label for the submodule belonging to the partition $\{3,1\}$ and gives the maximal vector of $ey_\tau T$ as

$$\begin{aligned} ey_\tau \beta_\tau &= \frac{1}{4}(id + (1\ 2))(y_\tau - (3\ 4)y_\tau)\beta_\tau \\ &= \frac{1}{4}(2y_\tau - 2y_{\tau'}(3\ 4))\beta_\tau = \frac{1}{2}(y_\tau \beta_\tau - y_{\tau'} \beta_{\tau'}) \\ &= \frac{1}{2}(y_\tau(v_1 \otimes v_1 \otimes v_1 \otimes v_2) - y_{\tau'}(v_1 \otimes v_1 \otimes v_2 \otimes v_1)) \\ &= 4(v_1 \otimes v_1 \otimes v_1 \otimes v_2 - v_1 \otimes v_1 \otimes v_2 \otimes v_1). \end{aligned}$$

As is shown in ([BBL], Chapter 2), the vectors $y_\tau \beta_\tau$ and $y_{\tau'} \beta_{\tau'}$ are maximal vectors for the summands $y_\tau T$ and $y_{\tau'} T$ respectively, and the calculation above gives the maximal vector $ey_\tau \beta_\tau$ as an explicit linear combination of them. A similar computation shows that $ey_{\tau'} \beta_{\tau'} = \frac{1}{2}(y_{\tau'} \beta_{\tau'} - y_\tau \beta_\tau) = -ey_\tau \beta_\tau$. The tableau labelling the summand corresponding to the partition $\{2,1^2\}$ is

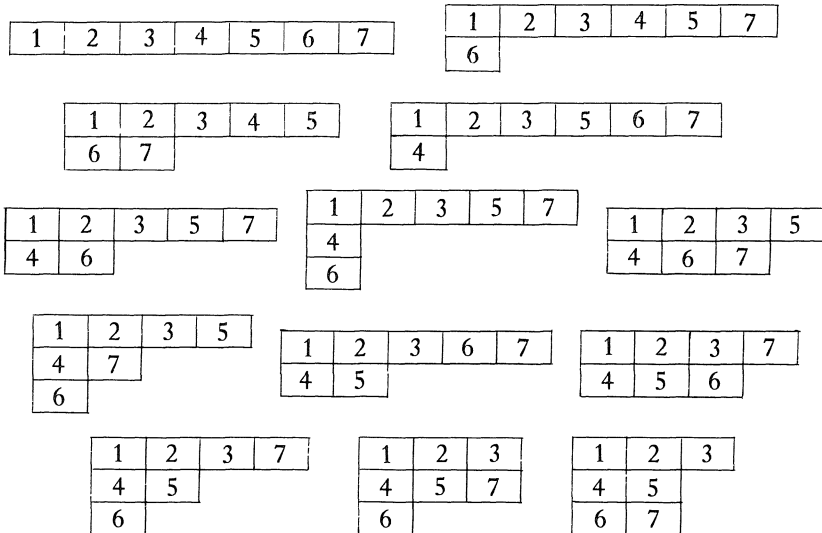
$$\tau^* = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array},$$

and hence, the centralizer algebra is given by $End_{\mathcal{G}}(eT) = ey_\tau e\mathcal{C}[S_4]e \oplus ey_{\tau^*} e\mathcal{C}[S_4]e$.

Remarks. When $\lambda^1 = \dots = \lambda^m = \{1\}$, every standard tableau of shape π , for π some partition of m , is a $(\lambda^1, \dots, \lambda^m)$ -LR tableau. In this particular case the idempotent e is just the identity, and (ii) reduces to the decomposition $\otimes^m V(\{1\}) = T = \bigoplus_{\tau \in \mathcal{S}(\mathcal{S}(m))} y_\tau T$ encountered earlier. Thus, Theorem 1.13

can be viewed as the natural generalization of that classical result.

When the partition λ^i has just one part, the row group attached to the canonically filled tableau ζ^i is the group S_{I_i} where $I_i = \{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_i\}$ and the column group is trivial. The idempotent e_i associated to λ^i is the idempotent $(1/n_i!) \sum_{\rho \in S_{I_i}} \rho$ defined by S_{I_i} . Thus, when all partitions λ^i have just one part, the idempotent e is the idempotent $|\mathcal{S}|^{-1} \sum_{\sigma \in \mathcal{S}} \sigma$ determined by the Young subgroup $\mathcal{S} = S_{I_1} \times S_{I_2} \times \dots \times S_{I_m}$ of S_M . For all $r \geq M$, the centralizer algebra $End_{g^l}(eT)$ equals the Hecke algebra $eC[S_M]e$ determined by \mathcal{S} , and Theorem 1.13 provides primitive idempotents e_τ in $eC[S_M]e$. In this special case, for each $(\lambda^1, \dots, \lambda^m)$ -LR tableau τ , the entries in $\{1, \dots, n_1 + \dots + n_i\}$ comprise a subtableau τ^i which is obtained from τ^{i-1} by adjoining a horizontal strip (no 2 boxes in the same column) having n_i boxes. The slant labelling requires that the numbers in the interval I_i be placed into the boxes of τ^i/τ^{i-1} beginning with the leftmost box on the bottom and proceeding to the rightmost box on the top. For example, when $\lambda^1 = \{3\}$, and $\lambda^2 = \{2\} = \lambda^3$, the idempotents $e_\tau = \xi_\tau^{-1} e y_\tau e$ in the Hecke algebra $eC[S_M]e$ correspond to the $(\lambda^1, \lambda^2, \lambda^3)$ -LR tableaux τ where τ is one of the following:



There are many other ways that the labelling of the second subscript in $\alpha_{i,*}$ could have been performed to yield a collection of tableaux satisfying the conclusions of Theorem 1.13. We have chosen the slant labelling because it facilitates the proof of Theorem 1.13.

§2. $e y_\tau$ is an Essential Idempotent

Recall that we are fixing certain standard tableaux ζ^1, \dots, ζ^m corresponding to the partitions $\lambda^1, \dots, \lambda^m$ and are assuming that the idempotent e_i is the symmetrizer associated to ζ^i . Suppose that \mathcal{C}_i and \mathcal{R}_i denote the column group and the row group of ζ^i , respectively, and let

$$\begin{aligned} \mathcal{C} &= \mathcal{C}_1 \times \dots \times \mathcal{C}_m \\ \mathcal{R} &= \mathcal{R}_1 \times \dots \times \mathcal{R}_m. \end{aligned}$$

Observe that e is some scalar multiple of

$$\left(\sum_{g \in \mathcal{C}} \text{sgn}(g)g \right) \left(\sum_{f \in \mathcal{R}} f \right) = \sum_{\substack{f \in \mathcal{R} \\ g \in \mathcal{C}}} \text{sgn}(g)gf.$$

As before, let $M_i = n_1 + \dots + n_i$ where $\lambda^i \vdash n_i$, and write $I_k = \{M_{k-1} + 1, \dots, M_k\}$. Then for $g \in \mathcal{C}$ and $f \in \mathcal{R}$ we have $g(I_k) = I_k = f(I_k)$ for $k = 1, \dots, m$.

Assume now that τ is a $(\lambda^1, \dots, \lambda^m)$ -LR tableau of shape π with row group \mathcal{R}_τ and column group \mathcal{C}_τ . Then

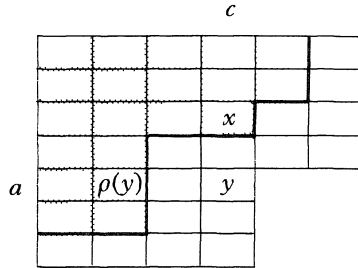
$$y_\tau = \left(\sum_{\rho \in \mathcal{R}_\tau} \rho \right) \left(\sum_{\substack{\gamma \in \mathcal{C}_\tau \\ \rho \in \mathcal{R}_\tau \\ \gamma \in \mathcal{C}_\tau}} \text{sgn}(\gamma)\gamma \right) = \sum_{\substack{\rho \in \mathcal{R}_\tau \\ \gamma \in \mathcal{C}_\tau}} \text{sgn}(\gamma)\rho\gamma.$$

The subtableau τ^p of τ consists of the boxes containing the entries in $\{1, \dots, M_p\} = I_1 \cup \dots \cup I_p$.

Lemma 2.1. *Suppose $\gamma \in \mathcal{C}_\tau$ and $\rho \in \mathcal{R}_\tau$, and assume that $\rho\gamma(I_k) = I_k$ for all $k = 1, \dots, m$. Then $\rho(I_k) = I_k = \gamma(I_k)$ for all k .*

Proof. Let $J_p = I_1 \cup \dots \cup I_p$. It suffices to show that $\gamma(J_p) = J_p$ for all $p = 1, \dots, m$. Let $K_0 = \emptyset$ and for $c \geq 1$, let $K_c = \{x \in J_p \mid x \text{ is in the first } c \text{ columns of } \tau\}$. Assume that $\gamma(K_{c'}) \subseteq K_{c'}$ for all $c' < c$, but $\gamma(K_c) \not\subseteq K_c$. Then there exists $x \in J_p$ in column c of τ such that $y = \gamma(x) \notin J_p$. Then x is an entry in the subtableau τ^p , but y is in $\tau^q - \tau^p$ for some $q > p$. Thus, y is in the same column as x but is below x . Assume that y is in row a of τ , and let A

denote the set of elements in row a of τ . Then $\rho(y) = \rho\gamma(x) \in J_p \cap A = K_{c-1} \cap A$ as portrayed below, where the shaded portion depicts τ^p .



However, $\rho(K_{c-1}) = \rho\gamma(K_{c-1}) \subseteq J_p$, and hence $\rho(K_{c-1} \cap A) \subseteq J_p \cap A = K_{c-1} \cap A$. Thus, $\rho(K_{c-1} \cap A) = K_{c-1} \cap A$ as ρ is a bijection. We have reached a contradiction, for $y \notin K_{c-1} \cap A$, yet $\rho(y) \in K_{c-1} \cap A$. Thus it must be that J_p , and hence I_p , are preserved by γ and by ρ as well. ■

Lemma 2.2. ([Mi], Lemma 4.2) *Let ζ be a standard tableau of shape λ where $\lambda \vdash n$. Then an element $\sigma \in C[S_n]$ can be written $\sigma = RC$ where $R \in \mathcal{R}_\zeta$ and $C \in \mathcal{C}_\zeta$ if and only if no two entries in the same row of ζ lie in the same column of $\sigma(\zeta)$.*

Theorem 2.3. *Let τ be a $(\lambda^1, \dots, \lambda^m)$ -LR tableau. Then in ey_τ the identity id occurs with coefficient equal to $|\mathcal{R}_\tau \cap \mathcal{R}| |\mathcal{C}_\tau \cap \mathcal{C}|$.*

Proof. Write $y_\tau = y_1 + y_2 + y_3$ where

- (i) y_1 is the sum of all $\rho\gamma$ such that $\rho \in \mathcal{R}_\tau \cap \mathcal{R}$ and $\gamma \in \mathcal{C}_\tau \cap \mathcal{C}$.
- (ii) y_2 is the sum of all $\rho\gamma$ such that $\rho\gamma(I_j) = I_j$ for all $j = 1, \dots, m$, but either $\rho \notin \mathcal{R}$ or $\gamma \notin \mathcal{C}$.
- (iii) y_3 is the sum of all $\rho\gamma$ such that $\rho\gamma(I_j) \not\subseteq I_j$ for some j .

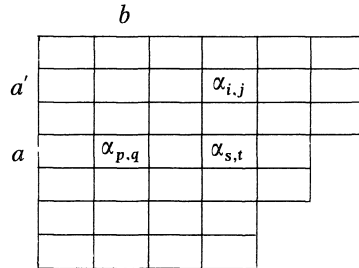
We consider the products ey_i . Each summand of ey_i is, up to sign, a term of the form $gf\rho\gamma$, where $g \in \mathcal{C}$, $f \in \mathcal{R}$, $\rho \in \mathcal{R}_\tau$, and $\gamma \in \mathcal{C}_\tau$. Observe that the identity id cannot occur in the expression for ey_3 . Indeed, each summand of ey_3 is $\pm gf\rho\gamma$, where for some I_j there exists an $x \in I_j$ with $\rho\gamma(x) \in I_k \neq I_j$. But then $gf\rho\gamma(x) \in I_k$ and so $gf\rho\gamma \neq id$. Next consider ey_1 and suppose that $sgn(g)sgn(\gamma)gf\rho\gamma = \pm id$ for $g \in \mathcal{C}$, $f \in \mathcal{R}$. Since $\rho \in \mathcal{R}_\tau \cap \mathcal{R}$ and $\gamma \in \mathcal{C}_\tau \cap \mathcal{C}$, this implies $f\rho = g^{-1}\gamma^{-1} \in \mathcal{R} \cap \mathcal{C} = \{id\}$. Thus $\rho = f^{-1} \in \mathcal{R}$, $\gamma = g^{-1} \in \mathcal{C}$, and $sgn(g) = sgn(\gamma)$. Consequently, id occurs in ey_1 with coefficient equal to $|\mathcal{R}_\tau \cap \mathcal{R}| |\mathcal{C}_\tau \cap \mathcal{C}|$. The proof of the theorem will be complete once we show that

no summand of $e\gamma_2$ is a multiple of $i\mathcal{d}$.

Suppose then that $gf\rho\gamma=i\mathcal{d}$ where $\rho\gamma(I_j)=I_j$ for all j , and either $\rho\notin\mathcal{R}$ or $\gamma\notin\mathcal{C}$. By Lemma 2.1 $\rho(I_j)=I_j=\gamma(I_j)$ for all j , so that all the permutations involved preserve the intervals I_j . Take k minimal with the property that either $\rho|_{I_k}\notin\mathcal{R}_k$ or $\gamma|_{I_k}\notin\mathcal{C}_k$. We show that it is impossible for $gf\rho\gamma|_{I_k}=i\mathcal{d}|_{I_k}$. Thus we focus on the skew tableau τ^k/τ^{k-1} . We assume that the entry in row i and column j of ζ^k is $\alpha_{i,j}$. Since τ^k is obtained from τ^{k-1} by taking the slant labelling of a component θ^k of $v^{k-1}\otimes\lambda^k$, we may assume that the entries of the skew tableau τ^k/τ^{k-1} are also the $\alpha_{i,j}$'s.

First assume that there exists $(\alpha_{h,j} \alpha_{i,j})\in\mathcal{C}_k$ such that $\gamma(\alpha_{h,j})=\alpha_{i,j}$. Then $\alpha_{h,j}$ and $\alpha_{i,j}$ lie in the same column of τ , and multiplying both sides of $\rho\gamma=f^{-1}g^{-1}$ by $(\alpha_{h,j} \alpha_{i,j})$ yields $\rho\gamma'=f^{-1}g'^{-1}$ where $\gamma'=\gamma(\alpha_{h,j} \alpha_{i,j})\in\mathcal{C}_\tau$ and $g'^{-1}=g^{-1}(\alpha_{h,j} \alpha_{i,j})\in\mathcal{C}$. By replacing γ by γ' if necessary and g by g' we may assume that no such transpositions exist. Similarly, by making suitable replacements we may assume that there does not exist a transposition $(\alpha_{i,h} \alpha_{i,j})$ in \mathcal{R}_k with $\rho(\alpha_{i,h})=\alpha_{i,j}$.

Since either $\rho|_{I_k}\notin\mathcal{R}_k$ or $\gamma|_{I_k}\notin\mathcal{C}_k$, the permutation $\rho\gamma$ is not the identity permutation when applied to ζ^k . Take the bottommost row of ζ^k , (say row p) on which $\rho\gamma$ does not act as the identity. From all the $\alpha_{p,r}$ with $\rho\gamma(\alpha_{p,r})\neq\alpha_{p,r}$, choose the one, say $\alpha_{p,q}$, which occurs leftmost in τ^k/τ^{k-1} . Suppose that $\rho\gamma(\alpha_{i,j})=\alpha_{p,q}$. Then $i\leq p$ by maximality of p . We consider the positions of $\alpha_{i,j}$ and $\alpha_{p,q}$ in τ . Suppose that $\alpha_{p,q}$ is in row a and column b of τ . If $\alpha_{i,j}$ is in a column to the right of column b , then necessarily it occupies some row a' with $a'\leq a$ by the semistandard property. Then γ moves $\alpha_{i,j}$ to some entry $\alpha_{s,t}$ in row a to the right of $\alpha_{p,q}$, as pictured below,

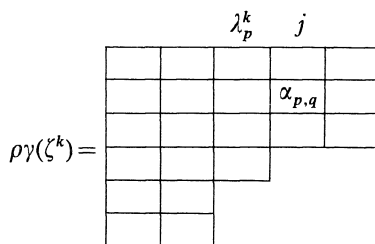


and $s\geq p\geq i$ must hold by semistandardness. Now if $s=p$, then $\rho(\alpha_{p,t})=\alpha_{p,q}$ to contradict the reductions made above. Hence, $s>p\geq i$ and $\rho\gamma(\alpha_{s,t})=\alpha_{s,t}$ holds by maximality. It is impossible for γ to displace $\alpha_{s,t}$ to a different

row of τ , because then ρ would not bring it back to $\alpha_{s,t}$. Thus γ must fix $\alpha_{s,t}$ and so must ρ . This contradicts the fact that $\rho(\alpha_{s,t}) = \alpha_{p,q}$.

Thus, we may assume that $\alpha_{i,j}$ lies in column b' where $b' \leq b$. Consider first the case that column j of ζ^k has length at least p so that $\alpha_{p,j}$ exists. We claim that $\alpha_{p,j}$ must lie in column c where $c < b$. Indeed, if $b' = b$, then by semistandardness no other entry of the form $\alpha_{p,q'}$ for $q' \neq q$ lies in column b , so that $i < p$. Then $\alpha_{p,j}$, which is to the left and below $\alpha_{i,j}$ because of the slant labelling, must be in column c for some $c < b$. Similarly if $b' < b$, then $\alpha_{p,j}$ is in column $c \leq b'$ by the slant labelling. Then $\alpha_{p,j}$ is fixed by $\rho\gamma$ by our choice of q . Thus in the j -th column of $\rho\gamma(\zeta^k)$ both $\alpha_{p,j}$ and $\alpha_{p,q} = \rho\gamma(\alpha_{i,j})$ occur. By Lemma 2.2 $\rho\gamma$ cannot equal $f^{-1}g^{-1}$ and have that property.

Thus, we may assume that column j of ζ^k has length less than p . Then $j > \lambda_p^k$ where λ_p^k is the p -th part of λ^k . In each of the first λ_p^k columns of $\rho\gamma(\zeta^k)$ there must occur a term of the form $\alpha_{p,t}$, because by assumption $\rho\gamma(\zeta^k) = f^{-1}g^{-1}(\zeta^k)$ and $f^{-1}g^{-1}(\zeta^k)$ has that property.



But since $\alpha_{p,q}$ lies in column $j > \lambda_p^k$ one of the first λ_p^k columns must be missing such a term. This contradiction shows that it is impossible under the assumptions placed on the permutations for $\rho\gamma$ to equal $f^{-1}g^{-1}$. Thus, no identity terms occur in the expression for $e\gamma_2$, and the identity terms in $e\gamma_\tau$ come only from $e\gamma_1$ and so $i\mathcal{d}$ has the coefficient claimed. ■

Corollary 2.4. *Let τ be a $(\lambda^1, \dots, \lambda^m)$ -LR tableau. Then $e\gamma_\tau$ is a nonnilpotent element in $\mathbb{C}[S_M]$.*

Proof. Consider the map $L_{e\gamma_\tau}: \mathbb{C}[S_M] \rightarrow \mathbb{C}[S_M]$ which is left multiplication by $e\gamma_\tau$. Relative to the basis $\{\sigma \in S_M\}$ of $\mathbb{C}[S_M]$ the transformation $L_{e\gamma_\tau}$ has a matrix with $\kappa = |\mathcal{R}_\tau \cap \mathcal{R}| \cdot |\mathcal{C}_\tau \cap \mathcal{C}|$ as its diagonal entries by Theorem 2.3. Thus, the trace of $L_{e\gamma_\tau}$ is $\kappa M!$ which is nonzero, so that $L_{e\gamma_\tau}$, and hence $e\gamma_\tau$, cannot be nilpotent. ■

- Proof of Theorem 1.13.* (i) ey_τ is an essential idempotent in $\mathcal{C}[S_M]$ for τ a $(\lambda^1, \dots, \lambda^m)$ -LR tableau of shape $\pi = \{\pi_1 \geq \dots \geq \pi_{\ell(\pi)} > 0\}$ and
(ii) $ey_\tau\beta_\tau$ is a maximal vector of the irreducible module $ey_\tau T = e_\tau T$.

To avoid detailed computations inside $\mathcal{C}[S_M]$ we use the Lie algebra $\mathcal{G} = \mathfrak{gl}(r, \mathcal{C})$ where $r \geq M$ and its action on $T = \bigotimes^M V(\{1\})$ to establish (i). Since e commutes with the action of \mathcal{G} on T , and since $y_\tau T$ is an irreducible \mathcal{G} -submodule of T , the space $ey_\tau T$ is (0) or isomorphic to $y_\tau T \cong V(\pi)$. If $\{v_i\}_{i=1}^r$ denotes the standard basis for $V(\{1\})$ as in statement of Theorem 1.13, then $ey_\tau(v_1 \otimes \dots \otimes v_M)$ is nonzero by Theorem 2.3, as $v_1 \otimes \dots \otimes v_M$ occurs as a summand in it with coefficient equal to $|\mathcal{R}_\tau \cap \mathcal{R}| |\mathcal{C}_\tau \cap \mathcal{C}| \neq 0$. Thus, $ey_\tau T \cong V(\pi)$. Likewise since $ey_\tau e$ commutes with \mathcal{G} , we have that $(ey_\tau)^2 T$ is (0) or isomorphic to $V(\pi)$. By Corollary 2.4 $(ey_\tau)^2 = \sum_\sigma c_\sigma \sigma \neq 0$, so that $(ey_\tau)^2(v_1 \otimes \dots \otimes v_M) \neq 0$. Thus, $(0) \neq (ey_\tau)^2 T \subseteq ey_\tau T$, and irreducibility forces $(ey_\tau)^2 T = ey_\tau T$.

Relative to the Cartan subalgebra of diagonal matrices in \mathcal{G} , the simple tensor $\beta = v_{i_1} \otimes \dots \otimes v_{i_M}$ has weight $\mu_1 \varepsilon_1 + \dots + \mu_r \varepsilon_r$ where ε_j denotes the projection of a matrix onto its (j, j) entry and μ_j records the number of subscripts i_t equal to j . (See for example, [BBL], Chap. 3 or [Be].) In particular, the vector β_τ in the statement of Theorem 1.13 has weight $\pi_1 \varepsilon_1 + \dots + \pi_r \varepsilon_r$, (where if $j > \ell(\pi)$, then by convention $\pi_j = 0$). As a shorthand notation we write π for the weight $\pi_1 \varepsilon_1 + \dots + \pi_r \varepsilon_r$, and say β_τ has weight π . Any other simple tensor β of weight π is obtainable from β_τ by applying a permutation σ to β_τ . Since ey_τ commutes with \mathcal{G} , and since $ey_\tau T \cong V(\pi)$, there must exist some β of weight π with $ey_\tau \beta \neq 0$. We may assume $\beta = \sigma(\beta_\tau)$. Now if $\sigma \notin \mathcal{C}_\tau$, then $y_\tau \sigma(\beta_\tau) = 0$. Thus, it must be that $\sigma \in \mathcal{C}_\tau$. But then

$$(2.5) \quad 0 \neq ey_\tau \beta = ey_\tau \sigma(\beta_\tau) = \text{sgn}(\sigma) ey_\tau \beta_\tau.$$

By completely analogous arguments, $(ey_\tau)^2 \beta_\tau \neq 0$. Now in the \mathcal{G} -module $(ey_\tau)^2 T = ey_\tau T$ there is a unique vector (up to scalar multiple) of weight π , as π is the highest weight of that module. Therefore

$$(2.6) \quad (ey_\tau)^2 \beta_\tau = \xi ey_\tau \beta_\tau$$

for some nonzero scalar $\xi \in \mathcal{C}$. The vector $ey_\tau \beta_\tau$ is a maximal vector for $ey_\tau T$ and so every vector in that module can be achieved by applying an element of the universal enveloping algebra $\mathcal{U} = \mathcal{U}(\mathcal{G})$ to $ey_\tau \beta_\tau$. Therefore for each

$v \in T$, we have $ey_\tau v = uey_\tau \beta_\tau$ for some $u \in \mathcal{U}$. Now if we apply ey_τ to both sides of that relation we see that

$$(2.7) \quad (ey_\tau)^2 v = u(ey_\tau)^2 \beta_\tau = \xi uey_\tau \beta_\tau = \xi ey_\tau v.$$

Consequently $(ey_\tau)^2 - \xi ey_\tau = 0$ on T . If $(ey_\tau)^2 - \xi ey_\tau = \sum_{\sigma \in S_M} d_\sigma \sigma$, then applying $(ey_\tau)^2 - \xi ey_\tau$ to $v_1 \otimes \cdots \otimes v_M$ shows that $d_\sigma = 0$ for all σ , and hence that $(ey_\tau)^2 = \xi ey_\tau$ as desired.

In the process of establishing of Theorem 1.13 (i) we have also shown that $ey_\tau \beta_\tau \neq 0$ is a maximal vector of weight π in $ey_\tau T \cong V(\pi)$ whenever the rank r of \mathcal{G} satisfies $r \geq M$. When $\ell(\pi) \leq r < M$, we can imbed \mathbf{C}^r in \mathbf{C}^M and view \mathcal{G} as imbedded in $g^\ell(M, \mathbf{C})$ in the natural way. Since the factors in the simple tensor β_τ , and also in $ey_\tau \beta_\tau$ are v_j for $j = 1, \dots, \ell(\pi)$, $ey_\tau \beta_\tau$, and hence $ey_\tau T$, must be nonzero when viewed relative to \mathcal{G} . Thus, these additional results hold for all $r \geq \ell(\pi)$, and (ii) of Theorem 1.13 follows. ■

§3. The Linear Independence of the Vectors $ey_\tau \beta_\tau$

In this section we show that the set $\{ey_\tau \beta_\tau\}$, where τ ranges over the $(\lambda^1, \dots, \lambda^m)$ -LR tableaux having no more than r rows, is linearly independent and use that result to complete the proof of Theorem 1.13. Our approach is to impose a certain ordering “ \succ ” on the $(\lambda^1, \dots, \lambda^m)$ -LR tableaux with the property that

$$(3.1) \quad \tau \succ \tau' \Rightarrow ey_\tau ey_{\tau'} = 0.$$

The independence of the set $\{ey_\tau \beta_\tau\}$ will follow from (3.1) and the essential idempotent property of the ey_τ 's. We begin with a useful observation.

Lemma 3.2. *Let τ and τ' be two tableaux (not necessarily standard) associated with partitions of M . If there is some transposition $(a \ b) \in \mathcal{C}_\tau \cap \mathcal{R}_{\tau'}$, then $y_\tau y_{\tau'} = 0$.*

Proof. If γ is any element in the column group of τ , then $y_\tau \gamma = \text{sgn}(\gamma) y_\tau$, and if ρ is any element in the row group of τ' , then $\rho y_{\tau'} = y_{\tau'}$. Thus, if $(a \ b) \in \mathcal{C}_\tau \cap \mathcal{R}_{\tau'}$, then

$$y_\tau y_{\tau'} = y_\tau (a \ b) y_{\tau'} = -y_\tau y_{\tau'}$$

to imply $y_\tau y_{\tau'} = 0$. ■

Suppose now that $\lambda^i \vdash n_i$ for $i=1, \dots, m$, and let τ be a $(\lambda^1, \dots, \lambda^m)$ -LR tableau. Recall that $M_i = n_1 + \dots + n_i$, and the boxes of τ containing the numbers in $\{1, \dots, M_p\}$ form a subtableau τ^p of shape ν^p for ν^p some partition of M_p . The tableau τ^p is obtained from the slant labelling of some component θ^p of $\nu^{p-1} \otimes \lambda^p$ having shape ν^p . (See Definition 1.12.) The order we define on two $(\lambda^1, \dots, \lambda^m)$ -LR tableaux τ, τ' of the same shape searches for the first two subtableaux τ^k and τ'^k which differ and compares the locations of numbers in the corresponding components θ^k and θ'^k . In this procedure, as in the slant labelling process, we assume that each entry i in a component has been changed to $\alpha_{i,*}$.

Definition 3.3. *Let τ and τ' be two $(\lambda^1, \dots, \lambda^m)$ -LR tableaux of the same shape gotten from components θ^j and θ'^j , $j=2, \dots, m$ respectively. Then $\tau > \tau'$ provided that for some k and ℓ the following three conditions hold:*

- (i) $\tau^{k-1} = \tau'^{k-1}$ but $\tau^k \neq \tau'^k$.
- (ii) For $i=1, \dots, \ell-1$, the location of the $\alpha_{i,*}$'s in θ_*^k and θ'^k are the same and ℓ is the largest integer with that property.
- (iii) If the rows of θ_*^k and θ'^k are read from top to bottom, in the topmost row where the number of $\alpha_{\ell,*}$'s differ, θ_*^k has more.

The pair (k, ℓ) is termed the *ordering pair* of τ and τ' .

Since for any $(\lambda^1, \dots, \lambda^m)$ -LR tableau τ , the first subtableau τ^1 is the canonically filled tableau ζ^1 associated to λ^1 , the ordering pair (k, ℓ) necessarily will have $k \geq 2$.

Lemma 3.4. *If τ and τ' are $(\lambda^1, \dots, \lambda^m)$ -LR tableaux of the same shape with $\tau > \tau'$, and if $g \in \mathcal{C} = \mathcal{C}_1 \times \dots \times \mathcal{C}_m$, and $f \in \mathcal{R} = \mathcal{R}_1 \times \dots \times \mathcal{R}_m$, then*

$$y_\tau g f y_{\tau'} = 0.$$

Proof. Assume that the conditions of the lemma are satisfied, and write $g = g_1 \cdots g_m$ with $g_i \in \mathcal{C}_i$ and $f = f_1 \cdots f_m$ with $f_i \in \mathcal{R}_i$. Let (k, ℓ) be the ordering pair for τ and τ' .

Our first observation is that if $\gamma \in \mathcal{C}_\tau \cap \mathcal{C}$, and $\rho \in \mathcal{R}_\tau \cap \mathcal{R}$, then

$$(3.5) \quad y_\tau g f y_{\tau'} = y_\tau \gamma^{-1} \gamma g f \rho \rho^{-1} y_{\tau'} = \text{sgn}(\gamma) y_\tau g' f' y_{\tau'}$$

where $g' = \gamma g \in \mathcal{C}$ and $f' = f \rho \in \mathcal{R}$. Therefore,

$$y_\tau g f y_{\tau'} = \pm y_\tau g' f' y_{\tau'} = \pm g' (g')^{-1} y_\tau g' f' y_{\tau'} (f')^{-1} f' = \pm g' y_{(g')^{-1} \tau} y_{f' \tau'} f'.$$

Thus, $y_\tau g f y_{\tau'} = 0$ if and only if $y_\tau g' f' y_{\tau'} = 0$ if and only if $y_{(g')^{-1} \tau} y_{f' \tau'} = 0$, for $g' = \gamma g$, $f' = f \rho$, $\gamma \in \mathcal{C}_\tau \cap \mathcal{C}$, and $\rho \in \mathcal{R}_{\tau'} \cap \mathcal{R}$. In particular since $\tau^1 = \tau'^1 = \zeta^1$, by taking $\gamma = g_1^{-1} \in \mathcal{C}_\tau \cap \mathcal{C}_1$ and $\rho = f_1^{-1} \in \mathcal{R}_{\tau'} \cap \mathcal{R}_1$ we can assume that $y_\tau g f y_{\tau'} = \pm y_\tau g' f' y_{\tau'}$, where g' and f' act as the identity on $\{1, \dots, M_1\}$.

Suppose then for some p with $2 \leq p \leq k$ that $y_\tau g f y_{\tau'} = \pm y_\tau g' f' y_{\tau'}$, where

$$(3.6) \quad g' \in (\mathcal{C}_\tau \cap \mathcal{C})g \text{ and } g' \text{ restricted to } \{1, \dots, M_{p-1}\} \text{ is the identity;}$$

$$(3.7) \quad f' \in f(\mathcal{R}_{\tau'} \cap \mathcal{R}) \text{ and the location of each value in } \{1, \dots, M_{p-1}\} \text{ is the same in } \tau \text{ as it is in } f'\tau'.$$

As discussed above, elements g' and f' satisfying (3.6) and (3.7) can be found when $p=2$. We assume $p \in \{2, \dots, k\}$ is maximal with those properties. From the collection of all possible such $g' \in (\mathcal{C}_\tau \cap \mathcal{C})g$ satisfying (3.6) pick one such that $|\{a \in I_p | g'(a) \neq a\}|$ is minimal (where I_p is, as before, the interval $\{M_{p-1} + 1, \dots, M_p\}$). We claim that this g' has the property that it does not map any entry of τ coming from I_p to another entry in the same column. Indeed if $g'(a) = b \neq a$ where a, b are in the same column of τ , then $(a \ b) \in \mathcal{C}_\tau \cap \mathcal{C}$. Since $g' \in \mathcal{C}$, and the elements of \mathcal{C} preserve the interval I_p , we have $b \in I_p$. The permutation $(a \ b)g'$ satisfies (3.6) but fixes more elements in I_p , contrary to the minimality assumption on g' . Thus, g' has the desired property.

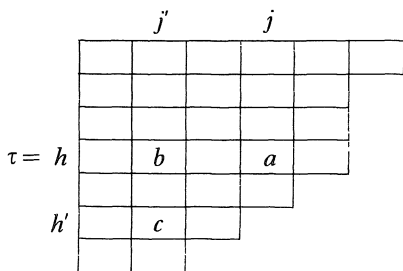
Let \mathcal{E}_s denote the set of entries in row s of the standard tableau ζ^p associated with λ^p . Suppose that t is the topmost row of ζ^p such that the entries in \mathcal{E}_t occupy different positions in τ and $f'\tau'$. If no such row exists, set $t = \ell(\lambda^p) + 1$. Now if $t \leq \ell(\lambda^p)$, and if the i th rows of τ and $f'\tau'$ contain the same subset of elements of \mathcal{E}_t but ordered differently, then there is some $\phi \in \mathcal{R}_{f' \tau'} \cap \mathcal{C}_p$ which permutes just those elements, so that τ and $\phi f'\tau'$ agree on the locations of the values from \mathcal{E}_t in their i th rows. But,

$$\phi \in \mathcal{R}_{f' \tau'} \cap \mathcal{R}_p \Leftrightarrow \phi' = (f')^{-1} \phi f' \in \mathcal{R}_{\tau'} \cap \mathcal{R}_p$$

and hence, $f' \phi' \in f(\mathcal{R}_{\tau'} \cap \mathcal{R})$. We could then replace f' by $f' \phi'$ and assume that τ and $f'\tau'$ agree on the entries from \mathcal{E}_t in their i th rows. Thus, we may assume that for our choice of $f' \in f(\mathcal{R}_{\tau'} \cap \mathcal{R})$ satisfying (3.7) that either τ and $f'\tau'$ agree on all entries from I_p (in which case we set $t = \ell(\lambda^p) + 1$) or else there is a smallest value of t with $t \leq \ell(\lambda^p)$ such that the entries from \mathcal{E}_t

occupy different rows in τ and $f'\tau'$. In the latter case, assume that the elements in \mathcal{E}_i in the first $i-1$ rows of τ and $f'\tau'$ are in precisely the same positions, and i is the largest integer with that property. When $t = \ell(\lambda^p) + 1$ set i equal to one more than the number of rows of τ .

We claim that if the restriction of g' to the elements in $\mathcal{E} = \mathcal{E}_1 \cup \dots \cup \mathcal{E}_i$ in first $i-1$ rows of τ is not the identity map, then $0 = \pm y_i g' f' y_{\nu'} = y_i g f y_{\nu}$, (and the proof of the lemma would be finished in this event). Indeed, suppose that $h \leq i-1$ is the smallest integer such that g' moves an element in \mathcal{E} that resides in row h of τ . Let a be the leftmost such entry so that $g'(a) = c \neq a$. Since $g' \in \mathcal{C}$, a and c are in the same column of ζ^p . Therefore, by the slant labelling of τ , it must be that either c is in the NE corner of a in τ , or a is in the NE corner of c . The first possibility can be eliminated by the minimality of h . Thus, c is in a row strictly below that of a in τ , and since g' does not map a to another entry in the same column of τ , the element c must be in a column strictly to the left of a . Hence, if a is in position (h, j) in τ , then c is in location (h', j') in τ where $h' > h$, and $j' < j$. Let b denote the entry in position (h, j') of τ as pictured below.



Then b occupies location (h, j') in both $(g')^{-1}\tau$ and in $f'\tau'$. Thus, the transposition $(a\ b)$ belongs to both $\mathcal{C}_{(g')^{-1}\tau}$ and $\mathcal{R}_{f'\tau'}$. Lemma 3.2 when applied to the tableaux $(g')^{-1}\tau$ and $f'\tau'$ gives

$$y_{\tau} g' f' y_{\nu'} = g' y_{(g')^{-1}\tau} y_{f'\tau'} f' = 0.$$

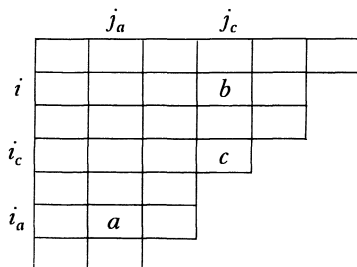
Thus, we can assume that g' restricted to the elements in \mathcal{E} which lie in the first $i-1$ rows of τ is the identity map.

If the locations of all the values in I_p are exactly the same in the two tableaux τ and $f'\tau'$, then it follows from the argument just given that g' is the identity on $\{1, \dots, M_p\}$. We would contradict the maximality of p unless $p = k$. But when $p = k$, the location of all the values in I_p cannot be the

same in τ and $f'\tau'$ by the definition of the ordering pair, so that this situation cannot occur.

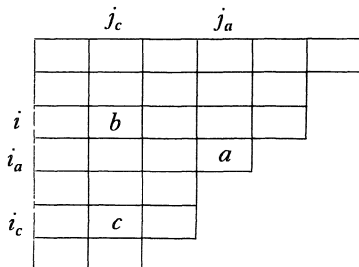
We claim that by similar reasoning we can also assume that g' when restricted to the elements in $\mathcal{E}' = \mathcal{E}_1 \cup \dots \cup \mathcal{E}_{t-1}$ in row i of τ is also the identity map. Suppose that $g'(a) = c \neq a$ for some $a \in \mathcal{E}'$ in row i of τ , and let a be the leftmost such element with those properties. Then c and a lie in the same column of ζ^p . If a is in location (i, j) , then c must be in position (i', j') for some $i' > i$ and $j' < j$ since g' restricted to elements in \mathcal{E}' in the first $i-1$ rows is the identity. The entry b in location (i, j') belongs to $\{1, \dots, M_{p-1}\}$ or to \mathcal{E}' , and so is fixed by g' . Thus, $(a\ b) \in \mathcal{C}_{(g')^{-1}\tau} \cap \mathcal{R}_{f'\tau}$, and consequently, $y_\tau g' f' y_{\tau'} = g' y_{(g')^{-1}\tau} y_{f'\tau} f' = 0$.

By our choice of f' , the i th row of τ and the i th row of $f'\tau'$ contain different subsets of elements from \mathcal{E}_t . Since $\tau > \tau'$, and since f' permutes the entries coming from the $\alpha_{t,*}$'s among themselves, there is an $a \in \mathcal{E}_t$ in position (i, j) say of $f'\tau'$, but not in the i th row of τ . Since τ and $f'\tau'$ agree on the entries in \mathcal{E}_t contained in their first $i-1$ rows, a must be in a lower row in τ . Assume that a is in position (i_a, j_a) of τ . If $j_a \geq j$, then by the semistandardness of the component, the entry d in position (i, j_a) of τ must come from $\{1, \dots, M_{p-1}\} \cup \mathcal{E}'$. However, d would have the same position in $f'\tau'$ by our assumptions, which is impossible. Hence, we can assume that $j_a < j$ and $i_a > i$. Suppose that $g'(a) = c \neq a$. Then since a and c are in the same column of ζ^p , we have by the slant labelling that either c is in the NE corner of a in τ or a is in the NE corner of c . If c is in the NE corner of a in τ , then c comes from labelling some $\alpha_{s,*}$ with $s < t$. Since τ and $f'\tau'$ agree on the entries in \mathcal{E}' , then c must have the same location in τ and $f'\tau'$, call it (i_c, j_c) . Necessarily $i_a > i_c > i$ must hold, since g' fixes the entries of \mathcal{E}' in the first i rows of τ . Let b be the entry of τ in position (i, j_c) as pictured below:



Then $b \in \{1, \dots, M_{p-1}\} \cup \mathcal{E}_1 \cup \dots \cup \mathcal{E}_{s-1}$, and so b has the same location in both

τ and $f'\tau'$. By our reductions in the previous paragraphs $g'(b)=b$. Thus $(a\ b)\in\mathcal{C}_{(g')^{-1}\tau}\cap\mathcal{R}_{f'\tau'}$, and we conclude from Lemma 3.2 that $y_\tau g' f' y_{\tau'} = g' y_{(g')^{-1}\tau} y_{f'\tau'} f' = 0$. Hence, the assertions in the lemma are true in these circumstances. On the other hand, if a is in the NE corner of c , and c is in location (i_c, j_c) , then $i_c > i_a > i$, and $j_c < j_a < j$. Let b be the (i, j_c) -entry of τ as displayed below:



Since the entry in location (i_a, j_c) of τ must belong to $\{1, \dots, M_{p-1}\} \cup \mathcal{E}$, we must have $b \in \{1, \dots, M_{p-1}\} \cup \mathcal{E}'$. Thus, b occupies the same position in $f'\tau'$, and $(a\ b) \in \mathcal{C}_{(g')^{-1}\tau} \cap \mathcal{R}_{f'\tau'}$. As before, $y_\tau g' f' y_{\tau'} = g' y_{(g')^{-1}\tau} y_{f'\tau'} f' = 0$, so the lemma holds in this situation. Finally, we suppose that $g'(a)=a$. The entry b in position (i, j_a) of τ is forced to belong to $\{1, \dots, M_{p-1}\} \cup \mathcal{E}'$, and so b is the (i, j_a) -entry of $f'\tau'$ as well. Then $(a\ b) \in \mathcal{C}_{(g')^{-1}\tau} \cap \mathcal{R}_{f'\tau'}$ once again, and $y_\tau g' f' y_{\tau'} = g' y_{(g')^{-1}\tau} y_{f'\tau'} f' = 0$. In each possible case we have shown that $y_\tau g f y_{\tau'} = \pm y_\tau g' f' y_{\tau'} = 0$ as claimed. ■

Since e is the sum of terms $\pm gf$ where $g \in \mathcal{C}$ and $f \in \mathcal{R}$, Lemma 3.4 allows us to conclude the following:

Corollary 3.8. *If τ and τ' are $(\lambda^1, \dots, \lambda^m)$ -LR tableaux of the same shape with $\tau > \tau'$ then $ey_\tau ey_{\tau'} = 0$.*

Corollary 3.9. *Let \mathcal{L} be the complete set of $(\lambda^1, \dots, \lambda^m)$ -LR tableaux having no more than r rows. Then the set $\{ey_\tau \beta_\tau \mid \tau \in \mathcal{L}\}$ is a linearly independent set.*

Proof. If the tableau τ has shape π , then the vector $ey_\tau \beta_\tau$ has weight $\pi = \pi_1 \varepsilon_1 + \dots + \pi_r \varepsilon_r$ relative to the Cartan subalgebra of \mathcal{G} of diagonal matrices. Since vectors of different weights are linearly independent, it suffices to consider the relation

$$\sum_{\tau'} a_{\tau'} e y_{\tau'} \beta_{\tau'} = 0,$$

where the sum is over all $\tau' \in \mathcal{L}$ of shape π , and the coefficients $a_{\tau'} \in \mathbf{C}$. Suppose τ is the largest tableau relative to the “ \succ ” ordering which appears in this expression with $a_{\tau} \neq 0$. Then applying $e y_{\tau}$ we get

$$0 = e y_{\tau} \left(\sum_{\tau'} a_{\tau'} e y_{\tau'} \beta_{\tau'} \right) = a_{\tau} (e y_{\tau})^2 \beta_{\tau} = \kappa a_{\tau} (e y_{\tau} \beta_{\tau})$$

for some nonzero $\kappa \in \mathbf{C}$ by Corollary 3.8 and (i) of Theorem 1.13. Since $e y_{\tau} \beta_{\tau} \neq 0$ (see Remarks 2.8), $a_{\tau} = 0$, contrary to assumption. Thus, the vectors must be linearly independent. ■

Conclusion of the Proof of Theorem 1.13. We have from Corollary 3.9 that the set $\{e y_{\tau} \beta_{\tau} | \tau \in \mathcal{L}\}$ is linearly independent. We know from Section 2 that each $e y_{\tau} T$ is an irreducible \mathcal{G} -module. Hence, the sum $\sum_{\tau \in \mathcal{L}} e y_{\tau} T \subseteq e T$ is direct. Since the number of summands is that given by the Littlewood-Richardson rule, $\bigoplus_{\tau \in \mathcal{L}} e y_{\tau} T = e T$, to give (iii) of Theorem 1.13. This completes the proof. ■

References

- [Be] Benkart, G., Partitions, tableaux, and stability in the representation theory of classical Lie algebras, *Lie Theory, Differential Equations and Representation Theory*, Proc. of the Annual Seminar of the Canad. Math. Soc., V. Hussin, ed. Centre de Recherches Mathématiques, (1990) 47–76.
- [BBL] Benkart, G.M., Britten, D.J., and Lemire, F.W., Stability in Modules for Classical Lie Algebras—a Constructive Approach, *Mem. Amer. Math. Soc.*, **430** (1990).
- [Bo] Boerner, H., *Representations of Groups*, North Holland Amsterdam, 1970.
- [C] Curtis, C.W., On centralizer rings and characters of representations of finite groups, *Math. Zeit.*, **107** (1968), 402–406.
- [J] James, G.D., *The Representation Theory of the Symmetric Groups*, Lecture Notes in Math., Springer-Verlag, Berlin, Heidelberg, and New York, **682** (1978).
- [K] Kilmoyer, R.W., *Some Irreducible Complex Representations of a Finite Group with BN Pair*, Ph.D. Thesis, M.I.T., 1969.
- [Ma] Macdonald, I.G., *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.
- [Mi] Miller, Jr., W., *Symmetry Groups and their Applications*, Pure and Appl. Math., **50**, (1972).
- [S] Schützenberger, M.-P., La correspondance de Robinson, *Combinatoire et Représentation du Groupe Symétrique*, Strasbourg 1976, *Springer Lect. Notes in Math.*, **579** (1977), 59–113.

- [T] Thomas, G.P., On Schensted's construction and the multiplication of Schur functions, *Adv. Math.*, **30** (1978) 8–32.
- [W] Weyl, H., *Classical Groups, Their Invariants and Representations*, 2nd ed., Princeton Mathematical Series **1**, Princeton University Press, Princeton, 1946.