Publ. RIMS, Kyoto Univ.
28 (1992), 1063–1073

On Orthogonality of Two Subfactors Constructed from Factor Maps

By

Takashi SANO*

Abstract

Two subfactors constructed from two factor maps are considered. A characterization is given for the angles between two subfactors to become trivial.

§ 1. Introduction

Motivated by the Jones Index theory ([4]), in [6] Watatani and the author introduced the notion of angles between two subfactors. On the other hand, subfactors constructed via the ergodic theory were studied in [2], [3]. In this paper we consider two (finite to one) factor maps of a (single) ergodic transformation and investigate relative positions between the resulting subfactors. We at first find a condition guaranteeing that the intersection of two subfactors becomes a subfactor of finite index. We then determine when the set of the angles is trivial (i.e., this set reduces to the singleton $\{\frac{\pi}{2}\}$. This is equivalent to the "commuting square" condition, see [6].) Our characterization is given in terms of two kinds of fibres obtained from factor maps.

After collecting basic definition in §2, we state and prove our main results in §3. Some examples (based on sofic systems) are explained in §4.

Acknowledgement

The author would like to express his sincere gratitude to Professor Toshihiro Hamachi for his helpful suggestions and constant encouragement

---

Commuticated by H. Araki, April 20, 1992.
1991 Mathematics Subject Classification: 46L37
* Department of Mathematics, Faculty of Science, Kyushu University, Fukuoka 812, Japan

and to Professor Hideki Kosaki for fruitful discussions and constant encouragement. This work had been completed while he was visiting RIMS as a research visitor. He acknowledges with gratitude the support and the hospitality from RIMS.

§2. Preliminaries

2.1. Let $\mathscr{H}$ be a Hilbert space and $\mathscr{M}$, $\mathscr{N}$ two subspaces with orthogonal projections p and q. When p and q do not commute, there exist a Hilbert space $\mathscr{K}$ and positive contractions s and c on $\mathscr{K}$ with null kernels and $s^2+c^2=1$ such that the two projections p and q are unitarily equivalent to

$$I_{(p\wedge q)\mathscr{H}} \oplus \begin{pmatrix} I_{\mathscr{K}} & 0 \\ 0 & 0 \end{pmatrix} \oplus I_{(p\wedge q^{\perp})\mathscr{H}} \oplus 0_{(p^{\perp}\wedge q)\mathscr{H}} \oplus 0,$$

$$I_{(p\wedge q)\mathscr{H}} \oplus \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \oplus 0_{(p\wedge q^{\perp})\mathscr{H}} \oplus I_{(p^{\perp}\wedge q)\mathscr{H}} \oplus 0,$$

respectively. Let Θ be the operator uniquely determined by $c=\cos\Theta$ and $s=\sin\Theta$ $(0\leq\Theta\leq\frac{\pi}{2})$.

Definition 1 ([6]). The angles between two subspaces denoted by $Ang(p, q)$ is the set in $[0, \frac{\pi}{2}]$ defined as

$$Ang(p, q)=\begin{cases} sp\Theta, & \text{if } pq\neq qp, \\ \{\frac{\pi}{2}\}, & \text{if } pq=qp. \end{cases}$$

2.2. Let us consider a finite factor L containing two subfactors M and N. Let ξ be a trace vector in $L^2(L)$ and we consider the two subspaces $\overline{M\xi}$ and $\overline{N\xi}$. The corresponding projections e_M and e_N are related to the conditional expectations E_M and E_N ([4]) via

$$e_M(x\xi)=E_M(x)\xi,\ e_N(x\xi)=E_N(x)\xi, \quad \text{for } x\in L.$$

Definition 2. The angles between two subfactors denoted by $Ang(M, N)$ is defined as $Ang(e_M, e_N)$.

(See [6] for related results.)

2.3. An (ergodic) dynamical system $(X, \mathscr{F}, \mu, T)$ consists of a probability Lebesgue space $(X, \mathscr{F}, \mu)$ and a nonsingular (ergodic) transformation T. A dynamical system $(Y, \mathscr{G}, \nu, S)$ is a factor of $(X, \mathscr{F}, \mu, T)$ if there exists a measurable (projection) map $\varphi: X \to Y$ such that $\mu \circ \varphi^{-1} = \nu$ and $\varphi \circ T = S \circ \varphi$. Then there exists a (unique) disintegration of $(X, \mathscr{F}, \mu)$ over $(Y, \mathscr{G}, \nu)$, that is, there exists a family of measures $\{m_y\}_{y \in Y}$ satisfying

$$m_y(\varphi^{-1}(y)) = 1 \text{ for } y \in Y,$$

for each $B \in \mathscr{F}$, the map $y \mapsto m_y(\varphi^{-1}(y) \cap B)$ is ($\mathscr{G}$-)measurable,

$$\mu(\cdot) = \int_Y m_y(\cdot) d\nu(y).$$

The existence of a finite invariant measure is not necessary in what follows. But for simplicity in the rest we assume the existence of such a measure. It is also possible to deal with a countable ergodic equivalence relation ([1]) instead of an ergodic transformation. However such generalizations are straightforward and we will deal with just the above mentioned case.

§3. Main Theorem

Let $(Y_i, \mathscr{F}_i, \mu_i, S_i)$ $(i = 1, 2)$ be a factor of an ergodic dynamical system $(X, \mathscr{F}, \mu_X, T)$ with the factor map π_i. We assume that π_i is a finite $(= m_i)$ to one map and μ_X is an invariant measure for T. We further assume the existence of a natural number m satisfying

$$(1) \qquad (\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)^m(x) = (\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)^{m+1}(x) \ \ a.e. x \in X.$$

(The meaning of this condition in subfactor set-up will be clarified later.)

Proposition 3. *The partition* $\xi = \{(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)^m(x);\, x \in X\}$ *is measurable under the above assumption* (1).

A partition ξ_0 is said to be measurable when there exist countable measurable sets $\{A_n\}_{n \in N}$ (called a basis for ξ_0) such that $\xi_0 = \{\bigcap_{n=1}^{\infty} B_n;\, B_n = A_n, A_n^c\}$. The proposition guarantees that the quotient space of X by ξ is Lebesgue (see [5] for instance).

Proof. Let $\{D_n\}_n$ be a basis for the Lebesgue space Y_1. Set

$B_1=Y_1$, $B_{2n+1}=D_n$ and $B_{2n}=D_n^c(n\geq 1)$. The assumption implies the invariance of the $(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)^m(x)$ under $\pi_1^{-1}\pi_1$ and $\pi_2^{-1}\pi_2$ and the existence of a natural number p satisfying $|\pi_1(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)^m(x)|\leq p$. Let us consider a family of countable measurable sets $\{A_{(n_1,n_2,\cdots,n_p)};\ (n_1,\ n_2,\cdots,\ n_p)\in \mathrm{N}^p\}\subseteq Y_1$ defined by

$$A_{(n_1,n_2,\cdots,n_p)}=B_{n_1}\cap B_{n_2}\cap\cdots\cap B_{n_p}.$$

We will show that $\{(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)^m(\pi_1^{-1}(A_{(n_1,n_2,\cdots,n_p)}));\ (n_1,\ n_2,\cdots,\ n_p)\in \mathrm{N}^p\}$ is a basis for the partition ξ. Remark that $(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)^m(\pi_1^{-1}(A_{(n_1,n_2,\cdots,n_p)}))$ is measurable because π_1 and π_2 are finite to one ([5]). Take an element $C(=(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)^m(x))\in\xi$ and consider a disjoint one C' with C. And let us write that $\pi_1(C')=\{y_1,\cdots,y_q\}$ $(q\leq p)$. Since the basis $\{D_n\}$ separates two distinct points, we can choose a set of natural numbers $\{k_j;\ 1\leq j\leq p\}$ such that $\pi_1(x)\in B_{k_j}$, $y_j\notin B_{k_j}$ $(1\leq j\leq q)$. (For $j(q+1\leq j\leq p)$, $k_j=1$.) Then $\{(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)^m(\pi_1^{-1}(A_{(k_1,k_2,\cdots,k_p)}))\}$ contains the element C but does not include C'. □

Lemma 4. *The function*

$$x\mapsto|\pi_1^{-1}\pi_1(x)\cap\pi_2^{-1}\pi_2(x)|\ (=:c(x))$$

is measurable and hence constant by the ergodicity.

Proof. We may assume $X\cong Y_i\times\{1,\ 2,\ \cdots,\ m_i\}$ $(i=1,\ 2)$. Define the map φ_i by $\varphi_i(y,\ s)=(y,\ s+1)$ (mod m_i in $\{1,\ 2,\ \cdots,\ m_i\}$). Consider the measurable sets

$$M_k=\{x\in X;\ (\varphi_1)^k(x)=(\varphi_2)^l(x)\ \text{for some}\ l\}.$$

Since

$$\{x;\ c(x)\geq m\}=\bigcup_{0=n_0<n_1\cdots<n_{m-1}}\ \bigcap_{k=n_0,n_1,\cdots,n_{m-1}} M_k,$$

$c(x)$ is a measurable function. □

We call this constant the crossing number (denoted by $c(\pi_1,\ \pi_2)$ for π_1 and π_2). The discussion so far allows us to give another factor system $(X/\xi,\ \mathscr{F}_3,\ \mu_3,\ S_3)$ with the finite $(=m_3)$ to one factor map π_3 over the Lebesgue space X/ξ.

Fig. 1

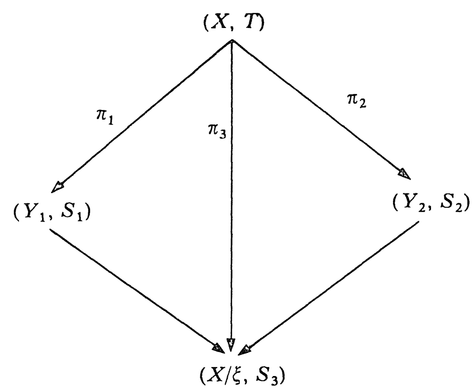

Fig. 1

Let M be the Krieger factor corresponding to $(X, \mathscr{F}, \mu_x, T)$. As in [3], the above three factor maps give rise to the three subfactors in M. For the readers convenience we briefly recall the construction described in [1], [2], and [3]:

Let $\mathscr{R}_T$ be the (countable) ergodic equivalence relation (on X) generated by T:

$$x \sim y \text{ if } y = T^n x \text{ for some } n \in \mathbf{Z}.$$

The left counting measure $(\mu_X)_l$ on $\mathscr{R}_T(d(\mu_X)_l(u, v) = d(\mu_X)(v))$ gives us the Hilbert space

$$\mathscr{H} = L^2(\mathscr{R}_T;\ (\mu_X)_l) = \{\xi;\ \text{a measurable function on } \mathscr{R}_T \text{ and}$$

$$\|\xi\|^2 = \int_{\mathscr{R}_T} |\xi(u, v)|^2 d(\mu_X)_l = \int_X \sum_{u \sim v} |\xi(u, v)|^2 d(\mu_X)(v) < \infty\}.$$

The factor $M = W^*(\mathscr{R}_T)$ is generated by the convolution operators L_f of "nice" functions f on $\mathscr{R}_T$ such as $(L_f\xi)(u, v) = \sum_{w \sim v} f(u, w)\xi(w, v)$.

Let N_1, N_2, and N_3 be the von Neumann subalgebras on $\mathscr{H}$ defined by

$$N_i = \{L_f \in M;\ f(u, v) = f(u', v') \text{ for } \pi_i(u) = \pi_i(u'),\ \pi_i(v) = \pi_i(v')\}.$$

(Note that $[M: N_i] = m_i$ (i=1, 2) ([3]).) From the construction, we have $N_1 \cap N_2 = N_3$. Actually N_i is a factor since it is isomorphic to $W^*(\mathscr{R}_{S_i})$

(see [3] for details). Remark that our previous assumption (1) is equivalent to that the index $[M: N_3]$ is finite $(=m_3)$. The corresponding orthogonal projection e_{N_i} is given by

$$(e_{N_i}\xi)(u, v)=\frac{1}{m_i}\sum_{\substack{u'\sim v'\\ \pi_i(u)=\pi_i(u')\\ \pi_i(v)=\pi_i(v')}}\xi(u', v').$$

Similarly the measurable partition $\{\pi_1^{-1}\pi_1(x)\cap\pi_2^{-1}\pi_2(x);\ x\in X\}$ gives us the subfactor N_0 containing N_1 and N_2. By Lemma 3.2. (2) ([6]), we have $Ang_M(N_1, N_2)=Ang_{N_0}(N_1, N_2)$. Hence we may restrict our attention to the case $c(\pi_1, \pi_2)=1$ by starting with N_0 instead of M.

Definition 5. A pair of factor maps π_1 and π_2 is said to be exclusive if there exists a measurable set A with $\mu_X(A)>0$ satisfying: For any $x\in A$, we have $x_1, x_2\in X$ such that $\pi_1(x)=\pi_1(x_1)$, $\pi_2(x)=\pi_2(x_2)$ and $\pi_1^{-1}\pi_1(x_2)\cap \pi_2^{-1}\pi_2(x_1)=\phi$.

We are ready to state our main theorem.

Theorem 6. *Let M be the factor and N_1 and N_2 the subfactors constructed above. Then the following are equivalent.*

(1) $e_{N_1}e_{N_2}=e_{N_2}e_{N_1}$ *(i.e., $Ang(N_1, N_2)=\{\frac{\pi}{2}\}$).*

(2) *The pair of π_1 and π_2 is not exclusive.*

(3) $(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)(x)=(\pi_2^{-1}\pi_2\pi_1^{-1}\pi_1)(x)$ *a.e.x.*

Proof. As was pointed out above, we may and do assume $c(\pi_1, \pi_2)=1$.

(2)⇒(3); The assumption means that $\pi_1^{-1}\pi_1(x_2)\cap\pi_2^{-1}\pi_2(x_1)\neq\phi$ *a.e.x* for any x_1 and x_2 satisfying $\pi_1(x)=\pi_1(x_1)$ and $\pi_2(x)=\pi_2(x_2)$. Hence we have (3).

(3)⇒(1); Obvious.

(3)⇒(2); Assume that the pair of π_1 and π_2 is exclusive. Then there exists a measurable set A such that for any $x\in A$, we have $x_1, x_2\in X$ such that $\pi_1(x)=\pi_1(x_1)$, $\pi_2(x)=\pi_2(x_2)$ and $\pi_1^{-1}\pi_1(x_2)\cap\pi_2^{-1}\pi_2(x_1)=\phi$. Hence, there exists at least one point in $\pi_2^{-1}\pi_2(x_1)$ which is never contained in $(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)(x)$.

(1)⇒(3); Suppose that there exists a measurable set A with $\mu_X(A)>0$

satisfying $(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)(x)\neq(\pi_2^{-1}\pi_2\pi_1^{-1}\pi_1)(x)$ for $x\in A$. Choose a cross section $A_0(\subseteq A)(\mu_X(A_0)>0)$ over the factor system $(X/\xi,\ \mu_3)$ with the factor map π_3 ([5]). Set $B=(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)(A_0)/(\pi_2^{-1}\pi_2\pi_1^{-1}\pi_1)(A_0)=\bigcup_{x\in A_0}(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)(x)/(\pi_2^{-1}\pi_2\pi_1^{-1}\pi_1)(x)$. Then we have

$$\begin{aligned}\mu_X(B)&=\int_{X/\xi}\chi_{\pi_3(A_0)}(z)m_z(B)d\mu_3(z)\\&=\int_{X/\xi}\chi_{\pi_3(A_0)}(z)\sum_{y\in C(z)}m_z(\{y\})d\mu_3(z)\\&\geq\frac{1}{m_3}\mu_3(\pi_3(A_0))>0,\end{aligned}$$

where $C(z)$ is the set $(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)(A_0\cap\pi_3^{-1}(z))/(\pi_2^{-1}\pi_2\pi_1^{-1}\pi_1)(A_0\cap\pi_3^{-1}(z))$. Let ξ be the function $\xi(u,\ v)=\chi_D(u,\ v)\chi_B(u)$, where $D=\{(u,\ u);\ u\in X\}$. For any $u\in A_0$, we have

$$(e_{N_1}e_{N_2}\xi)(u,\ u)=\sum_{v\in(\pi_2^{-1}\pi_2\pi_1^{-1}\pi_1)(u)}\xi(v,\ v)=0,$$

$$(e_{N_2}e_{N_1}\xi)(u,\ u)=\sum_{v\in(\pi_1^{-1}\pi_1\pi_2^{-1}\pi_2)(u)}\xi(v,\ v)>0.$$

Therefore, $e_{N_1}e_{N_2}\neq e_{N_2}e_{N_1}$. □

§4. Examples

A labeled graph means a directed graph $\Gamma=(V,\ A,\ i,\ t)$ together with label maps λ: $A\to\mathscr{A}$ and μ: $V\to\mathscr{B}$, where V (resp. A) is the set of all vertices (resp. arcs). Here $\mathscr{A}$, $\mathscr{B}$ are sets of labels and i, t are initial and terminal vertex maps (see the example presented below).

We will construct two factor maps from two labeled graphs Γ_1 and Γ_2 with the same directed graph $(V,\ A,\ i,\ t)$ but different label maps $(\lambda,\ \mu_1)$ and $(\lambda,\ \mu_2)$; λ: $A\to\mathscr{A}$, μ_j: $V\to\mathscr{B}_j$ $(j=1,\ 2)$. Set $X=\{(x_n)_n;\ x_n\in A,\ t(x_n)=i(x_{n+1})\}$, $Y_j=\{((y_n)_n,\ \beta)$; there exists a $(x_n)_n\in X$ satisfying $y_n=\lambda(x_n)$ and $\beta=\mu_j(i(x_0))\}$ and define the shifts T: $X\to X$ and S_j: $Y_j\to Y_j$ by

$$T((x_n)_n)=(x_{n+1})_n,$$
$$S_j((\lambda(x_n))_n,\ \mu_j(i(x_0)))=((\lambda(x_{n+1}))_n,\ \mu_j(i(x_1))).$$

In this setting the map π_j: $X \to Y_j(\pi_j((x_n)_n) = ((\lambda(x_n))_n, \mu_j(i(x_0)))$ is a factor map of (X, T) over (Y_j, S_j). Set $Y = \{(y_n)_n$; there exists a $(x_n)_n \in X$ *satisfying* $y_n = \lambda(x_n)\}$, $S((y_n)_n) = (y_{n+1})_n$, and $\pi((x_n)_n) = (\lambda(x_n))_n$. Notice that (Y, S) is a common factor of (Y_j, S_j).

To construct concrete Γ_1 and Γ_2, we now consider a finite group $G(=\{g_1, g_2, \cdots, g_n\})$ with $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_m\}$ as generators. Set $V = \{1, 2, \cdots, n\}$. Two vertices i, j are connected by a (unique) oriented arc α if $\gamma_k g_i = g_j$ for some (unique) $\gamma_k \in \Gamma$. We define the label map λ: $A = \{\alpha\text{'s}\} \to \mathscr{A} = \Gamma$ by $\lambda(\alpha) = \gamma_k$. To construct two different labeled graphs (two different μ_j's), we choose two subgroups H_1 and H_2. Choose a right coset representative $\mathscr{B}_j = \{g_\alpha^j\}$; $G = \cup_\alpha g_\alpha^j H_j$ $(j=1,2)$. We define the label map μ_j: $V \to \mathscr{B}_j$ as follows; $\mu_j(k) = g_\alpha^j$ if $g_k H_j = g_\alpha^j H_j$. For example, let G be the symmetric group S_3 on $\{1, 2, 3\}$ with the subgroup $H_1 = S_2$ on $\{1, 2\}$. Set $g_1 = e$, $g_2 = (12) = \gamma_1$, $g_3 = (23) = \gamma_2$, $g_4 = (13)$, $g_5 = (123)$, $g_6 = (132)$ $(H_1 = \{g_1, g_2\})$. The corresponding directed graph is as follows;

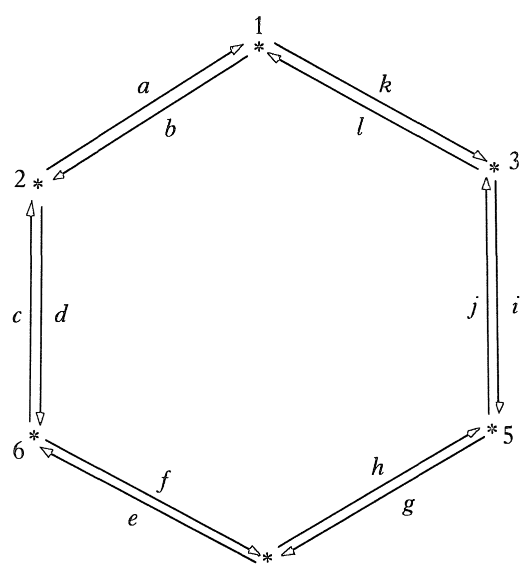

Fig. 2

And the labeled graph Γ_1 is drawn simply by Figure 3. Similarly the different subgroup $H_2 = \{g_1, g_3\}$ gives rise to the labeled graph Γ_2 described by Figure 4;

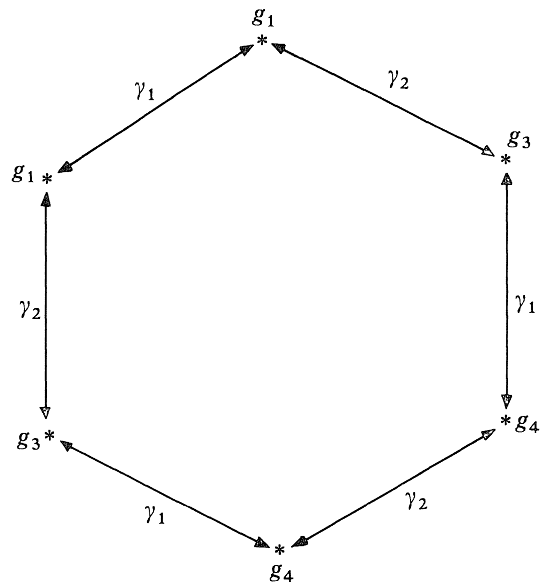

Fig. 3

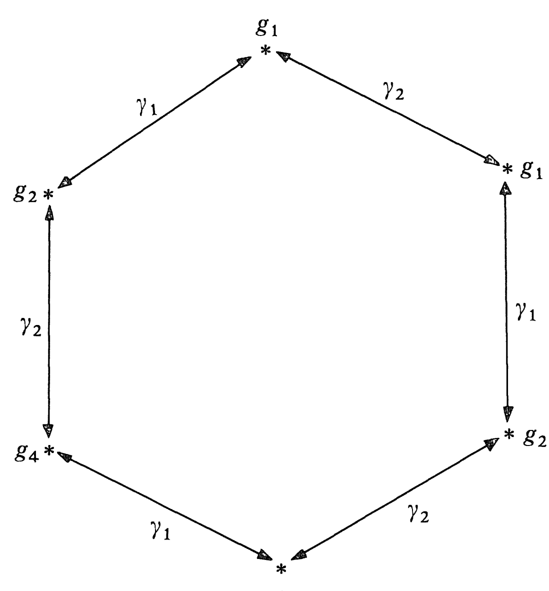

Fig. 4

We have constructed two labeled graphs with (λ, μ_1) and (λ, μ_2) from two subgroups, and as was explained earlier we have (X, T), (Y_1, S_1), $(Y_2,$

S_2), and (Y, S). (X, T) (resp. (Y, S)) have the unique ergodic probability measure $\mu_X(resp.\mu_Y)$ with maximal entropy $(\mu_Y=\mu_X\circ\pi_3^{-1})$ (see [3] for details). Hence, we have two factor systems (Y_1, μ_1, S_1) and (Y_2, μ_2, S_2) of the ergodic dynamical system (X, μ_X, T) (μ_1 and μ_2 are the image measures of μ_X). Following the same procedure explained in the previous section, we have two subfactors. Here, we apply the theorem to, for instance, the pair of the two subfactors N_1 and N_2 constructed from the subgroups $H_1=\{g_1, g_2\}$, $H_2=\{g_1, g_3\}(\subseteq G)$. It is straightforward to show that the conditions (2) and (3) of the theorem do not hold. Therefore, $Ang(N_1, N_2)\neq\{\frac{\pi}{2}\}$.

Remark 7. For simplicity, let us assume that $G=H_1\vee H_2$ and $H_1\cap H_2=\{e\}$. (The latter is equivalent to $c(\pi_1, \pi_2)=1$.) Then we may identify X with $Y\times G$ and T, π_j look like

$$T((y_n)_n, g)=((y_{n+1})_n, y_0g),$$

$$\pi_j(y, g)=(y, gH_j)\in Y\times G/H_j\cong Y\times\mathscr{B}_j.$$

If we consider the automorphisms $\{\alpha_g\}_{g\in G}\subseteq N[\mathscr{R}_T]$, the normalizers, on $Y\times G$ defined by

$$\alpha_g(y, h)=(y, hg^{-1}),$$

then they induce an outer action of G on $M=W^*(\mathscr{R}_T)$. It is easy to check that the fixed point algebra by H_i is precisely $N_i(i=1,2)$. Therefore the quadrilateral (M, N_1, N_2, N_3) can be identified with $(M, M^{H_1}, M^{H_2}, M^G)$. Then, by Proposition 7.9 ([6]), we have the trivial angle iff $|G|=|H_1|\times|H_2|$. Notice that this condition precisely means that two factor maps are not exclusive (because of the crossing number$=1$).

More generally, $Ang(N_1, N_2)$ might be calculated by looking at how two kinds of fibres meet. This seems to deserve further investigation.

References

[1] Feldman, J. and Moore, C.C., Ergodic equivalence relations, cohomology and von Neumann algebras I and II, *Trans. Amer. Math. Soc.*, **234** (1977), 289–324, 325–359.

[2] Feldman, J., Sutherland, C. and Zimmer, R., Subrelations of ergodic equivalence relations, *Ergod. Th. Dynam. Sys.*, **9** (1989), 239–269.

[3] Hamachi, T. and Kosaki, H., Orbital factor map, *preprint*.

[4] Jones, V., Index for subfactors, *Invent. Math.*, **72** (1983), 1–25.

[5] Rohlin, V.A., On the fundamental ideas of measure theory, *Amer. Math. Soc. Transl. Series* 1, **10** (1952), 1–54.
[6] Sano, T. and Watatani, Y., Angles between two subfactors, *preprint*.