

An Inclusion of Type III Factors with Index 4 Arising from an Automorphism

By

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Abstract

We consider an inclusion of type III factors with index 4 arising from an automorphism. It has the extended Coxeter-Dynkin diagram \tilde{A} or $A_{\infty, \infty}$ as the principal graph. We show the equivalence between classification of automorphisms and that of the corresponding subfactors, and compute conjugacy invariants for the subfactors. As an application, we show the existence of a pair of type III₁ factors which does not split into a type II₁ inclusion.

§0. Introduction

We shall consider an inclusion of type III factors with index 4 arising from an automorphism. More precisely, we consider the following inclusion:

$$M = P \otimes M_2(\mathbf{C}) \supset N = \left\{ \begin{bmatrix} x & \\ & \alpha(x) \end{bmatrix}; x \in P \right\},$$

where P is a type III factor and α is an automorphism of P . The main purposes in this article are to show the relation between classification of automorphisms and that of the corresponding subfactors, and to study the correspondence between the four outer conjugacy invariants of the automorphism and conjugacy invariants of the subfactor. As an application, we shall show the existence of an inclusion of type III₁ factors which does not split into a type II₁ inclusion.

In §1, we shall consider the relation between the outer conjugacy of an automorphism and the conjugacy of the corresponding subfactor, and show the “equivalence” of them. In §2, we shall show how the four outer

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conjugacy invariants of an automorphism appear as conjugacy invariants of the subfactor. In the index theory, the tower of the higher relative commutants is known to play important roles to study the structure of subfactors (see Ocneanu [20] and Popa [23]). In type III case, we need two towers of the higher relative commutants arising from type III inclusion and the associated type II inclusion. In particular, the latter contains many informations for the study of type III inclusion. Furthermore, we compute the mirroring introduced by Ocneanu [20] and the dual action which makes sense only type III case, and show that these systems include all informations of the four invariants of the automorphism. In order to calculate them, the canonical extension of automorphism in the sense of Haagerup-Størmer [4] is used.

In §3, we deal with inclusions of type III_1 factors as an application of this example. In this case, the known result on classification of subfactors is that if an inclusion of approximately finite dimensional type III_1 factors arises from a finite group action, then it splits into a type II_1 inclusion. In particular, all inclusions with index 2 and 3 split into type II_1 inclusions (Izumi [9] and Kawahigashi-Sutherland-Takesaki [13]). On the other hand, it has not been known, at least in the literature, whether there exists a pair of type III_1 factors which does not split into a type II_1 inclusion. We shall show the existence of such an inclusion. (In the case of type III_λ , $\lambda \neq 1$, see Hamachi-Kosaki [7], Izumi-Kawahigashi [10], Kosaki-Longo [17], Loi [18], [19].) The essential argument of the proof is to show the gap of the two towers of the higher relative commutants. It is characterized by the non-triviality of the dual action.

For type III_λ case ($0 < \lambda < 1$), inclusions of factors containing the above one have been considered in Loi [19], and their two kinds of derived towers have been computed.

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§1. Relation between Automorphisms and Subfactors

We recall the way to construct an inclusion of factors from Jones [11] (see also Pimsner-Popa [21] and Popa [22]).

Let P be a type III factor and α an automorphism of P . We set $\alpha_0 = \text{id}$, and $\alpha_1 = \alpha$, and denote by $\{e_{ij}\}_{i,j=0}^1$ the usual matrix units in $M_2(\mathbf{C})$. Then we define a factor M and a subfactor N by

$$M = P \otimes M_2(\mathbf{C}),$$

$$N = \left\{ \sum_{i=0}^1 \alpha_i(x) e_{ii} \mid x \in P \right\},$$

and define a faithful normal conditional expectation E by

$$E\left(\sum_{i,j=0}^1 x_{ij} e_{ij} \right) = \sum_{i=0}^1 \alpha_i(x) e_{ii},$$

where $x = \frac{1}{2}(x_{00} + \alpha^{-1}(x_{11}))$.

Since the following lemma follows from a standard argument, we leave its proof to the reader (see Hiai [8] and Kosaki [15]).

Lemma 1.1. *With the above notations, we have*

- (i) M and N are isomorphic to P ,
- (ii) $\text{Index } E = 4$ and E is the minimal conditional expectation for $M \supset N$.

Remark 1.2. Let $M \supset N$ be a pair of factors of type III with non-trivial relative commutant and let $E: M \rightarrow N$ be a faithful normal conditional expectation with index 4. Then it follows from the proof in Pimsner-Popa [21; Corollary 4.8] that there exist a type III factor P and an automorphism α of P such that $M \supset N$ is conjugate to $P \otimes M_2(\mathbf{C}) \supset \left\{ \sum_{i=0}^1 \alpha_i(x) e_{ii} \mid x \in P \right\}$.

Let P be a type III factor. For automorphisms α and β of P , we define the corresponding subfactors N_α and N_β , respectively.

Proposition 1.3. *The following conditions are equivalent.*

- (i) $M \supset N_\alpha$ is conjugate to $M \supset N_\beta$,
- (ii) α is outer conjugate to β or β^{-1} .

Proof.

(i) \rightarrow (ii): Suppose that there exists an automorphism Φ of M such that $\Phi(N_\alpha) = N_\beta$. Then there exists an automorphism θ of P such that

$$\Phi\left(\sum_{i=0}^1 \alpha_i(x)e_{ii}\right) = \sum_{i=0}^1 \beta_i(\theta(x))e_{ii}, \quad x \in P.$$

Since $\Phi(N'_\alpha \cap M) = N'_\beta \cap M$, after a suitable perturbation by an inner automorphism of $N'_\beta \cap M$ we may assume that there exists a bijection φ of $\{0, 1\}$ such that

$$\Phi(e_{ii}) = e_{\varphi(i)\varphi(i)}, \quad i = 0, 1.$$

We deal with the following two cases separately.

(1) The case of $\varphi(0) = 0$.

We compute, for any $x \in P$,

$$\begin{aligned} \Phi(xe_{00}) &= \Phi(e_{00} \cdot \left(\sum_{i=0}^1 \alpha_i(x)e_{ii}\right) \cdot e_{00}) \\ &= e_{00} \cdot \left(\sum_{i=0}^1 \beta_i(\theta(x))e_{ii}\right) \cdot e_{00} \\ &= \theta(x)e_{00} \end{aligned}$$

and

$$\begin{aligned} \Phi(\alpha(x)e_{11}) &= \Phi(e_{11} \cdot \left(\sum_{i=0}^1 \alpha_i(x)e_{ii}\right) \cdot e_{11}) \\ &= e_{11} \cdot \left(\sum_{i=0}^1 \beta_i(\theta(x))e_{ii}\right) \cdot e_{11} \\ &= \beta(\theta(x))e_{11}. \end{aligned}$$

Since there exists a unitary u in P such that $\Phi(e_{10}) = ue_{10}$, we also compute

$$\begin{aligned} \Phi(\alpha(x)e_{11}) &= \Phi(e_{10} \cdot \alpha(x)e_{00} \cdot e_{01}) \\ &= ue_{10} \cdot \theta(\alpha(x))e_{00} \cdot u^*e_{01} \\ &= u\theta(\alpha(x))u^*e_{11}. \end{aligned}$$

Hence we get

$$\beta(\theta(x)) = u\theta(\alpha(x))u^*, \quad x \in P.$$

This means that α is outer conjugate to β .

(2) The case of $\varphi(0) = 1$.

A similar calculation shows that α is outer conjugate to β^{-1} .

(ii) \rightarrow (i): It is sufficient to prove it in the following three cases.

(1) The case of $\beta = \theta \circ \alpha \circ \theta^{-1}$ for some automorphism θ of P .

If we set $\Phi = \theta \otimes \text{id}$., then Φ is an automorphism of M and

$$\begin{aligned} \Phi\left(\sum_{i=0}^1 \alpha_i(x)e_{ii}\right) &= \sum_{i=0}^1 \theta(\alpha_i(x))e_{ii} \\ &= \sum_{i=0}^1 \beta_i(\theta(x))e_{ii}, \quad x \in P. \end{aligned}$$

(2) The case of $\beta = \text{Ad} u \circ \alpha$ for some unitary u in P .

If we set $\Phi = \text{Ad}(e_{00} + ue_{11})$, then

$$\begin{aligned} \Phi\left(\sum_{i=0}^1 \alpha_i(x)e_{ii}\right) &= xe_{00} + u\alpha(x)u^*e_{11} \\ &= \sum_{i=0}^1 \beta_i(x)e_{ii}, \quad x \in P. \end{aligned}$$

(3) The case of $\beta = \alpha^{-1}$.

If we set $\Phi = \alpha^{-1} \otimes \text{Ad}(e_{01} + e_{10})$, then

$$\begin{aligned} \Phi\left(\sum_{i=0}^1 \alpha_i(x)e_{ii}\right) &= xe_{00} + \alpha^{-1}(x)e_{11} \\ &= \sum_{i=0}^1 \beta_i(x)e_{ii}, \quad x \in P. \end{aligned}$$

q.e.d.

Remark 1.4. In general, an automorphism of a type III factor is not outer conjugate to its inverse. Therefore we can not distinguish them in the subfactor level. But we may consider that classification of automorphisms up to outer conjugacy is “equivalent” to that of the corresponding subfactors up to conjugacy.

§2. Conjugacy Invariants for Subfactors

In this section, we shall show how the four outer conjugacy invariants of an automorphism of a type III factor appear as conjugacy invariants of the corresponding subfactor.

Let P be a type III factor and α an automorphism of P . Let $M \supset N$ be as in §1 and let $\tilde{M} \supset \tilde{N}$ be the canonical inclusion of type II_∞ von Neumann algebras with the dual action θ^M (Kosaki [16], see also [5], [6], [14]). We set, for a non-negative integer k , $P_k = M_2(\mathbb{C})$ with matrix units $\{e_{ij}^k\}_{i,j=0}^1$. Furthermore, we set

$$M_k = P \otimes P_0 \otimes P_1 \otimes \cdots \otimes P_k, \quad k \geq 0,$$

$$M_{-1} = N,$$

and

$$\tilde{M}_k = \tilde{P} \otimes P_0 \otimes P_1 \otimes \cdots \otimes P_k, \quad k \geq 0,$$

$$\tilde{M}_{-1} = \left\{ \sum_{i=0}^1 \tilde{\alpha}_i(x) e_{ii} \mid x \in \tilde{P} \right\}.$$

Here \tilde{P} means the crossed product $P \rtimes_{\sigma, \varphi} \mathbb{R}$ for some weight φ on P and $\tilde{\alpha}$ is the canonical extension of α in the sense of Haagerup-Størmer [4] ($\tilde{\alpha}_0$ id. and $\tilde{\alpha}_1 = \tilde{\alpha}$). We denote the dual action on \tilde{P} by $\theta^{\tilde{P}}$. We embed M_{2k-1} (resp. M_{2k}) into M_{2k} (resp. M_{2k+1}) by

$$\begin{aligned} & \sum x_{i_0 j_0 \cdots i_{2k-1} j_{2k-1}} e_{i_0 j_0}^0 \cdots e_{i_{2k-1} j_{2k-1}}^{2k-1} \in M_{2k-1} \rightarrow \\ & \sum \alpha_{i_{2k}}(x_{i_0 j_0 \cdots i_{2k-1} j_{2k-1}}) e_{i_0 j_0}^0 \cdots e_{i_{2k-1} j_{2k-1}}^{2k-1} e_{i_{2k} j_{2k}}^{2k}, \\ & \sum x_{i_0 j_0 \cdots i_{2k} j_{2k}} e_{i_0 j_0}^0 \cdots e_{i_{2k} j_{2k}}^{2k} \in M_{2k} \rightarrow \\ & \sum \alpha_{i_{2k+1}}^{-1}(x_{i_0 j_0 \cdots i_{2k} j_{2k}}) e_{i_0 j_0}^0 \cdots e_{i_{2k} j_{2k}}^{2k} e_{i_{2k+1} j_{2k+1}}^{2k+1}. \end{aligned}$$

Replacing the automorphism α by the canonical extension $\tilde{\alpha}$, we embed \tilde{M}_{2k-1} (resp. \tilde{M}_{2k}) into \tilde{M}_{2k} (resp. M_{2k+1}) as above. We define faithful normal conditional expectations $E_{2k-1}: M_{2k} \rightarrow M_{2k-1}$ and $E_{2k}: M_{2k+1} \rightarrow M_{2k}$ by

$$\begin{aligned} & E_{2k-1} \left(\sum x_{i_0 j_0 \cdots i_{2k} j_{2k}} e_{i_0 j_0}^0 \cdots e_{i_{2k} j_{2k}}^{2k} \right) \\ & = \frac{1}{2} \sum \alpha_{i_{2k}} \left(\sum_{l=0}^1 \alpha_l^{-1}(x_{i_0 j_0 \cdots i_{2k-1} j_{2k-1} u}) \right) e_{i_0 j_0}^0 \cdots e_{i_{2k} j_{2k}}^{2k}, \end{aligned}$$

$$\begin{aligned}
 E_{2k}(\sum x_{i_0j_0 \dots i_{2k+1}j_{2k+1}} e_{i_0j_0}^0 \dots e_{i_{2k+1}j_{2k+1}}^{2k+1}) \\
 = \frac{1}{2} \sum \alpha_{i_{2k+1}}^{-1} \left(\sum_{l=0}^1 \alpha_l(x_{i_0j_0 \dots i_{2k}j_{2k}ll}) \right) e_{i_0j_0}^0 \dots e_{i_{2k+1}j_{2k+1}}^{2k+1}.
 \end{aligned}$$

Similarly we define conditional expectations $\tilde{E}_k: \tilde{M}_{k+1} \rightarrow \tilde{M}_k, k \geq -1$, by using $\tilde{\alpha}$ instead of α .

Lemma 2.1. *With the above notations, there exists an isomorphism Φ from \tilde{M} onto \tilde{M}_0 such that $\Phi(\tilde{N}) = \tilde{M}_{-1}$ and*

$$\Phi \circ \theta_t^M = (\theta_t^P \otimes \text{id.}) \circ \Phi, \quad t \in \mathbf{R}.$$

Proof. Taking a state (or weight) τ on P such that $\tau \circ \alpha = \tau$, we set $\tilde{P} = P \otimes_{\sigma} \tau \mathbf{R}$. We define a state φ on N by

$$\varphi \left(\sum_{i=0}^1 \alpha_i(x) e_{ii} \right) = \tau(x), \quad x \in P,$$

and set $\psi = \varphi \circ E \in M_*^+$. Then we have $\psi = \tau \otimes \frac{1}{2} \text{Tr}$, where Tr is the usual trace on $M_2(\mathbf{C})$. Since $\tilde{M} = M \otimes_{\sigma} \psi \mathbf{R}$ is generated by

$$\begin{aligned}
 &x \otimes 1 \otimes 1, \quad x \in P, \\
 &1 \otimes e_{ij} \otimes 1, \quad i, j = 0, 1, \\
 &\Delta_t^{it} \otimes 1 \otimes \lambda_t, \quad t \in \mathbf{R},
 \end{aligned}$$

and $\tilde{M}_0 = \tilde{P} \otimes M_2(\mathbf{C})$ is generated by

$$\begin{aligned}
 &x \otimes 1 \otimes 1, \quad x \in P, \\
 &\Delta_t^{it} \otimes \lambda_t \otimes 1, \quad t \in \mathbf{R}, \\
 &1 \otimes 1 \otimes e_{ij}, \quad i, j = 0, 1,
 \end{aligned}$$

we get a natural isomorphism Φ from \tilde{M} onto \tilde{M}_0 which is given by changing the second and the third components. From the definition of the canonical extension and the dual action, it has the desired properties. q.e.d.

Proposition 2.2 (see Bisch [1], Loi [19], Popa [24]).

(i) $\{M_k\}_{k=-1}^\infty$ is the Jones tower of type III factors arising from $M \supset N$ and $\{E_k\}_{k=-1}^\infty$ is the sequence of the canonical conditional expectations. Furthermore, the Jones projection e_{k-1} arising from $E_{k-1}: M_k \rightarrow M_{k-1}$ is given by

$$e_{k-1} = \frac{1}{2} \sum_{i,j=0}^1 e_{ij}^k e_{ij}^{k+1}.$$

(ii) $\{\tilde{M}_k\}_{k=-1}^\infty$ is the canonical tower of type II_∞ von Neumann algebras arising from $M \supset N$ and $\{\tilde{E}_k\}_{k=-1}^\infty$ are the canonical conditional expectations.

(iii) The dual action θ^k on \tilde{M}_k is given by

$$\theta^P \otimes \text{id.} \otimes \text{id.} \otimes \cdots \otimes \text{id.}$$

Proof. The assertion (i) follows from the characterization of the basic extension [7; Theorem 8]. From Lemma 2.1, we obtain (ii) and (iii) as a consequence of (i) q.e.d.

For the indices $i_0, j_0, \dots, i_k, j_k$, we set

$$I(i_0, j_0, \dots, i_k, j_k) = \begin{cases} i_k - i_{k-1} + \cdots - i_1 + i_0 - j_0 + j_1 - \cdots + j_{k-1} - j_k, \\ \text{if } k: \text{ even,} \\ -i_k + i_{k-1} - \cdots - i_1 + i_0 - j_0 + j_1 - \cdots - j_{k-1} + j_k, \\ \text{if } k: \text{ odd.} \end{cases}$$

Lemma 2.3 (see Bisch [1], Loi [19], Popa [24]). *The relative commutants are given as follows:*

$$(i) \quad N' \cap M_k = \left\{ \sum_{\alpha^{I(i_0, j_0, \dots, i_k, j_k)} \in \text{Int } P} c_{i_0 j_0 \dots i_k j_k} u_{I(i_0, j_0, \dots, i_k, j_k)} e_{i_0 j_0}^0 \cdots e_{i_k j_k}^k \mid c_{i_0 j_0 \dots i_k j_k} \in \mathbf{C} \right\},$$

where $u_{I(i_0, j_0, \dots, i_k, j_k)}$ means an implementing unitary of $\alpha^{I(i_0, j_0, \dots, i_k, j_k)}$ in P . In particular, the inclusion $M \supset N$ has the extended Coxeter-Dynkin diagram \tilde{A} or $A_{\infty, \infty}$ as the principal graph,

$$(ii) \quad \tilde{N}' \cap \tilde{M}_k = \left\{ \sum_{I(i_0, j_0, \dots, i_k, j_k) \in N(\alpha)} c_{i_0 j_0 \dots i_k j_k} \tilde{u}_{I(i_0, j_0, \dots, i_k, j_k)} e_{i_0 j_0}^0 \cdots e_{i_k j_k}^k \mid c_{i_0 j_0 \dots i_k j_k} \in Z(\tilde{P}) \right\},$$

where $\tilde{u}_{I(i_0, j_0, \dots, i_k, j_k)}$ means an implementing unitary of $\tilde{\alpha}^{I(i_0, j_0, \dots, i_k, j_k)}$ in \tilde{P} .

Proof. We remark that the canonical extension $\tilde{\alpha}$ is free in the sense of [12] or inner ([4] or [25]). Thus the assertions follow from direct computations. q.e.d.

Theorem 2.4. (i) *Let (P, H, J, P^{\natural}) be a standard form of P and v the canonical implementation of α . We take and fix a cyclic and separating vector ξ for P in P^{\natural} .*

We represent M_k in $B(H \otimes C^2 \otimes C^2 \otimes \dots \otimes C^2)$ by

$$\left\{ \sum \alpha_{i_{2k+1}}^{-1} \alpha_{i_{2k}} \dots \alpha_{i_{k+1}}^{-1} (x_{i_0 j_0 \dots i_k j_k}) e_{i_0 j_0}^0 \dots e_{i_k j_k}^k e_{i_{k+1} i_{k+1}}^{k+1} \dots e_{i_{2k+1} i_{2k+1}}^{2k+1} \right. \\ \left. | x_{i_0 j_0 \dots i_k j_k} \in P \right\}$$

if k : even,

$$\left\{ \sum \alpha_{i_{2k+1}}^{-1} \alpha_{i_{2k}} \dots \alpha_{i_{k+1}} (x_{i_0 j_0 \dots i_k j_k}) e_{i_0 j_0}^0 \dots e_{i_k j_k}^k e_{i_{k+1} i_{k+1}}^{k+1} \dots e_{i_{2k+1} i_{2k+1}}^{2k+1} \right. \\ \left. | x_{i_0 j_0 \dots i_k j_k} \in P \right\}$$

if k : odd.

We define a vector ξ_k in the Hilbert space $H \otimes C^2 \otimes C^2 \otimes \dots \otimes C^2 = l^2(\{0,1\} \times \{0,1\} \times \dots \times \{0,1\}, H)$ by

$$\xi_k(i_0, i_1, \dots, i_{2k+1}) = \delta_{i_0, i_{2k+1}} \delta_{i_1, i_{2k}} \dots \delta_{i_k, i_{k+1}} \xi.$$

Here $\delta_{i,j}$ means the Kronecker symbol.

Then ξ_k is a cyclic and separating vector for M_k and the modular conjugation operator J_k arising from ξ_k is given by

$$J_k = \sum v_{i_{2k+1}}^* v_{i_{2k}} \dots v_{i_1}^* v_{i_0} J e_{i_0 i_{2k+1}}^0 e_{i_1 i_{2k}}^1 \dots e_{i_{2k+1} i_0}^{2k+1}.$$

(ii) *Let $(\tilde{P}, \tilde{H}, \tilde{J}, \tilde{P}^{\natural})$ be a standard form of \tilde{P} and \tilde{v} the canonical implementation of $\tilde{\alpha}$. We take and fix a cyclic and separating vector $\tilde{\xi}$ for \tilde{P} in \tilde{P}^{\natural} . Similarly we represent \tilde{M}_k in $B(\tilde{H} \otimes C^2 \otimes C^2 \otimes \dots \otimes C^2)$ by using $\tilde{\alpha}$ instead of α and define a vector $\tilde{\xi}_k$ in $\tilde{H} \otimes C^2 \otimes C^2 \otimes \dots \otimes C^2$ by using $\tilde{\xi}$ instead of ξ .*

Then $\tilde{\xi}_k$ is a cyclic and separating vector for \tilde{M}_k and the modular conjugation

operator \tilde{J}_k coming from $\tilde{\xi}_k$ is given by

$$\tilde{J}_k = \sum \tilde{v}_{i_{2k+1}}^* \tilde{v}_{i_{2k}} \cdots \tilde{v}_{i_1}^* \tilde{v}_{i_0} \tilde{J} e_{i_0 i_{2k+1}}^0 e_{i_1 i_{2k}}^1 \cdots e_{i_{2k+1} i_0}^{2k+1}.$$

Proof. We use the usual notations in modular theory ([26]).

(i) Let us assume that k is even.

For $X = \sum \alpha_{i_{2k+1}}^{-1} \alpha_{i_{2k}} \cdots \alpha_{i_{k+1}}^{-1} (x_{i_0 j_0 \cdots i_k j_k}) e_{i_0 j_0}^0 \cdots e_{i_k j_k}^k e_{i_{k+1} i_{k+1}}^{k+1} \cdots e_{i_{2k+1} i_{2k+1}}^{2k+1} \in M_k$, we have

$$(X \tilde{\xi}_k)(i_0, i_1, \dots, i_{2k+1}) = \alpha_{i_{2k+1}}^{-1} \alpha_{i_{2k}} \cdots \alpha_{i_{k+1}}^{-1} (x_{i_0 i_{2k+1} i_1 i_{2k} \cdots i_k i_{k+1}}) \tilde{\xi}.$$

Hence $\tilde{\xi}_k$ is a cyclic and separating vector for M_k .

From the definition, $S_k (= S_{\tilde{\xi}_k})$ is the closure of the operator $S_k^0: X \tilde{\xi}_k \rightarrow X^* \tilde{\xi}_k, X \in M_k$. Since we have

$$(X^* \tilde{\xi}_k)(j_0, j_1, \dots, j_{2k+1}) = \alpha_{j_{2k+1}}^{-1} \alpha_{j_{2k}} \cdots \alpha_{j_{k+1}}^{-1} (x_{j_{2k+1} j_0 j_{2k} j_1 \cdots j_{k+1} j_k})^* \tilde{\xi},$$

it follows that

$$S_k = \sum v_{i_{2k+1}}^* v_{i_{2k}} \cdots v_{i_{k+1}}^* S_{v_{i_{k+1}} \cdots v_{i_{2k}} v_{i_{2k+1}} \xi, v_{i_k} \cdots v_{i_1} v_{i_0} \xi} v_{i_k} \cdots v_{i_1}^* v_{i_0} e_{i_0 i_{2k+1}}^0 e_{i_1 i_{2k}}^1 \cdots e_{i_{2k+1} i_0}^{2k+1}.$$

Thus we obtain

$$\begin{aligned} \Delta_k &= S_k^* S_k \\ &= \sum v_{i_0}^* v_{i_1} \cdots v_{i_k}^* \Delta_{v_{i_{k+1}} \cdots v_{i_{2k}} v_{i_{2k+1}} \xi, v_{i_k} \cdots v_{i_1} v_{i_0} \xi} v_{i_k} \cdots v_{i_1}^* v_{i_0} \\ &\quad e_{i_{2k+1} i_{2k+1}}^0 e_{i_{2k} i_{2k}}^1 \cdots e_{i_0 i_0}^{2k+1}, \end{aligned}$$

and

$$\begin{aligned} J_k &= S_k \Delta_k^{-\frac{1}{2}} \\ &= \sum v_{i_{2k+1}}^* v_{i_{2k}} \cdots v_{i_1}^* v_{i_0} J e_{i_0 i_{2k+1}}^0 e_{i_1 i_{2k}}^1 \cdots e_{i_{2k+1} i_0}^{2k+1}. \end{aligned}$$

For any other case, we also get the consequence by a similar calculation.

q.e.d.

Now we shall compute the mirroring (and the canonical shift) introduced by Ocneanu [20] (see also [2] and [3]). We have two kinds of mirrorings $\{\gamma_k\}_{k \geq 0}$ and $\{\tilde{\gamma}_k\}_{k \geq 0}$ arising from type III inclusion and type II_∞ inclusion. From the definition, the mirroring γ_k is an anti-automorphism of $N' \cap M_{2k+1}$ given by

$$\gamma_k(x) = J_k x^* J_k, \quad x \in N' \cap M_{2k+1},$$

and the canonical shift Γ is a *-endomorphism of $\bigcup_{k=-1}^\infty (N' \cap M_k)$ given by

$$\Gamma(x) = \gamma_{k+1}(\gamma_k(x)), \quad x \in N' \cap M_{2k+1}.$$

Similarly, the mirrorings $\{\tilde{\gamma}_k\}_{k \geq 0}$ of $\{\tilde{N}' \cap \tilde{M}_{2k+1}\}_{k \geq 0}$ and the canonical shift $\tilde{\Gamma}$ of $\bigcup_{k=-1}^\infty (\tilde{N}' \cap \tilde{M}_k)$ are defined.

Let φ be a dominant weight on P such that $\varphi \circ \alpha = \varphi$ and for the continuous decomposition $P = P_\varphi \rtimes_\theta \mathbf{R}$,

$$\begin{aligned} \hat{\alpha} \circ \theta_t &= \theta_t \circ \hat{\alpha}, & t \in \mathbf{R}, \\ \begin{cases} \alpha(x) = \hat{\alpha}(x), & x \in P_\varphi, \\ \alpha(\lambda(t)) = \lambda(t), & t \in \mathbf{R}, \end{cases} \end{aligned}$$

where $\hat{\alpha}$ is the automorphism of P_φ induced by α (Sutherland-Takesaki [27; Lemma 5.11]). Then each of the invariants is calculated as follows: The module $\text{mod } \alpha$ of α is given by

$$\text{mod } \alpha = \hat{\alpha}|_{Z(P_\varphi)}.$$

Expressing $\alpha^h = \text{Ad } u_h \circ \bar{\sigma}_{c(h)}^\varphi$ for $h \in N(\alpha)$, we have

$$\begin{aligned} u_h u_k &= \mu(h, k) u_{hk}, & h, k \in N(\alpha), \\ \alpha^m(u_h) &= \lambda(m, h) u_h, & m \in \mathbf{Z}, h \in N(\alpha), \\ \theta_t(u_h) &= c(h, t) u_h, & t \in \mathbf{R}, h \in N(\alpha). \end{aligned}$$

Corollary 2.5. (i) *The canonical tower $\{\tilde{M}_k\}_{k=-1}^\infty$ of type II_∞ von Neumann algebras arising from $M \supset N$ is given by*

$$\tilde{M}_k = P_\varphi \otimes P_0 \otimes P_1 \otimes \cdots \otimes P_k, \quad k \geq 0,$$

$$\tilde{M}_{-1} = \tilde{N} = \left\{ \sum_{i=0}^1 \hat{\alpha}_i(x) e_{ii} \mid x \in P_\varphi \right\},$$

and the embedding is given as follows:

$$\begin{aligned} \sum x_{i_0 j_0 \cdots i_{2k-1} j_{2k-1}} e_{i_0 j_0}^0 \cdots e_{i_{2k-1} j_{2k-1}}^{2k-1} &\in \tilde{M}_{2k-1} \rightarrow \\ \sum \hat{\alpha}_{i_{2k}}(x_{i_0 j_0 \cdots i_{2k-1} j_{2k-1}}) e_{i_0 j_0}^0 \cdots e_{i_{2k-1} j_{2k-1}}^{2k-1} e_{i_{2k} j_{2k}}^{2k}, & \\ \sum x_{i_0 j_0 \cdots i_{2k} j_{2k}} e_{i_0 j_0}^0 \cdots e_{i_{2k} j_{2k}}^{2k} &\in \tilde{M}_{2k} \rightarrow \\ \sum \hat{\alpha}_{i_{2k+1}}^{-1}(x_{i_0 j_0 \cdots i_{2k} j_{2k}}) e_{i_0 j_0}^0 \cdots e_{i_{2k} j_{2k}}^{2k} e_{i_{2k+1} j_{2k+1}}^{2k+1}. & \end{aligned}$$

(ii) The tower of the relative commutants $\{\tilde{N}' \cap \tilde{M}_k\}_{k=-1}^\infty$ is given by

$$\begin{aligned} \tilde{N}' \cap \tilde{M}_k = \{ & \sum_{I(i_0, j_0, \dots, i_k, j_k) \in N(\alpha)} c_{i_0 j_0 \cdots i_k j_k} u_{I(i_0, j_0, \dots, i_k, j_k)} e_{i_0 j_0}^0 \cdots e_{i_k j_k}^k \\ & \mid c_{i_0 j_0 \cdots i_k j_k} \in Z(P_\varphi)\}. \end{aligned}$$

(iii) The mirrorings $\{\tilde{\gamma}_k\}_{k=0}^\infty$ are calculated by

$$\begin{aligned} &\tilde{\gamma}_k \left(\sum c_{i_0 j_0 \cdots i_{2k+1} j_{2k+1}} u_{I(i_0, j_0, \dots, i_{2k+1} j_{2k+1})} e_{i_0 j_0}^0 \cdots e_{i_{2k+1} j_{2k+1}}^{2k+1} \right) \\ &= \sum (\text{mod } \alpha)^{-j_0 + j_1 - \cdots + j_{2k+1}} (c_{i_0 j_0 \cdots i_{2k+1} j_{2k+1}}) \\ &\quad \lambda(-j_0 + j_1 - \cdots + j_{2k+1}, I(j_{2k+1}, i_{2k+1}, \dots, j_0, i_0)) \\ &\quad u_{I(j_{2k+1}, i_{2k+1}, \dots, j_0, i_0)} e_{j_{2k+1} i_{2k+1}}^0 \cdots e_{j_0 i_0}^{2k+1}, \end{aligned}$$

and the canonical shift $\tilde{\Gamma}$ is given by

$$\begin{aligned} &\tilde{\Gamma} \left(\sum c_{i_0 j_0 \cdots i_{2k+1} j_{2k+1}} u_{I(i_0, j_0, \dots, i_{2k+1} j_{2k+1})} e_{i_0 j_0}^0 e_{i_1 j_1}^1 \cdots e_{i_{2k+1} j_{2k+1}}^{2k+1} \right) \\ &= \sum c_{i_0 j_0 \cdots i_{2k+1} j_{2k+1}} u_{I(i_0, j_0, \dots, i_{2k+1} j_{2k+1})} e_{i_0 j_0}^2 e_{i_1 j_1}^3 \cdots e_{i_{2k+1} j_{2k+1}}^{2k+3}. \end{aligned}$$

(iv) The restriction of the dual actions $\{\theta^k\}_{k=-1}^\infty$ to the tower of the relative commutants $\{\tilde{N}' \cap \tilde{M}_k\}_{k=-1}^\infty$ are calculated by

$$\begin{aligned} & \theta_i^k(\sum c_{i_0 j_0 \dots i_k j_k} u_{I(i_0, j_0, \dots, i_k, j_k)} e_{i_0 j_0}^0 \dots e_{i_k j_k}^k) \\ &= \sum \theta_t(c_{i_0 j_0 \dots i_k j_k}) c(I(i_0, j_0, \dots, i_k, j_k), t) u_{I(i_0, j_0, \dots, i_k, j_k)} e_{i_0 j_0}^0 \dots e_{i_k j_k}^k, t \in \mathbf{R}. \end{aligned}$$

Proof. By [25], we may identify P_φ with \tilde{P} . Then (i) and (ii) follow from Proposition 2.2 and Lemma 2.3.

(iii) From Theorem 2.4, we compute

$$\begin{aligned} & \tilde{\gamma}_k(\sum c_{i_0 j_0 \dots i_{2k+1} j_{2k+1}} u_{I(i_0, j_0, \dots, i_{2k+1}, j_{2k+1})} e_{i_0 j_0}^0 \dots e_{i_{2k+1} j_{2k+1}}^{2k+1}) \\ &= \sum \tilde{v}_{j_0}^* \tilde{v}_{j_1} \dots \tilde{v}_{j_{2k}}^* \tilde{v}_{j_{2k+1}} \tilde{J} c_{i_0 j_0 \dots i_{2k+1} j_{2k+1}}^* u_{I(i_0, j_0, \dots, i_{2k+1}, j_{2k+1})}^* \\ & \quad \tilde{v}_{i_{2k+1}}^* \tilde{v}_{i_{2k}} \dots \tilde{v}_{i_1}^* \tilde{v}_{i_0} \tilde{J} e_{j_{2k+1} i_{2k+1}}^1 e_{j_{2k} i_{2k}}^0 \dots e_{j_0 i_0}^{2k+1} \\ &= \sum \tilde{v}_{j_0}^* \tilde{v}_{j_1} \dots \tilde{v}_{j_{2k}}^* \tilde{v}_{j_{2k+1}} c_{i_0 j_0 \dots i_{2k+1} j_{2k+1}} u_{I(i_0, j_0, \dots, i_{2k+1}, j_{2k+1})} \\ & \quad \tilde{v}_{i_{2k+1}}^* \tilde{v}_{i_{2k}} \dots \tilde{v}_{i_1}^* \tilde{v}_{i_0} e_{j_{2k+1} i_{2k+1}}^0 e_{j_{2k} i_{2k}}^1 \dots e_{j_0 i_0}^{2k+1} \\ &= \sum (\text{mod } \alpha)^{-j_0 + j_1 - \dots + j_{2k+1}} (c_{i_0 j_0 \dots i_{2k+1} j_{2k+1}}) \\ & \quad \lambda(-j_0 + j_1 - \dots + j_{2k+1}, I(j_{2k+1}, i_{2k+1}, \dots, j_0, i_0)) \\ & \quad u_{I(j_{2k+1}, i_{2k+1}, \dots, j_0, i_0)} e_{j_{2k+1} i_{2k+1}}^0 \dots e_{j_0 i_0}^{2k+1}. \end{aligned}$$

Furthermore, we get

$$\begin{aligned} &= \tilde{\gamma}_{k+1} \circ \tilde{\gamma}_k(\sum c_{i_0 j_0 \dots i_{2k+1} j_{2k+1}} u_{I(i_0, j_0, \dots, i_{2k+1}, j_{2k+1})} e_{i_0 j_0}^0 \dots e_{i_{2k+1} j_{2k+1}}^{2k+1}) \\ &= \sum c_{i_0 j_0 \dots i_{2k+1} j_{2k+1}} u_{I(i_0, j_0, \dots, i_{2k+1}, j_{2k+1})} e_{i_0 j_0}^2 \dots e_{i_{2k+1} j_{2k+1}}^{2k+3}. \end{aligned}$$

(iv) follows from Proposition 2.2.

q.e.d.

Corollary 2.6. *The mirrorings $\{\gamma_k\}_{k=0}^\infty$ of $\{N' \cap M_{2k+1}\}_{k=0}^\infty$ are given by*

$$\begin{aligned} & \gamma_k(\sum c_{i_0 j_0 \dots i_{2k+1} j_{2k+1}} u_{I(i_0, j_0, \dots, i_{2k+1}, j_{2k+1})} e_{i_0 j_0}^0 \dots e_{i_{2k+1} j_{2k+1}}^{2k+1}) \\ &= \sum c_{i_0 j_0 \dots i_{2k+1} j_{2k+1}} \lambda(-j_0 + \dots + j_{2k+1}, I(j_{2k+1}, i_{2k+1}, \dots, j_0, i_0)) \\ & \quad u_{I(j_{2k+1}, i_{2k+1}, \dots, j_0, i_0)} e_{j_{2k+1} i_{2k+1}}^0 \dots e_{j_0 i_0}^{2k+1}, \end{aligned}$$

and the canonical shift Γ of $\bigcup_{k=-1}^{\infty} (N' \cap M_k)$ is calculated by

$$\begin{aligned} &\Gamma\left(\sum c_{i_0 j_0 \dots i_{2k+1} j_{2k+1}} u_{I(i_0, j_0, \dots, i_{2k+1}, j_{2k+1})} e_{i_0 j_0}^0 \cdots e_{i_{2k+1} j_{2k+1}}^{2k+1}\right) \\ &= \sum c_{i_0 j_0 \dots i_{2k+1} j_{2k+1}} u_{I(i_0, j_0, \dots, i_{2k+1}, j_{2k+1})} e_{i_0 j_0}^2 \cdots e_{i_{2k+1} j_{2k+1}}^{2k+3}. \end{aligned}$$

Remark 2.7. A simple computation shows that the condition given in Sutherland-Takesaki [27; Theorem 5.14] corresponds to the commutativity of the mirroring and the dual action.

Remark 2.8. In general, it is known from Kosaki-Longo [17] that the relative commutant $N' \cap M_k$ is expressed as the fixed point algebra of the relative commutant $\tilde{N}' \cap \tilde{M}_k$ by the dual action. Moreover, since the mirroring $\tilde{\gamma}_k$ of $\tilde{N}' \cap \tilde{M}_{2k+1}$ commutes with the dual action, it induces the anti-automorphism of $N' \cap M_{2k+1}$. Then it is exactly the mirroring γ_k . (For our example, these facts follow from direct computations because we know all of them explicitly.) Therefore all informations of the tower of the relative commutants $\{N' \cap M_k\}_{k=-1}^{\infty}$ and the mirrorings $\{\gamma_k\}_{k=0}^{\infty}$ are included in $\{\tilde{N}' \cap \tilde{M}_k\}_{k=-1}^{\infty}$, $\{\tilde{\gamma}_k\}_{k=0}^{\infty}$ and $\{\theta^k\}_{k=-1}^{\infty}$.

Remark 2.9. Assuming that $M \supset N$ are approximately finite dimensional (equivalently so is P), we can show that the system consisting of the tower of the relative commutants $\{\tilde{N}' \cap \tilde{M}_k\}_{k=-1}^{\infty}$, the mirrorings $\{\tilde{\gamma}_k\}_{k=0}^{\infty}$ and the dual actions $\{\theta^k\}_{k=-1}^{\infty}$ is complete (see [13] and [27]).

§3. Application

At first we remark that since the tensor product of a type II factor and a type III factor is of type III, inclusions of type III factors contain the following inclusions which are completely reduced to type II₁ inclusions:

$$R \otimes M \supset Q \otimes M,$$

where $R \supset Q$ is a pair of type II₁ factors and M is a type III factor. An inclusion of the above form is said to split into a type II₁ inclusion. In this section, we shall show the existence of an inclusion of type III₁ factors which does not split into a type II₁ inclusion.

Lemma 3.1. *Let $M \supset N$ be type III₁ factors. If it splits into a type*

II_1 inclusion, then the two canonical towers of the higher relative commutants arising from $M \supset N$ coincide.

Proof. Let $M = R \otimes L \supset N = Q \otimes L$, where $R \supset Q$ are type II_1 factors and L is a type III factor. If we denote by $\{R_k\}_{k=-1}^\infty$ the Jones tower arising from $R \supset Q$ ($R_0 = R, R_{-1} = Q$), then the canonical towers $\{M_k\}$ and $\{\tilde{M}_k\}$ are given by

$$M_k = R_k \otimes L, \\ \tilde{M}_k = R_k \otimes \tilde{L},$$

where \tilde{L} is the crossed product $L \rtimes_{\sigma^\varphi} \mathbb{R}$ for some weight φ on L . The assertion follows from this. q.e.d.

In order to show the existence of a type III_1 inclusion without splitting into a type II_1 inclusion, it is sufficient to construct a pair whose two canonical towers differ. For our example, we have only to take an automorphism that makes a gap between the two towers. In type III_1 case, since the module is trivial, the outer conjugacy invariant of an automorphism is parametrized by (n, λ, t) , where n is a non-negative integer, λ is an n -th root of the unit and t is a real number. From Remark 2.8, there is a gap between the two towers if and only if the dual action is non-trivial, namely, the modular invariant is not trivial.

By the above arguments and Proposition 1.3, we have

Proposition 3.2. *Let P be a type III_1 factor and let $N_{(n,\lambda,t)}$ be the subfactor arising from the automorphism $\alpha_{(n,\lambda,t)}$ of P whose invariant is (n, λ, t) , $t \geq 0$.*

(i) $M \supset N_{(n_1,\lambda_1,t_1)}$ is conjugate to $M \supset N_{(n_2,\lambda_2,t_2)}$ if and only if $\alpha_{(n_1,\lambda_1,t_1)}$ is outer conjugate to $\alpha_{(n_2,\lambda_2,t_2)}$.

(ii) For non-zero t , $M \supset N_{(n,\lambda,t)}$ does not split into a type II_1 inclusion, and has $A_{\infty,\infty}$ as the type III principal graph and \tilde{A}_{2n-1} as the type II principal graph.

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Note added in proof: After submitting this manuscript, the author received a preprint of M. Izumi "On type II and type III principal graphs of subfactors". In it, the existence of a gap of their graphs for type III_λ case ($\lambda \neq 0$) has been characterized in terms of Longo's sectors.

