Publ. RIMS, Kyoto Univ.
28 (1992), 1029–1061

Operator Representations of $\boldsymbol{R}_q^2$

By

Konrad SCHMÜDGEN*

Abstract

We study the operator relation $ab=qba$, where $|q|=1$, for self-adjoint operators.

§1. Introduction

Suppose q is a fixed complex number of modulus one. Let $\boldsymbol{R}_q^2$ denote the *-algebra with unit which is generated by two hermitean elements a and b satisfying the relation

$$ab=qba. \tag{1.1}$$

(The precise mathematical definition is as follows: Let $\boldsymbol{C}\langle a, b\rangle$ be the free complex algebra with unit generated by two elements a and b and let $((ab-qba))$ be the two-sided ideal of $\boldsymbol{C}\langle a, b\rangle$ generated by the element ab-qba. We define an involution on the algebra $\boldsymbol{C}\langle a, b\rangle$ by the requirements $a^+:=a$ and $b^+:=b$, so that $\boldsymbol{C}\langle a, b\rangle$ becomes a *-algebra. Since $|q|=1$, we have $(ab-qba)^+=ba-\bar{q}ab=-\bar{q}(ab-qba)$. Hence the ideal $((ab-qba))$ is *-invariant and the quotient algebra $\boldsymbol{C}\langle a, b\rangle/((ab-qba))$ becomes a *-algebra. We denote this *-algebra by $\boldsymbol{R}_q^2$.)

For $q=1$, $\boldsymbol{R}_q^2$ is nothing but the commutative polynomial algebra $\boldsymbol{C}[a, b]$ in two hermitean variables a and b, that is, $\boldsymbol{R}_q^2$ is the coordinate algebra of $\boldsymbol{R}^2$. Therefore, one can think of $\boldsymbol{R}_q^2$ as the coordinate algebra of the "quantum two-dimensional real vector space" (cf. [3], Definition 8). Note that the quantum group $SL_q(2, \boldsymbol{R})$ acts on $\boldsymbol{R}_q^2$ in the obvious way by matrix

Communicated by H. Araki, March 26, 1992.
1991 Mathematics Subject Classification: 58H15, 47D40
* Fachbereich Mathematik/Informatik, Universität Leipzig, Augustusplatz 10, 0–7010 Leipzig, Federal Republic of Germany.

multiplication (see [6] or [3]), i.e. $\mathbb{R}_q^2$ is a quantum space for the quantum group $SL_q(2, \mathbb{R})$.

Now let us pass from the algebraic level to the Hilbert space level (in the terminology of [14]). That is, we want to study self-adjoint operators a and b which fulfill the relation (1.1). Let us consider for a moment the "classical" case $q=1$. Recall that the points of $\mathbb{R}^2$ are precisely (in one-to-one correspondence to) the equivalence classes of irreducible pairs $\{a, b\}$ of strongly commuting self-adjoint operators a and b. (Two self-adjoint operators a and b are said to be strongly commuting if their spectral projections commute or equivalently if $f(a)b \subseteq bf(a)$ for sufficiently many bounded functions f. For instance, one may take $f(x)=e^{itx}$ for all $t \in \mathbb{R}$ or $f(x)=(x-\lambda)^{-1}$ for one $\lambda \in \mathbb{C} \setminus \mathbb{R}$.) We shall try to proceed in a similar way in case of arbitrary q. First we have to define a concept of strong commutativity for the relation (1.1), that is, we have to select the "well-behaved" representations of (1.1). Adopting the terminology used in representation theory of Lie groups, we shall call these representations "integrable". Secondly, we have to classify the irreducible integrable representations of (1.1) up to unitary equivalence.

It seems that there is no canonical way to define integrable representations of the relation (1.1). Let us roughly explain the ideas of the approach proposed in this paper. From the relation $ab=qba$ in the algebra $\mathbb{R}_q^2$ we conclude immediately that $p(a)b=bp(qa)$ for each complex polynomial p. Having this fact and the definition of strong commutativity for $q=1$ in mind, one could try to define integrability by the requirement $f(a)b \subseteq bf(qa)$ for certain "nice" bounded functions f. In case when a is non-singular and either positive or negative, we shall use this method by taking the functions $f(x)=|x|^{it}$ for $t \in \mathbb{R}$ and we shall call the corresponding couples a-integrable. (The precise formulation is given in Definition 3.1.) For an arbitrary non-singular self-adjoint operator a we proceed as follows. Let $a=a_+ \oplus a_-$ be the decomposition of a into its positive part a_+ and its negative part a_-. We then assume that the operator b can be represented by a 2×2 operator matrix $\mathscr{b}=(b_{ij})$ with respect to the corresponding decomposition of the Hilbert space such that $b_{ij}^*=b_{ji}$, $i,j=1,2$. Such a matrix $\mathscr{b}$ will be called a self-adjoint operator matrix. Let $b_{12}=u|b_{12}|$ be the polar decomposition of b_{12}. Then the couple $\{a, b\}$ is said to be a-integrable if $\{a_+, b_{11}\}$ is a_+-integrable and $\{a_-, b_{22}\}$ is a_--integrable (both with respect to the parameter q), if $\{a_+, |b_{21}|\}$ is a_+-integrable and $\{a_-, |b_{12}|\}$ is a_--integrable (both with respect to the parameter $-q$) and if

$uu^*a_+uu^*=-ua_-u^*$. (A justification of this definition will be given in Section 4.) Finally, a pair $\{a, b\}$ is called integrable if $\{a, b\}$ is a-integrable and if $\{b, a\}$ is b-integrable.

It turns out that the operator relation (1.1) is more difficult to treat than one would expect from its rather simple structure. Also, it bears various interesting operator theoretic phenomena which might be surprising at first glance. As an illustration of this remark we mention the following result without proof which shows that there is a striking difference between (1.1) and Lie algebra relations: Let a and b be self-adjoint operators which satisfy the relation (1.1) on a dense invariant domain $\mathscr{D}$ of a Hilbert space. Suppose that $\ker a=\{0\}$, $\ker b=\{0\}$ and $q^2\neq 1$. If $\zeta\in\mathscr{D}$ is an analytic vector for a and b, then $\zeta=0$.

This paper is organized as follows. In Section 2 we collect a few basic definitions and some terminology which will be used freely throughout the sequel. In Section 3 we define and study a-integrable representations and integrable representations $\{a, b\}$ of (1.1) in case when the self-adjoint operators a resp. a and b are non-singular and either positive or negative. The structure of these representations can completely described in terms of operators e^Q and $e^{\alpha P}$, $\alpha\in\mathbf{R}$, where $P=i\frac{d}{dx}$ and $Q=x$ are the canonical operators. In Section 4 we derive our definition of a-integrability in the general case. We shall define this notion first for pairs $\{a, \mathscr{b}\}$, where $\mathscr{b}$ is a self-adjoint operator matrix of the form indicated above, and then for pairs $\{a, b\}$, where b is a self-adjoint operator. On the technical level, it is much easier and more convenient to work with a-integrable pairs $\{a, \mathscr{b}\}$ and we shall do this in a large part of this paper, even if the self-adjoint operator matrix $\mathscr{b}$ may not represent a self-adjoint operator. Section 5 is concerned with a model for a-integrable pairs $\{a, \mathscr{b}\}$, where $\mathscr{b}$ is a self-adjoint operator matrix. Criteria for the unitary equivalence and for the irreducibility of these pairs are obtained. Section 6 provides sufficient conditions which ensure that certain self-adjoint operator matrices give densely defined symmetric operators. The more difficult problem of when such a matrix represents a self-adjoint operator will be considered in a forthcoming paper. In Section 7 we study a general example. Among others it shows that there is a continuum of inequivalent irreducible a-integrable representations $\{a, b\}$ of (1.1) by self-adjoint operators a and b.

After completing the first draft of this paper the author was informed about the interesting work of Yu.S. Samoilenko and the Kiev school (see

[8] and the references therein) on pairs of self-adjoint operators satisfying quadratic relations. In particular, [8] contains a definition of integrability for such relations. However, as the authors of [8] remark on p. 18, there are two relations for which this definition might not be satisfactory, because all integrable representations are trivial. These two relations (denoted by (VII_0) and (VII_1) in [8]) can be reformulated as $ab=qba$ and $ab-qba=\frac{i}{2}(q+1)$, respectively, with $|q|=1$. The present paper could be considered as an attempt to define and to study integrability for the relation $ab=qba$.

It is my pleasure to thank A. Van Daele for many helpful and inspiring conversations on quantum groups and related matters. Also, I would like to thank Ju.S. Samoilenko and L.I. Vainermann for valuable discussions.

§2. Preliminaries

If T is an operator, we write $\mathscr{D}(T)$ for the domain of T and $\ker T$ for the kernel of T. Suppose A is a self-adjoint operator on a Hilbert space $\mathscr{H}$. We say that A is *non-singular* if $\ker A=\{0\}$, that A is *positive* if $\langle A\xi,\xi\rangle >0$ for all $\xi\in\mathscr{D}(A)$, $\xi\neq 0$, and that A is *negative* if $\langle A\xi,\xi\rangle <0$ for all $\xi\in\mathscr{D}(A)$, $\xi\neq 0$. We write $A>0$ if A is positive and $A<0$ if A is negative. Set $\mathscr{H}_+:=e((0,+\infty))\mathscr{H}$, $\mathscr{H}_-:=e((-\infty,0))H$ and $\mathscr{H}_0:=e(\{0\})\mathscr{H}$, where $e(.)$ are the spectral projections of A. The spaces $\mathscr{H}_+$, $\mathscr{H}_-$, $\mathscr{H}_0$ reduce A, so we can write A as $A=A_+\oplus A_-\oplus A_0$ relative to the direct sum $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-\oplus\mathscr{H}_0$. The operators A_+, A_- and A_0 are called the *positive part*, the *negative part* and the *null part* of A, respectively.

Let $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$ be an orthogonal sum of two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$. By a *self-adjoint operator matrix* (w.r.t. the decomposition $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$) we mean a 2×2 matrix $\mathscr{b}=(b_{ij})$ of densely defined linear operators b_{ij} of $\mathscr{H}_j$ into $\mathscr{H}_i$ such that $b_{ij}^*=b_{ji}$ for $i,j=1,2$. Let $\mathscr{b}=(b_{ij})$ be such a matrix. If $\mathscr{D}_j:=\mathscr{D}(b_{jj})\cap\mathscr{D}(b_{ij})$ is dense in $\mathscr{H}_j$ for $i,j\in\{1,2\}$, $i\neq j$, then $b(\xi_1,\ \xi_2):=(b_{11}\xi_1+b_{12}\xi_2,\ b_{21}\xi_1+b_{22}\xi_2)$ defines a symmetric linear operator on the dense domain $\mathscr{D}_1\oplus\mathscr{D}_2$ in $\mathscr{H}=\mathscr{H}_1\oplus\mathscr{H}_2$. Then we shall say that the *matrix represents the operator* b. By a slight abuse of this terminology, we also say that $\mathscr{b}$ represents the operator $\bar{b}$, the closure of b. Operator matrices are always denoted by small script letters, while their represented operators will be denoted by the corresponding letters in italics. Let $\tilde{\mathscr{b}}=(\tilde{b}_{ij})$ be another self-adjoint operator matrix on $\tilde{\mathscr{H}}=\tilde{\mathscr{H}}_1\oplus\tilde{\mathscr{H}}_2$ and let $T_j:\mathscr{H}_j\rightarrow\tilde{\mathscr{H}}_j$,

$j=1,2$, be bounded linear operators. We say that the operator $T:=T_1\oplus T_2$ of $\mathscr{H}$ into $\tilde{\mathscr{H}}$ *intertwines* $\mathscr{b}$ and $\tilde{\mathscr{b}}$ and write $T\mathscr{b}\subseteq\tilde{\mathscr{b}}T$ if $T_ib_{ij}\subseteq\tilde{b}_{ij}T_j$ for all $i,j=1,2$. Clearly, $T\mathscr{b}\subseteq\tilde{\mathscr{b}}T$ is not equivalent to $Tb\subseteq\tilde{b}T$, but we have the following simple lemma.

Lemma 2.1. *Suppose that* $\mathscr{b}$ *and* $\tilde{\mathscr{b}}$ *represent operators* b *and* $\tilde{b}$, *respectively.*

(i) *If* $T\mathscr{b}\subseteq\tilde{\mathscr{b}}T$, *then* $Tb\subseteq\tilde{b}T$.

(ii) *Suppose* $\mathscr{D}_j$ *is a core for the operators* b_{jj} *and* b_{ij}, $i,j\in\{1,2\}$, $i\neq j$. *Then* $Tb\subseteq\tilde{b}T$ *implies that* $T\mathscr{b}\subseteq\tilde{\mathscr{b}}T$.

The operator matrices $\mathscr{b}$ and $\tilde{\mathscr{b}}$ are called *unitarily equivalent* if there exist unitary operators U_j of $\mathscr{H}_j$ onto $\tilde{\mathscr{H}}_j$, $j=1,2$, such that $U_ib_{ij}U_j^{-1}=\tilde{b}_{ij}$ for $i,j=1,2$.

Let $\mathscr{G}$ and $\tilde{\mathscr{G}}$ be Hilbert spaces. We say that a linear operator T of $L^2(\boldsymbol{R})\otimes\mathscr{G}$ into $L^2(\boldsymbol{R})\otimes\tilde{\mathscr{G}}$ is *constant* if T is of the form $T=I\otimes\Lambda$, where $\Lambda:\mathscr{G}\to\tilde{\mathscr{G}}$. For notational simplicity, we shall write Λ instead of $I\otimes\Lambda$ for such an operator. The letters P and Q will always denote the canonical operators, i.e. P is the differential operator $i\dfrac{d}{dx}$ and Q is the multiplication operator by x. Also we shall write simply P, Q, $e^{\omega P}$ and $e^{\omega Q}$ for the operators $P\otimes I$, $Q\otimes I$, $e^{\omega P}\otimes I$ and $e^{\omega Q}\otimes I$, respectively, on a Hilbert space $L^2(\boldsymbol{R})\otimes\mathscr{G}$.

In what follows, q will denote a fixed complex number such that $|q|=1$ and $q^2\neq 1$ and φ will stand for the number of $(0,2\pi)$ such that $q=e^{i\varphi}$.

§3. Integrable Representations in Case when One Operator is Positive or Negative

Apart from Definition 3.2 below, we assume throughout this section that a is a non-singular self-adjoint operator which is either positive or negative.

Definition 3.1. Let $\{a, b\}$ be a pair of self-adjoint operators a and b on a Hilbert space. We shall say that the couple $\{a, b\}$ is an a-integrable representation of the relation (1.1) or briefly that $\{a, b\}$ is *a-integrable* if there exists an integer k such that

$$|a|^{it}\ b=e^{(-\varphi+2\pi k)t}b|a|^{it}\ \text{for}\ t\in\boldsymbol{R}. \tag{3.1}$$

In this case we shall write $\{a, b\} \in \mathscr{C}_k$.

Remarks.

1.) A motivation for the preceding definition was already given in the introduction.

2.) Note that the above definition refers to the relation (1.1) with a fixed parameter q. If relations (1.1) with different parameters occur (as in Section 4 or in Remark 4.) below) and confusion is possible, we will mention the parameter which appears in (1.1) instead of q.

3.) There are various arguments for taking only couples in $\mathscr{C}_0$ as a-integrable representations of (1.1). One reason for this would be that in the classical case $q=1$, $\varphi=0$ the above relation (3.1) is equivalent to the strong commutativity of a and b only if $k=0$. However, in this paper we shall consider $\mathscr{C}_k$ with arbitrary $k \in \mathbb{Z}$.

4.) One advantage of the above Definition 3.1 (with arbitrary integers k) is that the following fact is true: If $\{a, b\}$ is a-integrable, then for all $n,m \in \mathbb{N}$ such that $q^{nm} \neq \pm 1$ the couple $\{a^n, b^m\}$ is a^n-integrable with respect to the parameter q^{nm}. The preceding assertion is not valid in general if we define a-integrability only by the class $\mathscr{C}_0$ as indicated in Remark 2.).

Suppose for a moment we have already defined a-integrable representations of (1.1) for arbitrary self-adjoint operators a. (This will be done in Section 4.) Then we can give

Definition 3.2. A pair $\{a, b\}$ of self-adjoint operators a and b on a Hilbert space is called an *integrable representation* of (1.1) if $\{a, b\}$ is an a-integrable representation of the relation $ab=qba$ and if $\{b, a\}$ is a b-integrable representation of the relation $ba=\bar{q}ab$.

The next proposition provides a model for couples of the class $\mathscr{C}_k$.

Proposition 3.3. *Suppose that k is an integer.*

Let $\mathscr{H}_0$ be a Hilbert space and let a_0 be a self-adjoint operator on $\mathscr{H}_0$ such that $a_0>0$. Let $\mathscr{K}$ be another Hilbert space and let E be an orthogonal projection on $\mathscr{K}$. Let $\varepsilon \in \{+1, -1\}$. Define self-adjoint operators $\tilde{a}$ and $\tilde{b}$ on the Hilbert space $\mathscr{H} := L^2(\mathbb{R}) \otimes \mathscr{K} \oplus \mathscr{H}_0$ by

$$\tilde{a} := \varepsilon e^{Q} \oplus \varepsilon a_0 \quad \textit{and} \quad \tilde{b} := e^{(-\varphi+2\pi k)P}(2E-1) \oplus 0. \tag{3.2}$$

Then the couple $\{\tilde{a}, \tilde{b}\}$ is an a-integrable representation of (1.1) *and* $\{\tilde{a}, \tilde{b}\} \in \mathscr{C}_k$.

Conversely, each couple $\{a, b\}\in\mathscr{C}_k$ *is unitarily equivalent to a couple* $\{\tilde{a}, \tilde{b}\}$ *of the form described by* (3.2).

Proof. The first assertion is easily verified. We omit the details. To prove the second assertion, we assume that $\{a, b\}$ is a couple of the class $\mathscr{C}_k$. Then, by Definition 3.1, we have

$$|a|^{it}b|a|^{-it}=e^{(-\varphi+2\pi k)t}b,\ t\in\boldsymbol{R}. \tag{3.3}$$

We write b as an orthogonal direct sum $b=b_+\oplus b_-\oplus 0$ of its positive part b_+, its negative part b_- and its null part. Let $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-\oplus\mathscr{H}_0$ be the corresponding decomposition of the underlying Hilbert space $\mathscr{H}$. From (3.3) we conclude that the spaces $\mathscr{H}_+$, $\mathscr{H}_-$ and $\mathscr{H}_0$ reduce the unitary group $|a|^{it}\equiv e^{it\log|a|}$, hence they reduce its generator $\log|a|$ and so $|a|$. Since either $a>0$ or $a<0$ by assumption, these three spaces reduce the operator a itself. Thus we can write $a=\varepsilon a_1\oplus\varepsilon a_2\oplus\varepsilon a_0$, where $\varepsilon=$sign a and $a_j>0$ for $j=1,2,0$. Obviously, $\{a_1,b_+\}\in\mathscr{C}_k$ and $\{a_2,-b_-\}\in\mathscr{C}_k$. Definition 3.1, applied to the pair $\{a_1, b_+\}$, yields

$$a_1^{it}\ b_+\ a_1^{-it}=e^{(-\varphi+2\pi k)t}\ b_+,\ t\in\boldsymbol{R}.$$

Since the functional calculus for self-adjoint operators is invariant under unitary transformations and since $b_+>0$, the latter equality implies that

$$a_1^{it}\ b_+^{is}\ a_1^{-it}=(e^{i(-\varphi+2\pi k)t}\ b_+)^{is}=e^{i(-\varphi+2\pi k)ts}\ b_+^{is}$$

for s, $t\in\boldsymbol{R}$. Hence the unitary groups $U(t):=a_1^{it}$ and $V(s):=b_+^{i(-\varphi+2\pi k)^{-1}s}$ satisfy the Weyl relation $U(t)V(s)=e^{its}V(s)U(t)$, s, $t\in\boldsymbol{R}$. From the Stone-von Neumann uniqueness theorem (cf. [10], Theorem 4.3.1) it follows that there exists a Hilbert space $\mathscr{K}_+$ such that the pair $\{a_1, b_+\}$ on $\mathscr{H}_+$ is unitarily equivalent to the pair $\{e^Q, e^{(-\varphi+2\pi k)P}\}$ on the Hilbert space $L^2(\boldsymbol{R})\otimes\mathscr{K}_+$. By the same reasoning, applied to the pair $\{a_2,-b_-\}$, we conclude that there is a Hilbert space $\mathscr{K}_-$ such that $\{a_2,-b_-\}$ is unitarily equivalent to the pair $\{e^Q, e^{(-\varphi+2\pi k)P}\}$ on $L^2(\boldsymbol{R})\otimes\mathscr{K}_-$. Put $\mathscr{K}:=\mathscr{K}_+\oplus\mathscr{K}_-$ and let E be the orthogonal projection of $\mathscr{K}$ onto $\mathscr{K}_+$. Then, by the preceding, $\{a, b\}$ is unitarily equivalent to the pair $\{\tilde{a}, \tilde{b}\}$ on $L^2(\boldsymbol{R})\otimes\mathscr{K}\oplus\mathscr{H}_0$, where $\tilde{a}$ and $\tilde{b}$ are as in (3.2). □

An immediate consequence of Proposition 3.3 is

Corollary 3.4. *Suppose $k\in\mathbb{Z}$. Apart from the trivial one-dimensional representations in $\mathscr{C}_k$ (i.e. $a\in\mathbb{R}\backslash\{0\}$ and $b=0$ on $\mathscr{H}=\mathbb{C}$), there are precisely four irreducible pairwise inequivalent pairs of the class $\mathscr{C}_k$. Up to unitary equivalence, these are the pairs $\{\varepsilon_1 e^Q, \varepsilon_2 e^{(-\varphi+2\pi k)P}\}$ on the Hilbert space $L^2(\mathbb{R})$, where $\varepsilon_1, \varepsilon_2\in\{1, -1\}$.*

The a-integrable representations of (1.1) were originally defined only by relation (3.1). The next corollary shows that (3.1) implies (1.1) on a suitable domain.

Corollary 3.5. *Suppose that $\{a, b\}$ is an a-integrable representation of (1.1) on a Hilbert space $\mathscr{H}$. Then there exists a dense linear subspace $\mathscr{D}$ of $\mathscr{H}$ such that:*

(i) $\mathscr{D}\subseteq\mathscr{D}(a)\cap\mathscr{D}(b)$, $a\mathscr{D}\subseteq\mathscr{D}$, $b\mathscr{D}\subseteq\mathscr{D}$.

(ii) *$\mathscr{D}$ is a core for a and b.*

(iii) *$ab\psi=qba\,\psi$ for $\psi\in\mathscr{D}$.*

Proof. Let $\mathscr{F}_0$ be the linear span of functions $e^{-\delta x^2+\gamma x}$ in $L^2(\mathbb{R})$, where $\delta>0$ and $\gamma\in\mathbb{C}$. It is not difficult to prove that $\mathscr{F}_0\subseteq\mathscr{D}(e^{\alpha Q})$ is a core for the self-adjoint operator $e^{\alpha Q}$ for each $\alpha\in\mathbb{R}$. Since $e^{-\alpha Q}$ is unitarily equivalent to $e^{\alpha P}$ by the Fourier transform and $\mathscr{F}_0$ is obviously invariant under the Fourier transform, this implies that $\mathscr{F}_0\subseteq\mathscr{D}(e^{\alpha P})$ is also a core for $e^{\alpha P}$. Applying the Fourier transform, $e^{-\alpha Q}$ and then the inverse Fourier transform, it follows easily that $(e^{\alpha P}\eta)\ (x)=\eta(x+i\alpha)$ for $\eta\in\mathscr{F}_0$, $x\in\mathbb{R}$ and $\alpha\in\mathbb{R}$. Hence $\mathscr{F}_0$ is invariant under $e^{\alpha P}$ for $\alpha\in\mathbb{R}$ and $e^Q\, e^{(-\varphi+2\pi k)P}\eta=q\, e^{(-\varphi+2\pi k)P}\, e^Q\eta$ for $\eta\in\mathscr{F}_0$ and $k\in\mathbb{Z}$. It is trivial that $\mathscr{F}_0$ is invariant under e^Q.

If $\{a, b\}$ is an a-integrable representation of (1.1), we can assume, by Proposition 3.3, that the operators a and b are of the form (3.2). But then we see immediately from the preceding paragraph that the domain $\mathscr{D}:=\mathscr{F}_0\otimes\mathscr{K}\oplus\bigcap_{n=1}^{\infty}\mathscr{D}(a_0^n)$ has the desired properties (i)-(iii). □

Proposition 3.6. *Let $\{a, b\}$ be a pair of self-adjoint operators on a Hilbert space. Suppose that either $b>0$ or $b<0$. (Recall, as always in this section, we also have either $a>0$ or $a<0$.) Then the following four conditions are equivalent:*

(i) *There is an integer k such that the self-adjoint operators $A:=\log|a|$ and $B:=\log|b|$ satisfy the Weyl relation*

$$e^{itA}\quad e^{isB}=e^{i(-\varphi+2\pi k)ts}\quad e^{isB}\quad e^{itA},\quad s,\quad t\in R. \tag{3.4}$$

(ii) $\{a, b\}$ *is an a-integrable representation of* $ab=qba$.

(iii) $\{b, a\}$ *is a b-integrable representation of* $ba=\bar{q}ab$.

(iv) $\{a, b\}$ *is an integrable representation of* $ab=qba$.

Proof. (i)→(ii): By the Stone-von Neumann uniqueness theorem ([10], Theorem 4.3.1), there exists a Hilbert space $\mathcal{K}$ such that up to unitary equivalence we have $A=Q$ and $B=(-\varphi+2\pi k)P$, that is, $|a|=e^Q$ and $|b|=e^{(-\varphi+2\pi k)P}$, on the Hilbert space $L^2(R)\otimes\mathcal{K}$. Since we assumed that either $a>0$ or $a<0$ and that either $b>0$ or $b<0$, we get $a=\varepsilon_1 e^Q$ and $b=\varepsilon_2 e^{(-\varphi+2\pi k)P}$ with ε_1, $\varepsilon_2\in\{1,-1\}$. But the latter pair $\{a, b\}$ is obviously in $\mathscr{C}_k$, so $\{a, b\}$ is a-integrable.

(ii)→(i): Suppose $\{a, b\}\in\mathscr{C}_k$. Then, by Proposition 3.3, we can assume that a and b are of the form (3.2). Since $\ker b=\{0\}$ by assumption, $\mathscr{H}_0=\{0\}$ and hence $|b|=e^{(-\varphi+2\pi k)P}$ and also $|a|=e^Q$. From this (3.4) follows.

We have just shown the equivalence of (i) and (ii). Replacing $\{a, b\}$ by $\{b,a\}$ and q by $\bar{q}$, it follows that (i) and (iii) are equivalent. Recall that by Definition 3.2 (iv) means (ii) and (iii) together. Therefore, all four conditions are equivalent. □

§4. The Definition of a-Integrable Representations in the General Case

The main aim of this section is to derive a definition of a-integrability for a pair $\{a, b\}$ of arbitrary self-adjoint operators a and b on a Hilbert space $\mathscr{H}$. First let us note that we can assume without loss of generality that the operator a is non-singular, i.e. $\ker a=\{0\}$. Indeed, if a is arbitrary, let $\mathscr{H}_0:=\ker a$ and write $a=a_1\oplus 0$ on $\mathscr{H}=\mathscr{H}_0^\perp\oplus\mathscr{H}_0$. Then a_1 is non-singular. We shall say that $\{a, b\}$ is *a-integrable* if there are self-adjoint operators b_1 and b_0 on $\mathscr{H}_0^\perp$ and $\mathscr{H}_0$, respectively, such that $b=b_1\oplus b_0$ and $\{a_1,b_1\}$ is a_1-integrable. For the rest of this section we suppose that the self-adjoint operator a is *non-singular.*

Since $\ker a=\{0\}$, we can decompose a as an orthogonal direct sum $a=a_+\oplus a_-$ on $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$, where a_+ and a_- are the positive resp. negative part of a. Suppose that the self-adjoint operator b is represented by a self-adjoint operator matrix $\mathscr{b}=(b_{ij})_{i,j=1,2}$ relative to the decomposition

$\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$.

The following notations and formulas are often needed in the sequel. Let $b_{12}=u|b_{12}|$ be the polar decomposition of the closed operator b_{12} and let p_{12} and p_{21} denote the orthogonal projections of $\mathscr{H}_-$ resp. $\mathscr{H}_+$ onto $\ker b_{12}$ resp. $\ker b_{21}$. Recall that u is a partial isometry with initial space $\overline{|b_{12}|\mathscr{H}_-}=(\ker|b_{12}|)^\perp=(\ker b_{12})^\perp=(1-p_{12})\mathscr{H}_-$ in $\mathscr{H}_-$ and final space $\overline{b_{12}\mathscr{H}_-}=(\ker b_{12}^*)^\perp=(\ker b_{21})^\perp=(1-p_{21})\mathscr{H}_+$ in $\mathscr{H}_+$. In particular, we have that

$$u^*u=1-p_{12} \text{ and } uu^*=1-p_{21}. \tag{4.1}$$

In what follows we will develop some arguments which in turn lead to the precise definitions given below (cf. Definitions 4.1 and 4.2). For this reasoning we will ignore mainly domain questions for the corresponding operators. (Roughly speaking, everything will be correct on suitable domains.)

By matrix multiplication the relation $ab=qba$ is equivalent to the four equations

$$a_+b_{11}=q\ b_{11}a_+ \tag{4.2}$$

$$a_-b_{22}=q\ b_{22}a_- \tag{4.3}$$

$$a_+b_{12}=q\ b_{12}a_- \tag{4.4}$$

$$a_-b_{21}=q\ b_{21}a_+. \tag{4.5}$$

Recall that $a_+>0$ and $a_-<0$ and that b_{11} and b_{22} are self-adjoint operators, so the a_+-integrability of (4.2) and the a_--integrability of (4.3) are well-defined according to Definition 3.1. These are the first two requirements (D.1) and (D.2) in Definition 4.1 below.

To derive the other parts of Definition 4.1, we essentially work with (4.4) and (4.5). From these two relations we obtain

$$a_+|b_{21}|^2=a_+b_{21}^*b_{21}=a_+b_{12}b_{21}=q\ b_{12}a_-b_{21}=q^2\ b_{12}b_{21}a_+=$$
$$q^2\ b_{21}^*b_{21}a_+=q^2|b_{21}|^2a_+.$$

For the relation $a_+|b_{21}|^2=q^2|b_{21}|^2a_+$ we know already how to define a_+-integrability. It means that there exists a $k\in\mathbb{Z}$ such that

$$a_+^{it}|b_{21}|^2a_+^{-it}=e^{(-2\varphi+2\pi k)t}|b_{21}|^2,\quad t\in\mathbb{R}. \tag{4.6}$$

(Note that $q^2 \neq 1$ as assumed in Section 2, so that Definition 3.1 applies.) Taking the square roots on both sides of (4.6), we get

$$a_+^{it}|b_{21}|a_+^{-it} = e^{(-\varphi+\pi k)t}|b_{21}|, \ t \in \boldsymbol{R}. \tag{4.7}$$

We will show that the integer k in (4.7) can be taken as odd. For this we suppose that k is even, i.e. $k=2n$ with $n \in \boldsymbol{Z}$. But then (4.7) means that the pair $\{a_+|b_{21}|\}$ is a_+-integrable. Hence, by Corollary 3.5, there is a suitable domain such that

$$a_+|b_{21}| = q|b_{21}|a_+.$$

Since $b_{12} = |b_{12}^*|u = |b_{21}|u$ by general properties of the polar decomposition, this gives

$$a_+b_{12} = a_+|b_{21}|u = q|b_{21}|a_+u \quad \text{and} \quad qb_{12}a_- = q|b_{21}|ua_-.$$

Comparing these two relations with (4.4), we obtain

$$(1-p_{21})(a_+u - ua_-) = 0.$$

Using (4.1), this leads to

$$(1-p_{21})a_+(1-p_{21}) = ua_-u^*. \tag{4.8}$$

Let c denote the restriction $ua_-u^* \restriction (1-p_{21})\mathscr{H}_+$. Since $a_+ > 0$ and $a_- < 0$, we conclude from (4.8) that $c > 0$ and $c < 0$ on the Hilbert space $(1-p_{21})\mathscr{H}_+$. This is only possible if $p_{21}=1$, that is, if $|b_{21}|=0$. Thus we have shown that the integer k in (4.7) must be odd if $|b_{21}| \neq 0$. In case $|b_{21}|=0$ the relation (4.7) is trivially fulfilled for any k. Therefore, we can assume that k is odd. But then (4.7) says that the couple $\{a_+, |b_{21}|\}$ is a_+-integrable with respect to the parameter $e^{(\varphi-\pi)i} \equiv -q$. This is condition (D.3) in Definition 4.1 below. A similar reasoning leads to condition (D.4).

We still need a condition which connects the actions of the operators on $\mathscr{H}_+$ (in (D.1) and (D.3)) with the actions of the operators on $\mathscr{H}_-$ (in (D.2) and (D.4)). By (D.3), the pair $\{a_+, |b_{21}|\}$ is a_+-integrable with respect to $-q$. Therefore, we have

$$a_+|b_{21}| = -q|b_{21}|a_+$$

by Corollary 3.5. Combined with (4.4), this gives

$$a_+b_{12}=a_+|b_{21}|u=-q|b_{21}|a_+u=qb_{12}a_-=q|b_{21}|ua_-,$$

so that $(1-p_{21})(a_+u+ua_-)=0$.

By (4.1), this in turn yields

$$(1-p_{21})a_+(1-p_{21})\equiv uu^*a_+uu^*=u(-a_-)u^*$$

which is the last condition (D.5) in Definition 4.1.

We summarize the outcome of the preceding discussion in

Definition 4.1. Suppose a is a non-singular self-adjoint operator on a Hilbert space $\mathscr{H}$ with decomposition $a=a_+\oplus a_-$ on $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$ into its positive part a_+ and its negative part a_-. Suppose $\mathscr{b}=(b_{ij})_{i,j=1,2}$ is a self-adjoint operator matrix with respect to the decomposition $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$. Let $b_{12}=u|b_{12}|$ be the polar decomposition of b_{12}. We shall say that the pair $\{a, \mathscr{b}\}$ is an *a-integrable representation* of (1.1) if the following five conditions are fulfilled:

(D.1) $\{a_+,b_{11}\}$ is a_+-integrable with respect to the parameter q.
(D.2) $\{a_-,b_{22}\}$ is a_--integrable with respect to the parameter q.
(D.3) $\{a_+,|b_{21}|\}$ is a_+-integrable with respect to the parameter $-q$.
(D.4) $\{a_-,|b_{12}|\}$ is a_--integrable with respect to the parameter $-q$.
(D.5) $uu^*a_+uu^*=u(-a_-)u^*$.

Definition 4.2. Let a be as in Definition 4.1 and let b be another self-adjoint operator on $\mathscr{H}$. We say that the couple $\{a, b\}$ is an *a-integrable representation* of (1.1) if there exists a self-adjoint operator matrix $\mathscr{b}$ with respect to $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$ such that $\mathscr{b}$ represents the operator b (cf. section 2) and $\{a, \mathscr{b}\}$ is an a-integrable representation of (1.1).

Remarks.

1.) Obviously (D.5) is equivalent to

(D.5)′ $\omega^*ua_-u^*u=u^*(-a_+)u$.

That is, condition (D.5) is in fact symmetric with respect to a_+ and a_-.

2.) Often the following equivalent forms of (D.5) and (D.5)′ are more convenient:

$$(1-p_{21})a_+(1-p_{21})=u(-a_-)u^*,\quad (1-p_{12})a_-(1-p_{12})=u^*(-a_+)u. \tag{4.9}$$

The above formulation of Definition 4.1 is useful for applications, but it can be weakened. For this we introduce the following conditions:

(D.3)′ The unitary group $t \to a_+^{it}$ reduces $p_{21}H_+ \equiv \ker|b_{21}|$.

(D.4)′ The unitary group $t \to |a_-|^{it}$ reduces $p_{12}H_- \equiv \ker|b_{12}|$.

Proposition 4.3. *Equivalent formulations of Definition* 4.1 *are obtained if we replace* (D.3) *by* (D.3)′ *or if we replace* (D.4) *by* (D.4)′.

Proof. We carry out the proof for (D.4). The condition (D.4) says that there exists a $k \in \boldsymbol{Z}$ such that

$$|a_-|^{it}|b_{12}||a_-|^{-it} = e^{(-\varphi+\pi+2\pi k)t}|b_{12}| \text{ for } t \in \boldsymbol{R}, \tag{4.10}$$

hence (D.4) obviously implies (D.4)′.

Conversely, assume that (D.3), (D.4)′ and (D.5) are satisfied. By (D.3), there is an integer k such that

$$a_+^{it}|b_{21}|a_+^{-it} = e^{(-\varphi+\pi+2\pi k)t}|b_{21}| \quad \text{for} \quad t \in \boldsymbol{R}. \tag{4.11}$$

In order to prove (D.4), it suffices to verify (4.10). Since $|a_-|^{it}$ reduces $p_{12}\mathscr{H}_-$ by (D.4)′, (4.10) is trivially fulfilled for vectors in $p_{12}\mathscr{H}_-$. Thus it remains to verify (4.10) for vectors in $(I-p_{12})\mathscr{H}_-$. Let $\psi \in \mathscr{D}(|b_{12}|)$ be such a vector. From (D.5)′ (which is equivalent to (D.5) be the above remark) we conclude easily that

$$(1-p_{12})|a_-|^{it}(1-p_{12}) = u^* a_+^{it} u. \tag{4.12}$$

Recall that $b_{12} = |b_{21}|u$, $|b_{21}| = u|b_{12}|u^*$ and $u^*u|b_{12}| = |b_{12}|$ by general properties of the polar decomposition (see, for instance, [4], p.335). Using (4.11), (4.12) and these facts, we obtain

$$\begin{aligned} |a_-|^{it}|b_{12}|\psi &= (1-p_{12})|a_-|^{it}(1-p_{12})|b_{12}|\psi = u^* a_+^{it} u|b_{12}|\psi \\ &= u^* a_+^{it} b_{12}\psi = u^*(a_+^{it}|b_{21}|)u\psi \\ &= u^*(e^{(-\varphi+\pi+2\pi k)t}|b_{21}|a_+^{it})u\psi \\ &= u^* e^{(-\varphi+\pi+2\pi k)t} u|b_{12}|u^* a_+^{it} u\psi \\ &= e^{(-\varphi+\pi+2\pi k)t} u^* u|b_{12}|(1-p_{12})|a_-|^{it}(1-p_{12})\psi \\ &= e^{(-\varphi+\pi+2\pi k)t}|b_{12}||a_-|^{it}\psi. \end{aligned}$$

This completes the proof of (4.11) and so of (D.4). □

Remarks.

3.) Let $\{a, \mathscr{b}\}$ be an a-integrable pair in the sense of Definition 4.1, where $\mathscr{b}=(b_{ij})$ is a self-adjoint operator matrix on $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$. Suppose in addition that

$$\ker b_{12}=\{0\} \quad \text{and} \quad \ker b_{21}=\{0\}. \tag{4.13}$$

Then u is a unitary operator of $\mathscr{H}_-$ onto $\mathscr{H}_+$ and condition (D.5) means that $-a_+=ua_-u^*$. Therefore, if v denotes the unitary operator $1\oplus u$ of $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$ onto $\mathscr{H}_+\oplus\mathscr{H}_+$, we have $vav^*=a_+\oplus(-a_+)$ and

$$v\mathscr{b}v^*=\begin{pmatrix} b_{11} & b_{21} \\ b_{21} & ub_{22}u^* \end{pmatrix},$$

where $b_{21}>0$. In other words, if $\{a, \mathscr{b}\}$ is an a-integrable pair such that $\ker b_{12}=\{0\}$ and $\ker b_{21}=\{0\}$, then we can assume, after a unitary transformation, that $\mathscr{H}_+=\mathscr{H}_-$, $a_-=-a_+$ and $b_{12}=b_{21}>0$.

Proposition 4.4. *Let $\mathscr{K}$, $\mathscr{K}_1$ and $\mathscr{K}_2$ be Hilbert spaces and let E_1 and E_2 be orthogonal projections in $\mathscr{K}_1$ resp. $\mathscr{K}_2$. Suppose that $w_j: L^2(\mathbf{R})\otimes\mathscr{K}_j\to L^2(\mathbf{R})\otimes\mathscr{K}$, $j=1,2$, are isometries which interwine the unitary groups e^{itQ} (i.e. $w_je^{itQ}=e^{itQ}w_j$ for $t\in\mathbf{R}$, $j=1,2$). Let $k, k_1, k_2\in\mathbf{Z}$ and set $\alpha:=-\varphi+\pi+2\pi k$ and $\alpha_j:=-\varphi+2\pi k_j$, $j=1,2$. We define a self-adjoint operator a and a self-adjoint operator matrix $\mathscr{b}$ on the Hilbert space $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$, where $\mathscr{H}_+=\mathscr{H}_-:=L^2(\mathbf{R})\otimes\mathscr{K}$, by $a:=e^Q\oplus(-e^Q)$ and*

$$\mathscr{b}:=\begin{pmatrix} w_1e^{\alpha_1P}(2E_1-1)w_1^* & e^{\alpha P} \\ e^{\alpha P} & w_2e^{\alpha_2P}(2E_2-1)w_2^* \end{pmatrix}, \tag{4.14}$$

Then the pair $\{a, \mathscr{b}\}$ is a-integrable and (4.13) *is fulfilled.*

Conversely, each a-integrable pair $\{a, \mathscr{b}\}$ (according to Definition 4.1) *of a self-adjoint operator a and a self-adjoint operator matrix $\mathscr{b}$ which satisfies* (4.13) *is unitarily equivalent to a pair of the above form.*

Proof. The first assertion follows by a straightforward verification. We sketch the proof of the second assertion. For this let $\{a, \mathscr{b}\}$ be an a-integrable pair (in the sense of Definition 4.1) such that $\ker b_{12}=\{0\}$ and $\ker b_{21}=\{0\}$, where $\mathscr{b}=(b_{ij})$. By the above Remark 3.) we can assume without loss of generality that $\mathscr{H}_+=\mathscr{H}_-$, $a_-=-a_+$ and $b_{12}=b_{21}>0$. By (D.3), $\{a_+, b_{21}\}$

is a_+-integrable with respect to the parameter $-q$. Since $a_+>0$ and $b_{21}>0$, we conclude from Proposition 3.3 (or from the Stone-von Neumann uniqueness theorem, cf. Proposition 3.6) that there exist a Hilbert space $\mathscr{K}$ and an integer k such that $\{a_+, b_{21}\}$ is unitarily equivalent to the pair $\{e^Q, e^{(-\varphi+\pi+2\pi k)P}\}$ on the Hilbert space $L^2(\mathbb{R})\otimes\mathscr{K}$. For notational simplicity, let us identify $\mathscr{H}_+=\mathscr{H}_-$ with $L^2(\mathbb{R})\otimes\mathscr{K}$, a_+ with e^Q *and* $b_{21}=b_{12}$ with $e^{(-\varphi+\pi+2\pi k)P}$. By (D.1), the pair $\{a_+, b_{11}\}$ is a_+-integrable with respect to the parameter q, i.e. we have $\{a_+, b_{11}\}\in\mathscr{C}_{k_1}$ for some $k_1\in\mathbb{Z}$. Now we apply again Proposition 3.3. There are Hilbert spaces $\mathscr{K}_1$ and $\mathscr{H}_{01}$, an orthogonal projection E_1 on $\mathscr{K}_1$, a selfadjoint operator a_{01} *on* $\mathscr{H}_{01}$ and a unitary operator v_1 of $L^2(\mathbb{R})\otimes\mathscr{K}_1\oplus\mathscr{H}_{01}$ onto $\mathscr{H}_+=L^2(\mathbb{R})\otimes\mathscr{K}$ such that

$$v_1(e^Q\oplus a_{01})v_1^*=a_+\equiv e^Q \tag{4.15}$$

and

$$v_1(e^{(-\varphi+2\pi k_1)P}(2E_1-1)\oplus 0)v_1^*=b_{11}. \tag{4.16}$$

The restriction $w_1:=v_1\lceil L^2(\mathbb{R})\otimes\mathscr{K}_1$ is an isometry into $L^2(\mathbb{R})\otimes\mathscr{K}$ which obviously satisfies $w_1e^{(-\varphi+2\pi k_1)P}(2E_1-1)w_1^*=b_{11}$ by (4.16). From (4.15) we get $v_1(e^{itQ}\oplus a_{01}^{it})v_1^*=e^{itQ}$ and hence $w_1e^{itQ}=e^{itQ}w_1$ for $t\in\mathbb{R}$. Thus we have shown that the matrix elements b_{11}, b_{12} and b_{21} have the desired form. The proof for b_{22} is quite similar to the proof for b_{11}. □

§5. A Model for a-integrable Representations

The model mentioned in the heading is defined as follows:

Let $\mathscr{K}_+$, $\mathscr{K}_-$, $\mathscr{K}$, $\mathscr{K}_1$ and $\mathscr{K}_2$ be Hilbert spaces and let E_1 and E_2 be orthogonal projections on $\mathscr{K}_1$ and $\mathscr{K}_2$, respectively. Put $\mathscr{H}_+:=L^2(\mathbb{R})\otimes\mathscr{K}_+$ and $\mathscr{H}_-:=L^2(\mathbb{R})\otimes\mathscr{K}_-$. Let $w_{11}:L^2(\mathbb{R})\otimes\mathscr{K}_1\rightarrow\mathscr{H}_+$, $w_{22}:L^2(\mathbb{R})\otimes\mathscr{K}_2\rightarrow\mathscr{H}_-$ and $w_{12}:L^2(\mathbb{R})\otimes\mathscr{K}\rightarrow\mathscr{H}_+$ be isometries and let u be a partial isometry of $\mathscr{H}_+$ into $\mathscr{H}_-$ with initial space $w_{12}(L^2(\mathbb{R})\otimes\mathscr{K})$. Suppose that the operators w_{11}, w_{22}, w_{12} and u intertwine the unitary groups e^{itQ}, $t\in\mathbb{R}$. Let k, k_1, $k_2\in\mathbb{Z}$ and set $\alpha_{jj}:=-\varphi+2\pi k_j$ for $j=1,2$ and $\alpha_{12}=\alpha_{21}:=-\varphi+\pi+2\pi k$. Define a self-adjoint operator a and a self-adjoint operator matrix $\mathscr{b}\equiv(b_{ij})$ on $\mathscr{H}:=\mathscr{H}_+\oplus\mathscr{H}_-$ by

$$a:=e^Q\oplus(-e^Q) \tag{5.1}$$

and

$$b:=\begin{pmatrix} w_{11}e^{\alpha_{11}P}(2E_1-1)w_{11}^* & uw_{12}e^{\alpha_{12}P}w_{12}^* \\ w_{12}e^{\alpha_{21}P}w_{12}^*u^* & w_{22}e^{\alpha_{22}P}(2E_2-1)w_{22}^* \end{pmatrix}. \tag{5.2}$$

Then it is not difficult to see that the couple $\{a, \mathscr{b}\}$ fulfills the conditions in Definition 4.1, so $\{a, \mathscr{b}\}$ is an a-integrable representations of (1.1).

For $i,j=1,2$, let p_{ij} denote the orthogonal projection of the corresponding Hilbert space onto $\ker b_{ij}$. Clearly, $\ker b_{ij}$ is the orthogonal complement of the range of w_{ij}, so $1-p_{ij}=w_{ij}w_{ij}^*$. Also, it is easy to see that the positive part and the negative part of the operator b_{jj}, $j=1,2$, are

$$w_{jj}e^{\alpha_{jj}P}E_jw_{jj}^*\restriction w_{jj}(L^2(\mathbb{R})\otimes E_j\mathscr{K}_j)$$

and $$w_{jj}e^{\alpha_{jj}P}(1-E_j)w_{jj}^*\restriction w_{jj}(L^2(\mathbb{R})\otimes(1-E_j)\mathscr{K}_j),$$

respectively.

Remarks.

1.) The above model is more symmetric in the matrix entries of $\mathscr{b}$ than the model occuring in Proposition 4.4. Also, it is more general, since the operators b_{12} and b_{21} may have non-trivial kernels.

2.) Suppose that $\{a, \mathscr{b}\}$ is an a-integrable pair such that a is as stated in (5.1), i.e. $a_+=e^Q$ on $\mathscr{H}_+=L^2(\mathbb{R})\otimes\mathscr{K}_+$ and $a_-=-e^Q$ on $\mathscr{H}_-=L^2(\mathbb{R})\otimes\mathscr{K}_-$ for certain Hilbert spaces $\mathscr{K}_+$ and $\mathscr{K}_-$. Then it can be shown that $\mathscr{b}$ is of the form (5.2).

In order to obtain criteria for the irreducibility or the unitary equivalence of pairs of the above model, we study bounded linear operators which intertwine two pairs.

Theorem 5.1. *Let $\{a, \mathscr{b}\}$ and $\{\tilde{a}, \tilde{\mathscr{b}}\}$ be two pairs of the form described above. Suppose that T is a bounded linear operator from $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$ into $\tilde{\mathscr{H}}=\tilde{\mathscr{H}}_+\oplus\tilde{\mathscr{H}}_-$ such that $Ta\subseteq\tilde{a}T$. Then there are bounded linear operators $T_1:\mathscr{H}_+\to\tilde{\mathscr{H}}_+$ and $T_2:\mathscr{H}_-\to\tilde{\mathscr{H}}_-$ such that $T=T_1\oplus T_2$.*

Suppose in addition that $\alpha_{ij}=\tilde{\alpha}_{ij}$ for all $i,j\in\{1,2\}$. (Notations with tilde always refer to the pair $\{\tilde{a}, \tilde{\mathscr{b}}\}$.) Then the operator $T=T_1\oplus T_2$ intertwines the pairs $\{a, \mathscr{b}\}$ and $\{\tilde{a}, \tilde{\mathscr{b}}\}$ (that is, $Ta\subseteq\tilde{a}T$ and $T\mathscr{b}\subseteq\tilde{\mathscr{b}}T$) if and only if the following three conditions are fulfilled:

(i) *For* $j=1,2$, $T_j p_{jj}=\tilde{p}_{jj}T_j$ *and this operator intertwines the unitary groups* e^{itQ}, $t\in\boldsymbol{R}$.
(ii) *For* $j=1,2$, $\Lambda_j:=\tilde{w}_{jj}^* T_j w_{jj}$ *is constant and* $\Lambda_j E_j=\tilde{E}_j\Lambda_j$.
(iii) $T_1 u=\tilde{u}T_2$ *and* $\Lambda:=\tilde{w}_{12}^* T_2 w_{12}$ *is constant.*

The proof of Theorem 5.1 depends on the following two simple lemmas.

Lemma 5.2. *Suppose that* A_1 *and* A_2 *are non-negative self-adjoint operators on Hilbert spaces* $\mathscr{H}_1$ *resp.* $\mathscr{H}_2$, *where* A_1 *or* A_2 *is non-singular. If* B *is a bounded linear operator from* $\mathscr{H}_1$ *into* $\mathscr{H}_2$ *such that* $-BA_1\subseteq A_2B$, *then* $B=0$.

Proof. From $-BA_1\subseteq A_2B$ we get $-B^*A_2\subseteq -(A_2B)^*\subseteq (BA_1)^*=A_1B^*$ and so $|B|^2A_1=B^*BA_1\subseteq B^*(-A_2B)=(-B^*A_2)B\subseteq A_1B^*B=A_1|B|^2$, i.e. $|B|^2$ commutes with the self-adjoint operator A_1. Hence $|B|$ commutes with A_1 as well.

Let $B=U|B|$ be the polar decomposition of B. Fix $\eta\in\mathscr{D}(A_1)$. We have $-BA_1\eta=A_2B\eta$ and $|B|A_1\eta=A_1|B|\eta$, so that $-\langle A_2B\eta,\ B\eta\rangle=\langle BA_1\eta,\ B\eta\rangle=\langle U|B|A_1\eta,\ U|B|\eta\rangle=\langle |B|A_1\eta,\ |B|\eta\rangle=\langle A_1|B|\eta,\ |B|\eta\rangle$. Since $A_1\geqslant 0$, $A_2\geqslant 0$ and at least one of these operators is non-singular, the latter is only possible if $B\eta=0$ or $|B|\eta=0$. In either cases it follows that $B=0$. □

Lemma 5.3. *Let* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *be Hilbert spaces and let* B *be a bounded linear operator from* $L^2(\boldsymbol{R})\otimes\mathscr{G}_1$ *into* $L^2(\boldsymbol{R})\otimes\mathscr{G}_2$. *If* B *intertwines the unitary groups* $t\to e^{itQ}$ *and the self-adjoint operators* $e^{\beta P}$ *for some positive real number* β, *then* B *is constant.*

Proof. Put $\mathscr{G}:=\mathscr{G}_1\oplus\mathscr{G}_2$. We extend B to a bounded linear operator $\tilde{B}$ of the Hilbert space $L^2(\boldsymbol{R})\otimes\mathscr{G}$ into itself by setting $\tilde{B}=0$ on the subspace $L^2(\boldsymbol{R})\otimes\mathscr{G}_2$. Then $\tilde{B}$ obviously commutes with the unitary group e^{itQ} and with the self-adjoint operator $e^{\beta P}$ on the Hilbert space $L^2(\boldsymbol{R})\otimes\mathscr{G}$. Since $\tilde{B}e^{\beta P}\subseteq e^{\beta P}\,\tilde{B}$, $\tilde{B}$ commutes with the spectral projections and hence with each function of the self-adjoint operator $e^{\beta P}$ on $L^2(\boldsymbol{R})\otimes\mathscr{G}$, so in particular with $(e^{\beta P})^{is}\equiv e^{is\beta P}$ for all $s\in\boldsymbol{R}$. Thus $\tilde{B}\in(A\otimes\boldsymbol{C}\cdot 1)'$, where A denotes the *-algebra which is generated by the operators e^{itQ} and $e^{is\beta P}$ on $L^2(\boldsymbol{R})$. Since $A'=\boldsymbol{C}\cdot 1$, $(A\otimes\boldsymbol{C}\cdot 1)'=\boldsymbol{C}\cdot 1\otimes B(\mathscr{G})$ (see, for instance, [12], p. 184). Hence $\tilde{B}\in\boldsymbol{C}\cdot 1\otimes B(\mathscr{G})$ which means that $\tilde{B}$ is constant. Since $\tilde{B}=0$ on $L^2(\boldsymbol{R})\otimes\mathscr{G}_2$ by definition, B is constant. $\square$

Proof of Theorem 5.1. Writing T as a 2×2 operator matrix (T_{ij}) in the obvious way, the relation $Ta\subseteq\tilde{a}T$ leads to $-T_{12}(-a_-)\subseteq\tilde{a}_+T_{12}$ and $-T_{21}a_+\subseteq(-\tilde{a}_-)T_{21}$. Therefore, we conclude from Lemma 5.2 that $T_{12}=0$ and $T_{21}=0$, so that T is of the desired form $T=T_1\oplus T_2$, where $T_j:=T_{jj}$.

Now we turn to the proof of the second assertion of Theorem 5.1. *Necessity part*: By definition, the relation $T\mathscr{b}\subseteq\tilde{\mathscr{b}}T$ is equivalent to the four relations

$$T_jb_{jj}\subseteq\tilde{b}_{jj}T_j \quad \text{for} \quad j=1,2, \tag{5.3}$$

$$T_1b_{12}\subseteq\tilde{b}_{12}T_2 \quad \text{and} \quad T_2b_{21}\subseteq\tilde{b}_{21}T_1. \tag{5.4}$$

Fix $j\in\{1,2\}$. From $T_jb_{jj}p_{jj}=0\subseteq\tilde{b}_{jj}T_jp_{jj}$ by (5.3) we obtain

$$(1-\tilde{p}_{jj})T_jp_{jj}=0. \tag{5.5}$$

Since $\tilde{b}_{jj}$ is self-adjoint, $\tilde{p}_{jj}\tilde{b}_{jj}=0$ and hence $\tilde{p}_{jj}T_jb_{jj}\subseteq\tilde{p}_{jj}\tilde{b}_{jj}T_j=0$, so that

$$\tilde{p}_{jj}T_j(1-p_{jj})=0. \tag{5.6}$$

(5.5) and (5.6) together yield $T_jp_{jj}=\tilde{p}_{jj}T_j$.

Recall that $\{a_+, b_{11}\}$ is a_+-integrable and $\{a_-, b_{22}\}$ is a_--integrable. More precisely, by (5.1) and (5.2) we have $e^{itQ}b_{jj}e^{-itQ}=e^{\alpha_{jj}t}\,b_{jj}$ for $t\in\mathbf{R}$. In particular, this implies that p_{jj} commutes with e^{itQ} for all t. Further, $Ta\subseteq\tilde{a}T$ gives $T_1a_+\subseteq\tilde{a}_+T_1$ and $T_2a_-\subseteq\tilde{a}_-T_2$, so that $T_je^Q\subseteq e^QT_j$ by (5.1). From $T_je^Q\subseteq e^QT_j$ it follows that $T_je^{itQ}=e^{itQ}T_j$. (In order to see this, one can repeat some arguments from the proof of Lemma 5.3, where we showed that $Be^{\beta P}\subseteq e^{\beta P}B$ implies that $\tilde{B}e^{is\beta P}=e^{is\beta P}\tilde{B}$.) Thus we obtain $T_jp_{jj}e^{itQ}=e^{itQ}T_jp_{jj}$ for $t\in\mathbf{R}$ which completes the proof of (i).

In order to prove conditions (ii) and (iii), we essentially use the concrete form of the operators b_{ij} and $\tilde{b}_{ij}$ as described by (5.2). From $T_jb_{jj}\subseteq\tilde{b}_{jj}T_j$ by (5.3) and $\alpha_{jj}=\tilde{\alpha}_{jj}$ by assumption we get

$$T_jw_{jj}e^{\alpha_{jj}P}(2E_j-1)w_{jj}^*\subseteq\tilde{w}_{jj}e^{\alpha_{jj}P}(2\tilde{E}_j-1)\tilde{w}_{jj}^*T_j. \tag{5.7}$$

Recall that $w_{jj}^*w_{jj}=1$ and $\tilde{w}_{jj}^*\tilde{w}_{jj}=1$, since w_{jj} and $\tilde{w}_{jj}$ are isometries. Therefore, (5.7) implies that the operator $\Lambda_j=\tilde{w}_{jj}^*T_jw_{jj}$ satisfies

$$\Lambda_j e^{\alpha_{jj}P}(2E_j-1)\subseteq e^{\alpha_{jj}P}(2\tilde{E}_j-1)\Lambda_j, \tag{5.8}$$

so that

$$\Lambda_j e^{2\alpha_{jj}P}=\Lambda_j\big(e^{\alpha_{jj}P}(2E_j-1)\big)^2\subseteq\big(e^{\alpha_{jj}P}(2\tilde{E}_j-1)\big)^2\Lambda_j=e^{2\alpha_{jj}P}\Lambda_j.$$

We already noted in the preceding paragraph of this proof that $T_je^{itQ}=e^{itQ}T_j$ for $t\in\boldsymbol{R}$. By construction of the model, the operators w_{jj} and $\tilde{w}_{jj}$ intertwine the unitary groups e^{itQ}. Hence the operator $\Lambda_j=\tilde{w}_{jj}^*T_jw_{jj}$ also intertwines the unitary groups e^{itQ}. That is, Λ_j satisfies the assumptions of Lemma 5.3, so Λ_j is constant. Using this and applying (5.8) once more, we conclude that $\Lambda_jE_j=\tilde{E}_j\Lambda_j$. This proves (ii).
Finally, we verify condition (iii). Recall that $b_{12}=u|b_{12}|$, $|b_{12}|=b_{21}u$, $|\tilde{b}_{21}|=\tilde{u}\tilde{b}_{21}$ and $\tilde{b}_{12}=|\tilde{b}_{21}|\tilde{u}$ by general properties of the polar decomposition. Using these relations and (5.4) we obtain

$$\begin{aligned}(T_1u-\tilde{u}T_2)|b_{12}|&=T_1u|b_{12}|-\tilde{u}T_2|b_{12}|=T_1b_{12}-\tilde{u}T_2b_{21}u\subseteq\\ \tilde{b}_{12}T_2-\tilde{u}\tilde{b}_{21}T_1u&=|\tilde{b}_{21}|\tilde{u}T_2-|\tilde{b}_{21}|T_1u\subseteq|\tilde{b}_{21}|(\tilde{u}T_2-T_1u).\end{aligned} \tag{5.9}$$

Now we apply Lemma 5.2 to the operators $A_1:=|b_{12}|(1-p_{12})$ on $\mathscr{H}_1:=(1-p_{12})\mathscr{H}_-$, $A_2:=|\tilde{b}_{21}|(1-\tilde{p}_{21})$ on $\mathscr{H}_2:=(1-\tilde{p}_{21})\tilde{\mathscr{H}}_+$ and $B:=(1-\tilde{p}_{21})(T_1u-\tilde{u}T_2)\restriction(1-p_{12})\mathscr{H}_-$. From (5.9) we see immediately that $-BA_1\subseteq A_2B$. Lemma 5.2 yields that $B=0$. Since $u=u(1-p_{12})$ and $(1-p_{21})\tilde{u}=\tilde{u}$, this gives

$$(1-\tilde{p}_{21})T_1u=\tilde{u}T_2(1-p_{12}). \tag{5.10}$$

From $T_1b_{12}\subseteq\tilde{b}_{12}T_2$ it follows that $T_1b_{12}p_{12}=0\subseteq\tilde{b}_{12}T_2p_{12}$, so that $(1-\tilde{p}_{12})T_2p_{12}=0$. Hence

$$\tilde{u}T_2p_{12}=\tilde{u}\tilde{p}_{12}T_2p_{12}=0. \tag{5.11}$$

Further, $T_1b_{12}\subseteq\tilde{b}_{12}T_2$ leads to

$$b_{21}T_1^*=b_{12}^*T_1^*=(T_1b_{12})^*\supseteq(\tilde{b}_{12}T_2)^*\supseteq T_2^*\tilde{b}_{12}^*=T_2^*\tilde{b}_{21}$$

and $b_{21}T_1^*\tilde{p}_{21}\supseteq T_2^*\tilde{b}_{21}\tilde{p}_{21}=0$, so that $(1-p_{21})T_1^*\tilde{p}_{21}=0$ and $\tilde{p}_{21}T_1(1-p_{21})=0$. Therefore,

$$\tilde{p}_{21}T_1u=\tilde{p}_{21}T_1p_{21}u=0. \tag{5.12}$$

From (5.10)—(5.12) we conclude that $T_1u=\tilde{u}T_2$.

By (5.2), (5.4) and the assumption $\alpha_{12}=\tilde{\alpha}_{12}$, we have

$$T_1 b_{12}=T_1 u w_{12} e^{\alpha_{12}P} w_{12}^* \subseteq \tilde{b}_{12} T_2 = \tilde{u}\tilde{w}_{12} e^{\alpha_{12}P} \tilde{w}_{12}^* T_2,$$

so

$$\tilde{w}_{12}^* \tilde{u}^* T_1 u w_{12} e^{\alpha_{12}P} \subseteq e^{\alpha_{12}P} \tilde{w}_{12}^* T_2 w_{12}. \tag{5.13}$$

Since $T_1 u=\tilde{u} T_2$ as just shown, we have $\tilde{u}^* T_1 u=(1-\tilde{p}_{12}) T_2$ and so $\tilde{w}_{12}^* \tilde{u}^* T_1 u w_{12}=\tilde{w}_{12}^*(1-\tilde{p}_{12}) T_2 w_{12}=\tilde{w}_{12}^* T_2 w_{12}=\Lambda$. Therefore, (5.13) shows that the operator Λ satisfies the relation $\Lambda e^{\alpha_{12}P} \subseteq e^{\alpha_{12}P}\Lambda$. By a similar reasoning as in case of Λ_j, Λ intertwines the unitary groups e^{itQ}, $t\in\mathbb{R}$. Therefore, by Lemma 5.3, Λ is constant and condition (iii) is proved.

Sufficiency part : Suppose $T=(T_{ij})$ satisfies the above conditions. Let $j\in\{1,2\}$. Since $w_{jj}w_{jj}^*=1-p_{jj}$ *and* $\tilde{w}_{jj}\tilde{w}_{jj}^*=1-\tilde{p}_{jj}$, it follows from (i) and (ii) that

$$T_j = T_j p_{jj} + \tilde{w}_{jj}\Lambda_j w_{jj}^*.$$

Both summands intertwine the unitary groups e^{itQ}, the first one by assumption (i) and the second one because $\tilde{w}_{jj}$, Λ_j and w_{jj} have this property. Therefore, $T_j e^{itQ}=e^{itQ}T_j$ for $t\in\mathbb{R}$. This yields $T_jQ\subseteq QT_j$ (by differentiation at $t=0$) and hence $T_je^Q\subseteq e^QT_j$ (by power series expansion). Thus $Ta\subseteq\tilde{a}T$ by (5.1). In order to prove that $Tb\subseteq\tilde{b}T$, we have to verify the relations (5.3) and (5.4). We first show that $T_jb_{jj}\subseteq\tilde{b}_{jj}T_j, j=1,2$. From condition (i) we obtain

$$\tilde{p}_{jj}T_jb_{jj}=T_jp_{jj}b_{jj}=0 \quad\text{and}\quad \tilde{b}_{jj}T_jp_{jj}=\tilde{b}_{jj}p_{jj}T_j=0. \tag{5.14}$$

Since Λ_j is constant, $\Lambda_jE_j=\tilde{E}_j\Lambda_j$ and $\alpha_{jj}=\tilde{\alpha}_{jj}$, we have

$$\tilde{w}_{jj}^*T_jw_{jj}e^{\alpha_{jj}P}(2E_j-1)\subseteq e^{\tilde{\alpha}_{jj}P}(2\tilde{E}_j-1)\tilde{w}_{jj}^*T_jw_{jj}.$$

Multiplying this by $\tilde{w}_{jj}$ from the left and by w_{jj} from the right and using the definition of b_{jj}, we get

$$(1-\tilde{p}_{jj})T_jb_{jj}\subseteq\tilde{b}_{jj}T_j(1-p_{jj}).$$

Combined with (5.14), the latter implies that $T_jb_{jj}\subseteq\tilde{b}_{jj}T_j$.
Next we prove that $T_1b_{12}\subseteq\tilde{b}_{12}T_2$. From $T_1u=\tilde{u}T_2$, $up_{12}=0$ and $\tilde{p}_{21}\tilde{u}=0$ we conclude that

$$(1-\tilde{p}_{12})T_2p_{12}=\tilde{u}^*\tilde{u}T_2p_{12}=\tilde{u}^*T_1up_{12}=0$$

and $\tilde{p}_{21}T(1-p_{21})=\tilde{p}_{21}T_1uu^*=\tilde{p}_{21}\tilde{u}T_2u^*=0,$

so that

$$\tilde{b}_{12}T_2p_{12}=\tilde{b}_{12}\tilde{p}_{12}T_2p_{12}=0 \tag{5.15}$$

and

$$\tilde{p}_{12}T_1b_{12}=\tilde{p}_{21}T_1p_{21}b_{12}=0, \tag{5.16}$$

where the last equality follows from the fact that $p_{21}\mathscr{H}_+=\ker b_{21}\perp b_{21}^*\mathscr{H}_-=b_{12}\mathscr{H}_-$. Using $T_1u=\tilde{u}T_2$ once more, we have

$$\tilde{w}_{12}^*\tilde{u}^*T_1uw_{12}=\tilde{w}_{12}^*\tilde{u}^*\tilde{u}T_2w_{12}=\tilde{w}_{12}^*(1-\tilde{p}_{12})T_2w_{12}=\tilde{w}_{12}^*T_2w_{12}=\Lambda.$$

Since Λ is constant and $\alpha_{12}=\tilde{\alpha}_{12}$, the latter yields

$$\tilde{w}_{12}^*\tilde{u}^*T_1uw_{12}e^{\alpha_{12}P}\subseteq e^{\tilde{\alpha}_{12}P}\tilde{w}_{12}^*T_2w_{12}.$$

We multiply this relation by $\tilde{u}\tilde{w}_{12}$ from the left and by w_{12}^* from the right. Using the definitions of b_{12} and $\tilde{b}_{12}$, we then obtain $(1-\tilde{p}_{21})T_1b_{12}\subseteq \tilde{b}_{12}T_2(1-p_{12})$.
Combined with (5.15) and (5.16), this gives $T_1b_{12}\subseteq\tilde{b}_{12}T_2$.
A similar reasoning leads to $T_2b_{21}\subseteq\tilde{b}_{21}T_1$. □

Definition 5.4. Let $\{a, b\}$ and $\{\tilde{a}, \tilde{b}\}$ be two pairs of the form described at the beginning of this section.
(i) We say that $\{a, b\}$ is *unitarily equivalent* to $\{\tilde{a}, \tilde{b}\}$ if there are unitary operators U_1 of $\mathscr{H}_+$ onto $\tilde{\mathscr{H}}_+$ and U_2 of $\mathscr{H}_-$ onto $\tilde{\mathscr{H}}_-$ such that $UaU^{-1}=\tilde{a}$ for $U:=U_1\oplus U_2$ and $U_ib_{ij}U_j^{-1}=\tilde{b}_{ij}$ for $i,j=1,2$.
(ii) The couple $\{a, b\}$ is said to be *irreducible* if each projection T on $\mathscr{H}$ of the form $T=T_1\oplus T_2$, where $T_1:\mathscr{H}_+\to\tilde{\mathscr{H}}_+$ and $T_2:\mathscr{H}_-\to\tilde{\mathscr{H}}_-$, such that $Ta\subseteq\tilde{a}T$ and $Tb\subseteq\tilde{b}T$ is either 0 or 1.

We briefly discuss the preceding definition by a few

Remarks.

3.) Recall that if T is a bounded operator of $\mathscr{H}$ into $\mathscr{H}$ such that $Ta\subseteq\tilde{a}T$, then, by Theorem 5.1, T is of the form $T=T_1\oplus T_2$, where $T_1:\mathscr{H}_+\to\tilde{\mathscr{H}}_+$ and $T_2:\mathscr{H}_-\to\tilde{\mathscr{H}}_-$. This justifies the restriction to unitaries U and projections T of the form $U_1\oplus U_2$ resp. $T_1\oplus T_2$ in the above definition.

4.) The above definition applies also to pairs $\{a, b\}$ for which b does not represent a densely defined symmetric operator b (cf. Example 6.1).

5.) Suppose that $\mathscr{b}$ and $\tilde{\mathscr{b}}$ represent operators b and $\tilde{b}$, respectively. Obviously, if $\{a, \mathscr{b}\}$ is unitarily equivalent to $\{\tilde{a}, \tilde{\mathscr{b}}\}$, then $\{a, b\}$ and $\{\tilde{a}, \tilde{b}\}$ are unitarily equivalent. Also, if $\{a, b\}$ is irreducible (i.e. there is closed linear subspace except $\{0\}$ and $\mathscr{H}$ which reduces both a and b), then the couple $\{a, \mathscr{b}\}$ is irreducible. The converses are not valid in general, but they are true (for instance) if the assumption of Lemma 2.1, (ii), is satisfied for $\mathscr{b}$ and $\tilde{\mathscr{b}}$.

Having Theorem 5.1, it is straightforward to formulate necessary and sufficient criteria for the unitary equivalence and for the irreducibility of pairs of the above model. We write down these criteria only in the important special case where $\ker b_{12}=\{0\}$ and $\ker b_{21}=\{0\}$. Recall that by Proposition 4.4 such a pair is unitarily equivalent to a pair $\{a, \mathscr{b}\}$ of our model, where $\mathscr{K}=\mathscr{K}_+=\mathscr{K}_-$ (hence $\mathscr{H}_+=\mathscr{H}_-$), $w_{12}=1$ and $u=1$.

Corollary 5.5. *Suppose that $\{a, \mathscr{b}\}$ and $\{\tilde{a}, \tilde{\mathscr{b}}\}$ are two couples of the form described in Proposition 4.4. Then the couples $\{a, \mathscr{b}\}$ and $\{\tilde{a}, \tilde{\mathscr{b}}\}$ are unitarily equivalent if and only if there exists a unitary operator Λ of $\mathscr{K}$ onto $\tilde{\mathscr{K}}$ such that operator $\Lambda_j := \tilde{w}_j^* \Lambda w_j$ of $L^2(\mathbb{R})\otimes\mathscr{K}_j$ into $L^2(\mathbb{R})\otimes\tilde{\mathscr{K}}_j$ is constant, $\Lambda_j E_j=\tilde{E}_j\Lambda_j$ and $\Lambda p_{jj}=\tilde{p}_{jj}\Lambda$ for $j=1,2$,*

Corollary 5.6. *Let $\{a, \mathscr{b}\}$ be as defined in Proposition 4.4. The couple $\{a, \mathscr{b}\}$ is irreducible (in the sense of Definition 5.4) if and only if each orthogonal projection Λ on $\mathscr{K}$ such that the operator $\Lambda_j := w_j^* \Lambda w_j$ on $L^2(\mathbb{R})\otimes\mathscr{K}_j$ is constant, $\Lambda p_{jj}=p_{jj}\Lambda$ and $\Lambda_j E_j=E_j\Lambda_j$ for $j=1,2$ is either 0 or 1.*

We omit the (easy) proofs of these corollaries. As an illustration we consider the scalar case (i.e. $\mathscr{K}=\mathscr{K}_1=\mathscr{K}_2=\mathbb{C}$) in

Example 5.7. Let $\{a, \mathscr{b}\}$ be an a-integrable representation of (1.1) such that $\ker b_{ij}=\{0\}$ for $i, j=1,2$ and the spectral multiplicity of the operators a_+ and a_- is one. Then, up to unitary equivalence, the pair $\{a, \mathscr{b}\}$ is of the form stated in Proposition 4.4, where $\mathscr{K}=\mathscr{K}_1=\mathscr{K}_2=\mathbb{C}$. The projections E_j are either 0 or 1 and the operators w_j are multiplication operators by functions $w_j(x)\in L^\infty(\mathbb{R})$ such that $|w_j(x)|=1$ a.e. By Corollary 5.6, such a couple $\{a, \mathscr{b}\}$ is always irreducible.

Let $\{\tilde{a}, \tilde{\mathscr{b}}\}$ be another such couple. By Corollary 5.5, the pairs $\{a, \mathscr{b}\}$ and $\{\tilde{a}, \tilde{\mathscr{b}}\}$ are unitarily equivalent if and only if both functions $\tilde{w}_j(x)^{-1}w_1(x)$,

$j=1,2$, are constant a.e. on $\boldsymbol{R}$ and if $E_j=\tilde{E}_j$ for $j=1,2$. (As always, all notations with tilde refer to the couple $\{\tilde{a}, \tilde{\mathscr{b}}\}$.)

§6. Representation of the Operator Matrix $\mathscr{b}$ as a Symmetric Operator

We have seen in the preceding two sections how to define and how to work with a-integrable pairs $\{a, \mathscr{b}\}$, where $\mathscr{b}$ are self-adjoint operator matrices. Knowing that operator matrices with unbounded entries are a very delicate matter (cf. [7]), it is not surprising that in general such a matrix $\mathscr{b}$ does not give a densely defined operator as shown by the following

Example 6.1. Let A_1, A_2 and A be self-adjoint operators on a Hilbert space $\mathscr{K}$ and let $\mathscr{H}_+=\mathscr{H}_-:=L^2(\boldsymbol{R})\otimes\mathscr{K}$. Define a self-adjoint operator a and a self-adjoint operator matrix $\mathscr{b}$ on $\mathscr{H}:=\mathscr{H}_+\oplus\mathscr{H}_-$ by

$$a\equiv a_+\oplus a_-:=e^Q\oplus(-e^Q)$$

and
$$\mathscr{b}\equiv(b_{ij}):=\begin{pmatrix} e^{\alpha_1 P}A_1 & e^{\alpha P}A \\ e^{\alpha P}A & e^{\alpha_2 P}A_2 \end{pmatrix},$$

where $\alpha_j=-\varphi+2\pi k_j$ and $\alpha=-\varphi+\pi+2\pi k$ with $k, k_j\in\boldsymbol{Z}$. Clearly, $\{a, \mathscr{b}\}$ is an a-integrable pair (in the sense of Definition 4.1). Suppose that $\mathscr{D}(A)\cap\mathscr{D}(A_1)=\{0\}$. (Note that such operators A and A_1 exist: For each unbounded self-adjoint operator A there is a unitary operator U such that $A_1:=UAU^{-1}$ satisfies $\mathscr{D}(A)\cap\mathscr{D}(A_1)=\{0\}$, cf. [11], Section 5.). Then we have $\mathscr{D}(b_{11})\cap\mathscr{D}(b_{21})=\{0\}$, so the matrix $\mathscr{b}$ does not represent a densely defined operator. It is not difficult to impose other conditions on A_1, A_2 and A which ensure that $\mathscr{b}$ represents a (densely defined) symmetric or a self-adjoint operator.

The aim of this section is to formulate some conditions which imply that a self-adjoint operator matrix $\mathscr{b}$ of the form (4.14) represents a symmetric operator. For this some preliminaries are needed.

Throughout the rest of this section, we keep the notation of Proposition 4.4 and we assume that the Hilbert spaces $\mathscr{K}_1$, $\mathscr{K}_2$ and $\mathscr{K}$ occuring therein are separable. We freely use the terminology on direct integrals of Hilbert spaces and operators (see, for instance, [12], Chapter IV). For a separable Hilbert space $\mathscr{G}$, we always identify the tensor product $L^2(\boldsymbol{R})\otimes\mathscr{G}$ with the

Hilbert space $L^2_{\mathscr{G}}(\mathbb{R})$ of $\mathscr{G}$-valued square integrable measurable functions on $\mathbb{R}$ with respect to the Lebesgue measure.

Lemma 6.2. *The isometry $w_j: L^2_{\mathscr{K}_j}(\mathbb{R}) \to L^2_{\mathscr{K}}(\mathbb{R})$, $j=1,2$, in Proposition 4.4 is given by a measurable field $\mathbb{R} \ni x \to w_j(x)$ of isometries $w_j(x)$ from $\mathscr{K}_j$ into $\mathscr{K}$.*

Proof. We argue in a similar way as in the proof of Lemma 5.3. Let $\mathscr{G}_j := \mathscr{K}_j \oplus \mathscr{K}$. We extend w_j to a bounded linear operator $\tilde{w}_j$ on $L^2_{\mathscr{G}_j}(\mathbb{R})$ by defining $\tilde{w}_j = 0$ on $\mathscr{K}$. Since w_j intertwines the unitary groups $t \to e^{itQ}$ (see Proposition 4.4), $\tilde{w}_j$ commutes with the unitary group $t \to e^{itQ}$ and hence with all bounded functions of Q on $L^2_{\mathscr{G}_j}(\mathbb{R})$. Considering the Hilbert space $L^2_{\mathscr{G}_j}(\mathbb{R})$ as a direct integral of Hilbert spaces $\mathscr{H}_\lambda := \mathscr{G}_j$ for $\lambda \in \mathbb{R}$, the latter means that $\tilde{w}_j$ commutes with the algebra of diagonalizable operators. Therefore, $\tilde{w}_j$ is decomposable ([12], p. 259), i.e. $\tilde{w}_j$ is given by a measurable field $\mathbb{R} \ni x \to \tilde{w}_j(x)$ of bounded operators on $\mathscr{G}_j$. Since $\tilde{w}_j = 0$ on $\mathscr{K}$ and w_j is an isometry, $\tilde{w}_j(x) = 0$ on $\mathscr{K}$ a.e. and $w_j(x) := \tilde{w}_j(x) \upharpoonright \mathscr{K}_j$ are isometries a.e.. □

The next lemma is a Hilbert space valued version of the classical Paley-Wiener theorem. For $\beta \in \mathbb{R}$, let $I(\beta) := \{z \in \mathbb{C} : |\operatorname{Im} z| < |\beta|\}$.

Lemma 6.3. *Let $\mathscr{G}$ be a separable Hilbert space and let $\beta \in \mathbb{R}$. Suppose that $z \to \psi(z)$ is a holomorphic mapping of $I(\beta)$ into $\mathscr{G}$ such that $M := \sup_{|y|<|\beta|} \int_{-\infty}^{\infty} \|\psi(x+iy)\|^2 \, dx < \infty$. Then the $\mathscr{G}$-valued function $\psi(x) \in L^2_{\mathscr{G}}(\mathbb{R})$ belongs to $\mathscr{D}(e^{\beta P})$. Moreover, $(e^{\omega P}\psi)(x) = \psi(x+i\omega)$, $x \in \mathbb{R}$, for $\omega \in \mathbb{R}$, $|\omega| < |\beta|$.*

Proof. Take an orthonormal basis $(\eta_n)_{n \in I}$ of $\mathscr{G}$ and set $\psi_n(z) := \langle \psi(z), \eta_n \rangle$. By the above assumptions, $\psi_n(z)$ is a holomorphic function on $I(\beta)$ such that $\sup_{|y|<|\beta|} \int |\psi_n(x+iy)|^2 \, dx < \infty$. Therefore, by the Paley-Wiener theorem ([5]), the function $e^{|\beta||t|}\hat{\psi}_n(t)$ is in $L^2(\mathbb{R})$. As usual, $\hat{\psi}_n = F\psi_n$ denotes the Fourier transform of ψ_n. Let $\omega \in \mathbb{R}$, $|\omega| < |\beta|$. As shown in the proof of the Paley-Wiener theorem (see [5], p. 174), the function $\psi_n(x+i\omega)$ has the Fourier transform $e^{-\omega t}\hat{\psi}_n(t)$. Since $F e^{\omega P} F^{-1} = e^{-\omega Q}$, it follows that

$$\psi_n(x) \in \mathscr{D}(e^{\omega P}) \quad \text{and} \quad (e^{\omega P}\psi_n)(x) = \psi_n(x+i\omega) \quad \text{in} \quad L^2(\mathbb{R}). \tag{6.1}$$

Further, by the Plancherel theorem, we have

$$\begin{aligned}&\int \sum_n (e^{2\omega t}+e^{-2\omega t})\ |\hat{\psi}_n(t)|^2\ dt\\&=\sum_n \int (|\psi_n(x+i\omega)|^2+|\psi_n(x-i\omega)|^2)\ dx\\&=\int (\|\psi(x+i\omega)\|^2+\|\psi(x-i\omega)\|^2)\ dx \leqslant 2M<\infty.\end{aligned}$$

The left-hand side is monotonic in ω. Therefore, letting $\omega\uparrow|\beta|$ and applying the monotone convergence theorem, we get

$$\sum_n (\|e^{\beta t}\hat{\psi}_n(t)\|^2+\|e^{-\beta t}\hat{\psi}_n(t)\|^2)=\int \sum_n (e^{2\beta t}+e^{-2\beta t})\ |\hat{\psi}_n(t)|^2\ dt\ \leqslant 2M<\infty$$

which in turn yields

$$\sum_n \|e^{\beta P}\psi_n\|^2=\sum_n \|e^{-\beta t}\hat{\psi}_n\|^2<\infty. \tag{6.2}$$

Clearly, the map $\zeta\to(\langle\zeta(x),\ \eta_n\rangle)_{n\in I}$ is an isometry of $L^2_{\mathscr{G}}(\boldsymbol{R})$ onto the orthogonal direct sum $\sum_{n\in I}\oplus\mathscr{H}_n$, where $\mathscr{H}_n:=L^2(\boldsymbol{R})$. Therefore, it follows from (6.2) that $\psi(x)\in\mathscr{D}(e^{\beta P})$ in $L^2_{\mathscr{G}}(\boldsymbol{R})$. The formula for $e^{\omega P}\psi$ follows at once from (6.1). □

For $\rho\geqslant 0$, let $\mathscr{F}_\rho$ denote the linear span of functions $e^{-\delta x^2+\gamma x}$ in $L^2(\boldsymbol{R})$, where $\delta>\rho$ and $\gamma\in\boldsymbol{C}$.

Proposition 6.4. *Suppose that, for $j\in\{1,2\}$, there exists a positive number ρ_j, a dense linear subspace $\mathscr{E}_j$ of $\boldsymbol{K}$ and a family of (possibly unbounded) linear operators $\{v_j(z);z\in I(\alpha_j)\}$ of $\boldsymbol{K}$ into $\boldsymbol{K}_j$ such that:*

(i) $v_j(x)=w_j(x)^*$ *on* $\boldsymbol{R}$ a.e.,

(ii) *$z\to v_j(z)\eta$ is a holomorphic mapping of $I(\alpha_j)$ into $\boldsymbol{K}_j$ for all $\eta\in\mathscr{E}_j$ and*

(iii) $$\sup_{|y|<|\alpha_j|}\int_{-\infty}^{\infty} e^{-2\rho_j x^2}\ \|v_j(x+iy)\eta\|^2\ dx<\infty.$$

Then the self-adjoint operator matrix $\mathscr{b}$ from Proposition 4.4 *represents a symmetric operator whose domain contains the dense set $\mathscr{F}_{\rho_1}\otimes\mathscr{E}_1\oplus\mathscr{F}_{\rho_2}\otimes\mathscr{E}_2$ in $\mathscr{H}=\mathscr{H}_+\oplus\mathscr{H}_-$. (Here $\mathscr{F}_{\rho_j}\otimes\mathscr{E}_j$ means the algebraic tensor product of vector spaces.) Moreover, we have $w_j^*(\zeta\otimes\eta)\in\mathscr{D}(e^{\omega P})$ and $(e^{\omega P}w_j^*(\zeta\otimes\eta))(x)$ $=v_j(x+i\omega)\zeta(x+i\omega)\eta$, $x\in\boldsymbol{R}$, for $\omega\in\boldsymbol{R}$, $|\omega|<|\alpha_j|$, $\zeta\in F_{\rho_j}$ and $\eta\in\mathscr{E}_j$, where $\zeta(z)$*

denotes the holomorphic extension to $\mathbb{C}$ *of the function* $\zeta(x) \in \mathscr{F}_{\rho_j}$.

Proof. Obviously, $\mathscr{D}_j := \mathscr{F}_{\rho_j} \otimes \mathscr{E}_j$ is dense in $\mathscr{H}_+$ resp. $\mathscr{H}_-$ for $j=1$ resp. $j=2$. We show that $\mathscr{D}_j \subseteq \mathscr{D}(b_{jj})$ for $j=1,2$. Since $b_{jj} = w_j e^{\alpha_j P}(2E_j - 1)w_j^*$ by (4.14), this is equivalent to $w_j^* \mathscr{D}_j \subseteq \mathscr{D}(e^{\alpha_j P})$. In order to prove the latter, it suffices to check that for any $\delta > \rho_j$, $\gamma \in \mathbb{C}$ and $\eta \in \mathscr{E}_j$ the $\mathscr{G}$-valued function $\psi(z) := v_j(z) e^{-\delta z^2 + \gamma z}$ satisfies the assumptions of Lemma 6.3. By (ii), $\psi(z)$ is holomorphic on $I(\alpha_j)$. Further, we have $\|\psi(x+iy)\| \leqslant \text{const.}\ e^{-\rho_j x^2}$ $\|v_j(x+iy)\eta\|$ for $x+iy \in I(\alpha_j)$. Thus, by Lemma 6.3 applied with $\beta = \alpha_j$, $w_j^* \mathscr{D}'_j \subseteq \mathscr{D}(e^{\alpha_j P})$ and hence $\mathscr{D}'_j \subseteq \mathscr{D}(b_{jj})$. Since $b_{12} = b_{21} = e^{\alpha P}$, we have $\mathscr{D}'_j \subseteq \mathscr{D}(b_{21}) \cap \mathscr{D}(b_{12})$. Thus the matrix represents a (densely defined) symmetric operator b and $\mathscr{D}'_1 \oplus \mathscr{D}'_2 \subseteq \mathscr{D}(b)$. The formula for $e^{\omega P} w_j^*(\zeta \otimes \eta)$ follows at once from the corresponding formula in Lemma 6.3. □

Remark. Let a and b be as in Proposition 4.4 and keep the assumptions of Proposition 6.4. Then the domain $\mathscr{D}'_1 \oplus \mathscr{D}'_2$ is contained in $\mathscr{D}(ab) \cap \mathscr{D}(ba)$ and we have $ab\xi = qba\xi$ for $\xi \in \mathscr{D}'_1 \oplus \mathscr{D}'_2$.

§7. An Example

Throughout this section, let A_j and B_j, $j=1,2$, be (possibly unbounded) self-adjoint operators on a separable Hilbert space $\mathscr{K}$. We assume that A_j and B_j strongly commute, i.e. the spectral projections of A_j and B_j commute. Further, let $\alpha_j = -\varphi + 2\pi k_j$ and $\alpha = -\varphi + \pi + 2\pi k$ with $k_j, K \in \mathbb{Z}$ for $j=1,2$.

Our example will be a special case of the pairs in Propositions 4.4 and 6.4. In order to define it and to describe it by more explicit formulas, we need some preliminaries.

Fix $j \in \{1,2\}$. Let $A_j = A_{j,+} \oplus A_{j,-} \oplus A_{j,0}$ on $\mathscr{K} = \mathscr{K}_{j,+} \oplus \mathscr{K}_{j,-} \oplus \mathscr{K}_{j,0}$ be the orthogonal decomposition of A_j into positive part, negative part and null part. Let E_j and F_j denote the orthogonal projections of $\mathscr{K}_j := \mathscr{K}_{j,+} \oplus \mathscr{K}_{j,-}$ onto $\mathscr{K}_{j,+}$ and of $\mathscr{K}$ onto $\mathscr{K}_j$, respectively. Define operators C_j and B'_j on K_j by $C_j := \alpha_j^{-1} (\log A_{j,+} \oplus \log |A_{j,-}|)$ and $B'_j := B_j \restriction \mathscr{K}_j$. (Note that B'_j maps K_j into itself, since A_j and B_j strongly commute.)

In what follows, the multiplication operator by the independent variable on $\mathbb{R}$ in some Hilbert space $L^2_{\mathscr{G}}(\mathbb{R})$ is often denoted by x. By an operator such as $\beta x C_j + \gamma x^2 B'_j$, where $\beta, \gamma \in C$, we shall mean the closure of the operator $\beta(x \otimes C_j) + \gamma(x^2 \otimes B'_j)$ on the domain $(\mathscr{D}(x) \otimes \mathscr{D}(C_j)) \cap (\mathscr{D}(x^2) \otimes \mathscr{D}(B'_j))$ in the

Hilbert space $L^2_{\mathscr{K}_j}(R) \equiv L^2(R) \otimes \mathscr{K}_j$. A similar meaning is attached to operators like $\beta z C_j + \gamma z^2 B'_j + \delta z B'_j$, where $z = x + iy$ is interpreted as a sum of the multiplication operator x and the constant iy. Recall that the operators C_j and B'_j strongly commute on $\mathscr{K}_j$, hence $\beta x C_j + \gamma x^2 B'_j$ is self-adjoint for real β and γ and all such operators strongly commute. Define isometries $w_j: L^2_{\mathscr{K}_j}(R) \to L^2_{\mathscr{K}}(R)$ by $w_j(x) := e^{ixC_j + ix^2 B'_j}$.

Put $E_{j,n} := e_{C_j}([-n, n]) e_{B'_j}([-n,n])$ for $n \in N_0$ and $\mathscr{E}_j := \bigcup_{n=1}^{\infty} E_{j,n}\mathscr{K}_j$, where $e_T(\cdot)$ denote the spectral projections of a self-adjoint operator T. Clearly, $\mathscr{E}_j$ is a core for C_j and for B'_j. Let $\mathscr{D}'_j$ be the linear span of vectors $e^{\omega x B_j}(\zeta \otimes \eta)$ in $\mathscr{K}$, where $\omega \in C$, $\zeta \in \mathscr{F}_0$ and $\eta \in \mathscr{E}_j \oplus (\mathscr{K}_{j,0} \cap \mathscr{D}(e^{B_j^2}))$. Recall that $\mathscr{F}_0$ is spanned by the functions $e^{-\delta x^2 + \gamma x}$ in $L^2(R)$, where $\delta > 0$ and $\gamma \in \mathbb{C}$. As usual, let $\mathscr{b} \equiv (b_{rs})$ be the self-adjoint operator matrix defined by (4.14) and let $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$, where $\mathscr{H}_+ = \mathscr{H}_- = L^2_{\mathscr{K}}(R) \equiv L^2(R) \otimes \mathscr{K}$.

Proposition 7.1. *For* $j \in \{1,2\}$, *we have*:

(i) $b_{jj}\xi = w_j e^{\alpha_j P}(2E_j - 1) w_j^* \xi = A_j e^{(2x + i\alpha_j)\alpha_j B_j} e^{\alpha_j P} \xi$ (7.1)

for $\xi \in \mathscr{D}'_j$.

(ii) $\mathscr{D}'_j$ *is a core for the self-adjoint operator* b_{jj}.

Proof. Except for the Hilbert spaces $\mathscr{K}_j$ and $\mathscr{K}_{j,0}$ we shall omit the lower index j throughout the following proof.

(i): It suffices to prove (7.1) for a vector $\xi = e^{\omega x B}(\zeta \otimes \eta)$, where $\zeta(x) = e^{-\delta x^2 + \gamma x}$ with $\delta > 0$, $\gamma \in C$. If $\eta \in \mathscr{K}_{j,0}$, then both sides of (7.1) are obviously zero. Thus it is sufficient to treat the case where $\eta \in \mathscr{E}$, say $\eta \in E_n \mathscr{K}_j$. For $z = x + iy$, we define $\xi(z) := e^{\omega z B'} \zeta(z) \eta$ and $v(z) := e^{-izC - iz^2 B'} F$. For fixed $z \in C$, $v(z)$ is a linear operator from $\mathscr{K}$ into $\mathscr{K}_j$. From the spectral theorem we easily see that $\xi(z) \in \mathscr{D}(v(z))$ and that $\psi(z) := v(z)\xi(z) = \zeta(z)\ e^{-izC - iz^2 B' + \omega z B'}\eta$. Since $\eta \in E_n \mathscr{K}_j$, we have

$$\|(-iB')^{m-k}(\omega B' - iC)^k \eta\| \le n^{m-k} \sum_{r=0}^{k} \binom{k}{r} |\omega|^r \|B'^r C^{k-r}\eta\| \le (n(|\omega|+1))^m$$

for $m, k \in N_0$, $m \geqslant k$. Therefore, the power series expansion

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m} z^{2m-k}\ (k!(m-k)!)^{-1} (-iB')^{m-k} (\omega B' - iC)^k \eta$$

concerges absolutely in $\mathscr{K}_j$ for all $z\in\mathbf{C}$, so the mapping $z\to e^{-izC-iz^2B'+\omega zB'}\eta$ of $\mathbf{C}$ onto $\mathscr{K}_j$ is holomorphic. Thus $z\to\psi(z)$ is a holomorphic mapping of $\mathbf{C}$ into $\mathscr{K}_j$.

Let $\beta>0$. Recall that $\eta\in E_n\mathscr{K}_j=e_C([-n, n])e_{B'}([-n, n])\mathscr{K}_j$. Therefore, it follows from the functional calculus for self-adjoint operators combined with the strong commutativity of C and B' that for $z=x+iy\in I(\beta)$ we have

$$\|\psi(z)\|=|\zeta(z)|\ \|e^{-izC-iz^2B'+\omega zB'}\eta\|$$
$$\leq\text{const. } e^{-\delta x^2+|\gamma|x}\ e^{|x|(2\beta+|\omega|)n}\ \|\eta\|\leq\ \text{const. } e^{-\frac{1}{2}\delta x^2},$$

where the constants depend on β, γ, ω, η, m, but not on $x\in\mathbf{R}$. Hence the mapping $z\to\psi(z)$ satisfies the assumptions of Lemma 6.3 for any $\beta>0$. Lemma 6.3 yields

$$(e^{\alpha P}w^*\xi)(x)=v(x+i\alpha)\xi(x+i\alpha). \tag{7.2}$$

Similarly, we obtain

$$(e^{\alpha P}\xi)(x)=\xi(x+i\alpha). \tag{7.3}$$

Since $\eta\in\mathscr{E}$, $F\xi(x+i\alpha)=\xi(x+i\alpha)$. By construction, the projection E (of $\mathscr{K}_j$ onto $\mathscr{K}_{j,+}$) commutes with C and B'. Using these facts, we get

$$w(x)Ev(x+i\alpha)\xi(x+i\alpha)=E\ e^{ixC+ix^2B'}\ e^{-i(x+i\alpha)C-i(x+i\alpha)^2B'}\ F\xi(x+i\alpha)$$
$$=E\ e^{\alpha C}\ e^{(2x+i\alpha)\alpha B'}\ \xi(x+i\alpha)=A_+\ e^{(2x+i\alpha)\alpha B'}\ \xi(x+i\alpha). \tag{7.4}$$

Similarly,

$$w(x)(E-1)v(x+i\alpha)\xi(x+i\alpha)=A_-\ e^{(2x+i\alpha)\alpha B'}\ \xi(x+i\alpha). \tag{7.5}$$

Using (7.2), (7.4) and (7.5) and finally (7.3), we obtain

$$(b\xi)(x)=(w(2E-1)e^{\alpha P}w^*\xi)(x)=w(x)(2E-1)v(x+i\alpha)\xi(x+i\alpha)$$
$$=(A_+\oplus A_-)e^{(2x+i\alpha)\alpha B'}\ \xi(x+i\alpha)=A\ e^{(2x+i\alpha)\alpha B}\ (e^{\alpha P}\xi)(x)$$

which proves (7.1)

(ii): The non-singular part of the self-adjoint operator b is the operator $b_{ns}:=we^{\alpha P}(2E-1)w^*\lceil\mathscr{K}_j$ on the Hilbert space $\mathscr{K}_j$. We have $|b_{ns}|^{it}=w|e^{\alpha P}\ (2E-1)|^{it}\ w^*\lceil\mathscr{K}_j=we^{it\alpha P}w^*\lceil\mathscr{K}_j$ for $t\in\mathbf{R}$. Clearly, $e^{it\alpha P}$ is the translation operator by $-t\alpha$. The space $\mathscr{F}_0$ is invariant under translation. From the spectral theorem it is clear that the operators $e^{\omega C}$ and $e^{\omega B'}$ leave the dense domain $\mathscr{E}$ in $\mathscr{K}_j$ invariant. Putting all these facts together, we conclude that the domain $\mathscr{E}$ is invariant under the unitary group $|b_{ns}|^{it}$. By Lemma 7.2 below, $\mathscr{E}$ is a core for b_{ns}. Since $\mathscr{K}_{j,0}=\ker b$ and $\mathscr{K}_{j,0}\cap\mathscr{D}(e^{B^2})$ is dense in $\mathscr{K}_{j,0}$, $\mathscr{D}$ is a core for b. $\square$.

The following lemma is similar to a result of Poulsen [9].

Lemma 7.2. *Let T be a non-singular self-adjoint operator on a Hilbert space $\mathscr{G}$. If a linear subspace $\mathscr{D}$ of $\mathscr{D}(T)$ is dense in $\mathscr{G}$ and invariant under the unitary group $U(t):=|T|^{it}$, $t\in\boldsymbol{R}$, then $\mathscr{D}$ is a core for T.*

Proof. Fix a number $z\in\boldsymbol{C}\backslash\boldsymbol{R}$. Let $\mathscr{G}_z$ be the closure of $(|T|-z)\mathscr{D}$ in $\mathscr{G}$. Since $U(t)$ commutes with $|T|$ and since $U(t)\mathscr{D}\subseteq\mathscr{D}$ by assumption, $U(t)\mathscr{G}_z\subseteq\mathscr{G}_z$ for all real t. Consequently, $U(t)\mathscr{G}_z^\perp\subseteq\mathscr{G}_z^\perp$ for $t\in\boldsymbol{R}$. Set $V(t):=U(t)\lceil\mathscr{G}_z$ and $W(t):=U(t)\lceil\mathscr{G}_z^\perp$. By Stone's theorem, there are self-adjoint operators R and S on $\mathscr{G}_z$ and $\mathscr{G}_z^\perp$, respectively, such that $V(t)=e^{itR}$ and $W(t)=e^{itS}$, $t\in\boldsymbol{R}$. Since $U(t)=V(t)\oplus W(t)$ on $\mathscr{G}=\mathscr{G}_z\oplus\mathscr{G}_z^\perp$ and $U(t)=e^{it\log|T|}$ by definition, it follows that $\log|T|=R\oplus S$, so $|T|=e^R\oplus e^S$. Take a vector $\eta\in\mathscr{D}(e^S)\subseteq\mathscr{D}(|T|)$. For $\zeta\in\mathscr{D}$, we have $\langle|T|\zeta,\eta\rangle=\langle\zeta,\bar{z}\zeta\rangle$, hence $\langle|T|\zeta,\eta\rangle=\langle\zeta,\ |T|\eta\rangle=\langle\zeta,\ e^S\eta\rangle=\langle\zeta,\ \bar{z}\eta\rangle$. Since $\mathscr{D}$ is dense in $\mathscr{G}$, the latter gives $e^S\eta=\bar{z}\eta$. Since e^S is self-adjoint and $\bar{z}$ is not real, $\eta=0$. Thus $\mathscr{D}(e^S)=\{0\}$ which yields $\mathscr{G}_z^\perp=\{0\}$. From the preceding we conclude that $\mathscr{D}$ is a core for $|T|$. Since $\mathscr{D}(T)=\mathscr{D}(|T|)$ and $\|T.\|=\||T|.\|$, $\mathscr{D}$ is a core for T as well. $\square$

By Proposition 7.1, (i) and (ii), the self-adjoint operator b_{jj} coincides with the closure of the essentially self-adjoint operator $A_je^{(2x+i\alpha_j)\alpha_jB_j}\ e^{\alpha_jP}\lceil\mathscr{D}_j'$. We shall denote this closure again by $A_je^{(2x+i\alpha_j)\alpha_jB_j}\ e^{\alpha_jP}$. Thus the couple $\{a,\mathscr{b}\}$ in our example takes the following form:

$$a=e^Q\oplus -e^Q\equiv e^x\oplus -e^x \tag{7.6}$$

$$\mathscr{b}\equiv(b_{rs})=\begin{pmatrix} A_1e^{(2x+i\alpha_1)\alpha_1B_1}\ e^{\alpha_1P} & e^{\alpha P} \\ e^{\alpha P} & A_2e^{(2x+i\alpha_2)\alpha_2B_2}\ e^{\alpha_2P}\end{pmatrix} \tag{7.7}$$

It is clear that the operator matrix $\mathscr{b}$ in (7.7) satisfies the assumptions of Proposition 6.4.

Next we want to decide on the unitary equivalence and the irreducibility of the couples of our example. Before stating our result, let us note that B_jF_j is a self-adjoint operator on the Hilbert space $\mathscr{K}$, because A_j and B_j strongly and hence the projection F_j reduces B_j.

Proposition 7.3. *Suppose that* $\{a, \mathscr{b}\}$ *and* $\{\tilde{a}, \tilde{\mathscr{b}}\}$*are two couples of the form described above, i.e. the operators a and* $\tilde{a}$ *and the self-adjoint operator matrices* $\mathscr{b}$ *and* $\tilde{\mathscr{b}}$ *are given by* (7.6) *resp.* (7.7). *Suppose that* $\alpha=\tilde{\alpha}$ *and* $\alpha_j=\tilde{\alpha}_j$ *for* $j=1,2$. *(It is needless to say that the tilde refers to the couple* $\{\tilde{a}, \tilde{\mathscr{b}}\}$.)
(i) *The couples* $\{a, \mathscr{b}\}$ *and* $\{\tilde{a}, \tilde{\mathscr{b}}\}$ *are unitarily equivalent if and only if the* 4-*tuples* $\{A_1, A_2, B_1F_1, B_2F_2\}$ *and* $\{\tilde{A}_1, \tilde{A}_2, \tilde{B}_1\tilde{F}_1, \tilde{B}_2\tilde{F}_2\}$ *of self-adjoint operators on* $\mathscr{K}$ *resp.* $\tilde{\mathscr{K}}$ *are unitarily equivalent, i.e. there is a unitary operator* Λ *of* $\mathscr{K}$ *onto* $\tilde{\mathscr{K}}$ *such that* $\Lambda A_j\Lambda^{-1}=\tilde{A}_j$ *and* $\Lambda B_jF_j\Lambda^{-1}=\tilde{B}_j\tilde{F}_j$ *for* $j=1,2$.
(ii) *The pair* $\{a, \mathscr{b}\}$ *is irreducible if and only if the* 4-*tuple* $\{A_1, A_2, B_1F_1, B_2F_2\}$ *of self-adjoint operators on* $\mathscr{K}$ *is irreducible, i.e.* $\{A_1, A_2, B_1F_1, B_2F_2\}'=\mathbb{C}\cdot 1$.

Proof. We carry our the proof of (i). The assertions of (ii) follow by some slight modifications of the arguments. The if part of (i) is clear. Indeed, considering Λ as a constant operator of $L^2_{\mathscr{K}}(\mathbb{R})$ into $L^2_{\tilde{\mathscr{K}}}(\mathbb{R})$, the unitary operator $U:=\Lambda\oplus\Lambda$ of $\mathscr{H}$ onto $\tilde{\mathscr{H}}$ establishes the equivalence of $\{a, \mathscr{b}\}$ and $\{\tilde{a}, \tilde{\mathscr{b}}\}$.

We prove the only if part of (i). For this let $T=T_1\oplus T_2$ be a bounded linear operator of $\mathscr{H}$ into $\tilde{\mathscr{H}}$, where $T_1:\mathscr{H}_+\to\tilde{\mathscr{H}}_+$ and $T_2:\mathscr{H}_-\to\tilde{\mathscr{H}}_-$. We suppose that T intertwines $\{a, \mathscr{b}\}$ and $\{\tilde{a}, \tilde{\mathscr{b}}\}$. Then Theorem 5.1 applies, so the conditions (i)-(iii) stated therein are fulfilled. We freely use the notation from Theorem 5.1. By (iii), $\Lambda:=T_1=T_2$ is constant, i.e. Λ is an operator from $\mathscr{K}$ into $\tilde{\mathscr{K}}$. (i) yields $\Lambda(I-F_j)=(I-\tilde{F}_j)\Lambda$, so $\Lambda F_j=\tilde{F}_j\Lambda$. By (ii), there is a bounded linear operator $\Lambda_j:\mathscr{K}_j\to\tilde{\mathscr{K}}_j$ such that $\tilde{w}_j\Lambda_j=\tilde{F}_j\Lambda w_j$, i.e. $\tilde{w}_j(x)\ \Lambda_j=\tilde{F}_j\Lambda w_j(x)$ a.e.. The latter implies that

$$\langle\Lambda_j\eta,\ e^{-ix\tilde{C}_j-ix^2\tilde{B}'_j}\ \tilde{\eta}\rangle=\langle\tilde{F}_j\Lambda e^{+ixC_j+ix^2B'_j}\ \eta,\ \tilde{\eta}\rangle \tag{7.8}$$

for all $\eta\in\mathscr{E}_j$ and $\tilde{\eta}\in\tilde{\mathscr{E}}_j$. From the definitions of $\mathscr{E}_j$ and $\tilde{\mathscr{E}}_j$ it follows easily that both sides of (7.8) have power series expansions in x which converge on the whole real line (see also the above proof of Proposition 7.1, (i)). Comparing the constant terms, we get $\langle\Lambda_j\eta,\ \tilde{\eta}\rangle=\langle\tilde{F}_j\Lambda\eta,\ \tilde{\eta}\rangle$ for all $\eta\in\mathscr{E}_j$ and $\tilde{\eta}\in\tilde{\mathscr{E}}_j$, so that $\Lambda_j=\tilde{F}_j\Lambda\upharpoonright\mathscr{K}_j$. Since $\tilde{F}_j\Lambda=\Lambda F_j$, we obtain

$\Lambda_j=\Lambda\upharpoonright\mathscr{K}_j$. Comparing the linear terms and the quadratic terms in (7.8), we have

$$\langle\Lambda_j\eta,\ \tilde{C}_j\tilde{\eta}\rangle=\langle\Lambda_jC_j\eta,\ \tilde{\eta}\rangle \tag{7.9}$$

and

$$\langle\Lambda_j\eta,\ (\tilde{C}_j^2+2\tilde{B}'_j)\tilde{\eta}\rangle=\langle\Lambda_j(C_j^2+2B'_j)\eta,\ \tilde{\eta}\rangle, \tag{7.10}$$

where we used that $\Lambda_j=\tilde{F}_j\Lambda\upharpoonright\mathscr{K}_j$. Since $\tilde{\mathscr{E}}_j$ is a core for $\tilde{C}_j$, (7.9) implies that $\Lambda_jC_j\subseteq\tilde{C}_j\Lambda_j$. Using the latter and the invariance of $\tilde{\mathscr{E}}_j$ under $\tilde{C}_j$, (7.10) leads to $\langle\Lambda_j\eta,\ \tilde{B}'_j\tilde{\eta}\rangle=\langle\Lambda_jB'_j\eta,\ \tilde{\eta}\rangle$. Therefore, $\Lambda_jB'_j\subseteq\tilde{B}'_j\Lambda_j$, since $\mathscr{E}_j$ is also a core for B'_j.
To complete the proof of (i), we assume that T is unitary. Then $\Lambda:\mathscr{K}\to\tilde{\mathscr{K}}$ and $\Lambda_j:\mathscr{K}_j\to\tilde{\mathscr{K}}_j$ are unitary operators. From $\Lambda_j=\Lambda\upharpoonright\mathscr{K}_j$, $\Lambda(I-F_j)=(I-\tilde{F}_j)\Lambda$, $\Lambda_jC_j\subseteq\tilde{C}_j\Lambda_j$ and $\Lambda_jE_j=\tilde{E}_j\Lambda_j$ (by condition (ii) of Theorem 5.1) we conclude that $\Lambda A_j\Lambda^{-1}=\tilde{A}_j$. From $\Lambda_jB'_j\subseteq\tilde{B}'_j\Lambda_j$ we obtain that $\Lambda B_jF_j\Lambda^{-1}=\tilde{B}_j\tilde{F}_j$. □

We conclude with a discussion of the special case where $B_1=0$ and $B_2=0$. Then it is easy to check (using (7.1)) that the symmetric operator defined by the operator matrix $\mathscr{b}$ on the dense domain $\mathscr{D}'_1\oplus\mathscr{D}'_2$ in $\mathscr{H}$ is essentially self-adjoint. According to our terminology (see Section 2), this means that $\mathscr{b}$ represents the self-adjoint operator $\bar{b}$. Moreover, the domain $\mathscr{D}'_j$, $j=1,2$, is invariant under $e^{\omega Q}$ and $e^{\omega P}$ for any $\omega\in\mathbb{C}$ and under b_{jj} and $b_{12}=b_{21}$. Therefore, the self-adjoint operators a and b fulfill the relation (1.1) in operator theoretic sense on the dense invariant domain $\mathscr{D}'_1\oplus\mathscr{D}'_2$ (that is, we have $a\bar{b}\xi=p\bar{b}a\xi$ for $\xi\in\mathscr{D}'_1\oplus\mathscr{D}'_2$) which is a core for both operators. Further, $\mathscr{D}'_j$ is obviously a core for b_{jj} and for $b_{12}=b_{21}$. Hence unitary equivalence and irreducibility for the pairs $\{a,\ \mathscr{b}\}$ are equivalent to the corresponding notions for the couples $\{a,\ \bar{b}\}$ of self-adjoint operators, cf. Remarks 5.) and 3.) in Section 5. Thus Proposition 7.3 gives the following statements: The couple $\{a,\ \bar{b}\}$ of self-adjoint operators on $\mathscr{H}$ is irreducible if and only if $\{A_1,\ A_2\}$ is irreducible on $\mathscr{K}$. Two such pairs $\{a,\ \bar{b}\}$ and $\{\tilde{a},\ \bar{\tilde{b}}\}$ are unitarily equivalent if and only if $\{A_1,\ A_2\}$ and $\{\tilde{A}_1,\ \tilde{A}_2\}$ are. In particular, by setting $A_1:=\mathrm{Re}\ T$ and $A_2:=\mathrm{Im}\ T$, each generator T for the von Neumann algebra $B(\mathscr{K})$ gives us an irreducible pair $\{a,\ \bar{b}\}$ on $\mathscr{H}$. Thus this rather simple example produces already a continuum of inequivalent irreducible a-integrable representations (in the sense of Definition 4.2) of the relation (1.1) by self-adjoint operators.

We close this paper with

Example 7.4. Define self-adjoint operators a and b on the Hilbert space $\mathscr{H}:=L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ by

$$a=\begin{pmatrix} e^Q & 0 \\ 0 & -e^Q \end{pmatrix} \quad \text{and} \quad b=\begin{pmatrix} 0 & e^{\alpha P} \\ e^{\alpha P} & 0 \end{pmatrix} \tag{7.11}$$

on the domains $\mathscr{D}(a)=\mathscr{D}(e^Q)\oplus\mathscr{D}(e^Q)$ and $\mathscr{D}(b)=\mathscr{D}(e^{\alpha P})\oplus\mathscr{D}(e^{\alpha P})$. (Note that this is the special case $\mathscr{K}=\mathbb{C}$, $A_1=A_2=B_1=B_2=0$ of the above general example.)

Let U denote the unitary operator on $\mathscr{H}=L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ given by the 2×2 operator matrix (u_{ij}), where $u_{11}=u_{12}=u_{22}=-u_{21}:=2^{-1/2}$. Then we have

$$UaU^{-1}=\begin{pmatrix} 0 & -e^Q \\ -e^Q & 0 \end{pmatrix} \quad \text{and} \quad UbU^{-1}=\begin{pmatrix} e^{\alpha P} & 0 \\ 0 & -e^{\alpha P} \end{pmatrix} \tag{7.12}$$

on the respective domains. From (7.11) and (7.12) we conclude at once that the pair $\{a, b\}$ is a-integrable and that the pair $\{b, a\}$ is b-integrable (both in the sense of Definition 4.2), so $\{a, b\}$ is an integrable representation of (1.1) according to Definition 3.1. Note that the couple $\{a, b\}$ of self-adjoint operators is irreducible and that each of the non-singular operators a and b is neither positive nor negative.

References

[1] Van Daele, A., The operator $a\otimes b+b\otimes a^{-1}$ when $ab=\lambda ba$, Preprint, Leuven, 1990.

[2] Drinfeld, V.G., Quantum groups, in Proceedings of the ICM Berkeley 1986, *Amer. Math. Soc.*, Providence, (1987), 793–820.

[3] Faddeev, L.D., Reshetikhin, N.Yu. and Takhtajan, L.A., Quantization of Lie groups and Lie algebras, *Algebra i Analiz*, **1** (1989), 178–206. (Russian)

[4] Kato, T., *Pertubation theory for linear operators*, Springer-Verlag, Berlin, 1966.

[5] Katznelson, Y., *An introduction to harmonic analysis*, Dover Publ., New York, 1976.

[6] Manin, Yu. I, Quantum groups and non-commutative geometry, *Publications du CRM*, **1561**, Montreal, 1988.

[7] Nagel, R., Towards a "matrix theory" for unbounded operator matrices, *Math. Z.*, **201** (1989), 57–68.

[8] Ostrowski, V.L. and Samoilenko, Yu. S., Structure theorems for a pair of unbounded self-adjoint operators satisfying quadratic relations, Preprint 91.4, Kiev, 1991.

[9] Poulsen, N.S., On C^∞-vectors and intertwining bilinear forms for representations of Lie

groups, *J. Funct. Analysis*, **9** (1972), 87–120.

[10] Putnam, C.R., *Commutation properties of Hilbert space operators*, Springer-Verlag, Berlin, 1967.

[11] Schmüdgen, K., On domains of powers of closed symmetric operators, *J. Operator Theory*, **9** (1983), 53–75.

[12] Takesaki, M., *Theory of operator algebras I*, Springer-Verlag, New York, 1979.

[13] Woronowicz, S.L., Twisted SU(2). An example of noncommutative differential calculus,*Publ. RIMS Kyoto Univ.*, **23** (1987), 117–181.

[14] ———, Unbounded elements affiliated with C^*algebras and non-compact quantum groups, *Commun. Math. Phys.*,, **136** (1991), 399–432.