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Algebraic Aspects in Modular Theory

Dedicated to Nobuko on her sixtieth birthday

By

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Abstract

Algebraic features of modular theory in von Neumann algebras are discussed with the help of Haagerup's L^p-theory and some notational devices.

Preface

Nowadays, the so called modular theory for von Neumann algebras is well-established and fully utilized (see [17] for example) in the field of operator algebras. The symbols conventionally used there, however, does not seem to be so much expressive in some sense. One of the main purposes in the present article is a focussed account of the problem of this kind for modular theory in operator algebras.

In the past time, there had been already some suggestions on the improvement of notations for the modular theory but not in a thorough way. Among them, Woronowicz's approach [22] and new symbols introduced in [2] are worthy of attention. Around the same time of these works, the non-commutative L^p-theory for arbitrary von Neumann algebras came out and had been developped by several people such as Haagerup, Connes-Hilsum, Kosaki, and Araki-Masuda, and so on. It is worth pointing out the fact that, in this theory, the $1/p$-th power of a state of a von Neumann algebra is identified with an element in the relevant L^p-space.

Now we give a brief outline of the contents in this article. The first section surveys the background materials and the substantial parts start from the next section.

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In §2, after the introduction of formal powers of positive linear functionals on von Neumann algebras, we shall give an algebraic interpretation of the works mentioned above. For example, we can get clear understanding of Connes' converse theorem on Radon-Nikodym cocycles, Araki's multiple time KMS-condition, and so on.

In §3, 'relative modular theory' is introduced based on the formalism in §2, which enables us to incorporate the theory of operator-valued weights of Haagerup. The theory of operator-valued weights is utilized in the index theory for type III von Neumann algebras (cf. [10]) and the present formulation will give some perspective in that field.

As to the technical part of our exposition, we rely heavily on Haagerup's theory of non-commutative L^p-spaces. In some sense, our algebraic approach in modular theory is a natural generalization of L^p-theory to complex exponents.

Originally, the present research comes out from the investigation of the meaning of modular operators which appear in Sauvageot's relative tensor products. We can explain his construction as well as Connes' theory of spatial derivatives in the framework of the present approach. The appropriate language to treat such topics is, however, the 'modular theory' for bimodules and it deserves separate expositions because of the importance in the Ocneanu's approach to Jones' index theory.

We would like to return to this subject in a future paper.

§1. Preliminaries

This section is devoted to the description of standard facts as well as related technical tools which will be used in the succeeding sections. The main reference here is [20].

Notation and Convention

For a linear operator x in a Hilbert space $\mathscr{H}$, we denote by $\mathbb{s}(x)$ the support projection of x, i.e., the (orthogonal) projection in $\mathscr{H}$ corresponding to the closed subspace $\overline{D(x)} \ominus \overline{\ker x}$. More generally (and vaguely), any object x which is represented (or can be regarded) as a closable linear operator, the symbol $\mathbb{s}(x)$ is used to denote the support of the operator.

Resolutions of the identity in the spectral decomposition of self-adjoint operators are assumed to be left-continuous.

When one speaks of holomorphic functions f on a closed subset D of

$\mathbb{C}$, f is assumed to be a bounded continuous function which is 'integrable' in the following sense: For any closed path γ in D, the contour integral of f along γ vanishes whenever γ is homotopic (in D) to a point.

For unbounded linear operators A and B, we denote the usual addition and product by $A \dotplus B$ and $A \cdot B$, while the notation $A+B$ and AB is reserved for the strong sum and the strong product (if they exist).

Measurable Operators

In this part, we shall scratch the theory of measurable operators in the form appropriate for our purpose (so the description may be biassed). For the full account, we refer to [13] (also cf. [20]).

Let N be a semifinite von Neumann algebra realized in a Hilbert space $\mathscr{H}$ with a faithful normal semifinite trace τ.

A closed linear operator x in $\mathscr{H}$ affiliated with N is called **τ-measurable** if there is a projection p in N such that

(i) $p\mathscr{H}$ is contained in the definition domain of x and the restriction $x|_{p\mathscr{H}}$ is bounded (this situation is simply expressed as $\|xp\| < +\infty$),
(ii) $\tau(1-p) < +\infty$.

The set of τ-measurable operators affiliated with N is denoted by $\overline{N}^{\tau}$ (or simply $\overline{N}$). Any τ-measurable operator x is automatically densely defined and its adjoint x^* is again τ-measurable. Moreover, for two τ-measurable operators x, $y \in \overline{N}$, $x+y$ and xy (here the sum and the product should be understood in the usual sense of unbounded operator) are densely defined closable operators and their closures are again τ-measurable. In the following the closures $\overline{x+y}$ and $\overline{xy}$ are simply denoted by $x+y$ and xy respectively.

Theorem 1.1. *With the algebraic operations described above, $\overline{N}^{\tau}$ is a *-algebra.*

Remark. The condition of τ-measurability on a closed densely defined operator x is equivalent to the condition

$$\exists c > 0, \ \tau(1-e_c(|x|)) < +\infty,$$

where $\lambda \mapsto e_\lambda(|x|)$ denotes the spectral resolution of the positive self-adjoint operator $|x| = (x^*x)^{1/2}$.

The following topology (defined by the convergence) in $\overline{N}$ is introduced in [16] and fully utilized in [13].

Definition 1.2. *A sequence of operators* $\{x_n\}_{n\geq 1}$ *in* $\overline{N}^{\tau}$ *is called to converge in measure to* $x\in\overline{N}^{\tau}$ *if* $\exists$ *a sequence of projections* $\{p_n\}_{n\geq 1}$ *in* N *such that*

$$\lim_n \|(x_n-x)p_n\|=0 \quad \textit{and} \quad \lim_n \tau(1-p_n)=0.$$

Theorem 1.3. $\overline{N}^{\tau}$ *is a complete Hausdorff topological *-algebra in the measure topology which contains* N *as a dense *-subalgebra.*

Corollary 1.4. *As a completion of* N *with respect to the measure topology,* $\overline{N}^{\tau}$ *is independent of the choice of a specific representation of* N *on a Hilbert space.*

Remark. The above result other than the completeness is already proven in [16]. In [13], the completeness is proved in the following way: First complete N with respect to the measure topology and then the abstract space of completion is identified with the set of measurable operators. For a direct proof of the completeness of $\overline{N}$, see [20].

The following fact may be known. For the completeness we add its proof.

Lemma 1.5. *Let* φ, ψ *be two normal semifinite weights on* N *with* h, k *their Radon-Nikodym derivatives with respect to* τ. *If* h *and* k *are* τ*-measurable, then the sum* $\varphi+\psi$ *is semifinite and its Radon-Nikodym derivative with respect to* τ *is given by the* (*strong*) *sum* $h+k$.

$\because$) Suppose that N is the von Neumann algebra on a Hilbert space $\mathscr{H}$. Let $\{e_\lambda\}_{\lambda\in\boldsymbol{R}}$, $\{f_\lambda\}_{\lambda\in\boldsymbol{R}}$ be the spectral resolutions of h, k respectively and set $h_n=he_n$, $k_n=kf_n$.

Then $h_n\nearrow h$, $k_n\nearrow k$ and [14, Proposition 4.2] gives

$$\varphi=\lim_n \tau(h_n\cdot),\ \psi=\lim_n \tau(k_n\cdot).$$

Since h_n and k_n are bounded, [14, Proposition 4.1] assures

$$\tau((h_n+k_n)\cdot)=\tau(h_n\cdot)+\tau(h_k\cdot).$$

Thus, once we can show that $h_n+k_n \nearrow h+k$, [14, Proposition 4.2] gives

$$\begin{aligned}\tau((h+k)\cdot) &= \lim_n \tau((h_n+k_n)\cdot)\\ &= \lim_n (\tau(h_n\cdot)+\tau(k_n\cdot))\\ &= \varphi+\psi.\end{aligned}$$

The semifiniteness of $\varphi+\psi$ is clear from this expression.

To see the convergence, we first remark that the strong sum $h+k$ is self-adjoint by the theory of measurable operators. The real content of the above convergence is the convergence of $(h_n+k_n)(1+h_nk_n)^{-1}$ to $(h+k)(1+h+k)^{-1}$ in strong operator topology. Let D_n be the range of $e_n \wedge f_n$ for $n=1,2,\cdots$. Then D_n is increasing as $n\to\infty$ and the estimate

$$\tau(1-e_n\wedge f_n)\leq \tau(1-e_n)+\tau(1-f_n)$$

shows that $D=\cup_{n\geq 1} D_n$ is dense in $\mathscr{H}$. Since

$$(h+k)\xi=(h_n+k_n)\xi \quad \text{for } \xi\in D_m \text{ with } m\leq n,$$

the problem is reduced to check the convergence of $(1+h_n+k_n)^{-1}$ to $(1+h+k)^{-1}$ in strong operator topology. Since the closure of $(1+h+k)|_D$ coincides with $1+h+k$ (the uniqueness of extensions of measurable operators), $(1+h+k)(D)$ is dense in $\mathscr{H}$. Take $\xi\in D$. For sufficiently large n, we have $(1+h+k)\xi=(1+h_n+k_n)\xi$ and hence

$$\begin{aligned}(1+h+k)^{-1}(1+h+k)\xi=\xi &= (1+h_n+k_n)^{-1}(1+h_n+k_n)\xi\\ &= (1+h_n+k_n)^{-1}(1+h+k)\xi.\end{aligned}$$

Thus $(1+h_n+k_n)^{-1}$ converges to $(1+h+k)^{-1}$ on the dense subspace $(1+h+k)(D)$. Since the norms of $(1+h_n+k_n)^{-1}$ and $(1+h+k)^{-1}$ are uniformly bounded, the desired convergence $\lim_n(1+h_n+k_n)^{-1}=(1+h+k)^{-1}$ follows. □.

For $\varepsilon>0$, $\delta>0$, if we set

$$N(\varepsilon,\ \delta)=\{x\in\overline{N}^{\tau};\ \exists \text{ a projection } p \text{ in } N,\ \|xp\|\leq\varepsilon, \tau(1-p)\leq\delta\},$$

$\{N(\varepsilon,\ \delta)\}$ forms a fundamental set of neighborhoods in measure topology at 0.

The following description of $N(\varepsilon, \delta)$ is taken from [20] (see the remark after Theorem 1 for the notation).

Lemma 1.6.

$$N(\varepsilon, \delta)=\{x\in\overline{N}^{\tau};\ \tau(1-e_{\varepsilon}(|x|)\leq\delta\}.$$

The next result is [20] II. Lemma 18.

Lemma 1.7. *Let h be a positive self-adjoint operation in $\overline{N}^{\tau}$. Set $C_{++}=\{z\in C;\ \Re z>0\}$. For $z\in C_{++}$, h^z belongs to $\overline{N}^{\tau}$ and the map*

$$z\mapsto h^z\in\overline{N}^{\tau}$$

is differentiable on C_{++} if $\overline{N}^{\tau}$ is furnished with the measure topology.

Modular Theory

In this part, we shall review some of the key formulae in modular theory, which at the same time makes our notation fixed (although it is standard).

Let N be a von Neumann algebra with the predual N_* and its positive part N_*^+. We denote by $\overline{N}_*^+$ the set of normal semifinite weights on N and by $\overline{N}_*^{++}$ the set of normal faithful semifinite weights. Since the weights dealt with in this paper are always assumed to be normal, we often omit the adjective 'normal' in the following.

For $\varphi\in\overline{N}_*^+$, we denote by $L^2(N,\varphi)$ the GNS-construction (or representation) of N. If φ is faithful, we define a densely defined operator S_φ in $L^2(N, \varphi)$ by

$$S_\varphi([x]_\varphi)=[x^*]_\varphi,\ x\in N,\ \varphi(x^*x)<\infty,\ \varphi(xx^*)<\infty.$$

S_φ is closable and, if we denote by $J_\varphi\Delta_\varphi^{1/2}$ the polar decomposition of the closure $\overline{S_\varphi}$, we have (note that $S_\varphi^2=id$)

$$\begin{cases} J_\varphi^2=id, \\ J_\varphi\Delta_\varphi J_\varphi=\Delta_\varphi^{-1}. \end{cases}$$

The fundamental facts in Tomita-Takesaki theory are

(i) AdJ_φ transfers N onto its commutant in $L^2(N, \varphi)$.

(ii) $\forall\alpha\in C$, $Ad\Delta_\varphi^\alpha$ 'preserves' N in a certain sense.

In particular,

$$\sigma_t^{\varphi}(x) = \Delta_{\varphi}^{it} x \Delta_{\varphi}^{-it}, \quad x \in N$$

defines a 1-parameter *-automorphism group of N, called the **modular automorphism group** of φ.

For two φ, $\psi \in \overline{N_*^{++}}$, we can compare their modular automorphism groups by the 2×2 matrix method: Let $\theta \in \overline{Mat_2(N)_*^{++}}$ be defined by

$$\theta\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = \varphi(x_{11}) + \psi(x_{22}).$$

Let

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since h and hence

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(1+h), \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}(1-h)$$

are in the centralizer of θ, we can find a 1-parameter group of isometries $\sigma_t^{\varphi,\varphi}$, $\sigma_t^{\varphi,\psi}$, $\sigma_t^{\psi,\varphi}$, $\sigma_t^{\psi,\psi}$ such that

$$\sigma_t^{\theta}\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = \begin{bmatrix} \sigma_t^{\varphi,\varphi}(x_{11}) & \sigma_t^{\varphi,\psi}(x_{12}) \\ \sigma_t^{\psi,\varphi}(x_{21}) & \sigma_t^{\psi,\psi}(x_{22}) \end{bmatrix}.$$

By the KMS-condition, we can identify $\sigma_t^{\varphi,\varphi}$ with σ_t^{φ} and $\sigma_t^{\psi,\psi}$ with σ_t^{ψ}. On the other hand, since

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

is a partial isometry with initial and final projections

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

we can find a continuous family of unitaries $\{(D\psi : D\varphi)_t\}_{t \in \mathbf{R}}$ in N satisfying

$$\sigma_t^\theta\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ (D\psi:D\varphi)_t & 0 \end{bmatrix}.$$

Now, by matrix manipulations, we have

$$(1) \qquad (D\varphi:D\psi)_{t+s} = (D\varphi:D\psi)_t\,\sigma_t^\psi((D\varphi:D\psi)_s),\ s,t\in\mathbb{R},$$

$$(2) \qquad \sigma_t^\psi(x) = (D\varphi:D\psi)_t\,\sigma_t^\psi(x)((D\varphi:D\psi)_t)^*,\ t\in\mathbb{R},$$

$$(3) \qquad \sigma_t^{\varphi,\psi}(x) = (D\varphi:D\psi)_t\sigma_t^\psi(x),\ t\in\mathbb{R},$$

$$(4) \qquad \sigma_t^{\varphi,\psi} = (\sigma_t^{\psi,\varphi})^{-1},\ t\in\mathbb{R}.$$

Similarly, if we consider the 3×3 matrix of N, for φ_1, φ_2, $\varphi_3\in\overline{N}_*^{++}$,

$$(5) \qquad \sigma_t^{\varphi_1,\varphi_2}\sigma_t^{\varphi_2,\varphi_3} = \sigma_t^{\varphi_1,\varphi_3},\ t\in\mathbb{R},$$

which follows from

$$\sigma_t^\theta\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \sigma_t^\theta\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)\sigma_t^\theta\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right).$$

The above argument works for non-faithful weights: For $\varphi\in\overline{N}_*^+$, φ defines a faithful semifinite weight on the reduced algebra $\mathbf{s}(\varphi)N\mathbf{s}(\varphi)$, where $\mathbf{s}(\varphi)$ denotes the support projection of φ, and its modular automorphism group is defined to be this reduced one. Similarly, given two weights φ, $\psi\in\overline{N}_*^+$, 1-parameter group of isometries $\sigma_t^{\varphi,\psi}$ of $\mathbf{s}(\varphi)N\mathbf{s}(\psi)$ is defined. 1-parameter family of operators $\sigma_t^{\varphi,\psi}(\mathbf{s}(\varphi)\mathbf{s}(\psi))$, $t\in\mathbb{R}$, in $\mathbf{s}(\varphi)N\mathbf{s}(\psi)$ is denoted by $\{[D\varphi: D\psi]_t\}$ as well.

§2. Modular Algebras

For a von Neumann algebra N, we shall introduce, in this section, a *-algebra which contains N as a *-subalgebra and is generated by N and the symbols φ^α with $\varphi\in N_*^+$ and $\alpha\in\mathbb{C}_+=\{z\in\mathbb{C};\ \Re z\geq 0\}$ (this account is a bit inaccurate when N does not admit a faithful normal state). This algebra is called the **modular algebra** of N in this paper because all the essense of algebraic features of the modular theory is stuffed in it.

Modular Algebras—boundary case

Let us begin with the description of the algebra generated by purely imaginary powers (cf. [2]), which turns out to give a canonical realization of Takesaki's dual. For a von Neumann algebra N, let $N(i\boldsymbol{R})$ be the universal $*$-algebra generated algebraically by N and a set of symbols $\{\varphi^{it};\ \varphi\in\overline{N}_*^+,\ t\in\boldsymbol{R}\}$ with the relations

$$(1)\qquad \varphi^{is}\varphi^{it}=\varphi^{i(s+t)},\ (\varphi^{it})^*=\varphi^{-it},\ \varphi^{i0}=\mathbf{s}(\varphi),$$

$$(2)\qquad \varphi^{it}x\varphi^{-it}=\sigma_t^{\varphi}(x),\ x\in N,$$

$$(3)\qquad \varphi^{it}=[D\varphi:D\psi]_t\psi^{it},$$

for $\varphi,\ \psi\in\overline{N}_*^{++}$ and $s,\ t\in\boldsymbol{R}$. Here $\mathbf{s}(\varphi)$ stands for the support of φ. (More precisely, $N(i\boldsymbol{R})$ is defined to be the quotient of a formal $*$-algebra generated by N and φ^{it} by the $*$-ideal generated by the relations (1), (2), (3).)

The $*$-algebra $N(i\boldsymbol{R})$ is graded in the obvious way: If we set

$$N(it)=\sum_{\varphi}N\varphi^{it}\ \text{(summation is an algebraic one)}$$

for $t\in\boldsymbol{R}$, then $N(i\boldsymbol{R})=\bigoplus_{t\in\boldsymbol{R}}N(it)$ and

$$N(is)N(it)\subset N(i(s+t)),\ N(it)^*=N(-it).$$

Note that $N(it)=N\varphi^{it}=\varphi^{it}N$ for any $\varphi\in\overline{N}_*^{++}$.

As a consequence of the gradation, we can define a 1-parameter automorphism group $\{\theta_s\}_{s\in\boldsymbol{R}}$ (called **scaling automorphism group**) of $\hat{N}$ by

$$(4)\qquad \theta_s(x\varphi^{it})=e^{-ist}x\varphi^{it},\ x\in N,\ \varphi\in\overline{N}_*^+,\ s,\ t\in\boldsymbol{R}.$$

By a $*$-representation of $N(i\boldsymbol{R})$ in a Hilbert space H, we mean a $*$-homomorphism π of $N(i\boldsymbol{R})$ into the full operator algebra $\mathscr{B}(\mathscr{H})$ such that $\pi|_N$ is a normal representation of the von Neumann algebra N, $t\mapsto\pi(\varphi^{it})$ is weakly continuous for any $\varphi\in\overline{N}_*^+$, and θ is extended to a weakly continuous 1-parameter group of automorphisms of $\pi(N(i\boldsymbol{R}))''$.

Let $\hat{N}$ be the von Neumann algebra obtained as the closure of $N(i\boldsymbol{R})$ with respect to the topology induced from the set of $*$-representations of $N(i\boldsymbol{R})$. By the universality of the construction, θ is uniquely extended to a 1-parameter automorphism group of $\hat{N}$, which is again denoted by θ.

Lemma 2.1. *$(\hat{N}, \theta)$ is identified with the Takesaki's dual of N.*

$\because$) Take a faithful weight $\psi \in \overline{N}_*^{++}$ and construct the crossed product $N \rtimes_\sigma \boldsymbol{R}$ relative to the modular automorphism group $\{\sigma_t\}$ associated with ψ. Note that $N \rtimes_\sigma \boldsymbol{R}$ is a von Neumann algebra generated by N and a distinguished 1-parameter group of unitaries $\{\lambda_t\}_{t \in \boldsymbol{R}}$ which implements σ, i.e.,

$$\lambda_t x \lambda_t^{-1} = \sigma_t(x), \quad x \in N, \ t \in \boldsymbol{R}. \tag{5}$$

We can define a *-representation π of $N(i\boldsymbol{R})$ in $N \rtimes_\sigma \boldsymbol{R}$ by

$$\pi(x\varphi^{it}) = x(D\varphi: D\psi)_t \lambda_t. \tag{6}$$

In fact, (1) follows from (1–3) and (6), while (2), (3) is a consequence of (1–1), (1–2). The continuity properties of π are clear (note that θ is identified with the dual action of σ).

Since π is evidently faithful on N and intertwines θ to the dual action of $N \rtimes_\sigma \boldsymbol{R}$, a characterization of crossed product ([19], [12]) shows that π is an isomorphism. □

As a corollary of this fact and the corresponding result for crossed products, N is identified with a subalgebra of $\hat{N}$ and is characterized as the fixed point algebra of θ. More generally, we have

$$N(it) = \{\hat{x} \in \hat{N};\ \theta_s(\hat{x}) = e^{-ist}\hat{x},\ \forall s \in \boldsymbol{R}\}. \tag{7}$$

Furthermore,

$$\varepsilon(x) = \int_{-\infty}^{+\infty} ds\, \theta_s(x), \quad x \in \hat{N}_+ \tag{8}$$

defines a normal faithful semifinite N-valued weight ε on $\hat{N}_+$. By a general theory of operator-valued weight ([5]), $\hat{\varphi} = \varphi \circ \varepsilon$ gives a normal semifinite weight (**dual weight**) for $\varphi \in \overline{N}_*^+$ and their modular groups as well as Radon-Nikodym cocycles are expressed in terms of original weights:

$$\sigma_t^{\hat{\varphi}} = Ad\varphi^{it}, \tag{9}$$

$$(D\hat{\varphi}: D\hat{\psi})_t = (D\varphi: D\psi)_t. \tag{10}$$

From (9) and (10), one sees that $\exists^1$ a normal faithful semifinite trace τ on

$\hat{N}$ satisfying

(11) $$(D\hat{\varphi}:D\tau)_t=\varphi^{it},\ \forall t\in\boldsymbol{R},\ \forall\varphi\in\overline{N}_*^+$$

([18]). The Radon-Nikodym derivative h_φ of $\hat{\varphi}$ with respect to τ in the sense of [14] (h_φ is a positive self-adjoint operator affiliated with $\hat{N}$) is identified with the (exponentiated) generator of φ^{it}:

$$h_\varphi^{it}=\varphi^{it},\ t\in\boldsymbol{R}$$

Theorem 2.2 (Haagerup).
(i) *The map $\varphi\mapsto h_\varphi$ defines a bijection from the set of semifinite weights on N onto the set of positive self-adjoint operators h affiliated with $\hat{N}$ and satisfying*

$$\theta_s(h)=e^{-s}h,\ \forall s\in\boldsymbol{R}.$$

(ii) *For $\varphi\in\overline{N}_*^+$, φ is bounded, i.e., $\varphi\in\overline{N}_*^+$ iff h_φ is τ-measurable.*
(iii) *The map $N_*^+\ni\varphi\mapsto h_\varphi$ is uniquely extended to a linear isomorphism from the predual N_* onto the set of τ-measurable operators h affiliated with $\hat{N}$ and satisfying*

$$\theta_s(h)=e^{-s}h,\ \forall s\in\boldsymbol{R},$$

*which turns out to be an N-bimodule map preserving *-operation and polar decompositions: If $\varphi=u|\varphi|$ is the polar decomposition of φ as a linear form, then $h_\varphi=uh_{|\varphi|}$ gives the polar decomposition of h_φ as closed operator.*

Remark. The assertion (ii) in the theorem is assured by showing the following formula due to Haagerup:

$$\tau(1-e(\lambda))=\varphi(1)/\lambda,\quad\text{for }\lambda>0.$$

Here $\{e(\lambda)\}_{\lambda\geq 0}$ denotes the spectral resolution of φ as the positive self-adjoint operator affiliated with $\hat{N}$.

Considering the importance of the map $\varphi\to h_\varphi$ as it will be made clear succeedingly, we want to call it *Haagerup's correspondence* in this paper. It is not always convenient, however, to distinguish φ and h_φ and sometimes it is even unnatural, so we regard N_* as a subspace of the *-algebra $\overline{\hat{N}}$ of τ-measurable operators affiliated with $\hat{N}$ and use the same symbol to express them.

As an immediate application of the above results, one can prove the following fact which is already equivalent to a generalization of the Connes' converse theorem on Radon-Nikodym cocycles.

Lemma 2.3. *Let* $\mathbb{R}\ni t\mapsto u_t\in N(it)$ *be a weakly continuous* 1-*parameter group of partial isometries in* $\hat{N}$. *Then there is a unique weight* $\varphi\in\overline{N}_*^+$ *satisfying* $u_t=\varphi^{it}$ *for* $\forall t\in\mathbb{R}$.

$\because$) Let h be the generator of $\{u_t\}_{t\in\mathbb{R}}$: h is a positive self-adjoint operator affiliated with $\hat{N}$ and satisfying $h^{it}=u_t$. By the scaling property $\theta_s(u_t)=e^{-ist}u_t$, we see that

$$\theta_s(h)=e^{-s}h,\ \forall s\in\mathbb{R}.$$

Thus there is a unique weight $\varphi\in\overline{N}_*^+$ such that

$$\hat{\varphi}=\tau(h\cdot).$$

Taking the Radon-Nikodym cocycles of both sides with respect to τ, we have

$$\varphi^{it}=(D\hat{\varphi}\colon D\tau)_t=h^{it}=u_t.$$

The uniqueness of φ is clear. □

Modular Algebras—general case

For $\alpha\in\mathbb{C}_{++}=\{z\in\mathbb{C};\ \Re z>0\}$, let $N(\alpha)$ be the set of τ-measurable operators h affiliated with $\hat{N}$ and satisfying

$$\theta_s(h)=e^{-s\alpha}h,\qquad \forall s\in\mathbb{R}.$$

Note that this is an immediate extension of the definition of Haagerup's L^p-spaces.

We shall introduce a *-algebra which is an 'analytic continuation' of $N(i\mathbb{R})$: Since algebraic operations as well as operational calculus are applied for measurable operators, we can define the α-th power φ^α of $\varphi\in N_*^+$ $(\alpha\in\mathbb{C}_+)$ as a measurable operator in $\hat{\overline{N}}$ (note that φ^α is τ-measurable$\Leftrightarrow\varphi=0$ for $\Re\alpha<0$).

By the polar decomposition, $N(\alpha)$ is identified as

$$N(\alpha)=\{\textstyle\sum x_1\varphi_1^{z_1}\cdots x_n\varphi_n^{z_n};\ z_1\in\mathbb{C}_+,\cdots,z_n\in\mathbb{C}_+,\ \alpha=z_1+\cdots+z_n\}.$$

If we set

$$N(\boldsymbol{C}_+) = \sum_{\alpha \in \boldsymbol{C}_+} N(\alpha), \tag{12}$$

(the right hand side is an algebraic sum) $N(\boldsymbol{C}_+)$ is a *-subalgebra of $\bar{N}$. Furthermore, it is a graded *-algebra in the following sense:

$$N(\alpha)N(\beta) \subset N(\alpha+\beta), \quad N(\alpha)^* = N(\bar{\alpha}).$$

The summation in (12) is a direct sum because each component belongs to different spectral subspaces of θ. Note that $N(1) = N_*$ by Theorem (iii).

For a positive real number $s>0$, the set of positive τ-measurable operators in $N(s)$ is called the **positive cone** and is denoted by $N_+(s)$.

The following is immediate from operational calculus in the *-algebra of τ-measurable operators.

Proposition 2.4. *Let $s>0$ be a positive real number.*

(i) $\forall r \geq 0$, $\forall \rho \in N_+(s)$, $\rho^r \in N_+(rs)$.

(ii) $\forall t \in \boldsymbol{R}$, $\forall \xi \in N(s+it)$, $\exists^1 \rho \in N_+(s)$ *and a partial isometry* $u \in N(it)$ *such that*

$$\xi = u\rho, \quad u^*u = \mathbf{s}(\rho^{1/s}).$$

The following is a trivial generalization of a result in [20].

Lemma 2.5. *For a sequence $\{\varphi_n\}_{n\geq 1}$ and an element φ in N_*, the following two conditions are equivalent.*

(i) $\lim_n \varphi_n = \varphi$ *in the norm topology as the predual of N.*

(ii) $\lim_n \varphi_n = \varphi$ *in the measure topology as τ-measurable operators.*

$\because$) Taking the difference $\varphi_n - \varphi$, we may assume that $\varphi = 0$ (note that three topologies are translation-invariant). Let $\varphi_n = u_n|\varphi_n|$ be the polar decomposi-tion and $\{e_n(\lambda)\}_{\lambda \in \boldsymbol{R}}$ be the spectral resolution of $|\varphi_n|$ as τ-measurable operator.

(i)$\Rightarrow$(ii): Since $|\varphi_n|(1) = \|\varphi_n\| \to 0$ as $n \to +\infty$, we can find a sequence of positive numbers $\{\lambda_n\}$ such that

$$\lim \lambda_n = 0 \quad \text{and} \quad \lim |\varphi_n|(1)/\lambda_n = 0.$$

Set $p_n=e_n(\lambda_n)\in\hat{N}$. Clearly $\|\varphi_n p_n\|_{\hat{N}}\leq\||\varphi_n|p_n\|_{\hat{N}}\leq\lambda_n\to 0$. Moreover, by the formula of Haagerup, we have $\tau(1-p_n)=|\varphi_n|(1)/\lambda_n\to 0$ as $n\to+\infty$. Thus $\lim_n\varphi_n=0$ in the measure topology.

(ii)→(i): Take a sequence of projections $\{p_n\}_{n\geq 1}$ in $\hat{N}$ such that $\lim_n\tau(1-p_n)=0$, $\varphi_n p_n\in\hat{N}$, and $\lim_n\varphi_n p_n=0$ in the norm-topology of $\hat{N}$.

Let $c=\sup_{n\leq 1}\|\varphi_n p_n\|_{\hat{N}}$ and set $q_n=e_n(c/2)$. Then by the spectral decomposition of $|\varphi_n|$, we have $(1-q_n)\wedge p_n=0$. In fact, for $\xi\in(1-q_n)\wedge p_n H$, $\|\varphi_n\xi\|^2=\|\varphi_n p_n\xi\|^2\leq c^2\|\xi\|^2$, while

$$\|\varphi_n\xi\|^2=\||\varphi_n|\xi\|^2=\||\varphi_n|(1-q_n)\xi\|^2\geq(c/2)^2\|\xi\|^2$$

implies $\xi=0$. Hence $1-q_n\prec 1-p_n$. Again, by the formula of Haagerup, we have

$$\|\varphi_n\|_{N_*}=|\varphi_n|(1)=(c/2)\tau(1-q_n)\leq(c/2)\tau(1-p_n)\to 0 \text{ as } n\to+\infty.$$

□

Corollary 2.6 (Multiple KMS condition). *Let* $x_1,\cdots,x_n\in N$ and $\varphi_1,\cdots,\varphi_{n+1}\in N_*^+$. *Then the function*

$$(z_1,\cdots,z_n)\mapsto\varphi_1^{z_1}x_1\cdots\varphi_n^{z_n}x_n\varphi_{n+1}^{1-z_1-\cdots-z_n}$$

is a bounded holomorphic function on the tube

$$\{(z_1,\cdots,z_n)\in\mathbf{C}^n;\ z_1,\cdots,z_n\in\mathbf{C}_+,\ \Re z_1+\cdots+\Re z_n\leq 1\},$$

where N_* *is furnished with the norm topology as the predual of* N.

∵) The function is holomorphic on the domain in question by Lemma 5. The repeated application of three line theorem gives the boundedness. □

According to Haagerup, the evaluation map $N_*\ni\varphi\mapsto\varphi(1)$ is denoted by *tr* and is called **trace**. This terminology is justified by

Proposition 2.7.

(i) *Let* $\alpha\in\mathbf{C}$ *be in the strip* $0\leq\Re\alpha\leq 1$. *For* $\xi\in N(\alpha)$ *and* $\eta\in N(1-\alpha)$,

$$tr(\xi\eta)=tr(\eta\xi).$$

(ii) *Let* $s>0$ *be a positive real number and* $t\in\mathbf{R}$. *For* $\xi\in N(s+it)$,

$$tr((\xi^*\xi)^{1/2s}) = \mathrm{tr}((\xi\xi^*)^{1/2s}).$$

∵) The following is a slight modification of the proof of Proposition II.21 in [20].

(i) By Proposition 4 (polar decomposition), we can write $\xi = x\varphi^\alpha$, $\eta = y\psi^{1-\alpha}$ with $x, y \in N$ and $\varphi, \psi \in N_*^+$. Since

$$\begin{aligned} tr(x\varphi^{it}y\psi^{1-it}) &= \psi(x\varphi^{it}y\psi^{-it}) \\ &= \psi(\psi^{-it}x\varphi^{it}y) \quad \text{(by the } \sigma_t^\psi\text{-invariance of } \psi) \\ &= tr(y\psi(\psi^{-it}x\varphi^{it})) \\ &= tr(y\psi^{1-it}x\varphi^{it}), \end{aligned}$$

the analytic continuation of this relation from it to α gives the desired formula.

(ii) This is a special case of (i) by the polar decomposition of ξ and the operational calculus for $(\xi^*\xi)^{1/2s}$ and $(\xi\xi^*)^{1/2s}$. □

L^p-Spaces

Let $s > 0$ and $t \in \boldsymbol{R}$. For $\xi \in N(s+it)$, set

$$\|\xi\|_{s+it} = tr(\xi^*\xi)^{1/2s}.$$

Following the argument in [13], we can prove that $\|\cdot\|_{s+it}$ is a norm in $N(s+it)$: We first prove Hölder's inequality

$$|tr(\xi\eta)| \leq \|\xi\|_{s+it}\|\eta\|_{1-s-it} \text{ for } \xi = x\varphi^{s+it} \in N(s+it),\ \eta = y\psi^{1-s-it} \in N(1-s-it)$$

by the three-line theorem applied to the holomorphic function

$$z \mapsto tr(x\varphi^z y\psi^{1-z}), \quad 0 \leq \Re z \leq 1.$$

Then the identification

$$\|\xi\|_{s+it} = \sup\{|tr(\xi\eta)|;\ \eta \in N(1-s-it),\ \|\eta\|_{1-s-it} \leq 1\}$$

shows that $\|\cdot\|_{s+it}$ is a norm.

For $p \geq 1$, $N(1/p)$ with the norm $\|\cdot\|_{1/p}$ is called the L^p-**space** associated with a von Neumann algebra N (in the sense of Haagerup) and usually denoted by $L^p(N)$.

The following result is an extension of Lemma 5.

Lemma 2.8. *Let $s+it\in \boldsymbol{C}_{++}$. For a sequence $\{\xi_n\}_{n\geq 1}$ and an element ξ in $N(s+it)$, the following three conditions are equivalent.*

(i) $\lim_n \|\xi_n-\xi\|_{s+it}=0$.

(ii) $\lim_n \xi_n=\xi$ *in measure topology.*

∵) By a similar argument in the proof of Lemma 5. □

Now we sketch proofs of some of the properties of $N(s+it)$ (see [20] for details). Since the set $\overline{\hat{N}}$ of τ-measurable operators affiliated with the Takesaki's dual $\hat{N}$ of N is a complete topological vector space with respect to the measure topology and since $N(s+it)$ is identified with a closed linear subspace of $\overline{\hat{N}}$, $(N(s+it), \|\cdot\|_{s+it})$ is a Banach space. To see the reflexivity of these Banach spaces, we first prove Clarkson's inequality for $s\leq 1/2$:

$$\|\xi+\eta\|_{s+it}{}^{1/2s}+\|\xi-\eta\|_{s+it}{}^{1/2s}\leq 2^{(1-s)/s}(\|\xi\|_{s+it}{}^{1/2s}+\|\eta\|_{s+it}{}^{1/2s})$$

for ξ, $\eta\in N(s+it)$, by 2×2-matrix and three-line techniques, which enables us to show that $(N(s+it), \|\cdot\|_{s+it})$ is uniformly convex for $s\leq 1/2$. Then by a general theorem on uniformly convex Banach spaces (see [24] for example), we see that the Banach space $N(s+it)$ is reflexive for $s\leq 1/2$. Finally, $N(s+it)$ with $s\geq 1/2$ is reflexive as the dual space of the reflexive Banach space $(N(1-s-it), \|\cdot\|_{1-s-it})$. In this way, we obtain the first part of the next theorem.

Theorem 2.9.

(i) *For $\alpha\in \boldsymbol{C}$ with $0\leq \Re\alpha<1$, $(N(\alpha), \|\cdot\|_\alpha)$ is a Banach space and is identified with the dual Banach space of $(N(1-\alpha), \|\cdot\|_{1-\alpha})$ by the pairing*

$$N(\alpha)\times N(1-\alpha)\ni(\xi,\eta)\mapsto tr(\xi\eta)\in \boldsymbol{C}.$$

Here $\|\cdot\|_{it}$ for $t\in \boldsymbol{R}$ means the restriction of the (operator) norm in $\hat{N}$ to the closed subspace $N(it)$.

(ii) *$*$-operation is an isometry of $N(\alpha)$ onto $N(\bar{\alpha})$ for $0\leq\Re\alpha\leq 1$.*

(iii) *For α, $\beta\in \boldsymbol{C}$, with $0<\Re\alpha$, $0<\Re\beta$, and $\Re\alpha+\Re\beta\leq 1$, we have*

$$\|\xi\eta\|_{\alpha+\beta}\leq\|\xi\|_\alpha\|\eta\|_\beta.$$

(iv) *For $0\leq s\leq 1$, positive cones $N(s)_+$ and $N(1-s)_+$ are polars of each other* *under the pairing given by (i).*

∵) (ii) is a consequence of Proposition 7 (ii). (iii) follows from the duality in (i). To see (iv), first note that $tr(\xi\eta)=tr(\xi^{1/2}\eta\xi^{1/2})\geq 0$ for $\xi\in N(s)_+$ and $\eta\in N(1-s)_+$. Conversely, let $\xi\in N(s)$ be such that $tr(\xi\eta)\geq 0$ for all $\eta\in N(1-s)_+$. Considering the Jordan decomposition of ξ, one sees that ξ has only the positive component. □

The following is an easy modification of Proposition II.35 in [20].

Proposition 2.10. *For α, $\beta\in \boldsymbol{C}$ with $0\leq\Re\alpha\leq\Re\beta\leq 1$, let $T:N(\alpha)\to N(\beta)$ be a bounded linear map satisfying $T(a\xi)=aT(\xi)$ for $a\in N$, $\xi\in N(\alpha)$. Then there is an element $\eta\in N(\beta-\alpha)$ such that $T(\xi)=\xi\eta$ for $\xi\in N(\alpha)$.*

Problem. Extend this result to the case $0\leq\Re\alpha\leq\Re\beta$.

Remark. For $0\leq\Re\beta\leq\Re\alpha\leq 1$, it may happen that we cannot find a non-trivial T. An abelian von Neumann algebra provides such an example. On the other hand, when N is infinite-dimensional, all left N-modules $N(\alpha)$ are isomorphic and hence there are many intertwiners.

Standard Space

The space of half power $N(1/2)$ is a Hilbert space and provides a (canonical) standard space of N. In fact, N acts on N from the left by multiplication, the set $N(1/2)_+$ forms the self-dual positive cone (Theorem 9 (iv)), and the modular conjugation is given by * in $N(1/2)$.

In this part, we shall describe the position of weights in standard spaces.

Recall that for unbounded linear operators A and B, the usual product is denoted by $A\cdot B$, while the notation AB is reserved for the strong product.

Lemma 2.11. *Let A be a densely defined positive self-adjoint operator on a Hilbert space $\mathscr{H}$ and $a\in\mathscr{B}(\mathscr{H})$.* Let *$B=a^*Aa\in\mathscr{B}(\mathscr{H}))^+$ be the quadratic multiplication in the extended positive part of $\mathscr{B}(\mathscr{H})$. Then (i) $D(B^{1/2})=D(A^{1/2}\cdot a)$, (ii) $A^{1/2}\cdot a$ is closed, and (iii) $B=(A^{1/2}a)^*(A^{1/2}a)$ when $A^{1/2}a$ is densely-defined.*

∵) For an extended positive operator T on $\mathscr{H}$, denote the associated quadratic form by

$$(T^{1/2}\xi|T^{1/2}\xi)\in[0, +\infty], \quad \xi\in\mathscr{H}.$$

By the definition of a^*Aa,

$$(\xi|B\xi)=(a\xi|Aa\xi), \quad \xi\in\mathscr{H},$$

which gives (i).

The closedness of $A^{1/2}\cdot a$ is obvious because of the boundedness of a.

Suppose that $A^{1/2}a$ is densely defined. Then B is a positive self-adjoint operator. If we denote by $uC^{1/2}$ the polar decomposition of $A^{1/2}a$, then $D(B^{1/2})=D(C^{1/2})$ and we have

$$(C^{1/2}\xi|C^{1/2}\xi)=(B^{1/2}\xi|B^{1/2}\xi)$$

for $\xi\in D(B^{1/2})=D(C^{1/2})$. Thus $B^{1/2}\xi\mapsto C^{1/2}\xi$ gives rise to a partial isometry v and we have $C^{1/2}=vB^{1/2}$. Since v intertwines the support projections of $B^{1/2}$ and $C^{1/2}$, we concluded that $C=B^{1/2}v^*vB^{1/2}=B$. □

Proposition 2.12. *For* $\varphi\in\overline{N}_*^+$, *let* $\mathfrak{n}_\varphi=\{x\in N;\ \varphi(x^*x)<+\infty\}$ *as usual. Through the Haagerup's correspondence, we regard* φ *as a densely defined positive self-adjoint operator affiliated with* $\hat{N}$.

(i) *For* $a\in N$, $a\in\mathfrak{n}_\varphi$ *if and only if* $a\varphi a^*$ *is a* τ*-measurable densely defined operator.*

(ii) $\forall a\in\mathfrak{n}_\varphi$, $a\cdot\varphi^{1/2}$ *is a densely defined closable operator and its closure* $a\varphi^{1/2}$ *belongs to* $N(1/2)$.

(iii) $(a\varphi^{1/2})^*=\varphi^{1/2}a^*$ *and* $(\varphi^{1/2}a^*)^*(\varphi^{1/2}a^*)$ *is an operator in* $N(1)$ *which represents (via Haagerup's map) the positive linear form* $N\ni x\mapsto\varphi(a^*xa)$.

(iv) *For* $\varphi\in\overline{N}_*^+$, *the map*

$$L^2(N, \varphi)\ni[x]_\varphi\mapsto x\varphi^{1/2}\in L^2(N), \ x\in\mathfrak{n}_\varphi$$

is isometrically extended to unitary map from $L^2(N, \varphi)$ *onto* $L^2(N)\mathbf{s}(\varphi)$.

∵) Let $a\in N$. It is easy to see that the weight $\varphi(a^*(\cdot)a)$ is represented by an operator $a\varphi a^*$ in the extended positive part of $\hat{N}$. Since $a\varphi a^*$ is τ-measurable iff the associated weight is bounded, (i) follows from this. Let $a\in\mathfrak{n}_\varphi$ and set $\omega=a\varphi a^*$. Then $a\varphi a^*$ is densely defined and hence $\varphi^{1/2}a^*$ is densely defined as well. Since $D(\varphi^{1/2}a^*)=D(\omega^{1/2})$ and $\omega^{1/2}$ is τ-measurable,

$\varphi^{1/2}a^*$ is τ-measurable. This, with the relative invariance $\theta_s(\varphi)=e^{-s}\varphi$ and $\theta_s(a)=a$, implies that $\varphi^{1/2}a^*\in N(1/2)$.

On the other hand, from the general relation $(a\varphi^{1/2})^*\supset\varphi^{1/2}a^*$ and the fact that $\varphi^{1/2}a^*$ is τ-measurable, we have $(a\varphi^{1/2})^*=\varphi^{1/2}a^*$ (the fundamental theorem of measurable operators). Now (ii) and (iii) follows from Lemma 11.

(iv): From (iii) just proved, the map in question is isometric. Since its image is a left N-module, $\exists$ a projection $e\in N$ such that $\overline{\mathfrak{n}_\varphi\varphi^{1/2}}=L^2(N)e$. By $\varphi^{1/2}\mathbf{s}(\varphi)=\varphi^{1/2}$, $e\leq\mathbf{s}(\varphi)$. To see the opposite inclusion, we first restrict ourselves to the case $\varphi\in N_*^+$. This case, $\varphi^{1/2}(1-e)=0$ and hence taking the inner product with $\varphi^{1/2}$, $\langle\varphi,\ 1-e\rangle=0$. Thus $1-e\leq 1-\mathbf{s}(\varphi)$, i.e., $\mathbf{s}(\varphi)\leq e$, proving $e=\mathbf{s}(\varphi)$.

Now we go into the general case. Express $\varphi=\sum_i\omega_i$ with $\omega_i\in N_*^+$. Since $\mathbf{s}(\varphi)=\bigvee_i\mathbf{s}(\omega_i)$, the problem is reduced to the case just proved by the following lemma. □

Lemma 2.13. *Let $\varphi\in\overline{N}_*^+$ and $\omega\in N_*^+$ with $\omega\leq\varphi$. Then $\exists^1$ $a\in\mathfrak{n}_\varphi\mathbf{s}(\varphi)$ such that $\omega^{1/2}=a\varphi^{1/2}$.*

∵) Let $e\in N$ be the projection to the closed subspace $\overline{\varphi^{1/2}\mathfrak{n}_\varphi^*}\subset L^2(N)$ as before. Since

$$\varphi^{1/2}x^*\mapsto\omega^{1/2}x^*,\ x\in\mathfrak{n}_\varphi$$

is norm-decreasing and commutes with the right action of N, $\exists a\in eNe$ such that

$$\omega^{1/2}x^*=a(\varphi^{1/2}x^*),\ \forall x\in\mathfrak{n}_\varphi.$$

Let p_n and q_n be an increasing sequence of spectral projections of φ and ω respectively. For $x\in\mathfrak{n}_\varphi$, *let* $r_n(x)=1-\mathbf{s}((1-p_n\wedge q_n)x^*)$. Then $(p_n\wedge q_n)x^*r_n(x)=x^*r_n(x)$ and hence $\omega^{1/2}=a\cdot\varphi^{1/2}$ on $x^*r_n(x)\mathscr{H}$. Since $\sum_{x\in\mathfrak{n}_\varphi}x^*r_n(x)\mathscr{H}$ is dense in $(p_n\wedge q_n)\mathbf{s}(\varphi)\mathscr{H}$ $(\mathbf{s}(\omega)\leq\mathbf{s}(\varphi))$ and since $\omega^{1/2}$ and $\varphi^{1/2}$ are bounded on $(p_n\wedge q_n)\mathscr{H}$, we conclude that

$$\omega^{1/2}=a\varphi^{1/2}\ \text{on}\ (p_n\wedge q_n)\mathscr{H}.$$

Thus $a\cdot\varphi^{1/2}$ is τ-measurable and its closure $a\varphi^{1/2}$ is equal to $\omega^{1/2}$.

The uniqueness of a is clear. □

Corollary 2.14. *For $\xi\in L^2(N)$,*

$$\overline{N\xi}=L^2(N)\mathbf{s}(\xi).$$

∵) This is a combination of (iv) and the polar decomposition of ξ. □

Matrix Amplification and Reduction

For a projection e in N, the space of weights of reduced algebra eNe is naturally identified with

$$\{\varphi\in\overline{N}_*^+;\ \mathbf{s}(\varphi)\le e\}.$$

Starting from this fact, we can construct identifications $\widehat{(eNe)}=e\hat{N}e$ and $(eNe)(\alpha)=eN(\alpha)e$ for $\alpha\in\boldsymbol{C}_+$ (use the KMS-condition). These are called **reductions**.

We shall apply this identification to the construction of matrix amplification. Let I be a set which describes the size of matrix and denote by $Mat_I(N)$ the matrix amplification of N indexed by I. Through the pairing

$$(\varphi,\ x)\mapsto \mathrm{trace}((\varphi_{ij})(x_{kl}))=\sum_{i,j\in I}\varphi_{ij}(x_{ji}),$$

the predual of $Mat_I(N)$ is identified with $Mat_I(N_*)$ (under the suitable convergence condition). Cutting down $Mat_I(N)(\alpha)$ $(\alpha\in\boldsymbol{C}_+)$ to matrix components and then doing polar decomposition, we see that this space is also identified with $Mat_I(N(\alpha))$ (again under the suitable convergence condition) and the algebraic structure for $Mat_I(N)(\boldsymbol{C}_+)$ is recognized as the matrix amplification of $N(\boldsymbol{C}_+)$.

Similarly, given two index sets I and J, we can define $Mat_{I\times J}(N(\alpha))$ as the subspace of $Mat_{I\cup J}(N(\alpha))$. In particular, the space of column vectors (resp. row vectors) $Mat_{I\times\{1\}}(N(\alpha))$ (resp. $Mat_{\{1\}\times I}(N(\alpha))$ is denoted by $N(\alpha)^{\oplus I}$ (resp. $\bigoplus_I N(\alpha)$).

In the obvious way, $Mat_I(N(\alpha))$ multiplies on $\bigoplus_I N(\beta)$ from the left, producing $\bigoplus_I N(\alpha+\beta)$. Similarly for the multiplications of other types.

Here is the explicit description of important cases, $\alpha=0,1/2,1$:

$$\bigoplus_I N=\{\{x_i\}_{i\in I};\ x_i\in N,\ \sum_i x_i^*x_i\in N\},$$

$$N^{\oplus I}=\{\{x_i\}_{i\in I};\ x_i\in N,\ \sum_i x_ix_i^*\in N\},$$

$$\bigoplus_I L^2(N)=\{\{\xi_i\}_{i\in I};\ \sum_i(\xi_i|\xi_i)<+\infty\}=L^2(N)^{\oplus I},$$

$$\bigoplus_I N_*=\{\{\varphi_i\}_{i\in I};\ \varphi_i\in N_*, \exists c>0,\ |\sum_i \varphi_i(x_i)|\le c\|\sum_i x_i x_i^*\|^{1/2},\ \forall\{x_i\}\in N^{\oplus I}\},$$

$$N_*^{\oplus I}=\{\{\varphi_i\}_{i\in I};\ \varphi_i\in N_*, \exists c>0,\ |\sum_i \varphi_i(x_i)|\le c\|\sum_i x_i^* x_i\|^{1/2},\ \forall\{x_i\}\in \bigoplus_I N\}.$$

Note that $\bigoplus_I L^2(N)$ and $L^2(N)^{\oplus I}$ are Hilbert spaces and there is a natural pairing between them, i.e., they are the conjugate spaces of each other. Similarly, there is a pairung between $\bigoplus_I N$ and $N_*^{\oplus I}$, through which $\bigoplus_I N$ is identified with the dual space of $N_*^{\oplus I}$.

Remark. Note that the conjugation (=*-operation) gives isometric (conjugate-linear) isomorphisms $\bigoplus_I N\cong N^{\oplus I}$, $\bigoplus_I L^2(N)\cong L^2(N)^{\oplus I}$, and $\bigoplus_I N_*\cong N_*^{\oplus I}$. Though the analytic conditions are the same for $\bigoplus_I L^2(N)$ and $L^2(N)^{\oplus I}$, they should be discriminated by this reason.

§3. Relative Modular Algebras

In this section, we consider the relative notion for modular algebra. Recall that modular algebras developped in §2 incorporates the modular theory for weights. Analogously, 'relative' modular algebra is closely related with the modular theory for operator-valued weights. So let us begin with the relevant definitions of operator-valued weights.

Consider a von Neumann algebra M and its subalgebra N and denote by $\overline{M}_+$ and $\overline{N}_+$ their extended positive parts. Then an operator-valued weight (or an N-valued weight on M) ε is an $\overline{N}_+$-valued weight on M_+ satisfying $\varepsilon(yxy^*)=y\varepsilon(x)y^*$ for $x\in M_+$ and $y\in N$. The normality and semifiniteness are defined for operator-valued weights exactly by the same way for ordinary weights. Since we do not consider operator-valued weights other than normal and semifinite ones, these adjectives are often omitted in this paper. Let φ be a (normal semifinite) weight on N. Then φ is uniquely extended to a weight on $\overline{N}_+$ (again denoted by φ) and $\varphi\circ\varepsilon$ gives a weight on M. Note that $\varphi\circ\varepsilon$ is normal and semifinite because ε is normal and semifinite.

For an operator-valued weight ε: $M^+\to N^+$, its support is defined by

$$\mathbf{s}(\varepsilon)=\bigvee_{\varphi\in N_*^+}\mathbf{s}(\varphi\circ\varepsilon)\in M\cap N'.$$

ε is called faithful if $\mathbf{s}(\varepsilon)=1$. Since ε induces a faithful operator-valued weight $\mathbf{s}(\varepsilon)M^+\mathbf{s}(\varepsilon)\to\mathbf{s}(\varepsilon)N^+$ in the obvious way, the study of operator-valued weights is reduced to faithful ones in some sense. In particular, from the corresponding result on faithful operator-valued weights ([6]), we have

$$(D\varphi\circ\varepsilon\colon D\psi\circ\varepsilon)_t=\mathbf{s}(\varepsilon)(D\varphi\colon D\psi)_t$$

for φ, $\psi\in\overline{N}_*^+$.

Using the above relation we see that the map

$$N(it)\ni x\varphi^{it}\mapsto x(\varphi\circ\varepsilon)^{it}\in M(it)$$

is well-defined and gives rise to a *-homomorphism from $N(i\mathbf{R})$ into $M(i\mathbf{R})$. By the universality of crossed products, this map is further extended to a normal *-homomorphism ε_*: $\hat{N}\to\hat{M}$, which is N-linear and intertwines the scaling automorphisms (dual actions). The homomorphism $\varepsilon_*:\hat{N}\to\hat{M}$ is 1–1 if and only if the central support of $\mathbf{s}(\varepsilon)$ in N' is equal to 1 (use the Landstad's characterization of crossed product in terms of dual action). Note that ε_* preserves identities iff ε is faithful.

Let τ_M (resp. τ_N) be the canonical traces on $\hat{M}$ (resp. $\hat{N}$) described in the construction of Takesaki's dual. By [5, Theorem 2.7], there is a unique normal faithful semifinite operator-valued weight $\hat{\varepsilon}$ from $\hat{M}$ to $\hat{N}$ such that

$$\tau_M=\tau_N\circ\hat{\varepsilon}.$$

$\hat{\varepsilon}$ is called the modular extension of ε. By the uniqueness, $\hat{\varepsilon}$ preserves the scaling automorphisms.

$\hat{\varepsilon}$ is the transposed of ε_* in the following sense.

Proposition 3.1. *We have*

$$\langle\varepsilon_*(\hat{y})\tau_M,\ \hat{x}\rangle=\langle\hat{y}\tau_N,\ \hat{\varepsilon}(\hat{x})\rangle,\ \hat{x}\in\hat{M}^+,\ \hat{y}\in\hat{N}^+.$$

∵) See (the proof of) [5, Theorem 2.7]. □

Corollary 3.2. *Let $\mathfrak{m}\subset M$ be the defining *-subalgebra of ε. Then the defining subalgebra of $\hat{\varepsilon}$ contains a *-subalgebra defined by*

$$\sum_{\varphi\in\overline{N}_*^+} \mathfrak{m}(\varphi\circ\varepsilon)^{it}$$

on which $\hat{\varepsilon}$ *is given by*

$$\hat{\varepsilon}(x(\varphi\circ\varepsilon)^{it})=\varepsilon(\mathrm{x})\varphi^{it}.$$

If we denote by ε_M (resp. ε_N) the operator-valued weight from $\hat{M}$ (resp. $\hat{N}$) to M (resp. N) canonically associated to scaling automorphism group in the dual construction, then we obtain the following commutative diagrams (cf. [6] and [11, §2.1]):

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{\varepsilon}} & \hat{N} \\ {\scriptstyle\varepsilon_M}\downarrow & & \downarrow{\scriptstyle\varepsilon_N} \\ M & \xrightarrow[\varepsilon]{} & N \end{array} \qquad\qquad \begin{array}{ccc} \hat{M} & \xrightarrow{\hat{\varepsilon}} & \hat{N} \\ \uparrow & & \uparrow \\ M & \xrightarrow[\varepsilon]{} & N \end{array}$$

Conversely, any normal N-linear *-homomorphism $\hat{N}\to\hat{M}$ which extends the inclusion $N\subset M$ and preserves scaling automorphisms arises in this way (use the characterization of φ^{it} as a 1-parameter group of unitaries and the Haagerup's existence theorem on operator-valued weights).

Relative Modular Algebras—boundary case

For $t\in\boldsymbol{R}$, set

$$M/N(it)=Hom({}_N N(it)_{N,N}M(it)_N).$$

Note that $M/N(0)=M\cap N'$, $M/\boldsymbol{C}(it)=M(it)$, and $M/M(it)=Z(M)$ (the center of M). Define a multiplication

$$M/N(is)\times M/N(it)\ni(S,\ T)\mapsto ST\in M/N(i(s+t))$$

so that

$$(ST)(ab)=S(a)T(b),\ \text{for}\ a\in N(is)\ \text{and}\ b\in N(it).$$

To see this being well-defined, take $\varphi\in\overline{N}_*^{++}$ and define a linear map U: $N(i(s+t))\to M(i(s+t))$ by

$$U(y\varphi^{i(s+t)})=yS(\varphi^{is})T(\varphi^{it}).$$

For $a=y\varphi^{is}\in N(is)$ and $b=z\varphi^{it}\in N(it)$ with $y,\ z\in N$, we have

$$\begin{aligned} U(ab) &= y\sigma_s^{\varphi}(z)S(\varphi^{is})T(\varphi^{it}) \\ &= S(y\sigma_s^{\varphi}(z)\varphi^{is})T(\varphi^{it}) \\ &= S(y\varphi^{is}z)T(\varphi^{it}) \\ &= S(y\varphi^{is})zT(\varphi^{it}) \\ &= S(y\varphi^{is})T(z\varphi^{it}) \\ &= S(a)T(b). \end{aligned}$$

Hence $U=ST$ and U is N-linear.

Clearly the multiplication defined in this way is associative. For $T\in M/N(it)$, define the adjoint $T^*\in M/N(-it)$ by

$$T^*(a)=T(a^*)^*.$$

With these operations, $\{M/N(it)\}_{t\in\mathbb{R}}$ becomes a *-algebraic bundle in the sense of Fell. (Note that $T^*T\geq 0$ for $T\in M/N(it)$.)

We call $\{M/N(it)\}_{t\in\mathbb{R}}$ (the boundary of) the **relative modular algebra** associated to the inclusion $N\subset M$. Note that due to the algebraic structure $\mathbb{s}(M/N)\equiv\{t\in\mathbb{R};\ M/N(it)\neq\{0\}\}$ is a subgroup of $\mathbb{R}$, which turns out to be a conjugacy invariant of inclusion relations. The subgroup $\mathbb{s}(M/N)$ can be taken fairy arbitrary (see the proofs in [6, Proposition 5.8] and [4, Lemma 1]).

Suppose that there exists an operator-valued weight ε: $M^+\to N^+$. For $t\in\mathbb{R}$, define $\varepsilon^{it}\in M/N(it)$ by

$$\varepsilon^{it}:\ y\varphi^{it}\mapsto y(\varphi\circ\varepsilon)^{it}.$$

(The left N-linearity is clear and the right N-linearity follows from $\sigma^{\varphi\circ\varepsilon}|_N=\sigma^{\varphi}$. Note that $\varepsilon^{i0}=\mathbb{s}(\varepsilon)$.)

Then $t\mapsto\varepsilon^{it}$ gives a *-homomorphic section of $\{M/N(it)\}_{t\in\mathbb{R}}$, i.e.,

$$\varepsilon^{is}\varepsilon^{it}=\varepsilon^{i(s+t)},\ (\varepsilon^{it})^*=\varepsilon^{-it}, \qquad s,\ t\in\mathbb{R}$$

which is continuous in the following sense: For any $\varphi\in\overline{N}_*^+$, $\varepsilon^{it}(\varphi^{it})\in M(it)\subset\hat{M}$ is weakly continuous in $t\in\mathbb{R}$.

Conversely, any continuous *-homomorphic section of $\{M/N(it)\}_{t\in\mathbb{R}}$ is of this form.

Now the following properties on modular automorphisms associated with operator-valued weights are easily established.

Proposition 3.3. *Let $U, V \in M/N(it)$ and $\varphi, \psi \in \overline{N}_*^+$. Then for $x \in M/N(0) \subset M$ and $y \in N$, we have*

$$U(\varphi^{it})yxV(\psi^{it})^* = (\varphi^{it}y\psi^{-it})(UxV^*) \quad \text{in } M(i\boldsymbol{R}).$$

(Here note that $\varphi^{it}\psi^{-it} \in N \subset M$ and $UxV^* \in M/N(0) \subset M$.)

∵) From the definition of the *-algebra structure in $M/N(i\boldsymbol{R})$, we calculate as follows:

$$\begin{aligned} U(\varphi^{it})yxV(\psi^{it})^* &= U(\varphi^{it}y)x(1)V^*(\psi^{-it}) \\ &= (UxV^*)(\varphi^{it}y\psi^{-it}). \end{aligned}$$

□

Corollary 3.4. *For faithful normal semifinite N-valued weight E and F on M and faithful normal semifinite weights φ, ψ on N, we have*

(i) $\sigma_t^{\varphi\circ E}(y) = \sigma_t^{\varphi}(y)$, $\forall y \in N$,

(ii) $(\varphi\circ E)^{it}(\psi\circ E)^{-it} = \varphi^{it}\psi^{-it}$,

(iii) $(\varphi\circ E)^{it}x(\varphi\circ E)^{-it} = E^{it}xE^{-it}$, $\forall x \in M\cap N'$,

(iv) $(\varphi\circ E)^{it}(\varphi\circ F)^{-it} = E^{it}F^{-it}$,

for all $t \in \boldsymbol{R}$.

When there is a faithful operator-valued weight ε: $M^+ \to N^+$, we have

$$M/N(it) = (M\cap N')\varepsilon^{it} = \varepsilon^{it}(M\cap N').$$

In fact, the inclusion $(M\cap N')\varepsilon^{it} \subset M/N(it)$ is trivial. To see the reverse relation, let $T \in M/N(it)$, $\varphi \in \overline{N}_*^{++}$ and define $x \in M$ by

$$T(\varphi^{it}) = x(\varphi\circ\varepsilon)^{it}$$

($\varphi\circ\varepsilon$ is faithful because ε is faithful). Then by the N-linearity of T, $x \in M\cap N'$ and hence $T = x\varepsilon^{it}$.

This observation suggests similarity between $M/N(it)$ and $(M\cap N')(it)$. We shall discuss on this subject below.

Relative Modular Algebras—general case

Given an inclusion of von Neumann algebras and $\alpha \in \mathbb{C}$ with $\Re\alpha \geq 0$, set

$$M/N(\alpha) = Hom({}_N N(\alpha)_N,\ {}_N M(\alpha)_N),$$

where the right hand side denotes the set of algebraic N-N linear maps from $N(\alpha)$ into $M(\alpha)$.

Quite similarly as in the boundary case, we can introduce a *-algebra structure in the totality $M/N(\mathbb{C}_+) = \sum_{\alpha\in\mathbb{C}_+} M/N(\alpha)$.

An operator-valued weight Φ: $M_+ \to \overline{N}_+$ is called **bounded** if $\Phi(M_+) \subset N_+$. In this case, Φ admits a pre-transposed map Φ_*: $N_* \to M_*$ and a *-homomorphism Φ_*: $\hat{N} \to \hat{M}$ is extended to an N-linear *-homomorphism $N(\mathbb{C}_+) \to M(\mathbb{C}_+)$ in such a way that the extension commutes with the scaling automorphisms, which is again denoted by Φ_*:

$$\Phi_*(y\varphi^\alpha) = y(\varphi \circ \Phi)^\alpha.$$

Conversely any N-linear *-homomorphisms preserving the scaling automorphisms are of this form. The restriction of Φ_* to $N(\alpha)$ is denoted by Φ^α. Then Φ^α belongs to $M/N(\alpha)$ and satisfies

$$\Phi^\alpha \Phi^\beta = \Phi^{\alpha+\beta}, \quad (\Phi^\alpha)^* = \Phi^{\bar{\alpha}}.$$

Thus $(M \cap N')\Phi^\alpha \subset M/N(\alpha)$. Note that, for $0 \leq \Re\alpha \leq 1$, Φ^α is bounded as a linear map between Banach spaces and its norm is majorized by $\|\Phi(1)\|$ (use the polar decomposition in $N(\alpha)$).

Now consider the converse inclusion. To this end, we need the following known fact:

Lemma 3.5. *Let $s > 0$, $\varphi_1, \varphi_2 \in N_*^+$ and suppose that $\varphi_1^{2s} \leq \varphi_2^{2s}$. Then there exists an N-valued w*-continuous function F on the strip $\{z \in \mathbb{C};\ -s \leq \mathrm{Im} z \leq 0\}$ which is analytic in the interiour of the strip and satisfies (i) $F(t) = \varphi_2^{it}\varphi_1^{-it}$ and (ii) $\|F(z)\| \leq 1$ for $z \in \mathbb{C}$ in the strip.*

Let T: $N(\alpha) \to M(\alpha)$ be an N-N linear map. For $\varphi \in N_*^+$, let

$$T(\varphi^\alpha) = u(\varphi)\psi^\alpha$$

be the polar decomposition of $T(\varphi^\alpha)$ in $M(\alpha)$, i.e., $u \in M$ is a partial isometry and $\psi \in M_*^+$ with $u^*u = \mathbf{s}(\psi)$. Note that $\mathbf{s}(\psi) \leq \mathbf{s}(\varphi)$. Since

$$(*) \qquad T(\varphi^\alpha)y=\sigma^\varphi_{-is}(y)T(\varphi^\alpha)$$

for $y\in N$ in the domain $D(\sigma^\varphi_{-is})$ of σ^φ_{-is}, we have

$$\psi y=\sigma^\varphi_{-2is}(y) \quad \text{for } y\in D(\sigma^\varphi_{-2is}).$$

By a result on analytic generators in von Neumann algebras ([5, Theorem 3.2 and Lemma 4.4]), we deduce that

$$\sigma^\psi_t(y)\mathbf{s}(\psi)=\sigma^\varphi_t(y)\mathbf{s}(\psi) \quad \forall y\in N.$$

Then, again by the existence theorem on operator-valued weights in [5], we can find an N-valued weight Φ_φ on M satisfying

$$\varphi\circ\Phi_\varphi=\psi.$$

We claim that $\Phi_{\varphi_1}(\cdot)=\Phi_{\varphi_2}(\mathbf{s}(\varphi_1)\cdot\mathbf{s}(\varphi_1))$ whenever $\varphi_1^{2s}\leq\varphi_2^{2s}$. This can be seen as follows: Put $u_j=u(\varphi_j)$ and $\Phi_j=\Phi_{\varphi_j}$ $(j=1,2)$. Since $(\varphi_2\circ\Phi)^{-it}(\varphi_1\circ\Phi_2)^{it}=\varphi_2^{-it}\varphi_1^{it}$ for $t\in\boldsymbol{R}$, the analytic continuation of this relation gives $(\varphi_1\circ\Phi_2)^\alpha=(\varphi_2\circ\Phi_2)^\alpha(\varphi_2^{-\alpha}\varphi_1^\alpha)$ with $\|\varphi_2^{-\alpha}\varphi_1^\alpha\|\leq 1$. Thus

$$\begin{aligned} u_2(\varphi_1\circ\Phi_2)^\alpha &= T(\varphi_2^\alpha)(\varphi_2^{-\alpha}\varphi_1^\alpha) \\ &= T(\varphi_1^\alpha) \quad (\varphi_2^{-\alpha}v_1^\alpha\in N) \\ &= u_1(\varphi_1\circ\Phi_1)^\alpha. \end{aligned}$$

Since $T(\varphi_1^\alpha)=(\varphi_1^\alpha\varphi_2^{-\alpha})T(\varphi_2^\alpha)$, the right support of $T(\varphi_1^\alpha)$ is majorized by the right support of $T(\varphi_2^\alpha)$. Hence the above relation implies $(\varphi_1\circ\Phi_2)^{2s}=(\varphi_1\circ\Phi_1)^{2s}$, i.e., $\varphi_1\circ\Phi_2=\varphi_1\circ\Phi_1$. By the uniqueness theorem on operator-valued weight, we conclude the desired assertion.

From the claim just proved, we see that the family $\{\Phi_\varphi\}_{\varphi\in N_*^+}$ is patched into a single N-valued weight Φ on M satisfying

$$\varphi\circ\Phi=\psi$$

for any $\varphi\in N_*^+$ and the accompanied $\psi\in M_*^+$. Similarly we can glue $\{u(\varphi)\}_{\varphi\in N_*^+}$ into a partial isometry U in M with $U^*U=\mathbf{s}(\Phi)$. Since $\varphi\circ\Phi$ is bounded for any $\varphi\in N_*^+$, Φ is bounded as well. Then we can consider the bounded N-N linear map Φ^α: $N(\alpha)\to M(\alpha)$ and we have

$$T(\varphi^\alpha)=\mathrm{U}\Phi^\alpha(\varphi^\alpha) \quad \forall\varphi\in N_*^+.$$

Now, combining

$$\Phi^\alpha(\varphi^\alpha)y = \sigma^\varphi_{-is}(y)\Phi^\alpha(\varphi^\alpha) \quad \forall y \in D(\sigma^\varphi_{-is})$$

with (*), we obtain

$$yU = Uy\mathbb{s}(\Phi), \quad \forall y \in N.$$

Since the support $\mathbb{s}(\Phi)$ of Φ is in the relative commutant $M \cap N'$, we finally have $yU = Uy$ for all $y \in N$, i.e., $U \in M \cap N'$.

Thus we have proved the following:

Theorem 3.6. *Let $N \subset M$ be an inclusion relation of von Neumann algebras with common unit and α be a complex number with $0 < \Re\alpha$. Then for any $T \in M/N(\alpha)$, there exists a unique pair of a partial isometry U in $M \cap N'$ and a bounded N-valued weight Φ on M satisfying* (*i*) $T = U\Phi^\alpha$ *and* (*ii*) $U^* U = \mathbb{s}(\Phi)$.

Corollary 3.7. *If $0 \leq \Re\alpha \leq 1$, any N-N linear map from $N(\alpha)$ into $M(\alpha)$ is bounded as a linear map between Banach spaces.*

As an application of automatical boundedness, we can introduce $Z(N)$-valued trace on the relative modular algebra $M/N(\mathbb{C}_+)$:

$$tr_{M/N}\colon M/N(1) = Hom({}_N N_{*N}, {}_N M_{*N}) \ni T \mapsto {}^t T(1_M).$$

Here ${}^t T$ denotes the transposed (bounded) linear map from M to N.

Regularity on Inclusion Relations

Definition 3.8. *An inclusion relation $N \subset M$ of von Neumann algebras is called* **regular** *if there is a faithful operator-valued weight Φ: $M \to N$ such that the restriction of Φ to the relative commutant $M \cap N'$ is semifinite.*

For a regular inclusion relation $N \subset M$, every (normal and semifinite of course) operator-valued weight from M to N has the semifinite restriction to the relative commutant and, via this correspondence, the set of N-valued weights on M is bijectively parametrized by $Z(N)$-valued weights on $M \cap N'$ ([6]). In particular, there are plenty of bounded N-valued weights on M. Now we can clarify the similarity problem between M/N and $M \cap N'/Z(N)$.

Theorem 3.9. *If $N \subset M$ is a regular inclusion relation of von Neumann algebras, then the relative modular algebra $M \cap N'/Z(N)(\boldsymbol{C}_+)$ is canonically isomorphic to the relative modular algebra $M/N(\boldsymbol{C}_+)$.*

$\because$) In fact $M \cap N'/Z(N)(\alpha) \ni U(\Phi|_{M \cap N'})^\alpha \mapsto U\Phi^\alpha$ with Φ bounded N-valued weight on M, gives the isomorphism. The well-definedness is checked by Theorem 3.6 and other algebraic properties are immediate. $\square$

Theorem 3.10. *Let $N \subset M$ be a regular inclusion of von Neumann algebras.*

(i) *For $\alpha \in \boldsymbol{C}$ with $0 \leq \Re\alpha < 1$, $M/N(\alpha)$ is a Banach space with operator norm and is identified with the 'dual' Banach space of $M/N(1-\alpha)$ with respect to the $Z(N)$-valued pairing*

$$M/N(\alpha) \times M/N(1-\alpha) \ni (\xi, \eta) \mapsto tr_{M/N}(\xi\eta) \in Z(N).$$

(ii) *$*$-operation is an isometry from $M/N(\alpha)$ onto $M/N(\bar{\alpha})$ for $0 \leq \Re\alpha \leq 1$.*

(iii) *For α, $\beta \in \boldsymbol{C}$, with $0 < \Re\alpha$, $0 < \Re\beta$, and $\Re\alpha + \Re\beta \leq 1$, we have*

$$\|ST\|_{\alpha+\beta} \leq \|S\|_\alpha \|T\|_\beta \quad \text{for } S \in M/N(\alpha) \text{ and } T \in M/N(\beta).$$

(iv) *For $0 \leq s \leq 1$, positive cones $M/N(s)_+$ and $M/N(1-s)_+$ are polars of each other under the pairing in (i).*

$\because$) All the assertions are easy to check except for the duality in the pairing of (i). So, given a bounded N-linear map f: $M/N(\alpha) \to Z(N)$, we need to show that there is an element T in $M/N(1-\alpha)$ such that $f(S) = tr_{M/N}(ST)$ for all $S \in M/N(\alpha)$. By the previous theorem, the problem is reduced to the case when N is a central subalgebra of M. This case, we have the natural identification $M/N(\alpha) \otimes_N N(\alpha)$ given by $u\Phi^\alpha \otimes_N \varphi^\alpha \mapsto u(\varphi \circ \Phi)^\alpha$, where $u \in M$, $\varphi \in N_*^+$, and Φ is a bounded N-valued weight on M. Note that if $u\psi^\alpha$ is a polar decomposition in $M(\alpha)$ and E: $M \to N$ be the conditional expectation defined by $\psi \circ E = \psi$, then we have $\|u\psi^\alpha\| = \|uE^\alpha\| \|(\psi|_N)^\alpha\|$. By a direct sum decomposition of N, we may assume that N admits a faithful finite trace τ. Then we can define a linear functional on $M(\alpha)$ by

$$M/N(\alpha) \otimes_N N(\alpha) \ni S \otimes_Z \varphi^\alpha \mapsto (\tau^{1-\alpha}\varphi^\alpha)(f(S)).$$

The norm relation remarked above shows that this functional is bounded by the norm of f. Thus we can find an element ξ in $M(1-\alpha)$ satisfying

$tr(\xi(S\otimes_N\varphi^\alpha))=(\tau^{1-\alpha}\varphi^\alpha)(f(S))$. Let $\xi=u\psi^{1-\alpha}$ be the polar decomposition and define a conditional expectation E: $M\to N$ *by* $\psi\circ E=\psi$. Then $\xi=uE^{1-\alpha}\otimes_N (\psi|_N)^{1-\alpha}$ and we have

$$(\tau^{1-\alpha}\varphi^\alpha)(f(S))=((\psi|_N)^{1-\alpha}\varphi^\alpha)(tr_{M/N}(uE^{1-\alpha}S)),$$

for $S\in M/N(\alpha)$ and $\varphi\in N_*^+$. If we put $S=yE^\alpha$ with $y\in N$ in the above formula, we get $(\tau^{1-\alpha}\varphi^\alpha)(y)=((\psi|_N)^{1-\alpha}\varphi^\alpha)(y)$. Since $y\in N$ and $\varphi\in N_*^+$ are arbitrary, we conclude that the Radon-Nikodym derivative $h=(\psi|_N)^{1-\alpha}\tau^{-(1-\alpha)}$ is in N. Thus we have

$$f(S)=tr_{M/N}(huE^{1-\alpha}S)$$

for $S\in M/N(\alpha)$. In other words, we can put $T=huE^{1-\alpha}$. □

On the algebraic tensor product $L^2(M/N)\odot_{Z(N)} L^2(N)$ we can introduce an (ordinary) inner product defined by

$$(S\odot\xi|T\odot\eta)=tr(\xi^*\langle S,T\rangle\eta).$$

Let $L^2(M/N)\otimes_{Z(N)} L^2(N)$ be its completion. Then we have

Corollary 3.11. *Let* $N\subset M$ *be a regular inclusion, then* $S\odot\xi\mapsto S(\xi)$ *is extended to an isometric isomorphism* $L^2(M/N)\otimes_{Z(N)} L^2(N)\to L^2(M)$ *between Hilbert spaces.*

As a special case of bounded operator-valued weights, a conditional expectation from M to N has the following characterization (cf. [11, §2.1]):

Theorem 3.12. *For a n.f.s. operator-valued weight* ε: $M_+\to N^+$, *the following conditions are equivalent.*

(i) ε *is a conditional expectation.*

(ii) $\hat{\varepsilon}$ *is a conditional expectation.*

(iii) $\tau_M\circ\varepsilon_*=\tau_N$.

∵) (ii)⇒(i): Because $\hat{\varepsilon}$ commutes with θ.

(i)⇒(iii): Since $\tau_M\circ\varepsilon_*$ and τ_N coincide on a dense *-subalgebra of holomorphic sections of $N(\mathbb{C}_+)$, we have their equality by a result in [14].

(iii)⇒(ii): A classical result (Umegaki-Dixmier) on the existence of conditional expectations. □

Since τ_N is the restriction of τ_M to $\hat{N}$ for a conditional expectation ε, the associated imbedding ε_*: $\hat{N} \to \hat{M}$ induces a *-homomorphism $N(\boldsymbol{C}_+) \to M(\boldsymbol{C}_+)$ given by

$$N(\alpha) \ni x\varphi^\alpha \mapsto x(\varphi \circ \varepsilon)^\alpha \in M(\alpha).$$

Noting that $\varepsilon(1)=1$, this map turns out to be an isometry for $0 \leq \Re\alpha \leq 1$. As the transposed map of $\varepsilon^{1-\alpha}$, we obtain a norm 1 projection $M(\alpha) \to N(\alpha)$ for $0 \leq \Re\alpha \leq 1$ as well.

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