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The Cauchy Problem for Hyperbolic Operators with Triple Involutive Characteristics

By

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§ 0. Introduction, Notations and Main Results

It is well known that the Cauchy problem in the C^∞ category for hyperbolic linear differential operators whose principal symbol has characteristics of varying multiplicities is, in general, not well-posed unless some Levi conditions on the lower order terms are imposed. When the characteristics have order at most three some results have been proved in [6] for operators generalizing the effectively hyperbolic case and in [2] where, on the other hand, an extension of the non effectively hyperbolic case has been dealt with. In these results the operator was assumed to have a localisation at triple characteristic points which factors out as the product of a real linear hyperbolic form and a quadratic form. In this paper we want to state necessary and sufficient conditions for the Cauchy problem to be well posed for a model equation in which this factorization property does not hold anymore, i.e. in which we have genuine triple characteristics. This equation represents the microlocal model of a general class of operators for which in [3] the propagation of Gevrey singularities has been studied. To be precise let us consider in $\boldsymbol{R}^{n+1}$ with coordinates $x=(x_0, x_1, \cdots, x_n)$ the following linear differential operator:

$$P(x, D)=D_0^3-a(D_1^2+D_2^2)D_0+bD_1^3+p_2(x, D)+p_1(x, D)+p_0(x, D). \tag{0.1}$$

Here $D_j=(1/i)\partial_{x_j}$, $j=0, 1, \cdots, n$; $a, b\in\boldsymbol{R}$. We shall assume that $p_3(x, \xi)$ is hyperbolic with respect to $(0, e_0)$, i.e. $(4/27)a^3-b^2>0$. p_3 vanishes exactly of the third order on the involutive submanifold Σ of $\dot{T}^*\boldsymbol{R}^{n+1}$, $\Sigma=\{(x, \xi)\mid\xi_0=\xi_1=\xi_2=0\}$, which has been assumed of codimension 3 for the sake of simplicity. The necessary conditions for well-posedness of Ivrii-Petkov [5] now assert that $p_{2|\Sigma}=0$, which however is not enough in this case. In fact our result can be stated as follows.

Theorem 1. *Let P as in* (0.1). *Then a necessary and sufficient condition in*

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order that the Cauchy problem for P be well posed in $S_0=\{x\in\boldsymbol{R}^{n+1}\mid x_0<0\}$ is that:

$$p_{2|\Sigma}=0, \qquad H_{p_{2|\Sigma}}=0, \qquad p_{1|\Sigma}=0\,. \tag{0.2}$$

We shall prove that $H_{p_{2|\Sigma}}=0$ by first establishing that $H_{p_2}(0;e_n)\in\Gamma^{\sigma}_{p_3}$, the dual cone with respect to $\sigma=d\xi\wedge dx$, the symplectic two form in the cotangent bundle, of the hyperbolicity cone Γ_{p_3} of p_3. Without loss of generality we may assume that $p_2(x, D)=(c_0D_0+c_1D_1+c_2D_2)D_n$, hence localizing at $(0, -e_n)$ we obtain that $H_{p_{2|\Sigma}}=0$. All necessary conditions are proved as in Hörmander [4] by means of suitable symplectic dilatations and conjugations with phase functions giving rise to algebraic equations of degree 3 whose complex roots are now important. As far as sufficiency is concerned we remark that once (0.2) holds true it is possible to prove energy estimates for P forming as usual $2i\Im\langle Pu, Mu\rangle$ where $u\in C_0^\infty$ and M is a suitable hyperbolic operator with double characteristics to be properly chosen. We also would like to remark that the results proved in [1] can be applied to the operator P satisfying the Levi conditions (0.2), thus giving propagation of C^∞ singularities along $\Gamma^{\sigma}_{p_3}$ on the leaves of the natural foliation of Σ.

§1. Necessity

Let us recall from [4] that if the Cauchy problem for P is well posed in S_0 then $\forall K$ compact subset of $\boldsymbol{R}^{n+1}$ $\exists\mu>0$ such that $\forall u\in C_0^\infty(K)$

$$\|u\|^{-}_{(-\mu)}\leqq C\|Pu\|^{-}_{(\mu)}\,. \tag{1.1}$$

(See e.g. [4] for a definition of the quotient Sobolev norms $\|\cdot\|^{-}_{(s)}$). In order to prove that (0.2) is necessary let us construct asymptotic null solutions of $Pu=0$ in order to violate (1.1). Therefore starting from P of the form:

$$P=D_0^3-a(D_1^2+D_2^2)D_0+bD_1^3+(c_0D_0+c_1D_1+c_2D_2)D_n+kD_n \tag{1.2}$$

let us perform the symplectic dilation

$$(x_0, x_1, x_2, x', x_n)\longrightarrow\left(x_0, \frac{1}{\rho}x_1, \frac{1}{\rho}x_2, x', \frac{1}{\rho}x_n\right),$$

$x'=(x_3, \cdots, x_{n-1})$ and conjugate with $E_\rho(x)=\exp(i\rho^2x_n+i\rho^2x_1\xi_1+i\rho^2x_2\xi_2+i\rho\varphi(x))$ we have:

$$\begin{aligned}\rho^{-3}P_\rho=\Big[&(D_0+\rho\varphi_0)^3-\frac{a}{\rho^2}\Big((D_1+\rho\varphi_1+\rho^2\xi_1)^2+(D_2+\rho\varphi_2+\rho^2\xi_2)^2\Big)(D_0+\rho\varphi_0)\\&+\frac{b}{\rho^3}(D_1+\rho\varphi_1+\rho^2\xi_1)^3+\Big(c_0(D_0+\rho\varphi_0)+\frac{c_1}{\rho}(D_1+\rho\varphi_1+\rho^2\xi_1)\\&+\frac{c_2}{\rho}(D_2+\rho\varphi_2+\rho^2\xi_2)\Big)(D_n+\rho\varphi_n+\rho^2)+k(D_n+\rho\varphi_n+\rho^2)\Big]\end{aligned} \tag{1.3}$$

$$=\{\varphi_0^3-a(\xi_1^2+\xi_2^2)\varphi_0+b\xi_1^3+\cdots+c_0\varphi_0+c_1\xi_1+c_2\xi_2\}$$

$$+\rho^{-1}\Big\{3\varphi_0^2D_0+\frac{3}{i}\varphi_0\varphi_{00}-a(\xi_1^2+\xi_2^2)D_0-2a(\varphi_1\xi_1+\varphi_2\xi_2)\varphi_0+3b\varphi_1\xi_1^2$$

$$+c_0D_0+c_1\varphi_1+c_2\varphi_2+(c_0\varphi_0+c_1\xi_1+c_2\xi_2)\varphi_n+k\Big\}+O(\rho^{-2})$$

where by $O(\rho^{-2})$ we denote differential operators with coefficients bounded uniformly after multiplication by ρ^{-2}. Let us now show that the c_j's are real. Assume the contrary and consider the eikonal equation:

$$\varphi_0^3-a(\xi_1^2+\xi_2^2)\varphi_0+b\xi_1^3+\cdots+c_0\varphi^0+c_1\xi_1+c_2\xi_2=0\,. \tag{1.4}$$

Since $\xi_1, \xi_2\in\boldsymbol{R}$, (1.4) certainly has a simple root ζ with $\Im\zeta<0$. Therefore $\varphi(x)=\zeta x_0+i(x_1^2+x_2^2+\cdots+x_n^2)$ solves (1.4) and $\Im\varphi(x)\geqq C|x|^2$ in $x_0\leqq 0$. Since the transport equations coming from (1.3) are non characteristics, the standard arguments in [4] prove the thesis. Let us now turn to the second part of the proof, i.e. let us show that the Hamilton vector field of the linear form $c_0\xi_0+c_1\xi_1+c_2\xi_2$ belongs to the propagation cone of the principal symbol. Let us turn back to the eikonal equation (1.4). It is of the form $p_3(\varphi_0, \xi_1, \xi_2)+p_1(\varphi_0, \xi_1, \xi_2)=0$ where p_j is a homogeneous polynomial of degree j and p_3 has real roots in φ_0 for every ξ_1, ξ_2. It is easily verified that the equation $\varphi_0\to(p_3+p_1)(\varphi_0, \xi_1, \xi_2)=0$ has real roots in φ_0 $\forall\xi_1, \xi_2$ iff $\nabla_{\varphi_0,\xi_1,\xi_2}p_1$ belongs to the euclidean polar of the hyperbolicity cone of p_3. Arguing as before shows that $\varphi(x)=\zeta x_0+i(x_1^2+x_2^2+\cdots+x_n^2)$ solves (1.4), $\Im\varphi(x)\geqq C|x|^2$ and we finally obtain $H_{p_2}\in\Gamma_{p_3}^{\sigma}$. Since P is differential this eventually yields that p_2 must vanish of order 2 on Σ. Therefore let us show that p_1 has to vanish on Σ. The operator can now be thought of the form:

$$P=D_0^3-a(D_1^2+D_2^2)D_0+bD_1^3+A(D_0, D_1, D_2)+kD_n \tag{1.5}$$

where A denotes a quadratic form in its argument. Performing the symplectic dilation $(\xi_0, \xi_1, \xi_2, \xi_3, \cdots, \xi_{n-1}, \xi_n)\to(\rho^N\xi_0, \rho^N\xi_1, \rho^N\xi_2, \xi_3, \cdots, \xi_{n-1}, \rho^{3N}\xi_n)$, together with its dual one, we can localize P in the following way:

$$P_\rho=D_0^3-a(D_1^2+D_2^2)D_0+bD_1^3+kD_n+O(\rho^{-N})\,. \tag{1.6}$$

With another symplectic dilation we can finally "eliminate" D_1 and D_2 from (1.6) and, after conjugation with $E_\rho(x)=e^{i\rho^3x_n+\rho\varphi(x)}$, we eventually obtain the eikonal equation:

$$\varphi_0^3+k=0\,. \tag{1.7}$$

Hence if $k\neq 0$, (1.7) certainly admits a complex root ζ with $\Im\zeta<0$ and again the standard arguments in constructing an asymptotic solution for the equation $P_\rho u_\rho\sim 0$ apply, thus proving that (0.2) is necessary.

§ 2. Sufficiency

Here we shall prove that (0.2) is sufficient for the well-posedness of the Cauchy problem for P, which we may assume to be of the following form:

$$P=D_0^3-a(D_1^2+D_2^2)D_0+bD_1^3+c_0D_0^2+2c_1D_1D_0+2c_2D_2D_0 \tag{2.1}$$
$$+2c_3D_1D_2+c_4D_1^2+c_5D_2^2+c_6D_0+c_7D_1+c_8D_2+c_9\,.$$

As a standard multiplier let us choose $M=D_0^2-(a/3)D_1^2$ and compute:

$$2i\mathfrak{I}\langle Pu,\ Mu\rangle=2i\mathfrak{I}\Big\langle\Big(D_0M-\frac{2a}{3}D_1^2D_0-aD_2^2D_0+bD_1^3+c_0M \tag{2.2}$$
$$+\Big(c_4-\frac{a}{3}\Big)D_1^2+c_5D_2^2+2c_1D_1D_0+2c_2D_2D_0+2c_3D_1D_2$$
$$+c_6D_0+c_7D_1+c_8D_2+c_9\Big)u,\ \Big(D_0^2-\frac{a}{3}D_2^2\Big)u\Big\rangle.$$

Let us first assume that the c_j's are real, $j=1,\cdots,9$. Then,

$$2i\mathfrak{I}\langle Pu,\ Mu\rangle=D_0\left\{|Mu|^2+\mathfrak{R}\left\langle\begin{bmatrix}2a^2/9 & -b\\ -b & 2a/3\end{bmatrix}\begin{bmatrix}D_1^2u\\ D_1D_0u\end{bmatrix},\ \begin{bmatrix}D_1^2u\\ D_1D_0u\end{bmatrix}\right\rangle\right. \tag{2.3}$$
$$+a|D_2D_0u|^2+\frac{a^2}{3}|D_1D_2u|^2-2\Big(c_4-\frac{a}{3}\Big)\mathfrak{R}\langle D_1^2u,\ D_0u\rangle$$
$$-2c_5\mathfrak{R}\langle D_2^2u,\ D_0u\rangle-2c_1\langle D_1D_0u,\ D_0u\rangle-2c_2\langle D_2D_0u,\ D_0u\rangle$$
$$-\frac{2ac_2}{3}c_2\langle D_2u,\ D_1^2u\rangle-\frac{2ac_2}{3}c_1\langle D_1u,\ D_1^2u\rangle$$
$$-2\mathfrak{R}\langle 2c_3D_1D_3u,\ D_0u\rangle-c_6|u|^2-2\mathfrak{R}c_7\langle D_1u,\ D_0u\rangle$$
$$\left.-2\mathfrak{R}c_8\langle D_2u,\ D_0u\rangle-2\mathfrak{R}c_9\langle u,\ D_0u\rangle\right\}$$
$$=D_0\{|Mu|^2+E+R\}$$

where E denotes the positive leading part:

$$\mathfrak{R}\left\langle\begin{bmatrix}2a^2/9 & -b\\ -b & 2a/3\end{bmatrix}\begin{bmatrix}D_1^2u\\ D_1D_0u\end{bmatrix},\ \begin{bmatrix}D_1^2u\\ D_1D_0u\end{bmatrix}\right\rangle+a|D_2D_0u|^2+\frac{a^2}{3}|D_1D_2u|^2. \tag{2.4}$$

Let us multiply (2.3) by $ie^{-2\tau x_0}$ and integrate for $x_0<0$ which gives:

$$C\int_{-\infty}^0|Pu|^2e^{-2\tau x_0}dx\geqq\int_{x_0=0}(|Mu|^2+E+R)dx' \tag{2.5}$$
$$+\tau\int_{-\infty}^0 e^{-2\tau x_0}(|Mu|^2+E+R)dx\,.$$

Let us now recall from [4] the following energy estimate for M:

$$C\int_{-\infty}^{0}|Mu|^2e^{-2\tau x_0}dx \geqq \tau^2\int_{-\infty}^{0}(|D_0u|^2+|D_1u|^2)e^{-2\tau x_0}dx + \tau^4\int_{-\infty}^{0}(|u|^2)e^{-2\tau x_0}dx\,. \tag{2.6}$$

It is now a simple matter to verify that every term in R can be estimated using Cauchy-Schwartz inequality by (2.6) and E. This eventually yields that, with $\tau>0$ sufficiently large,

$$\begin{aligned} C\int_{-\infty}^{0}|Pu|^2e^{-2\tau x_0}dx & \\ \geqq \tau\int_{-\infty}^{0}&(|D_1D_0u|^2+|D_1^2u|^2+|D_2D_0u|^2+|D_1D_2u|^2)e^{-2\tau x_0}dx \\ &+\tau^3\int_{-\infty}^{0}(|D_0u|^2+|D_1u|^2)e^{-2\tau x_0}dx+\tau^5\int_{-\infty}^{0}(|u|^2)e^{-2\tau x_0}dx\,. \end{aligned} \tag{2.7}$$

A similar argument holds if the c_j's are complex and from (2.7) it is now straightforward to derive existence and uniqueness results. We also would like to remark that, although our theorem has been proved in the particular case given by (0.1), all the basic features of a more general setting are preserved and in fact the following result could be proved precisely in the same way:

Theorem 2. *Let $P=p_m+p_{m-1}+p_{m-2}+\cdots$ be a linear differential operator of order m with C^∞ coefficients in $\boldsymbol{R}^{n+1}$. Assume that P is hyperbolic with respect to dx_0, p_m vanishes of order three on a C^∞ involutive submanifold $\Sigma\subset T^*\boldsymbol{R}^{n+1}$ and that $\forall\rho\in\Sigma$ the localization at ρ of p_m, $p_{m,\rho}(\delta z)=\lim_{t\to 0}p_m(\rho+t\delta z)$ is a homogeneous polynomial of degree three which is strictly hyperbolic with respect to $(0, e_0)$. Then a necessary and sufficient condition in order that the Cauchy problem for P be well posed in $S_0=\{x\in\boldsymbol{R}^{n+1}\,|\,x_0<0\}$ is that:*

$$p_{m-1}(\rho)=0, \qquad H_{p_{m-1}}(\rho)=0, \qquad p_{m-2}(\rho)=0 \quad \forall\rho\in\Sigma\,.$$

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