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An Explicit Realization of a GNS Representation in a Krein-Space

By

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Abstract

On the tensor-algebra over the basic space $\mathbf{C}^2$, a P-functional is constructed. Using methods which are due to J.P. Antoine and S. Ôta, a Krein-space theory (i.e., a *-algebra of operators which are defined on a common, dense, and invariant domain in a Krein-space $\mathcal{K}$) is obtained via the GNS construction. It is shown that $\mathcal{K}$ does not contain any π-invariant dual pairs. This gives an answer to a problem first posed by J.P. Antoine and S. Ôta.

The theory so obtained describes the complex superposition of two harmonic oscillators. With this in mind, the annihilation and creation operators, the operator of total electric charge, and the gauge group are explicitly given.

§1. Introduction

Within the algebraic approach to quantum field theory (QFT) one is led to consider certain positive linear functionals W (Wightman functionals) on tensor-algebras (Borchers algebras), [12], [29]. This is probably the most elegant version of axiomatic QFT. In the case of massless or gauge fields, the positivity condition on W has to be abandoned ([27]), and the GNS (Gelfand, Neumark, Segal) construction will give a state space $\mathcal{K}$ with an indefinite metric. This was developed by P.J.M. Bongaarts in [9]. Using these methods, the free quantized electromagnetic field in its various gauges was analyzed in detail ([10], [11]).

Quantum field theory with indefinite metric was also considered by D.A. Dubin and J. Tarski in [14], and there it was shown that the one-particle sector of the free massless field in two dimensions is a Pontryagin-space. Further, a quantum theory based on a Fock-space with indefinite metric was constructed

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and investigated by K.R. Ito ([19]).

In order to make the theory mathematically manageable, it is desirable to obtain for $\mathcal{K}$ a Krein-space ([6], [23]). For achieving that goal, W has to satisfy certain special conditions of positivity. These are Krein-positivity ([22]), α-positivity ([21]), and generalized α-positivity ([3]).

The aim in this note is to give an explicit realization of the GNS construction leading to a Krein-space, using methods which are due to J.P. Antione and S. Ôta ([3], [28]). This will answer a problem first posed in [3; p. 274].

Let us mention that a very similar model to that which is obtained in this note was investigated by H. Araki in [4]. Among others a one-parameter family of indefinite inner products such that the spaces of state vectors are Krein-spaces was considered in [4].

In this note we consider the easiest case, namely a free-field like, α-positive linear functional T on the tensor-algebra $(\boldsymbol{C}^2)_\otimes$ over the basic space $\boldsymbol{C}^2$ (two-dimensional complex plane). It is proved that T is a P-functional ([3]) on $(\boldsymbol{C}^2)_\otimes$. Constructing then the GNS representation, all relevant objects are given explicitly. Among others it is shown that the *-algebra of (field) operators $\mathcal{A}$ is generated by the operators of multiplication A_1, A_2 that are defined on their common, dense and globally invariant domain $\boldsymbol{C}[t_1, t_2]$ (algebra of polynomials in two (commuting) variables). Here it is

$$A_j p(t_1, t_2) = t_j p(t_1, t_2)\,,$$

$j=1, 2$, $p(t_1, t_2) \in \boldsymbol{C}[t_1, t_2]$, and the vacuum-vector is given by the constant polynomial $1 \in \boldsymbol{C}[t_1, t_2]$. Further, the Krein-space $\mathcal{K}$ and the spectral decomposition $J=P_+-P_-$ of its symmetry J are described. If follows also that $\mathcal{K}$ does not contain any π-invariant dual pairs ([3; Definition 4]), and hence, our reconstructed theory is not only a direct sum of two independent Hilbert space representations.

The theory so obtained describes the complex superposition of two independent harmonic oscillators. With this in mind, the annihilation and creation operators, the operator of total electric charge $Q=i(t_2\partial_1-t_1\partial_2)$, $\partial_j=\partial/\partial t_j$ $(j=1, 2)$, the gauge group $\mathcal{G}=U(1)$, and the sectors of electric charge are described.

The motivation of the investigations of the present note is the following: i) Showing by an example that the method of P-functionals ([3], or §2 of this note) applies also to tensor-algebras, and thus it is possible to describe theories of the algebraic approach to QFT by this method (see §5 of this note).

ii) Giving a direct proof of the α-positivity of the Wightman-type functional ϕ defined in §3, i.e., the proof does not use the method of functional integrals. (In the context of this note, this means proving inequality (12′) without using (21).) Let us mention that even in the case of massive free fields, it is not trivial to show directly the positivity of the corresponding Wightman functionals (see [26; Remark to Theorem II.15], [5]). iii) Applying the methods developed in §§3, 4 of this note to the Wightman functional of the free quantized electromagnetic field in a forthcoming paper.

The pattern of the present note is as follows. The main definitions and concepts of the theory of unbounded representations in Krein-spaces and of the theory of tensor-algebras are recalled in §2. §3 is devoted to an explicit construction of a P-functional on $(\boldsymbol{C}^2)_\otimes$ (see Theorem 3). Further, it is shown that there is an abstract conditional expectation (Lemma 1). Using the P-functional from §3, the aim of §4 is to give a realization of the GNS representation. §5 is devoted to some applications to physics.

§2. Preliminaries

Let us recall some concepts of the theory of indefinite inner product spaces ([6], [23]). Let us be given a Hilbert space $\mathcal{K}$, a symmetry J on $\mathcal{K}$ (i.e., $J=J^*=J^{-1}\in\mathcal{B}(\mathcal{K})$), and a sesquilinear form

$$[\xi, \eta]_J = (J\xi, \eta),$$

$\xi, \eta\in\mathcal{K}$, where $(\cdot, \cdot)$ denotes the scalar product of $\mathcal{K}$. Then, $\mathcal{K}$ equipped with $[\cdot, \cdot]_J$ is called a *Krein-space* (or J-space).

Let $\mathfrak{A}$, $\mathfrak{B}$ be *-algebras with unit $\mathbf{1}$ such that $\mathbf{1}\in\mathfrak{B}\subset\mathfrak{A}$, and suppose $\mathfrak{B}$ is a *-subalgebra of $\mathfrak{A}$. A linear mapping P of $\mathfrak{A}$ onto $\mathfrak{B}$ is called *abstract conditional expectation* ([25]), if

i) $P(\mathbf{1})=\mathbf{1}$,
ii) $P(axb)=aP(x)b$, for all $a, b\in\mathfrak{B}$, $x\in\mathfrak{A}$,
iii) $P(x^*)=P(x)^*$, for all $x\in\mathfrak{A}$.

Following [3], an hermitean linear functional ϕ on $\mathfrak{A}$ is said to be a *P-functional* if it satisfies the condition

i) $\phi(P(x))=\phi(x)$,
ii) $\phi((\alpha(x^*))x)\geqslant 0$

for all $x\in\mathfrak{A}$, where $\alpha(x)=2\,P(x)-x$. Further, the cone

$$K = \text{convex hull}\ (\{(\alpha(x^*))x;\ x\in\mathfrak{A}\})$$

will be refered as the cone of *generalized α-positivity*.

For $(\mathfrak{A}, \mathfrak{B}, P)$ and ϕ as above, the GNS representation for T consists of unbounded operators acting on a Krein-space. In order to decide whether or not the representation so obtained is the direct sum of two independent Hilbert space representations (and thus the indefinite metric is irrelevant), S. Ôta introduced the concept of *invariant dual pairs*, see [3; Definition 4], [28; Definition 4.1].

To recall some definitions from the theory of tensor-algebras let us be given a (complex) vector space E, and let

$$E_n = E\otimes E\otimes\cdots\otimes E$$

stand for the n-fold (algebraic) tensor product of E by itself, $n\in\boldsymbol{N}$. The *tensor-algebra* $E_\otimes$ over the basic space E is then defined by

$$E_\otimes = \boldsymbol{C}\oplus E_1\oplus E_2\oplus\cdots \qquad \text{(direct sum)}\,,$$

i.e., the elements $f\in E_\otimes$ are terminating sequences

$$f = (f_0, f_1, \cdots, f_N, 0, 0, \cdots)\,,$$

where $f_n\in E_n$, $n=0, 1, 2, \cdots$ ($E_0=\boldsymbol{C}$, $E_1=E$). Defining algebraic operations by

$$(f+g)_n = f_n+g_n\,,$$
$$(cf)_n = cf_n\,,$$
$$(fg)_n = \sum_{r+s=n} f_r\otimes g_s,\ (f_0\otimes g_n = g_n\otimes f_0 = f_0 g_n)\,,$$

for $f, g\in E_\otimes$, $c\in\boldsymbol{C}$, $n=0, 1, 2, \cdots$, $E_\otimes$ becomes an (associative) algebra with unit $\mathbb{1}=(1, 0, 0, \cdots)$.

In the following let $\check{\cdot}$ denote the canonical embedding of $\cdot$ in $E_\otimes$, i.e.,

$$\check{f}_n = (0, \cdots, 0, f_n, 0, 0, \cdots)\in E_\otimes\,,$$

$f_n\in E_n$, $n\in\boldsymbol{N}^*(=\boldsymbol{N}\cup\{0\})$.

If an antilinear bijection "*" satisfying $f_1^{**}=f_1$, $f_1\in E$, is given on E, then let us define antilinear mappings on E_n (which are also denoted by *) by

$$(f_n)^* = g^{(n)*}\otimes\cdots\otimes g^{(1)*}$$

for $f_n=g^{(1)}\otimes\cdots\otimes g^{(n)}\in E_n$, $n=1, 2, 3, \cdots$. Using antilinearity, the mappings

"*" are extended to an involution on $E_{\otimes}$. Notice that $E_{\otimes}$ becomes a *-algebra.

If $\dim(E)=n\in\boldsymbol{N}$, then there is a *-isomorphism μ between $E_{\otimes}$ and the *-algebra of polynomials $\boldsymbol{C}\{t_1,\cdots,t_n\}$ in n *non-commuting* variables $t_1,\cdots,t_n$, where μ is given as follows. Let $\{e^{(1)},\cdots,e^{(n)}\}$ be a basis for E such that $e^{(i)}=e^{(i)*}\in E$, $i=1,2,\cdots,n$. Define then

$$\mu(ce^{(i_1)}\otimes\cdots\otimes e^{(i_m)})=ct_{i_1}t_{i_2}\cdots t_{i_m}\in\boldsymbol{C}\{t_1,\cdots,t_n\},$$

where $c\in\boldsymbol{C}$, $i_j\in\{1,2,\cdots,n\}$ $(j=1,2,\cdots,m)$.

For further investigations in tensor-algebras the reader is refered to [17].

§3. Definition of a Free-Field Like P-Functional

Let us consider the tensor-algebra $(\boldsymbol{C}^2)_{\otimes}$ over the basic space $\boldsymbol{C}^2$. Let

$$\mathfrak{x}=\begin{pmatrix}1\\0\end{pmatrix},\quad \mathfrak{y}=\begin{pmatrix}0\\1\end{pmatrix}$$

denote the canonical basis of $\boldsymbol{C}^2$. On $\boldsymbol{C}^2$ let us be given two involutive mappings *, α_1 by

$$(\xi\mathfrak{x}+\eta\mathfrak{y})^*=\bar{\xi}\mathfrak{x}+\bar{\eta}\mathfrak{y}\quad\text{(antilinear)},\tag{1}$$

$$\alpha_1(\xi\mathfrak{x}+\eta\mathfrak{y})=-\xi\mathfrak{x}+\eta\mathfrak{y}\quad\text{(linear)},\tag{1'}$$

$\xi,\eta\in\boldsymbol{C}$. Define then antilinear and linear mappings on $(\boldsymbol{C}^2)_n$ by setting

$$(g^{(1)}\otimes\cdots\otimes g^{(n)})^*=g^{(n)*}\otimes\cdots\otimes g^{(1)*}$$

and

$$\alpha_n(g^{(1)}\otimes\cdots\otimes g^{(n)})=\alpha_1(g^{(1)})\otimes\cdots\otimes\alpha_1(g^{(n)}),\quad\text{respectively,}$$

where $g^{(1)},\cdots,g^{(n)}\in\boldsymbol{C}^2$, $n\in\boldsymbol{N}$. Putting $f^*=(\bar{f}_0,f_1^*,f_2^*,\cdots)$, $\alpha(f)=(f_0,\alpha_1(f_1),\alpha_2(f_2,),\cdots)$, $f\in(\boldsymbol{C}^2)_{\otimes}$, the mappings *, α are defined on $(\boldsymbol{C}^2)_{\otimes}$. Then, the cone of generalized α-positivity is given by

$$K=\{\sum_{i=1}^{M}(\alpha(f^{(i)*}))f^{(i)};f^{(i)}\in(\boldsymbol{C}^2)_{\otimes},\ M\in\boldsymbol{N}\}.$$

Recall that a basis of $(\boldsymbol{C}^2)_n$ is given by the 2^n elements

$$d^{(1)}\otimes\cdots\otimes d^{(n)},\tag{2}$$

where $d^{(s)}\in\{\mathfrak{x},\mathfrak{y}\}$, $s=1,2,\cdots,n$. Let $h_n=d^{(1)}\otimes\cdots\otimes d^{(n)}$, $\hat{h}_n=\hat{d}^{(1)}\otimes\cdots\otimes\hat{d}^{(n)}$ be elements of the basis (2). Further, let z_n (resp. $\hat{z}_n$) denote the number of elements $\mathfrak{x}$ occuring in h_n (resp. $\hat{h}_n$). Then, let h_n stand before $\hat{h}_n$ if i) $z_n<\hat{z}_n$, or ii) $z_n=\hat{z}_n$ and the word $d^{(1)}\cdots d^{(n)}$ stands before $\hat{d}^{(1)}\cdots\hat{d}^{(n)}$ with respect to lexiographic order. Using this ordering, let us number the basis (2) by

$$h_n^{(\nu)},\ \nu=1,2,\cdots,2^n. \tag{2'}$$

Further, let $z_n^{(\nu)}$ denote the number of elments $\mathfrak{x}$ occuring in $h_n^{(\nu)}$.

For every mapping α_n let us consider the projection P_n given by

$$P_n(h_n^{(\nu)})=\frac{1}{2}((-1)^{z_n^{(\nu)}}+1)\,h_n^{(\nu)}. \tag{3}$$

Define

$$\mathfrak{B}_n=\text{range}\,(P_n)=\text{span}\,\{h_n^{(\nu)};\ z_n^{(\nu)}\text{ is even}\},$$

and notice that

$$\dim\,(\mathfrak{B}_n)=\frac{1}{2}\dim\,(E_n)=2^{n-1},$$

$n=1,2,3,\cdots$. Further, let $P_0=I$ on $(\boldsymbol{C}^2)_0=\mathfrak{B}_0=\boldsymbol{C}$.

Lemma 1. a) $\mathfrak{B}=\bigoplus_{n=0}^{\infty}\mathfrak{B}_n$ *is a *-subalgebra of* $(\boldsymbol{C}^2)_\otimes$.
b) $P=(P_0,P_1,P_2,\cdots)$ *is an abstract conditional expectation of* $(\boldsymbol{C}^2)_\otimes$ *onto* $\mathfrak{B}$.

Proof. a) is obvious.
b) i) Property i) holds because of $P(\mathbb{1})=(P_0(1))^\vee=\check{1}=\mathbb{1}$. ii) Let $a=(a_0,a_1,\cdots)$, $b=(b_0,b_1,\cdots)\in\mathfrak{B}$, $x=(x_0,x_1,\cdots)\in(\boldsymbol{C}^2)_\otimes$. Further, let all the homogeneous components of a, b, x be given with respect to the basis (2′):

$$a_k=\sum_{\nu=1}^{2^k}\delta_\nu^{(k)}h_k^{(\nu)},\quad b_k=\sum_{\mu=1}^{2^k}\beta_\mu^{(k)}h_k^{(\mu)},$$

$$x_k=\sum_{\rho=1}^{2^k}\xi_\rho^{(k)}h_k^{(\rho)},\quad k=1,2,3,\cdots,$$

where $\delta_\nu^{(k)}=0$ and $\beta_\mu^{(k)}=0$ if $z_k^{(\nu)}$ and $z_k^{(\mu)}$ are odd, respectively. Then,

$$\begin{aligned}(P(axb))_n&=P_n\Big(\sum_{k+l+s=n}a_k\otimes x_l\otimes b_s\Big)\\&=\sum_{k+l+s=n}\ \sum_{\nu,\mu,\rho}(\delta_\nu^{(k)}\beta_\mu^{(s)}\xi_\rho^{(l)})\,P_n(h_l^{(\nu)}\otimes h_l^{(\rho)}\otimes h_s^{(\mu)})\\&=\sum_{k+l+s=n}\ \sum_{\nu,\mu,\rho}(\delta_\nu^{(k)}\beta_\mu^{(s)}\tilde{\xi}_\rho^{(l)})\,(h_k^{(\nu)}\otimes h_l^{(\rho)}\otimes h_s^{(\mu)}),\end{aligned}$$

where

$$\tilde{\xi}_p^{(l)} = \begin{cases} \xi_p^{(l)} & \text{if the number of } \mathfrak{x} \text{ occuring in } h_k^{(\nu)} \otimes h_l^{(p)} \otimes h_s^{(\mu)} \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$P_l(\delta_\nu^{(k)} \beta_\mu^{(s)} x_l) = \sum_p \delta_\nu^{(k)} \beta_\mu^{(s)} \tilde{\xi}_p^{(l)} h_l^{(p)}.$$

Hence,

$$(P(axb))_n = \sum_{k+l+s=n} a_k \otimes P_l(x_l) \otimes b_s = (a\, P(x)\, b)_n$$

are implied, $n=0, 1, 2, \cdots$.

iii) It is sufficient to prove iii) for the elements of the basis (2′). For any given $h_n^{(\nu)} = d^{(1)} \otimes \cdots \otimes d^{(n)}$, let $h_n^{(\tilde{\nu})} = d^{(n)} \otimes \cdots \otimes d^{(1)}$. Then, $z_n^{(\nu)} = z_n^{(\tilde{\nu})}$, (3) and $h_n^{(\nu)*} = h_n^{(\tilde{\nu})}$ imply

$$P_n(h_n^{(\nu)*}) = \frac{1}{2}((-1)^{z_n^{(\tilde{\nu})}} + 1)\, h_n^{(\tilde{\nu})} = \frac{1}{2}((-1)^{z_n^{(\nu)}} + 1)\, h_n^{(\nu)*} = (P_n(h_n^{(\nu)}))^*,$$

$n=0, 1, 2, \cdots$. This completes the proof.

Let us define a linear functional ϕ_2 on $(\boldsymbol{C}^2)_2$ by

$$-\phi_2(\mathfrak{x} \otimes \mathfrak{x}) = \phi_2(\mathfrak{y} \otimes \mathfrak{y}) = 1\,,$$
$$\phi_2(\mathfrak{x} \otimes \mathfrak{y}) = \phi_2(\mathfrak{y} \otimes \mathfrak{x}) = 0\,.$$

Notice that

$$\phi_2(\alpha_2(g_2)) = \phi_2(g_2)\,, \tag{4}$$

$g_2 \in (\boldsymbol{C}^2)_2$. As in the case of free fields, define now

$$\phi_{2n}(d^{(1)}) \otimes \cdots \otimes d^{(2n)}) = \sum_{(i,j)} \prod_{r=1}^{n} \phi_2(d^{(i_r)} \otimes d^{(j_r)})\,, \tag{5}$$

where $d^{(s)} \in \{\mathfrak{x}, \mathfrak{y}\}$ $(s=1, 2, \cdots, 2n)$, and the sum is over all the $(2n)!/(2^n n!)$ permutations $\{i_1, j_1, i_2, j_2, \cdots, i_n, j_n\}$ of $\{1, \cdots, 2n\}$ such that $i_1 < i_2 < \cdots < i_n$ and $i_1 < j_1$, $i_2 < j_2, \cdots, i_n < j_n$, $n \in \boldsymbol{N}$. Let us now consider the functional

$$\phi(f) = \sum_{n=0}^{\infty} \phi_{2n}(f_{2n})\,, \tag{6}$$
$$\phi_0(f_0) = f_0\,, \quad f \in (\boldsymbol{C}^2)_\otimes\,.$$

Considering $\phi_{2n}(f_k \otimes g_l)$, $f_k \in (\boldsymbol{C}^2)_k$, $g_l \in (\boldsymbol{C}^2)_l$, $k+l=2n$, let us distinguish between pairings of 1st and 2nd order in (5). A pairing is called to be of 1st order (resp. of 2nd order) if ϕ_2 applies only either to elements of f_k or to such of g_l

(resp. ϕ_2 applies to one element of f_k and one of g_l). (E.g., if $f_2=d^{(1)}\otimes d^{(2)}$, $g_2=d^{(3)}\otimes d^{(4)}$, then

$$\phi_4(f_2\otimes g_2) = \phi_2(d^{(1)}\otimes d^{(2)})\,\phi_2(d^{(3)}\otimes d^{(4)})+\phi_2(d^{(1)}\otimes d^{(3)})\phi_2(d^{(2)}\otimes d^{(4)}) + \phi_2(d^{(1)}\otimes d^{(4)})\,\phi_2(d^{(2)}\otimes d^{(3)})\,,$$

and the two pairings of the first summand of the right-hand side of the equation given above are of 1st order while the remaining pairings are of 2nd order.)

Let us introduce operators $\psi_{k,2s}\colon (\boldsymbol{C}^2)_k\to(\boldsymbol{C}^2)_{k-2s}$, $k\in\boldsymbol{N}$, $s=0, 1, \cdots, \left[\frac{k}{2}\right]$, by

$$\psi_{k,2s}(d^{(1)}\otimes\cdots\otimes d^{(k)}) = \sum_{\substack{i_1<\cdots<i_s\\ i_r<j_r}} \prod_{r=1}^{s} \phi_2(d^{(i_r)}\otimes d^{(j_r)})d^{(\nu_1)}\otimes\cdots\otimes d^{(\nu_{k-2s})} \tag{7}$$

$\nu_1<\nu_2<\cdots$, $\nu_t\in\{1, 2, \cdots, k\}\setminus\{i_1, \cdots, i_s, j_1, \cdots, j_s\}$ $(t=1, 2, \cdots, k-2s)$, $[\cdot]$ and denotes the integral part of $\cdot$. Notice that $\psi_{k,2s}$ constructs s pairings of 1st order in each summand of (7).

Let us define bilinear forms $\chi_\mu\colon (\boldsymbol{C}^2)_\mu\times(\boldsymbol{C}^2)_\mu\to\boldsymbol{C}$ by

$$\chi_\mu(d^{(1)}\otimes\cdots\otimes d^{(\mu)},\ d^{(\mu+1)}\otimes\cdots\otimes d^{(2\mu)}) = \sum_{\pi}\prod_{r=1}^{\mu}\phi_2(d^{(r)}\otimes d^{(\pi_r)})\,, \tag{8}$$

where $\pi=(\pi_1, \cdots, \pi_\mu)$ runs through the set of all the permutations of $\{\mu+1, \cdots, 2\mu\}$, $\mu=1, 2, 3, \cdots$. Note that (5) implies

$$\phi_{2n}(f_k\otimes g_l) = \sum_{s=0}^{\sigma}\chi_{k-2s}(\psi_{k,2s}(f_k),\ \psi_{l,l-k+2s}(g_l)) \tag{9}$$

where $\sigma=\left[\frac{k}{2}\right]$, and $\psi_{k,s}=0$ if $s<0$ or $k<0$ or $k-s<0$.

Let us introduce the diagonalized block-matrix

$$A = (a_{ij})_{i,j=1}^{2^\mu} = \operatorname{diag}[A_0, A_1, \cdots, A_\mu]\,, \tag{10}$$

where $A_m=(-1)^m\,(a_{rs}^{(m)})_{r,s=1}^{\rho}$, $\rho=\binom{\mu}{m}$, $a_{rs}^{(m)}=(\mu-m)!\,m!$. Further, let $\langle\cdot,\cdot\rangle$ denote the scalar-product in $(\boldsymbol{C}^2)_\mu=\boldsymbol{C}^{2^\mu}$.

Lemma 2. a) *Using the basis* (2′), *it holds*

$$\chi_\mu(f_\mu, g_\mu) = \langle f_\mu^*, A\,g_\mu\rangle,\ f_\mu, g_\mu\in(\boldsymbol{C}^2)_\mu\,.$$

b) *It is* $\chi_\mu(\alpha_\mu(f_\mu^*), f_\mu)\geqslant 0$ *for all* $f_\mu\in(\boldsymbol{C}^2)_\mu$, $\mu=1, 2, \cdots$.

Proof. a) Let us consider $h_\mu^{(\nu)}=d^{(1)}\otimes\cdots\otimes d^{(\mu)}$, $h_\mu^{(\nu')}=d^{(\mu+1)}\otimes\cdots\otimes d^{(2\mu)}$

from (2′). Notice that $z_\mu^{(\nu)} \neq z_\mu^{(\nu')}$ yields $\chi_\mu(h_\mu^{(\nu)}, h_\mu^{(\nu')})=0$. Further, if $m=z_\mu^{(\nu)}=z_\mu^{(\nu')}$, then there are $(\mu-m)!\,m!$ non-vanishing summands in (8). Since each of these non-vanishing summands is equal to $(-1)^m$, it is

$$\chi_\mu(h_\mu^{(\nu)}, h_\mu^{(\nu')}) = (-1)^m\,(\mu-m)!\,m!$$

implied. Noticing that there are $\rho=\binom{\mu}{m}$ elements $h_\mu^{(\nu')}$ such that $m=z_\mu^{(\nu')}$, a) follows.

b) Notice that (1′) implies $\alpha_\mu(h_\mu^{(\nu)})=(-1)^m\,h_\mu^{(\nu)}$, where $m=z_\mu^{(\nu)}$. Using a), it follows now that

$$\chi_\mu(\alpha_\mu(f_\mu^*), g_\mu) = \langle f_\mu, \tilde{A}\,g_\mu\rangle\,, \tag{11}$$

$\tilde{A}=(\tilde{a}_{ij})_{i,j=1}^{2^\mu}=\mathrm{diag}[\tilde{A}_0, \tilde{A}_1, \cdots, \tilde{A}_\mu]$, $\tilde{A}_m=(a_{rs}^{(m)})_{r,s=1}^{\rho}$, $m=0, 1, \cdots, \mu$. Checking the minors $\Delta_t=\det((a_{ij})_{i,j=1}^{t})$, $t=1, 2, \cdots 2^\mu$, one gets

$$\Delta_1 = \mu!,\quad \Delta_2 = \mu!(\mu-1)!,\quad \Delta_3 = \Delta_4 = \cdots = \Delta_{2^\mu} = 0\,.$$

Hence, $\tilde{A}\geqslant 0$. This yields $\langle f_\mu, \tilde{A}f_\mu\rangle\geqslant 0$ for all $f_\mu\in(\boldsymbol{C}^2)_\mu$. Using (11), the assertion to be shown follows.

Theorem 3. *ϕ is a P-functional.*

Proof. i) Apply ϕ_{2n} to the elements $h_{2n}^{(\nu)}$, $\nu=1, 2, \cdots, 2^{2n}$, taken from the basis (2′). If $z_{2n}^{(\nu)}$ is odd, then (3) and (5) imply $P_{2n}(h_{2n}^{(\nu)})=0$ and $\phi_{2n}(h_{2n}^{(\nu)})=0$, respectively. Hence, $\phi_{2n}(h_{2n}^{(\nu)})=\phi_{2n}(P_{2n}(h_{2n}^{(\nu)}))$.

Assume now that $z_{2n}^{(\nu)}$ is even. Using again (3), $P_{2n}(h_{2n}^{(\nu)})=h_{2n}^{(\nu)}$ and $\phi_{2n}(h_{2n}^{(\nu)})=\phi_{2n}(P_{2n}(h_{2n}^{(\nu)}))$ are implied.

Finally, (6) yields

$$\phi(Pf) = \sum_{n=0}^{\infty} \phi_{2n}(P_{2n}(f_{2n})) = \sum_{n=0}^{\infty} \phi_{2n}(f_{2n}) = \phi(f)\,,$$

$f\in(\boldsymbol{C}^2)_\otimes$.

ii) Using (6) and (9),

$$\begin{aligned}\phi((\alpha(f^*))f) &= \sum_{n=0}^{\infty} \sum_{k+l=2n} \phi_{2n}(\alpha_k(f_k^*)\otimes f_l) \\ &= \sum_{n=0}^{\infty} \sum_{l+k=2n} \sum_{s=0}^{\infty} \chi_{k-2s}(\psi_{k,2s}(\alpha_k(f_k^*)),\ \psi_{l,l-k+2s}(f_l)) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \chi_{k-2s}(\psi_{k,2s}(\alpha_k(f_k^*)),\ \psi_{2n-k,2(n-k+s)}\,(f_{2n-k}))\,,\end{aligned} \tag{12}$$

are implied, where $f=(f_0, \cdots, f_N, 0, 0, \cdots)\in(\boldsymbol{C}^2)_\otimes$ yields that all the sums considered in (12) are actually finite ones, and $\chi_\mu=0$ for $\mu<0$. Introducing new indices $\mu=k-2s$, $t=n-k+s$ for n, k, one gets

$$\phi(\alpha(f^*)f) = \sum_{\mu=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \chi_\mu(\psi_{\mu+2s,2s}(\alpha_{\mu+2s}(f_{\mu+2s}^*)),\ \psi_{\mu+2t,2t}(f_{\mu+2t}))\,.$$

Set $g_\mu = \sum_{s=0}^{\infty} \psi_{\mu+2s,2s}(f_{\mu+2s})$ and notice that

$$\alpha_\mu(g_\mu^*) = \sum_{s=0}^{\infty} \psi_{\mu+2s,2s}(\alpha_{\mu+2s}(f_{\mu+2s}^*))\,,$$

$\mu=0, 1, 2, \cdots$. Applying now Lemma 2b),

$$\phi(\alpha(f^*)f) = \sum_{\mu=0}^{\infty} \chi_\mu(\alpha_\mu(g_\mu^*),\ g_\mu) \geqslant 0 \tag{12'}$$

follows for all $f\in(\boldsymbol{C}^2)_\otimes$. This completes the proof.

§4. GNS Representation for the Functional ϕ

Using the functional ϕ introduced above, let us do the construction described by Antoine and Ôta in [3; Section 3].

Let $n\in\boldsymbol{N}$, $n\geqslant 2$. Using the basis (2′), let us define

$$\begin{aligned} N_n = \operatorname{span}\{&d^{(1,\nu)}\otimes\cdots\otimes d^{(n,\nu)} - d^{(\pi(1),\nu)}\otimes\cdots\otimes d^{(\pi(n),\nu)}; \\ &\pi = (\pi(1), \cdots, \pi(n))\in\Pi_n,\ \nu = 1, 2, \cdots, 2^n\}, \end{aligned} \tag{13}$$

where $h_n^{(\nu)}=d^{(1,\nu)}\otimes\cdots\otimes d^{(n,\nu)}$, $d^{(\cdot,\cdot)}\in\{\mathfrak{x}, \mathfrak{y}\}$, and Π_n denotes the set of all the permutations of $\{1, 2, \cdots, n\}$. These spaces will enable us to quotient out the left kernel of the state ϕ.

If $z_n^{(\nu)}=m$, then there are $\frac{n!}{m!\,(n-m)!}-1$ linearly independent elements $d^{(1,\nu)}\otimes\cdots\otimes d^{(n,\nu)} - d^{(\pi(1),\nu)}\otimes\cdots\otimes d^{(\pi(n),\nu)}$, $\pi\in\Pi_n$, occuring in (13). Hence,

$$\dim(N_n) = \sum_{m=0}^{n}\left(\frac{n!}{m!\,(n-m)!}-1\right) = 2^n-n-1\,. \tag{14}$$

Using (8), $f_n\in N_n$ implies

$$\chi_n(g_n, f_n) = 0 \tag{15}$$

for all $g_n\in(\boldsymbol{C}^2)_n$.

Furthermore, let us introduce the symmetrization operator S_n by

$$S_n(d^{(1)}\otimes\cdots\otimes d^{(n)}) = \frac{1}{n!}\sum_{\pi\in\Pi_n} d^{(\pi(1))}\otimes\cdots\otimes d^{(\pi(n))}\,,$$

$d^{(j)} \in \{\mathfrak{x}, \mathfrak{y}\}$, $j=1, 2, \cdots, n$. Notice that

$$\dim(\text{range}(S_n)) = n+1 . \tag{16}$$

Using that $h_n^{(\nu)} - S_n(h_n^{(\nu)}) \in N_n$, $\nu=1, 2, \cdots, 2^n$, (14) and (16) imply

$$\text{range}(I_n - S_n) = N_n , \tag{16'}$$

where I_n denotes the identity operator on $(\boldsymbol{C}^2)_n$.

Lemma 4. *Let $n \in \boldsymbol{N}$, $n \geqslant 2$. Then, the following are equivalent:*

i) $f_n \in N_n$,

ii) $\psi_{n,2s}(f_n) \in N_{n-2s}$, $s=0, 1, \cdots, \left[\frac{n}{2}\right]$,

iii) $S_n(f_n)=0$.

Proof. i)⇒ii): Take some element

$$w_n = d^{(1)} \otimes \cdots \otimes d^{(n)} - d^{(\pi(1))} \otimes \cdots \otimes d^{(\pi(n))} ,$$

$d^{(j)} \in \{\mathfrak{x}, \mathfrak{y}\}$, $j=1, 2, \cdots, n$, $\pi \in \Pi_n$. Let us consider

$$\psi_{n,2}(w_n) = \psi_{n,2}(d^{(1)} \otimes \cdots \otimes d^{(n)}) - \psi_{n,2}(d^{(\pi(1))} \otimes \cdots \otimes d^{(\pi(n))}) .$$

Using (7) for $s=1$, there is a one-to-one correspondence δ between all the $\binom{n}{2}$ summands of the right-hand side (r.h.s.) of (7) for $\psi_{n,2}(d^{(1)} \otimes \cdots \otimes d^{(n)})$ and those for $\psi_{n,2}(d^{(\pi(1))} \otimes \cdots \otimes d^{(\pi(n))})$. δ is given as follows.

Take any summand

$$A = \phi_2(d^{(i)} \otimes d^{(j)})\, d^{(1)} \otimes \cdots \otimes \check{d}^{(i)} \otimes \cdots \otimes \check{d}^{(j)} \otimes \cdots \otimes d^{(n)}$$

of the r.h.s. of (7) for $\psi_{n,2}(d^{(1)} \otimes \cdots \otimes d^{(n)})$, where $1 \leqslant i < j \leqslant n$, and $\check{\cdot}$ denotes that $\cdot$ does not occur. Then there is exactly one summand

$$B = \phi_2(d^{(\pi(s))} \otimes d^{(\pi(t))})\, d^{(\pi(1))} \otimes \cdots \otimes \check{d}^{(\pi(s))} \otimes \cdots \otimes \check{d}^{(\pi(t))} \otimes \cdots \otimes d^{(\pi(n))}$$

of the r.h.s. of (7) for $\psi_{n,2}(d^{(\pi(1))} \otimes \cdots \otimes d^{(\pi(n))})$ such that either $\pi(s)=i$, $\pi(t)=j$ or $\pi(t)=i$, $\pi(s)=j$. Define now $\delta(A)=B$.

Recalling that $\phi_2(d, d')=\phi_2(d', d)$, $d,d' \in \{\mathfrak{x}, \mathfrak{y}\}$, it follows from (13) that $A-\delta(A) \in N_{n-2}$. Hence, $\psi_{n,2}(w_n) \in N_{n-2}$. Noticing that the linear span of the set of all the w_n considered above is just N_n,

$$\psi_{n,2}(N_n) \subset N_{n-2}$$

is implied. Using finally that $\psi_{n,2s} = \psi_{n-2(s-1),2} \circ \cdots \circ \psi_{n-2,2} \circ \psi_{n,2}$, the proof of i)⇒

ii) is completed.

ii)⇒i): Setting $s=0$ in ii), i) is implied. i)⇒iii): The implication under consideration follows readily from (13). iii)⇒i): Assuming iii),

$$(I_n - S_n)(f_n) = I_n(f_n) = f_n$$

follow. (16′) implies now $f_n \in N_n$. This completes the proof.

Let us consider the left kernel of ϕ:

$$\mathfrak{N}_\phi = \{x \in (C^2)_\otimes;\ \phi(y^* x) = 0 \quad \text{for all } y \in (C^2)_\otimes\}.$$

Set $N_0 = 0$, $N_1 = 0$. Let us define the symmetrization operator

$$S(x) = (x_0, x_1, S_2(x_2), S_3(x_3), \cdots)$$

for $x = (x_0, x_1, \cdots) \in (C^2)_\otimes$, (and S_n, $n = 2, 3, \cdots$, is given beforehand of (16)).

Lemma 5. *It holds* $\mathfrak{N}_\phi = \bigoplus_{m=0}^{\infty} N_m$.

Proof. a) Let $x_l \in N_l$, $y_k \in (C^2)_k$, $k + l = 2n$. Applying (9), Lemma 4 i)⇔ ii), and (15), it follows that

$$\phi_{2n}(y_k^* \otimes x_l) = \sum_{s=0}^{\sigma} \chi_{k-2s}(\psi_{k,2s}(y_k^*), \psi_{l,l-k+2s}(x_l)) = 0, \tag{17}$$

where $\sigma = \left[\frac{k}{2}\right]$. For $x = (x_0, x_1, \cdots) \in \bigoplus_{m=0}^{\infty} N_m$, $y = (y_0, y_1, \cdots) \in (C^2)_\otimes$,

$$\phi(y^* x) = \sum_{n=0}^{\infty} \sum_{k+l=2n} \sum_{s=0}^{\sigma} \chi_{k-2s}(\psi_{k,2s}(y_k^*), \psi_{l,l-k+2s}(x_l)) = 0$$

is implied by (6) and (17). Hence, $\bigoplus_{m=0}^{\infty} N_m \subset \mathfrak{N}_\phi$.

b) Let $0 \neq x = (x_0, \cdots, x_N, 0, 0, \cdots) \in \mathfrak{N}_\phi$, and $x \notin \bigoplus_{m=0}^{\infty} N_m$. Consider $\tilde{x} = S(x)$. Then

$$\tilde{\tilde{x}} = x - \tilde{x} \in \bigoplus_{m=0}^{\infty} N_m$$

due to Lemma 4 i)⇔iii). Using a),

$$\phi(y\tilde{\tilde{x}}) = 0 \tag{18}$$

follows for all $y \in (C^2)_\otimes$. Further, $x \notin \bigoplus_{m=0}^{\infty} N_m$ implies $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_M, 0, 0, \cdots) \neq 0$. Let $\tilde{x}_M \neq 0$. Using (12′), Lemma 2b), $S_M(x_M) = \tilde{x}_M$ and (11),

$$\phi(\alpha(\tilde{x}^*)\tilde{x}) \geqslant \chi_M(\alpha_M(x_M^*), x_M) > 0 \tag{18'}$$

follow. (18), (18′) yield

$$\phi(\alpha(\tilde{x}^*)x) = \phi(\alpha(\tilde{x}^*)\tilde{\tilde{x}})+\phi(\alpha(\tilde{x}^*)\tilde{x})>0$$

which is a contradiction to the assumptions of b). This completes the proof of the lemma under consideration.

It is straightforward to prove the following characterization of $\mathfrak{N}_\phi$.

Remark. It holds $\mathfrak{N}_\phi=\{x\in(\boldsymbol{C}^2)_\otimes;\ \phi(\alpha(x^*)x)=0\}$.

Let us consider the quotient space

$$(\boldsymbol{C}^2)_\otimes/\mathfrak{N}_\phi = \boldsymbol{C}\oplus\boldsymbol{C}^2\oplus(\boldsymbol{C}^2)_2/N_2\oplus(\boldsymbol{C}^2)_3/N_3\oplus\cdots .$$

Observe that $S(x)=S(y)$ if and only if $x-y\in\mathfrak{N}_\phi$. Further, $S(x)\in\eta(x)$, and $\eta(x)$ denotes the residue class of x in $(\boldsymbol{C}^2)_\otimes/\mathfrak{N}_\phi$. Hence, there is a linear isomorphism

$$(\boldsymbol{C}^2)_\otimes/\mathfrak{N}_\phi\cong\boldsymbol{C}\oplus\boldsymbol{C}^2\oplus S_2((\boldsymbol{C}^2)_2)\oplus S_3((\boldsymbol{C}^2)_3)\oplus\cdots \quad =: D(\pi)\,. \tag{19}$$

For each $y\in(\boldsymbol{C}^2)_\otimes$ let us define the (field) operator $\pi(y)$ acting on $D(\pi)$ by

$$\pi(y)(S(x)) = S(yx)\,,$$

$x\in(\boldsymbol{C}^2)_\otimes$. Further, define sesquilinear forms

$$(S(x), S(y)) = \phi(\alpha(y^*)x)\,,$$
$$[S(x), S(y)] = \phi(y^*x)\,,$$

$x, y\in(\boldsymbol{C}^2)_\otimes$. Noticing that $(\cdot,\cdot)$ defines a (positive definite) scalar product, the completion $D(\pi)^\sim$ becomes a Hilbert space $\mathcal{H}$.

Let us consider the two subspaces of the right-hand side of (19):

$$D^+ = \boldsymbol{C}\oplus D_1^+\oplus D_2^+\oplus\cdots,$$
$$D^- = D_1^-\oplus D_2^-\oplus\cdots,$$

where

$$D_n^+ = \text{span}\,\{S_n(h_n^{(\nu)});\ z_n^{(\nu)} \text{ is even},\ \nu = 1, 2, \cdots, 2^n\},$$
$$D_n^- = \text{span}\,\{S_n(h_n^{(\nu)});\ z_n^{(\nu)} \text{ is odd},\ \nu = 1, 2, \cdots, 2^n\},$$

$n=1, 2, \cdots$. Observe that $D(\pi)=D^+\oplus D^-$ is an (algebraic) direct sum. Let $P_+: D(\pi)\to D^+$, $P_-: D(\pi)\to D^-$ denote the corresponding projections. Using $D^+\perp D^-$ with respect to the scalar product $(\cdot, \cdot)$, $\|a+b\|\geqslant\frac{1}{\sqrt{2}}(\|a\|+\|b\|)$ for $a, b\in D(\pi)$ with $(a, b)=0$, $\|\cdot\|^2=(\cdot, \cdot)$, it follows the orthogonal decomposition

$$\mathcal{H} = \tilde{D}^+ \oplus \tilde{D}^- .$$

If $\bar{P}_+$ and $\bar{P}_-$ denote the projections concerning this decomposition of $\mathcal{H}$, then

$$J = \bar{P}_+ - \bar{P}_- \in \mathcal{B}(\mathcal{H})$$

satisfies $[S(x), S(y)] = (S(x), JS(y))$, $x, y \in (\boldsymbol{C}^2)_\otimes$.

Consider now the symmetrized basis

$$\{\tilde{h}_n^{(m)};\ m = 0, 1, 2, \cdots, n\}$$

on $S_n((\boldsymbol{C}^2)_n)$, $n=2, 3, 4, \cdots$, where $\tilde{h}_n^{(m)} = S_n(h_n^{(\nu)})$ with $z_n^{(\nu)} = m$, and $h_n^{(\nu)}$ is taken from (2′). Using this basis, J is given on $D(\pi)$ by

$$J = J_0 \oplus J_1 \oplus J_2 \oplus \cdots ,$$

where

$$J_n = \begin{pmatrix} 1 & & & & & \\ & -1 & & & \mathbf{0} & \\ & & 1 & & & \\ & & & -1 & & \\ \mathbf{0} & & & & \ddots & \\ & & & & & \pm 1 \end{pmatrix}$$

denotes an $(n+1, n+1)$-matrix, $n=0, 1, 2, \cdots$.

Using the isomorphism μ between $(\boldsymbol{C}^2)_\otimes$ and $\boldsymbol{C}\{t_1, t_2\}$, there is a more convenient description for the objects considered above. Let

$$\mu(\check{x}) = t_1,\ \mu(\check{y}) = t_2 . \tag{20}$$

Note that $\mu(\mathfrak{N}_\phi) = \mathcal{J}$, where $\mathcal{J}$ denotes the two-sided ideal generated by $t_1 t_2 - t_2 t_1$ in $\boldsymbol{C}\{t_1, t_2\}$. Hence,

$$\hat{\mu}((\boldsymbol{C}^2)_\otimes / \mathfrak{N}_\phi) = \boldsymbol{C}\{t_1, t_2\} / \mathcal{J} \cong \boldsymbol{C}[t_1, t_2] ,$$

where the *-isomorphism between $(\boldsymbol{C}^2)_\otimes / \mathfrak{N}_\phi$ and $\boldsymbol{C}[t_1, t_2]$ (resp. between $D(\pi)$ and $\boldsymbol{C}[t_1, t_2]$), which is induced by μ, is denoted by $\hat{\mu}$. The action of the operator $\pi(x)$ is now given by multiplication with the polynomial $\hat{\mu}(S(x))$, i.e.,

$$\hat{\mu}(\pi(x)\, S(y)) = \hat{\mu}(S(x))\, \hat{\mu}(S(y)) ,$$

$x, y \in (\boldsymbol{C}^2)_\otimes$.

Recalling the solution of the (two-dimensioal) problem of monents

$$T(t_1^{2r} t_2^{2s}) = (2r)!\,(2s)!/(2^{r+s}\, r!\, s!),\quad r, s = 0, 1, 2, \cdots,$$
$$T(t_1^l t_2^m) = 0 \quad \text{if } l \text{ or } m \text{ are odd } (l, m \in \mathfrak{N}^*),$$

one obtains

$$T(p) = \int_{\boldsymbol{R}^2} p(t_1, t_2)\, d\rho(t_1, t_2)$$

for $p \in \boldsymbol{C}[t_1, t_2]$, $d\rho(t_1, t_2) = \frac{1}{2\pi} \exp\{-(t_1^2+t_2^2)/2\}\, dt_1\, dt_2$, (e.g., see [26; I.1]).

Let us now introduce automorphisms ι, ϑ on the algebra $\boldsymbol{C}[t_1, t_2]$ by setting

$$[\iota(p)](t_1, t_2) = p(it_1, t_2)\,,$$
$$[\vartheta(p)](t_1, t_2) = p(-t_1, t_2)\,,$$

$p \in \boldsymbol{C}[t_1, t_2]$. Noticing that (5) implies

$$\phi_{2n}(d^{(1)} \otimes \cdots \otimes d^{(2n)}) = (2r)!\,(2s)!/(2^{r+s}\, r!\, s!)\,(-1)^r,$$

where $2r$ (resp. $2s$) denotes the number of $\mathfrak{x}$ (resp. $\mathfrak{y}$) occuring in $\{d^{(1)}, \cdots, d^{(2n)}\}$, $r+s=n$, one gets

$$\phi(\hat{\mu}^{-1}(p)) = T(\iota(p))\,.$$

Recalling (1), (1′), it follows

$$\hat{\mu}(\alpha(\tilde{x})^*) = \overline{\vartheta(\hat{\mu}(\tilde{x}))}\,, \quad \iota(\hat{\mu}(\tilde{x}^*)) = \overline{\vartheta(\iota(\hat{\mu}(\tilde{x})))}\,,$$
$$\iota(\hat{\mu}(\alpha(\tilde{x}^*))) = \overline{\iota(\hat{\mu}(\tilde{x}))}\,,$$

$\tilde{x} \in S((\boldsymbol{C}^2)_\otimes)$. Setting $\tilde{x}=S(x)$, $\tilde{y}=S(y)$,

$$(\tilde{x}, \tilde{y}) = \phi(\alpha(x)^*y) = T(\iota\hat{\mu}(\alpha(\tilde{x})^*\tilde{y})) = \int \overline{\iota(\hat{\mu}(\tilde{x}))}\, \iota\hat{\mu}(\tilde{y}))\, d\rho$$
$$= \int \overline{[\hat{\mu}(\tilde{x})](it_1, t_2)}\, [\hat{\mu}(\tilde{y})](it_1, t_2)\, d\rho(t_1, t_2)\,, \tag{21}$$
$$[\tilde{x}, \tilde{y}] = \phi(x^*y) = T(\iota\hat{\mu}(\tilde{x}^*\tilde{y})) = \int \overline{\vartheta\iota(\hat{\mu}(\tilde{x}))}\, \iota\hat{\mu}(\tilde{y})\, d\rho$$
$$= \int \overline{[\hat{\mu}(\tilde{x})](-it_1, t_2)}\, [\hat{\mu}(\tilde{y})](it_1, t_2)\, d\rho(t_1, t_2) \tag{22}$$

$x, y \in (\boldsymbol{C}^2)_\otimes$, are implied. Further, (21) yields

$$\mathcal{H} \cong L^2(\boldsymbol{R}^2, d\rho)\,.$$

Notice also that the cyclic vacuum vector is given by

$$\Omega = \hat{\mu}(S(\mathbf{1})) = 1 \in \boldsymbol{C}[t_1, t_2]$$

in $L^2(\boldsymbol{R}^2, d\rho)$. For further investigations on the so-called Q-space method the reader is refered to [19; §9].

Let us now show that the indefinite metric is intrinsic for the theory constructed above. Thus, one is led to answer the question whether or not there are π-invariant dual pairs.

For a given subspace $\mathcal{M}\subset D(\pi)$, let us define

$$E(\mathcal{M}) = \{x\in(\boldsymbol{C}^2)_\otimes;\ S(x)\in\mathcal{M}\}.$$

Recall that if $\{\mathcal{M}, \mathcal{N}\}$ is a π-invariant dual pair, then

i) $(\boldsymbol{C}^2)_\otimes = E(\mathcal{M}) + E(\mathcal{N})$,

ii) $E(\mathcal{M})$, $E(\mathcal{N})$ are left-ideals in $(\boldsymbol{C}^2)_\otimes$,

iii) $\phi(x^*x)>0$ (resp. $\phi(x^*x)<0$) for all $x\in E(\mathcal{M})$ (resp. $x\in E(\mathcal{N})$), $x\notin\mathfrak{N}_\phi$, (see [3; Lemma 6]).

Proposition 6. *There are no π-invariant dual pairs in the theory constructed above.*

Proof. Assume that $\{\mathcal{M}, \mathcal{N}\}$ is a π-invariant dual pair. Because of $[\check{\mathfrak{y}}, \check{\mathfrak{y}}] = \phi(\check{\mathfrak{y}}^*\check{\mathfrak{y}}) = \phi_2(\mathfrak{y}\otimes\mathfrak{y}) = 1$, there is a $z\in E(\mathcal{M})$ with $z\notin\mathfrak{N}_\phi$. Choose $w\in(\boldsymbol{C}^2)_\otimes$ so that

$$\hat{\mu}(\tilde{w}) = \vartheta\hat{\mu}(\tilde{z}),$$

where $\tilde{w}=S(w)$, $\tilde{z}=S(z)$. ii) yields now

$$\check{\mathfrak{x}}wz\in E(\mathcal{M}).$$

Applying (20), (22), a contradiction to iii) follows from

$$\begin{aligned}[\check{\mathfrak{x}}wz, \check{\mathfrak{x}}wz] &= \phi((\check{\mathfrak{x}}wz)^*\check{\mathfrak{x}}wz)\\ &= -\int t_1^2\,|\,p(it_1, t_2)\,p(-it_1, t_2)\,|^2\,d\rho<0\,,\end{aligned}$$

where $p(t_1, t_2)=\hat{\mu}(\tilde{z})\in\boldsymbol{C}[t_1, t_2]$, and

$$\vartheta\iota\hat{\mu}(S(\check{\mathfrak{x}}wz)) = -\iota\hat{\mu}(S(\check{\mathfrak{x}}wz) = -it_1\,p(-it_1, t_2)\,p(it_1, t_2)$$

were applied. This completes the proof.

Remark. a) Proposition 6 implies that the indefinite metric is intrinsic for the theory constructed above, see [3; Chapter 4]. b) For the contrary, if one considers the Wightman-functional of a free field with positive mass, then the GNS representation consists of (unbounded) operators acting on a Hilbert space. Thus, there is no indefinite metric in this theory. For details of this GNS representation the reader is refered to [5], [8; Exercise 3.4.6].

§5. Applications to Physics

a) *Harmonic Oscillator of Quantum Mechanics* ([13; *Ch*.2.3], [15; *Ch*.1.5])

Let us recall some results from quantum theory with positive definite metric. Let us consider the tensor-algebra over the complex plane $\boldsymbol{C}$:

$$\boldsymbol{C}_{\otimes} = \boldsymbol{C}\oplus\boldsymbol{C}\oplus\boldsymbol{C}\oplus\cdots ,$$

and the linear functional $T=(T_0, T_1, T_2, \cdots)\in\boldsymbol{C}'_{\otimes}$, where

$$T_{2n+1} = 0, \quad T_{2n} = (2n)!/(2^n\, n!) ,$$

$n=0, 1, 2, \cdots$. Solving the corresponding problem of moments, one gets

$$T(f) = \int_{-\infty}^{+\infty} [\mu(f)](t)\, d\rho'(t) , \tag{23}$$

where $d\rho'(t)=(2\pi)^{-1/2} \exp(-t^2/2)\, dt$ (Gaussian measure), and $\mu(f)(t)=\sum_{j=0}^{\infty} f_j t^j$, $f=(f_0, f_1, \cdots)\in\boldsymbol{C}_{\otimes}$, is given in §2. Applying (23), the positivity of T is implied.

Considering the scalar product

$$(f, g) = T(g^*f) ,$$

$f, g \in \boldsymbol{C}_{\otimes}$, an ortho-normal basis of $\boldsymbol{C}[t]$ is given by the Hermite polynomials $\{P_m; m=0, 1, 2, \cdots\}$, $P_m(t)=(2^m m!)^{-1/2} H_m(t/\sqrt{2})$, $H_m(t)=(-1)^m \exp(t^2)$ (d^m/dt^m) $(\exp(-t^2))$.

Let us construct the GNS representation for T. It follows for the left ideal

$$I = \{f\in\boldsymbol{C}_{\otimes};\ T(f^*f) = 0\} = \{0\} .$$

Hence, the field operators $\pi(g)$, $g\in\boldsymbol{C}_{\otimes}$, are defined on $D=\boldsymbol{C}_{\otimes}$. Consider the operator $\varphi=\pi(\check{e})$, where $e=e^*$ is a (complex) basis of $\boldsymbol{C}$. If the basis $\{t^n;$ $n=0, 1, 2, \cdots\}$ is used in $\boldsymbol{C}[t]$, then φ is given by

$$\varphi p(t) = t\, p(t) , \tag{24}$$

$p(t)\in\boldsymbol{C}[t]$. Note that the annihilation and creation operators are defined on $\boldsymbol{C}[t]$ by

$$a = d/dt,\ a^* = t-d/dt , \tag{24'}$$

respectively. Notice that the Hilbert space of state vectors is given by

$$\mathcal{H}' = (\boldsymbol{C}[t])^{\sim} \cong L_2(\boldsymbol{R}, d\rho') .$$

The theory so obtained describes the harmonic oscillator of quantum mechanics in one degree of freedom.

b) *Complex Superposition of Two Harmonic Oscillators*

Let us consider the field

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_2 + i\varphi_1),$$

where $\varphi_j = a_j + a_j^*$ are harmonic oscillators satisfying $\varphi_j^* = \varphi_j$ $(j=1,2)$, $\varphi_1\varphi_2 - \varphi_2\varphi_1 = 0$. Observe that (20) implies $\varphi = \frac{1}{\sqrt{2}}\pi(\check{\mathfrak{y}} + i\check{\mathfrak{x}})$. Hence, the theory reconstructed in §4 describes the complex superposition of two independent harmonic oscillators.

Defining the annihilation and creation operators of electric charge by

$$A(\pm 1) = \frac{1}{\sqrt{2}}(a_2 \pm i a_1),$$

$$A^*(\pm 1) = \frac{1}{\sqrt{2}}(a_2^* \mp i a_1^*), \quad \text{respectively},$$

one obtains $\varphi = A^*(-1) + A(1)$, where $a_j = \partial/\partial t_j$, $a_j^* = t_j - \partial/\partial t_j$ $(j=1,2)$. Now, the operator Q of total electric charge is given by

$$Q\, A^*(\sigma_1)\cdots A^*(\sigma_n)\Omega = (\sigma_1 + \cdots + \sigma_n)\, A^*(\sigma_1)\cdots A^*(\sigma_n)\Omega,$$

where $\sigma_j \in \{1, -1\}$ $(j=1,2,\cdots,n)$, and $\Omega = 1 \in \boldsymbol{C}[t]$ denotes the vacuum. It follows that

$$Q = i(a_2^* a_1 - a_1^* a_2) = i(t_2\partial_1 - t_1\partial_2).$$

Introducing new variables $\tau_1 = \frac{1}{\sqrt{2}}(t_2 - it_1)$, $\tau_2 = \frac{1}{\sqrt{2}}(t_2 + it_1)$, the following sectors of electric charge

$$D^{(0)} = \operatorname{span}\{(\tau_1\tau_2)^n;\ n = 0, 1, 2, \cdots\} \text{ (vacuum sector)},$$

$$D^{(r)} = \operatorname{span}\{\tau_1^{r+n}\tau_2^n;\ n = 0, 1, 2, \cdots\},$$

$$D^{(-r)} = \operatorname{span}\{\tau_1^n\tau_2^{r+n};\ n = 0, 1, 2, \cdots\},$$

$r = 1, 2, 3, \cdots$, may be considered. Notice that $f \in D^{(s)}$ yields $Qf = sf$, $s = 0, \pm 1, \pm 2, \cdots$. Obviously, the following decomposition of the domain of the field operators holds:

$$D(\pi) = \cdots \oplus D^{(-1)} \oplus D^{(0)} \oplus D^{(1)} \oplus \cdots.$$

Further, the operators φ and φ^* act as multiplication by τ_2 and τ_1, respectively.

Noting that Q is essentially self-adjoint on $D(\pi)$, let us introduce a one-parameter group of operators

$$V(\lambda) = e^{-i\lambda Q},$$

$\lambda \in \boldsymbol{R}$. Then, a straightforward calculation yields the following gauge transformations of the first kind:

$$\varphi \to \varphi' = V(\lambda)\, \varphi\, V(\lambda)^{-1} = e^{i\lambda}\, \varphi,$$
$$\varphi^* \to \varphi^{*\prime} = V(\lambda)\, \varphi^*\, V(\lambda)^{-1} = e^{-i\lambda}\, \varphi^*.$$

Hence, the gauge group is given by $\mathcal{G}=U(1)$, and the *-algebra of observables (i.e., the gauge invariant elements of the *-algebra of field operators, see [7; Chapter 10.1.B]) is generated by $\varphi\varphi^*$, $\varphi^*\varphi$.

Following [4], let us consider the operator

$$H_0 = a_1 a_1^* + a_2^* a_2 = -\partial_1^2 - \partial_2^2 + t_1\partial_1 + t_2\partial_2 + 1$$

in $L_2(\boldsymbol{R}^2, d\rho)$. Setting

$$\psi_0(n, m) = P_n(t_1)\, P_m(t_2),$$

and using the differential equations $P_n'' - tP_n' + nP_n = 0$ for Hermite polynomials, it follows that $\{\psi_0(n, m);\, n, m \in \boldsymbol{N}^*\}$ is a complete set of vectors such that

$$H_0\psi_0(n, m) = (n+m+1)\, \psi_0(n, m), \tag{25}$$

[16; Chapter A.5]. Let us mention that (25) is a particular case of Theorem 1(1) from [4].

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