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Topes of Oriented Matroids and Related Structures

By

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Abstract

An oriented matroid can be viewed as a combinatorial abstraction of the facial incidence relations of the polyhedral cones induced by a finite arrangement of oriented hyperplanes in $\mathbf{R}^d$ through the origin. "Topes" of an oriented matroid correspond to maximal polyhedral cones. This paper discusses three structures related to topes of oriented matroids, namely, acycloids, L^1-systems and median systems. It is shown that L^1-systems are closely related to convex geometries. Median systems are introduced as an equivalent notion of median graphs, and they are, in particular, applied to characterize median graphs. Perturbations of acycloids and L^1-systems are studied.

§ 1. Introduction

Let E be a finite set and let $\mathbf{A}$ be an $m \times |E|$ real matrix having $\mathbf{A}^e$ as the column vector of $\mathbf{A}$ indexed by $e \in E$. For each vector $\mathbf{x} \in \mathbf{R}^E$, $\sigma(\mathbf{x})$ denotes the signed vector of $\mathbf{x}$, that is, $\sigma(\mathbf{x}) \in \{-, 0, +\}^E$ and $\sigma(\mathbf{x})_e$ is the sign of component $\mathbf{x}_e$. Let $\mathbf{V}$ be the row space of $\mathbf{A}$, i.e., $\mathbf{V} = \{\mathbf{xA}: \mathbf{x} \in \mathbf{R}^m\}$. Then the set $\sigma(\mathbf{V}) = \{\sigma(\mathbf{v}): \mathbf{v} \in \mathbf{V}\}$ represents the partition of $\mathbf{R}^m$ by polyhedral cones induced by the subspaces $\{\mathbf{x} \in \mathbf{R}^m: \mathbf{xA}^e = 0\}$ $(e \in E)$, and also represents the facial incidence relations of the polyhedral cones. An oriented matroid is defined by a set of signed vectors satisfying certain axioms (*face axioms*) that are trivially satisfied by $\sigma(\mathbf{V})$. Besides this, an oriented matroid can be viewed as abstractions of many different concepts in linear space, see [9, 10, 26] for the basic theory and applications.

"Topes" [27, 39] of an oriented matroid correspond to maximal polyhedral cones in the above setting. Topes can be also considered an abstraction of some properties of acyclic reorientations of loopless directed graphs, and further-

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more an abstraction of combinatorial properties of partitions $\{S, E-S\}$ of a finite subset $E \subseteq \mathbb{R}^d$ such that there is a hyperplane in $\mathbb{R}^d$ separating S strictly from $E-S$. In this paper, we investigate oriented matroids through their topes, and introduce and study three structures related to topes of oriented matroids, namely, acycloids [50], L^1-systems [28] and median systems.

L^1-systems are defined by the reorientation property of topes of oriented matroids, and *acycloids* by the negativity closedness property in addition to the reorientation property. These two structures are useful to characterize topes and tope graphs of oriented matroids. On the other hand, they are also interesting when they are viewed from the corresponding graphs. Indeed, L^1-systems are essentially equivalent to graphs isometrically embeddable in a hypercube [19, 31, 36], and acycloids such graphs with antipodality [5, 32].

Median systems are essentially equivalent to median graphs [1, 42, 44], and they constitute a broad class of L^1-systems. Median graphs have been studied under various names or viewpoints, e.g., median algebras [4] etc., median semilattices [47], median interval structures [43, 46] and maximal Helly copair hypergraphs [43], etc. Median systems, which we introduce in this paper, axiomatize median graphs by signed vectors. Since signed vectors are easy to deal with, one can give very simple proofs for propositions on median graphs.

This paper consists of 8 sections. In Section 2, we recall basic notions of oriented matroids, which we need to explain the related structures. In Section 3, we review the definitions of acycloids and L^1-systems, and we briefly summarize their applications [28, 34, 35] to oriented matroids.

A *convex geometry* [24] is a structure which combinatorially abstracts the notion of the convex hull of a finite set of points in Euclidean space In Section 4, it is mainly shown that a convex geometry is essentially equivalent to the set of positive "closed acyclons" of an acyclic L^1-system.

In Section 5, several propositions on median graphs are proved or reproved by using properties of median systems. In particular, a simple characterization of median graphs, similar to Djoković's theorem [19], is obtained. In general, their proofs are shorter and easier to understand than direct proofs by the graph language or properties. Every median system is shown to be an L^1-system, and hence we should note that the results on L^1-systems hold in median systems, too.

In Section 6, we extend the point perturbation theorem [27, 39] of oriented matroids to acycloids and L^1-systems. As an application, we obtain a way to transfer (perturb) an oriented matroid to a non-matroidal acycloid. This section is a joint work with Fukuda, [29].

In Section 7, some examples of non-matroidal acycloids are given. It is shown that there are exactly 4 non-matroidal acycloids on the 5-element set up to reorientation. We conclude this paper in Section 8.

Through this paper, we assume graphs have neither loops nor multiple edges. We denote the vertex-set, the edge-set and the distance function of a graph G by $V(G)$, $E(G)$ and d_G, respectively.

§ 2. Oriented Matroids

In this section, we will recall the basic notions of oriented matroid, which we will need in this paper.

Through this paper, let E be a finite set. A *signed vector* X on E is an element of $\{-, 0, +\}^E$, that is, X is a vector $(X_e: e\in E)$ with $X_e\in\{-, 0, +\}$. The zero vector is denoted by $\mathbf{0}$. The *negative* $-X$ of a signed vector X is defined in the trivial way. For $X\in\{-, 0, +\}^E$ and $S\subseteq E$, we denote the restriction of X to $E-S$ by $X\backslash S$. For $X, Y\in\{-, 0, +\}^E$, we define $D(X, Y)=\{e\in E: X_e=-Y_e\neq 0\}$ and $X\circ Y=(X_e$ if $X_e\neq 0$, and Y_e otherwise: $e\in E)$. Here $X\circ Y$ is called the *composition* of X and Y. Several concepts are used to define oriented matroids. In this paper we will start with the definition by faces [27, 39].

An *oriented matroid* (*on* E) is a pair $M=(E, \mathcal{F})$ where E is a finite set and $\mathcal{F}$ is a set of signed vectors on E, called the *faces* of M, satisfying

(F1) $\mathbf{0}\in\mathcal{F}$, and $X\in\mathcal{F}$ implies $-X\in\mathcal{F}$;

(F2) if $X, Y\in\mathcal{F}$ then $X\circ Y\in\mathcal{F}$; and

(F3) if $X, Y\in\mathcal{F}$ and $f\in D(X, Y)$, there exists $Z\in\mathcal{F}$ such that $Z_f=0$ and $Z\backslash D(X, Y)=(X\circ Y)\backslash D(X, Y)$.

We denote by ( ) the zero vector on the empty set $\emptyset$, and define for convenience that $M=(\emptyset, \{(\ )\})$ is an oriented matroid. A typical example is obtained from the row space $\mathbf{V}$ of a real matrix $\mathbf{A}$ as we mentioned in Section 1. We will denote by $M_{Lin}(\mathbf{A})$ this oriented matroid. An oriented matroid $M=(E, \mathcal{F})$ is *linear* if there is a linear subspace $\mathbf{V}$ of $\mathbf{R}^E$ such that $\mathcal{F}=\sigma(\mathbf{V})$.

For $X, Y\in\{-, 0, +\}^E$, X *conforms* to Y, $X\leq Y$, if $Y_e=X_e$ for all e with $X_e\neq 0$. The notation $X<Y$ denotes $X\leq Y$ and $X\neq Y$. This relation $\leq$ is clearly a partial order on $\{-, 0, +\}^E$. For $\mathcal{X}\subseteq\{-, 0, +\}^E$, we denote the set of minimal elements of $\mathcal{X}$ by $\mathrm{Min}\,\mathcal{X}$, i.e. $\mathrm{Min}\,\mathcal{X}=\{X\in\mathcal{X}: X\geq Y\in\mathcal{X}$ implies $X=Y\}$. $\mathrm{Max}\,\mathcal{X}$ is similarly defined.

Let $M=(E, \mathcal{F})$ be an oriented matroid. A *vertex* of M is a minimal nonzero vector of $\mathcal{F}$, and a *tope* of M is a maximal vector of $\mathcal{F}$. We denote by $\mathcal{V}$ and $\mathcal{T}$ the sets of vertices and topes of M, that is, $\mathcal{V}=\mathrm{Min}(\mathcal{F}-\{\mathbf{0}\})$ and $\mathcal{T}=\mathrm{Max}\,\mathcal{F}$. Note that the poset $\hat{\mathcal{F}}=(\mathcal{F}\cup\{1\},\leq)$, where 1 is the greatest element, forms a lattice, called the *face lattice* of M. For $X, Y\in\mathcal{F}$, their join in $\hat{\mathcal{F}}$ equals $X\circ Y$ if $D(X, Y)=\emptyset$, and equals 1 otherwise. The lattice $\hat{\mathcal{F}}$, furthermore, has the Jordan-Dedekind (J-D) chain property, i.e., $\hat{\mathcal{F}}$ is graded by the height function, see [37, Thm. 1.1]. The height of coatoms of $\hat{\mathcal{F}}$ is called the *rank* of M. (For lattice theory terminology, see e.g. [7]). The sets $\mathcal{V}$ and $\mathcal{T}$ are the sets of atoms and coatoms of $\hat{\mathcal{F}}$, respectively, and the set $\mathcal{F}$ can be described by them as follows; cf. [9,10],

$$\begin{aligned}\mathcal{F} &= \{X: X = Y^1\circ Y^2\circ\cdots\circ Y^k \text{ for some elements } Y^1, Y^2, \cdots, Y^k\in\mathcal{V}\}\cup\{\mathbf{0}\}\\ &= \{X: X\circ Y\in\mathcal{T} \text{ for all } Y\in\mathcal{T}\}\ .\end{aligned}$$

Hence, an oriented matroids is uniquely determined by its vertices, and also by its topes. The vertex axioms of an oriented matroid are given in the next theorem. The tope axioms will be presented in Section 3.

Theorem 2.1([10]). *A set $\mathcal{V}$ of signed vectors on E is the set of vertices of another oriented matroid on E if and only if it satisfies*

(O1) *$\mathbf{0}\notin\mathcal{V}$, and $X\in\mathcal{V}$ implies $-X\in\mathcal{V}$;*

(O2) *if $X, Y\in\mathcal{V}$ and $X\leq Y$, then $X=Y$; and*

(O3) *(elimination property) if $X, Y\in\mathcal{V}$, $X\neq -Y$ and $f\in D(X, Y)$, there exists $Z\in\mathcal{V}$ such that $Z_f=0$ and $Z\backslash D(X, Y)\leq(X\circ Y)\backslash D(X, Y)$.*

For a signed vector X on E, the set $\underline{X}\equiv\{e\in E: X_e\neq 0\}$ is called the *support* of X. Note that in Theorem 2.1, we may replace (O2) with

(O2') if $X, Y\in\mathcal{V}$ and $\underline{X}\subseteq\underline{Y}$, then $X=\pm Y$.

Signed vectors X and Y on E are said to be *orthogonal*, denoted by $X*Y$, if either $\underline{X}\cap\underline{Y}=\emptyset$, or $D(X, Y)\neq\emptyset$ and $D(X, -Y)\neq\emptyset$. For an oriented matroid $M=(E, \mathcal{F})$, the set $\mathcal{C}=\mathrm{Min}\,\{X: X\neq\mathbf{0}, \text{and } X*Y \text{ for all } Y\in\mathcal{F}\}$ is called the set of *circuits* of M. This set $\mathcal{C}$ determines $\mathcal{F}$ by $\mathcal{F}=\{X: X*Y \text{ for all } Y\in\mathcal{C}\}$. Thus the circuits also determine the oriented matroid. Its axiom system is the same as vertex axioms (O1)~(O3), which is the original definition of oriented matroids by Bland and Las Vergnas [10]. In [10], a vertex is called a cocircuit and the set of faces is called the span of cocircuits. The reader should note that the set $\mathcal{V}$ of vertices of an oriented matroid M is the set of circuits of another oriented matroid M^*. This oriented matroid M^* is called the *dual* of M and it has $\mathcal{C}$ as the set of vertices, see [10]. Hence we have $(M^*)^*=M$. In the linear oriented matroid $M_{Lin}(\mathbb{A})$ on $E=\{e_1, \cdots, e_n\}$, the set of circuits is the set Min $\{\sigma(\lambda): \lambda=(\lambda_1, \cdots, \lambda_n)\neq\mathbf{0}$ and $\sum_{i=1}^n \lambda_i \mathbb{A}^{e_i}=\mathbf{0}\}$, and $M_{Lin}(\mathbb{A})^*=(E, \sigma(\mathbb{V}^*))$, where $\mathbb{V}^*$ is the orthogonal complement of the row space $\mathbb{V}$ of $\mathbb{A}$, cf. [10].

Finally, we look at topes of oriented matroids from some different viewpoints.

Maximal cells of sphere systems

Let H be a finite collection of oriented hyperplanes in $\mathbb{R}^d$ through the origin. When we consider the facial incidence relations of the polyhedral cones induced by H, we may restrict our attention to the unit sphere $\mathbb{S}^d=\{\mathbf{x}\in\mathbb{R}^{d+1}: ||\mathbf{x}||=1\}$. Such a restriction is generalized to a *sphere system* [26, 39], which is equivalent to an oriented matroid.

A *d-sphere* is a topological space homeomorphic to $\mathbb{S}^d$. A subset S' of a d-sphere S is a *hypersphere* of S if there exists a homeomorphism f from $\mathbb{S}^d$ to S with $\mathrm{S}'=f(\{\mathbf{x}\in\mathbb{S}^d: \mathbf{x}_{d+1}=0\})$. The two components of $\mathrm{S}-\mathrm{S}'$ are called the *sides* of S'. A *sphere system* [39] is a triple $(\mathrm{S}, E, \mathcal{H})$ where E is a finite set, S is a d-sphere and $\mathcal{H}$ is a collection $\{s_e^i: e\in E, i\in\{-, 0, +\}\}$ of subsets of S satisfying

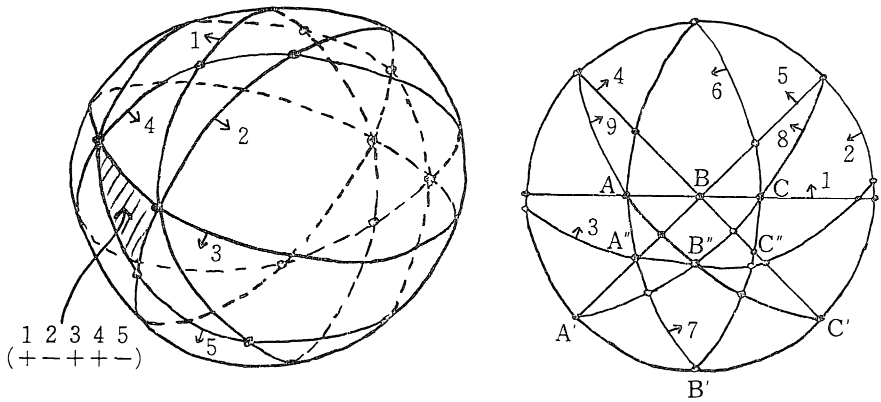
(S1) for every $e\in E$, either $(s_e^-, s_e^0, s_e^+)=(\emptyset, \mathrm{S}, \emptyset)$ or s_e^0 is a hypersphere of S with sides s_e^- and s_e^+;

(S2) for every subset A of E, $\bigcap\{s_e^0: e\in A\}$ is a sphere (possibly empty), called a *flat*; and

(S3) for every flat F and hypersphere s_e^0 not containing it, $F\cap s_e^0$ is a hypersphere of F with sides $F\cap s_e^+$ and $F\cap s_e^-$.

For a sphere system $(\mathrm{S}, E, \mathcal{H})$, we define the map σ from S to $\{-, 0, +\}^E$ by $\sigma(\mathbf{x})_e=i$ if and only if $\mathbf{x}\in s_e^i$. The "topological representation theorem" [26, 39] says: given a sphere system $(\mathrm{S}, E, \mathcal{H})$, the set $\sigma(\mathrm{S})\cup\{\mathbf{0}\}$ is the set of faces of an oriented matroid on E, and conversely every oriented matroid is obtained this way.

A sphere system $(\mathrm{S}, E, \mathcal{H})$ is said to be *linear* if $\mathrm{S}=\mathbb{S}^d$, and for each $e\in E$, s_e^0 is either $\mathbb{S}^d$ or a linear hypersphere. An oriented matroid is linear if and only if it can be represented by a linear sphere system We give an example of linear sphere system in Fig. 2.1 (a), and that of non-linear one in Fig. 2.1 (b), called the Non-Pappus sphere system. Here note that (b) is shown by drawing only a half of it, i.e., $s_2^+\cup s_2^0$. In general, when we draw a sphere system, it is sufficient to draw its closed "hemisphere" $s_f^+\cup s_f^0$ for some f because hemispheres s_f^+ and s_f^- are symmetric.



(a) (b)

Fig. 2.1

For $X\in\{-, 0, +\}^E$, we define $X^+=\{e\in E: X_e=+\}$ and similarly define X^-. We denote by ${}_{\bar{S}}X$, $S\subseteq E$, the signed vector on E obtained from X by reversing signs on S. For $\mathcal{X}\subseteq\{-, 0, +\}^E$ and $S\subseteq E$, define ${}_{\bar{S}}\mathcal{X}=\{{}_{\bar{S}}X: X\in\mathcal{X}\}$.

Acyclic reorientation

An oriented matroid is *acyclic* if it has the positive tope $(++\cdots+)$, i.e., it has no positive circuits (circuits X with $X^-=\emptyset$). Let $M=(E, \mathcal{T})$ be an oriented matroid and let $S\subseteq E$. Then ${}_{\bar{S}}M=(E, {}_{\bar{S}}\mathcal{T})$ is also clearly an oriented matroid, the *reorientation* of M *by* S. The sets of topes, vertices and circuits of ${}_{\bar{S}}M$ are described in the obvious way. If an oriented matroid M has no circuits X with $|\underline{X}|=1$, then the set $\mathcal{T}$ of topes of M can be denoted by

$$\mathcal{T} = \{X\in\{-, +\}^E: {}_{X^-}M \text{ is acyclic}\} .$$

This has a very natural meaning in examples of graphs. Let $G=(V, E)$ be a directed graph. Let $\mathbb{A}$ be the $(0, \pm 1)$-vertex-edge incidence matrix of G, and put $M(G)=M_{Lin}(\mathbb{A})$. This oriented matroid $M(G)$ is called the *oriented cycle matroid* [10, Ex.3.3] of G. For a subset $S\subseteq E$, denote by ${}_{\bar{S}}G$ the directed graph obtained from G by reversing directions of edges in S. Then the set of topes of $M(G)$ is given by $\mathcal{T}=\{X\in\{-, +\}^E: {}_{X^-}G \text{ is acyclic}\}$.

Non-Radon partition

Let $\mathbb{A}$ be an $m\times|E|$ real matrix and let $\hat{\mathbb{A}}$ be the $(m+1)\times|E|$ matrix obtained from $\mathbb{A}$ by adding as a row the vector $(1, \cdots, 1)\in\mathbb{R}^E$. Put $M=M_{Lin}(\hat{\mathbb{A}})$. This oriented matroid M is said to be *determined by affine dependence over* $\mathbb{R}$ and denoted by $M_{Aff}(\mathbb{A})$, see [10, Ex 3.5]. Here note that $M_{Aff}(\mathbb{A})$ is acyclic. We identify the set $\{\mathbb{A}^e: e\in E\}$ of column vectors of $\mathbb{A}$ with the index set E, and call a partition $\{S, E-S\}$ of E a *non-Radon partition* of E if there is a hyperplane in $\mathbb{R}^m$ separating S strictly from $E-S$. The set of topes of $M_{Aff}(\mathbb{A})$ is then given by $\mathcal{T}=\{X\in\{-,+\}^E: \{X^-, X^+\}$ is a non-Radon partition of $E\}$, see [14, 15]. Remark that it is a well-known open problem to characterize the non-Radon partitions of finite subsets of $\mathbb{R}^m$, and that this problem is equivalent to the characterization problem of linear oriented matroids in terms of topes, cf. [6, 20].

Other papers related to topes of oriented matroids can be seen in [11, 16, 17, 30, 45].

§ 3. Acycloids, L^1-Systems and Their Applications

Let $\mathcal{T}$ be the set of topes of an oriented matroid M on E. An element of E is called a *loop* of M if it is not contained in the support of any tope, and the set of loops of M is denoted by E_0. Two distinct elements $e, f\in E-E_0$ are *parallel* if either $X_e=X_f$ for all $X\in\mathcal{T}$ or $X_e=-X_f$ for all $X\in\mathcal{T}$. The set of elements which are parallel to e is denoted by $[e]$ and called the *parallel class* containing e. It is well-konwn that the set $\mathcal{T}$ satisfies the following three properties [13, 23, 27, 39]:

(T1) $X, Y\in\mathcal{T}$ implies $\underline{X}=\underline{Y}$;

(T2) $X\in\mathcal{T}$ implies $-X\in\mathcal{T}$; and

(T3) (*reorientation property*) if $X, Y\in\mathcal{T}$ and $X\neq Y$, there exists $f\in D(X, Y)$ such that $\overline{{}_{[f]}}X\in\mathcal{T}$.

The property (T3) is the most essential property of topes and it is closely related to isometric-embeddability in hypercube.

An *acycloid* [50, 51] is a pair $A=(E, \mathcal{T})$ where E is a finite set and $\mathcal{T}$ is a nonempty set of signed vectors on E, called the *topes* of A, satisfying (T1)~(T3). An acycloid is *simple* if it has no loops and every parallel class is a singleton set. For non-matroidal acycloids, i.e. ones which are not oriented matroids, see Section 7.

The *tope graph* G_A of an acycloid $A=(E, \mathcal{T})$ is a graph such that $V(G_A)=\mathcal{T}$ and such that $X, Y\in V(G_A)$ are adjacent if and only if $D(X, Y)$ is a parallel class.This definition is the same as in oriented matroids, [8, 27, 39].

L^1-systems [28] are defined by the reorientation property of topes for simple oriented matroids: an *L^1-system* is a pair $A=(E, \mathcal{T})$, where E is a finite set and

$\mathcal{T}$ is a nonempty set of elements of $\{-, +\}^E$, called the *topes* of A, satisfying

(L1) if $X, Y \in \mathcal{T}$ and $X \neq Y$, there exists $f \in D(X, Y)$ such that ${}_{\bar{f}}X \in \mathcal{T}$; and

(L2) for every $e \in E$, there exist $X, Y \in \mathcal{T}$ such that $X_e \neq Y_e$.

The condition (L2) is not essential but we include it for simplicity. The *tope graph* G_A of an L^1-system A is similarly defined to that of an acycloid: $V(G_A) = \mathcal{T}$ and $E(G_A) = \{[X, Y]: X, Y \in \mathcal{T}$ and $|D(X, Y)| = 1\}$. Fig. 3.1 (a), (b) show examples of tope graphs of L^1-systems.

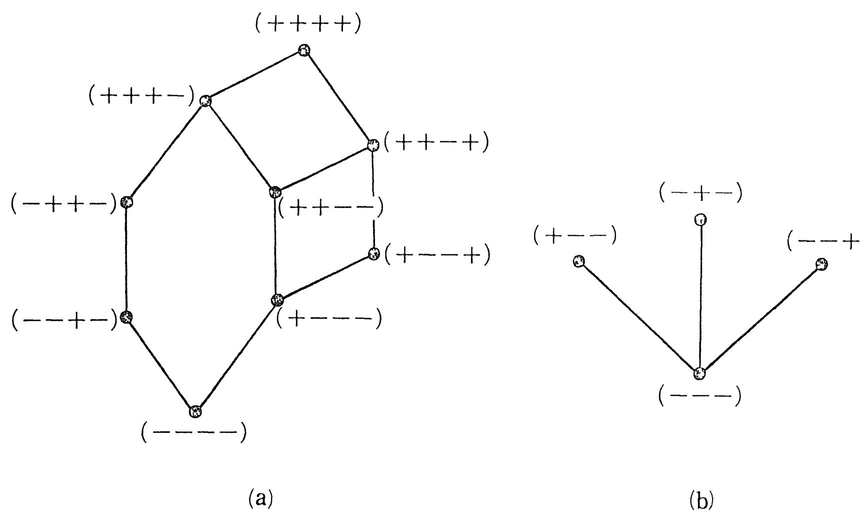


Fig. 3.1

The *hypercube* $Q(E)$ on E is the graph that has $\{-, +\}^E$ as a vertex-set and $\{[X, Y]: |D(X, Y)| = 1\}$ as an edge-set, cf. [36]. For two connected graphs G and G', G is *isometrically embeddable* in G' if there exists an injection f: $V(G) \to V(G')$, called an *isometric embedding* of G into G', such that $d_G(u, v) = d_{G'}(f(u), f(v))$ for all $u, v \in V(G)$. It is clear that the tope graph G_A of an L^1-system A on E is isometrically embeddable in $Q(E)$. Conversely, if G is a graph isometrically embeddable in some hypercube and if we choose an isometric embedding f: $G \to Q(E)$ such that E is minimal, then $A_G = (E, f(V(G)))$ is an L^1-system and the tope graph of A_G is exactly G. Hence we have

Proposition 3.1. ([28]) *A graph G is isomorphic to the tope graph of an L^1-system if and only if G is isometrically embeddable in some hypercube.*

Note that the tope graph of an L^1-system determines the L^1-system uniquely up to reorientation, see [28, Note].

For graphs isometrically embeddable in a hypercube, Djoković's theorem [19] is well-known. Let G be a connected graph. A subset $X \subseteq V(G)$ is *convex* in G if for all $u, v \in X$ all shortest (u, v)-paths are contained in the subgraph induced by X. For each $[a, b] \in E(G)$, define $C(a, b) = \{x \in V(G) : d_G(a, x) < d_G(b, x)\}$.

Theorem 3.2 (Djoković [19]). *A graph G is isometrically embeddable in some hypercube if and only if G satisfies*

(1) *G is connected bipartite, and*

(2) *$C(a, b)$ is convex for all $[a, b] \in E(G)$.*

Some other versions of this theorem can be seen in [2, 12, 28, 52, 53], etc.

In [28, 34, 35], we have applied acycloids and L^1-systems to oriented matroids. In the following we will briefly review the main results in them.

For $X \in \{-, 0, +\}^E$, $S \subseteq E$ and $i \in \{-, 0, +\}$, the notation $X + S^i$ denotes the signed vector on E obtained from X by replacing X_e by i for all $e \in S$. Let $A = (E, \mathcal{T})$ be an acycloid. For $e \in E$, let $\mathcal{T}/e = \{(X + [e]^0) \backslash e : X, \overline{{}_{[e]}}X \in \mathcal{T}\}$, where if e is a loop then consider $[e] = \emptyset$. Then we define $A/e = (E - e, \mathcal{T}/e)$ and, for an ordered subset $S = \{e_1, e_2, \cdots, e_n\}$ of E, inductively define $A/S = (A/(S - e_n))/e_n$, the *contraction* of *A by S*. The contraction A/S satisfies the conditions (T1) and (T2), but does not always satisfy (T3), see [35, 50], also see Example 7.2. The following theorem, which characterizes oriented matroids in terms of topes, was obtained by proving a conjectrue of Tomizawa [50]: if every contraction of an acycloid A is an acycloid, then A is matroidal.

Theorem 3.3 ([35]). *An oriented matroid is a pair $M = (E, \mathcal{T})$ where E is a finite set and $\mathcal{T}$ is a nonempty set of signed vectors on E, satisfying* (T1)~(T3) *and*

(T4) *every contraction of M has the reorientation property.*

For other characterizations, see [6, 18, 38].

In the subsequent paper [34], we have suggested another characterization, which uses the relation between "faces" and "coboundaries" [50, 51] of acycloids.

Let $\mathcal{X} \subseteq \{-, 0, +\}^E$, $X \in \{-, 0, +\}^E$ and $S \subseteq E$. Then we define $\mathcal{X}(X) = \{Y \in \mathcal{X} : X \preceq Y\}$ and $\mathcal{X} \backslash S = \{X \backslash S : X \in \mathcal{X}\}$. A signed vector X is a *face* of an acycloid $A = (E, \mathcal{T})$ if $X \circ Y \in \mathcal{T}$ for all $Y \in \mathcal{T}$; and a *coboundary* of A if X conforms to a tope and $\mathcal{T}(X) \backslash \underline{X}$ is closed under negativity. We denote the sets of faces and coboundaries of A by $\mathcal{F}$ and $\mathcal{B}^{\perp}$, respectively. Between these two sets, the inclusion $\mathcal{F}\subseteq\mathcal{B}^{\perp}$ holds, see [34]. For a geometric interpretation of coboundaries, see [34, 50].

For an acycloid $A=(E, \mathcal{T})$ and $S\subseteq E$, the pair $A-S=(E-S, \mathcal{T}\backslash S)$ is also an acycloid. This acycloid is called the *deletion* of A *by* S; in particular, the *elementary deletion* if S is a singleton set.

Proposition 3.4([34]). *An acyciloid A is matroidal if and only if $\mathcal{F}=\mathcal{B}^{\perp}$ and every elementary deletion of A is matroidal.*

Since faces and coboundaries are simply defined by topes, by this proposition, we obtain the following characterization.

Proposition 3.5. *An oriented matroid is a pair $M=(E, \mathcal{T})$ where E is a finite set and $\mathcal{T}$ is a nonempty set of signed vectors on E, satisfying* (T1)~(T3) *and*

(T5) *if X conforms to a tope and $\mathcal{T}(X)\backslash\underline{X}$ is closed under negativity, then $X\circ Y\in\mathcal{T}$ for all $Y\in\mathcal{T}$, and*

(T6) *every deletion of A satisfies* (T5).

In relation to Proposition 3.4, we proposed an open question and conjectured that it is negative.

Question([34]). *If an acycloid A satisfies $\mathcal{F}=\mathcal{B}^{\perp}$, then A is matroidal?*

Conjecture([34]). *There exists a non-matroidal acycloid satisfying $\mathcal{F}=\mathcal{B}^{\perp}$.*

Independently, da Silva [18] has also thought of the same problem above, although she does not use the terminology of acycloids. Contrary to our conjecture, she conjectures that the question has the affirmative answer.

Next we consider the problem to characterize tope graphs of oriented matroids. This problem is fundamental because its complete answer will lead to an axiomatization of oriented matroids using only the graph language. Note that such an axiomatization of linear oriented matroids is equivalent to that of the 1-skeltons of zonotopes, see e.g. [25] for zonotopes. In [28], Fukuda and the author characterized tope graphs of acycloids and those of oriented matroids of rank at most three.

A graph G, which contains at least one edge, is *antipodal* [5, 32] if for any $v\in V(G)$, there exists a unique $\bar{v}\in V(G)$, the *antipode* of v, such that $d_G(v, u)\leq d_G(v, \bar{v})$ for all neighbours u of $\bar{v}$. For convenience, we define a one-vertex graph K_1 is antipodal.

Theorem 3.6([28]). *A graph G is isomorphic to the tope graph of an acycloid if and only if G is an antipodal graph isometrically embeddable in some hypercube.*

Theorem 3.7([28]). *A graph G is isomorphic to the tope graph of an oriented matroid of rank at most three if and only if G is antipodal, planar and isometrically embeddable in some hypercube.*

This characterization enables us to test in a polynomial time whether a given graph is isomorphic to a graph representing adjacent relations of regions of an arrangement of pseudolines in the real projective palne $\mathbf{P}^2$, for arrangements, see e.g. [25, 33].

Finally, for L^1-systems and acycloids, we will define some similar concepts to those of oriented matroids. Let $A=(E, \mathcal{T})$ be an L^1-system or an acycloid. The sets of *faces*, *acyclons* and *circuits* of A are defined by

$$\mathcal{F} = \{X\colon X\circ Y\in\mathcal{T} \text{ for all } Y\in\mathcal{T}\}\ ,$$
$$\mathcal{A} = \{X\colon X\leq Y \text{ for some } Y\in\mathcal{T}\}\ , \text{ and}$$
$$\mathcal{C} = \mathrm{Min}\ \{X\colon X\not\leq Y \text{ for all } Y\in\mathcal{T}\}\ ,$$

respectively. By these definitions, we immediately obtain

$$\mathcal{T} = \mathrm{Max}\ \mathcal{A} = \mathrm{Max}\ \mathcal{F}\ ,$$
$$\mathcal{A} = \{X\colon X\not\geq Y \text{ for all } Y\in\mathcal{C}\}\ , \text{ and}$$
$$\mathcal{C} = \mathrm{Min}\ (\{-, 0, +\}^E-\mathcal{A})\ .$$

If A is an acylcoid, we have moreover

(1) $E_0=\{e\in E\colon \underline{X}=\{e\} \text{ for some } X\in\mathcal{C}\}$,
(2) $[e]=\{e\}\cup\{e'\in E\colon \{e, e'\}=\underline{X} \text{ for some } X\in\mathcal{C}\}$ $(e\in E-E_0)$,
(3) $\mathcal{F}=\{X\colon X*Y \text{ for all } Y\in\mathcal{C}\}$, and
(4) $\mathcal{C}=\mathrm{Min}\ \{X\colon X\neq\mathbf{0}, \text{ and } X*Y \text{ for all } Y\in\mathcal{F}\}$
$=\mathrm{Min}\ \{X\colon X\neq\mathbf{0}, \text{ and } X*Y \text{ for all } Y\in\mathcal{T}\}$.

To check the above relation between $\mathcal{F}$ and $\mathcal{C}$ is an easy exercise.

One can immediately verify that the set $\mathcal{F}$ of faces of an acycloid satisfies the conditions (F1) and (F2). By (F2), it follows that the poset $\hat{\mathcal{F}}=(\mathcal{F}\cup\{1\}, \leq)$ forms a lattice, the *face lattice* of A, where 1 is the greatest element. Unlike the case of oriented matroids, however, this lattice does not always satisfy the J-D chain property, see Example 7.1. Hence the notion of "rank" is not introduced in acycloids. Also, we do not define "vertices" of an acycloid and the "dual" of that. Because, as we will see in Example 7.2, the set Min $(\mathcal{F}-\{\mathbf{0}\})$

does not determine the acycloid.

§ 4. L^1-Systems and Convex Geometries

A function ϕ: $\mathcal{P}(E)\to\mathcal{P}(E)$, where E is a finite set, is called a *closure* (*operator*) if it satisfies

(1) $S\subseteq\phi(S)=\phi(\phi(S))$; and

(2) $R\subseteq S$ implies $\phi(R)\subseteq\phi(S)$,

for all $R, S\subseteq E$. A subset S of E is said to be *closed* if $\phi(S)=S$. For the purpose of simplicity, the empty set is assumed to be closed. A closure ϕ is said to be *anti-exchange* [21] if ϕ satisfies

(3) given a closed set S and two distinct elements e, f of $E-S$, then $e\in\phi(S\cup f)$ implies $f\notin\phi(S\cup e)$.

The anti-exchange closure is a generalization of the order ideals of a poset and it has many natural examples, such as the convex hull on finite points in $\mathbb{R}^n$, the transitive closure on the edges of an acyclic directed graph, the tree closure on the edges of a tree, etc, see [21]. The collection of closed sets of an anti-exchange closure has been studied under the name of a convex geometry.

A *convex geometry* [24] is a pair $(E, \mathbb{G})$ where E is a finite set and $\mathbb{G}\subseteq\mathcal{P}(E)$ satisfying

(G1) $\emptyset, E\in\mathbb{G}$;

(G2) $\mathbb{G}$ is closed under intersection; and

(G3) if $S\in\mathbb{G}$ and $S\neq E$, then there exists $f\in E-S$ such that $S\cup f\in\mathbb{G}$.

Anti-exchange closures ϕ on E and convex geometries $(E, \mathbb{G})$ are equivalent under the following correspondences:

$$\mathbb{G} = \text{the collection of closed sets of } \phi;$$
$$\phi(S) = \bigcap\{R\in\mathbb{G}\colon S\subseteq R\}\ (S\subseteq E).$$

In this section, we mainly show that a convex geometry is essentially equivalent to the set of positive "closed acyclons" of an acyclic L^1-system.

Given two signed vectors $X, Y\in\{-, 0, +\}^E$, define their *intersection* by

$$X\cap Y = (X_e \text{ if } X_e = Y_e, \text{ and } 0 \text{ otherwise}\colon e\in E).$$

Let $A=(E, \mathcal{I})$ be an L^1-system with acyclons $\mathcal{A}$. For an acyclon $X\in\mathcal{A}$, we define $cl(X)=\bigcap\mathcal{I}(X)$ and call it the *closure* of X. It is easy to see that this operator cl on $\mathcal{A}$ satisfies the following properties:

(i) $X\preceq cl(X)=cl(cl(X))$,

(ii) if $X \leq Y$, then $cl(X) \leq cl(Y)$.

In particular, $cl(\mathbf{0})=\mathbf{0}$. We say an acyclon $X \in \mathcal{A}$ is *closed* if $cl(X)=X$, and we denote by $\mathcal{D}$ the set of closed acyclons of A, i.e., $\mathcal{D}=\{X \in \mathcal{A}: cl(X)=X\}$. Then it is easily checked that $X^1, X^2 \in \mathcal{D}$ implies $X^1 \cap X^2 \in \mathcal{D}$. Hence for $X \in \mathcal{A}$, $cl(X)$ is the smallest closed acyclon to which X conforms, and we can describe $cl(X)=\cap \mathcal{D}(X)$. Note that $\mathcal{G}=\mathrm{Max}\, \mathcal{D}$ and $\mathcal{F} \subseteq \mathcal{D}$ hold.

Lemma 4.1. *For every $X \in \mathcal{D}$, there exists a unique minimal acyclon, denoted by $\hat{X}$, whose closure is X.*

Proof. Put $\mathcal{Y}=\mathrm{Min}\,\{Y \in \mathcal{A}: cl(Y)=X\}$. Since $X \in \mathcal{Y}$, $\mathcal{Y} \neq \emptyset$. Suppose that Y^1 and Y^2 are distinct two elements of $\mathcal{Y}$. Let $e \in \underline{Y}^1 - \underline{Y}^2$. By the minimality of Y^1, $cl(Y^1+e^0) \lneq X$. Since $cl(Y^1)=X$, $Y^1 \not\leq cl(Y^1+e^0)$. So there are $Z^1, Z^2 \in \mathcal{G}(cl(Y^1+e^0))$ such that $Z^1_e=-Z^2_e=X_e$. Since $Y^1 \leq Z^1$, $X=cl(Y^1) \leq Z^1$ holds. By the repeated application of the axiom (L1) to Z^2 and Z^1, we get $Z^3 \in \mathcal{G}(X+e^0)$ such that $Z^3_e=-X_e$. Then $Y^2 \leq X+e^0=X \cap Z^3 \in \mathcal{D}$, a contradiction. Hence $|\mathcal{Y}|=1$. This completes the proof. ∎

Now we consider the poset $L(\mathcal{D})=(\mathcal{D} \cup \{1\}, \leq)$, where 1 is an imaginary greatest element, i.e., an element such that $X \leq 1$ for all $X \in \mathcal{D}$. This poset $L(\mathcal{D})$ forms a lattice as in the following theorem. We show in Fig. 4.1 the lattice $L(\mathcal{D})$ of the L^1-system in Fig. 3.1 (a).

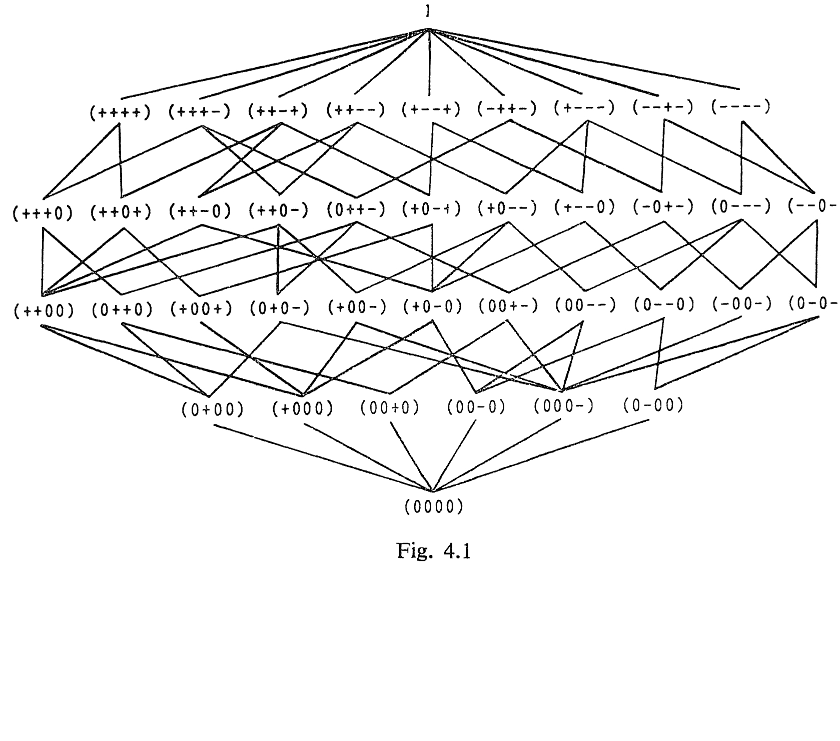


Fig. 4.1

Theorem 4.2. *Let $A=(E, \mathcal{T})$ be an L^1-system with closed acyclons $\mathcal{D}$. Then the poset $L(\mathcal{D})$ forms a coatomic lattice, in which the meet $X \wedge Y$ and the join $X \vee Y$ are defined by*

$$X \wedge Y = X \cap Y,$$
$$X \vee Y = \cap \{Z \in \mathcal{D} \cup \{1\} : X \leq Z \text{ and } Y \leq Z\}$$

for $X, Y \in \mathcal{D} \cup \{1\}$, where consider $X \wedge 1 = X$ for $X \in \mathcal{D} \cup \{1\}$. Moreover $L(\mathcal{D})$ has the J-D chain property and the height function h is given by $h(X) = |\underline{X}|$ for $X \in \mathcal{D}$.

Proof. It is clear that $L(\mathcal{D})$ is a coatomic lattice. By Lemma 4.1, it follows that for any $X \in \mathcal{D}$ and $e \in \underline{X}$, $X + e^0 \in \mathcal{D}$ if and only if $e \in \hat{\underline{X}}$. Hence if Y^1, $Y^2 \in \mathcal{D}$ satisfies $Y^1 < Y^2$, then since $\hat{Y}^2 \nleq Y^1$, there is $e \in \underline{Y^2} - \underline{Y^1}$ such that $Y^2 + e^0 \in \mathcal{D}$. Thus the second statement of the theorem follows. ■

In the case where A is a simple acycloid, the above theorem was proved by Tomizawa [50]. In simple acycloids, moreover, the set of atoms of $L(\mathcal{D})$ is given by $\mathrm{Min}(\mathcal{D} - \{0\}) = \{X \in \{-, 0, +\}^E : |\underline{X}| = 1\}$ and $L(\mathcal{D})$ is also atomic [50].

We denote the signed vectors $(++\cdots+)$ and $(--\cdots-)$ on E by $\tilde{+}$ and $\tilde{-}$ for short, respectively. Also we define, for $\mathcal{X} \subseteq \{-, 0, +\}^E$, $\mathcal{X}^+ = \{X \in \mathcal{X} : X^- = \emptyset\}$ and $\underline{\mathcal{X}} = \{\underline{X} : X \in \mathcal{X}\}$. An L^1-system A is *acyclic* if it has the positive tope $\tilde{+}$, or equivalently it has no positive circuits.

Proposition 4.3. *Let A be an acyclic L^1-system with closed acyclons $\mathcal{D}$. Then the pair $(E, \underline{\mathcal{D}^+})$ is a convex geometry.*

Proof. Immediate from Theorem 4.2. ■

Next, we show that every convex geometry is obtained this way. By the definition of convex geometries and by [21, Lemma 3.2], we have the following lemma.

Lemma 4.4. *Let $(E, \mathbf{G})$ be a convex geometry. Then the poset $L = (\mathbf{G}, \subseteq)$ is a lattice with the J-D chain property and the height function h satisfies $h(S) = |S|$ for all $S \in \mathbf{G}$.*

Proposition 4.5. *Let $(E\ \mathbf{G})$ be a convex geometry and put $\mathcal{G} = \{X \in \{-, +\}^E : X^+ \in \mathbf{G}\}$. Then the pair $A = (E, \mathcal{G})$ is an acyclic L^1-system with $\mathbf{G} = \underline{\mathcal{D}^+}$, where $\mathcal{D}$ is the set of closed acyclons of A.*

Proof. To show the condition (L1), let X, Y be distinct elements of $\mathcal{G}$. If $X^+ \subseteq Y^+$, then, by Lemma 4.4, there is $e \in Y^+ - X^+$ such that ${}_{\bar{e}}X \in \mathcal{G}$. Otherwise, consider the signed vector $Z = (X \cap Y) + D(X, Y)^-$. By (G2), $Z^+ = X^+ \cap Y^+ \in \mathbf{G}$, and so we have $Z \in \mathcal{G}$. By Lemma 4.4, there is $f \in X^+ - Z^+ \subseteq D(X, Y)$ such that ${}_{\bar{f}}X \in \mathcal{G}$. Thus $A = (E, \mathcal{G})$ satisfies (L1). Since $\mathbf{0}, \tilde{+} \in \mathcal{G}$, A satisfies (L2) and acyclic. Hence A is an acyclic L^1-system.

Next, we put $\mathcal{G}' = \{X \in \{0, +\}^E : X^+ \in \mathbf{G}\}$ and show that $\mathcal{G}' = \mathcal{D}^+$ holds. If $X \in \mathcal{G}'$, then $X \circ \tilde{-} \in \mathcal{G}$, so $X = (X \circ \tilde{-}) \cap \tilde{+} \in \mathcal{D}^+$. Conversely, if $X \in \mathcal{D}^+$, then for every $e_i \in E - \underline{X}$ there is $T^i \in \mathcal{G}$ such that $T^i_{e_i} = -$. By (G2), $\underline{X} = \cap_i (T^i)^+ \in \mathbf{G}$, and so $X \in \mathcal{G}'$. Hence $\mathcal{G}' = \mathcal{D}^+$, and hence $\mathbf{G} = \underline{\mathcal{D}^+}$. This completes the proof. ∎

Denote by $\mathcal{K}$ the set of all convex geometries, and by $\mathcal{K}_{L1}$ the set of convex geometries obtained from L^1-systems as in Proposition 4.3. As an immediate consequence of Propositions 4.3 and 4.5, we have

Theorem 4.6. $\mathcal{K}_{L1} = \mathcal{K}$.

For an acyclic oriented matroid M with circuits $\mathcal{C}$, Las Vergnas [37] defined the following closure, called the *convex hull* in M;

$$\mathrm{Conv}_M(S) = S \cup \{e \in E - S : \exists X \in \mathcal{C} \text{ such that } X^- = \{e\} \text{ and } X^+ \subseteq S\} \quad (S \subseteq E).$$

This closure is a generalization of the notion of convex hull in $\mathbf{R}^n$ and the closed sets are called the *convex sets* of M. Edelman showed in [22] that if M is simple, this closure is anti-exchange, and hence the convex sets of M froms a convex geometry.

The following proposition shows that the related anti-exchange closure of the convex geometry obtained from an L^1-system is a natural extension of the convex hull Conv_M in M above.

Proposition 4.7. *Let A be an acyclic L^1-system with closed acyclons $\mathcal{D}$ and circuits $\mathcal{C}$ and let $\underline{cl}$ be the anti-exchange closure associated with the convex geometry $(E, \underline{\mathcal{D}^+})$. Then, for all $S \subseteq E$, we have*

$$\underline{cl}(S) = S \cup \{e \in E - S : \exists X \in \mathcal{C} \textit{ such that } X^- = \{e\} \textit{ and } X^+ \subseteq S\}.$$

Proof. Denote the right hand of the equation by $\mathrm{Conv}_A(S)$. Since $\underline{cl}(\emptyset) = \mathrm{Conv}_A(\emptyset) = \emptyset$, we assume $S \neq \emptyset$. If $e \in \mathrm{Conv}_A(S) - S$, there is $X \in \mathcal{C}$ such that $X^- = \{e\}$ and $X^+ \subseteq S$. Hence $Y \in \mathcal{G}$ and $S \subseteq Y^+$ imply $Y_e = +$, and hence $e \in \underline{cl}(S)$. Thus $\mathrm{Conv}_A(S) \subseteq \underline{cl}(S)$. Conversely, let $e \in \underline{cl}(S) - S$. Since $\mathbf{0} + S^+$

$\in\mathcal{A}$ and $(\mathbb{0}+S^+)+e^-\notin\mathcal{A}$, there is $X\in\mathcal{C}$ such that $X^-=\{e\}$ and $X^+\subseteq S$. Hence $e\in\mathrm{Conv}_A(S)$. Thus $\underline{cl}(S)=\mathrm{Conv}_A(S)$ follows. ∎

What convex geometries airse from the convex sets of some acyclic simple oriented matroid [22, 24]? This open problem can be now restated as follows: characterize the set $\mathcal{K}_{om}$ of convex geometries obtained from simple oriented matroids as in Proposition 4.3.

Since the lattice $L(\mathcal{D})$ of closed acyclons of a simple acycloid is atomic, in particular, we have

Proposition 4.8. *If a convex geometry $(E, \mathbb{G})$ is an element of $\mathcal{K}_{om}$, then the lattice $(\mathbb{G}, \subseteq)$ is atomic.*

By this proposition, we know that the interval $[\mathbb{0}, \tilde{+}]$ of the lattice in Fig. 4.1 belongs to $\mathcal{K}_{L1}-\mathcal{K}_{om}$ $(=\mathcal{K}-\mathcal{K}_{om})$. An atomic example belonging to $\mathcal{K}-\mathcal{K}_{om}$ is given in [21]. The above mentioned problem is still open.

§ 5. Median Systems and Median Graphs

A graph G is *median* [1, 42, 44] if G is connected, and for any three vertices x, y, z there exists a unique vertex u such that u lies on a shortest (x, y)-path, a shortest (y, z)-path, and a shortest (z, x)-path. This vertex u is denoted by $m(x, y, z)$ and called the *median* of x, y and z. All trees, and all undirected Hasse diagrams of distributive lattices are median. Median systems, introduced in this section, axiomatize median graphs by signed vectors.

Before introducing median systems, we need the following theorem by Mulder [41, Thm. 1; Lemma 2]. We will present below a simple proof using Djoković's theorem (Theorem 3.2).

Theorem 5.1 (Mulder). *A graph G is median if and only if G is isometrically embeddable in some hypercube Q such that for any three vertices of G their median in Q is also a vertex of G.*

Proof. The if part is clear. Let G be a median graph. It is trivial that G is bipartite and connected. Let $[a, b]\in E(G)$ and suppose $C(a, b)\equiv\{x\in V(G)\colon d_G(a, x)<d_G(b, x)\}$ is not convex. Then there are two vertices $u, v\in C(a, b)$ such that some shortest (u, v)-path μ is not contained in $C(a, b)$. Choose such a shortest (u, v)-path μ so that the length of μ is minimum. Since all shortest paths between a and any vertex $x\in C(a, b)$ are contained in $C(a, b)$, we have $u\neq a$ and $v\neq a$. Also by the minimality of μ, there are $u', v'\in C(b, a)$

such that $[u, u'], [v, v'] \in E(G)$ and such that μ is denoted by $\mu = [u, u', \mu', v', v]$, where the subpath μ' is contained in $C(b, a)$.

Now consider the median $m(a, u, v)$. This vertex $m(a, u, v)$ is contained in $C(a, b)$ because it lies on a shortest (a, u)-path. Hence there is at least one shortest (u, v)-path τ in $C(a, b)$. Let w be the neighbour of v on τ and consider the median $m(u, w, v')$. Since G is bipartite, $d(w, v')=2$. By $d(u, w)<d(u, v)$ and by the choice of u and v, all shortest (u, w)-paths are contained in $C(a, b)$ and they are not through v. Hence we have $m(u, w, v')=w$, and so $d(u, v')= d(u, w)+d(w, v')$, which contradicts $d(u, v')=d(u, w)$. Thus $C(a, b)$ is convex. Therefore by the Djoković's theorem, G is isometrically embeddable in a hypercube Q. Now it is immediate that the median in Q of any three vertices of G is also a vertex of G. ∎

Note that every hypercube $Q(E)$ is median, and that the median of $X, Y, Z \in V(Q(E))=\{-, +\}^E$ is the signed vector U such that, for all $e \in E$, $U_e=i$ if and only if at least two of X_e, Y_e and Z_e are i. We will denote this signed vector U by $\langle X, Y, Z \rangle$.

A *median system* is a pair $A=(E, \mathcal{T})$ where E is a finite set and $\mathcal{T}$ is a non-empty set of elements of $\{-, +\}^E$, called the *topes* of A, satisfying

(M1) $X, Y, Z \in \mathcal{T}$ implies $\langle X, Y, Z \rangle \in \mathcal{T}$; and

(M2) A is simple, i.e., for every $e \in E$, there exist $X, Y \in \mathcal{T}$ such that $X_e \neq Y_e$, and for every distinct $e, f \in E$, there exist $X, Y \in \mathcal{T}$ such that $X_e = X_f$ and $Y_e = -Y_f$.

We define for convenience that $A=(\emptyset, \{(\ )\})$ is a median system. A connected graph G is *diametrical* [41] if each vertex v of G has a unique vertex v' such that $d_G(v, v')$ equals the diameter of G. Every antipodal graph is diametrical, see [5]. By [41, Cor. 5], a graph G is isomorphic to the hypercube $Q(E)$ if and only if G is diametrical and median with diameter $|E|$. Since the tope graph of an acycloid is antipodal, it follows that the pair $(E, \mathcal{T}=\{-, +\}^E)$ is the only

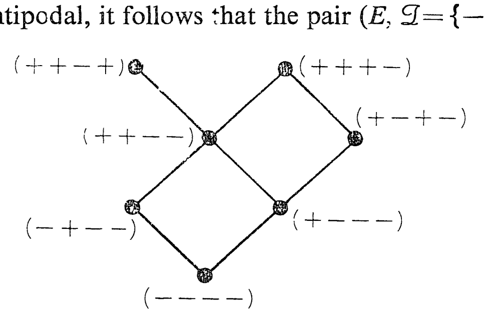


Fig. 5.1

acycloid (oriented matroid) on E which is a median system on E.

For a median system $A=(E, \mathcal{T})$ and a subset $S\subseteq E$, the pair $A-S=(E-S, \mathcal{T}-S)$, where $\mathcal{T}-S=\mathcal{T}\backslash S$, is also a median system, called the *deletion* of A *by* S. The *tope graph* G_A of a median system A is a graph with $V(G_A)=\mathcal{T}$ and $E(G_A)=\{[X, Y]: X, Y\in\mathcal{T} \text{ and } |D(X, Y)|=1\}$. An example of such a graph is given in Fig. 5.1.

Proposition 5.2. *Every median system $A=(E, \mathcal{T})$ is an L^1-system.*

Proof. Suppose that there are topes $X, Y\in\mathcal{T}$ which do not satisfy the condition (L1), and we choose such X, Y so that $|D(X, Y)|$ is minimum. Then there are two distinct elements $f, g\in D(X, Y)$. By (M2), there is $Z\in\mathcal{T}$ such that either $Z_f=X_f$ and $Z_g=Y_g$ or $Z_f=Y_f$ and $Z_g=X_g$. By the minimality of $|D(X, Y)|$, X and $\langle X, Y, Z\rangle$ ($\in\mathcal{T}$ by (M1)) satisfy (L1). Hence there is $h\in D(X, \langle X, Y, Z\rangle)\subseteq D(X, Y)$ such that ${}_{\bar{h}}X\in\mathcal{T}$, a contradiction. Therefore A satisfies (L1). By (M2), A satisfies (L2), too. ■

Median systems are equivalent to median graphs by the following proposition (cf. Proposition 3.1).

Proposition 5.3. *A graph G is isomorphic to the tope graph of a median system if and only if G is median.*

Proof. If G is a median graph and if f: $G\to Q(E)$ is an isometric ebmedding, where E is minimal for this property, then $A_G=(E, f(V(G))$ is a median system and the tope graph of A_G is exactly G. The median of x, y and $z\in V(G)$ in G corresponds to the signed vector $\langle f(x), f(y), f(z)\rangle$ by Theorem 5.1.

Conversely, if a graph G is isomorphic to the tope graph G_A of a median system $A=(E, \mathcal{T})$, then by Proposition 5.2, G is isometrically embeddable in $Q(E)$. Moreover, by (M1), G is median. ■

Let $A=(E, \mathcal{T})$ be an L^1-system. Let $\mathcal{T}_1, \mathcal{T}_2\subseteq\mathcal{T}$ be such that $\mathcal{T}_1\cup\mathcal{T}_2=\mathcal{T}$ and $\mathcal{T}_1\cap\mathcal{T}_2\neq\emptyset$, and such that for any $X\in\mathcal{T}_1-\mathcal{T}_2$ there exists no $e\in E$ with ${}_{\bar{e}}X\in\mathcal{T}_2-\mathcal{T}_1$. Let $p\notin E$, and put $\mathcal{T}'=\{X+p^+: X\in\mathcal{T}_1\}\cup\{X+p^-: X\in\mathcal{T}_2\}$, where $X+p^i$ ($i\in\{-, +\}$) denotes the signed vector Z on $E\cup\{p\}$ with $Z_p=i$ and $Z_e=X_e$ for other element e. (Note that the notation $X+p^i$ can be used for either cases of $p\in E$ and $p\notin E$.) Then we call the pair $A'=(E\cup\{p\}, \mathcal{T}')$ the *expansion* of A with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. The expansion A' is called *convex* if there exist closed acyclons X^1 and X^2 of A such that $\mathcal{T}_1=\mathcal{T}(X^1)$ and $\mathcal{T}_2=\mathcal{T}(X^2)$.

Lemma 5.4. *If $A=(E, \mathcal{T})$ is a median system, any convex expansion A' of A is a median system.*

Proof. We use the same notations $\mathcal{T}_1$, $\mathcal{T}_2$, p, $\mathcal{T}'$, X^1, X^2 as in the above definitions, and denote by $\mathcal{D}$ the set of closed acyclons of A. Clearly, A' satisfies (M2). To show (M1), let X', Y', $Z' \in \mathcal{T}'$. Then we may assume, without loss of generality, that at least two of X', Y' and Z', say X' and Y', are elements of the set $\{X+p^+: X \in \mathcal{T}_1\}$. Now for some X, $Y \in \mathcal{T}_1$, $X'=X+p^+$ and $Y'=Y+p^+$. Let $Z'=Z+p^i$ ($Z \in \mathcal{T}$, $i \in \{-, +\}$). Since $X^1 \leq X \cap Y$, $X^1 \leq \langle X, Y, Z \rangle$, so $\langle X, Y, Z \rangle \in \mathcal{T}_1$. Hence $\langle X', Y', Z' \rangle = \langle X, Y, Z \rangle + p^+ \in \mathcal{T}'$. This completes the proof. ∎

Proposition 5.5. *A pair $A=(E, \mathcal{T})$ of a finite set E and $\emptyset \neq \mathcal{T} \subseteq \{-, +\}^E$ is a median system if and only if A can be obtained from the smallest median system $(\emptyset, \{(\ )\})$ by a sequence of convex expansions.*

Proof. The elementary deletion of a median system A is exactly the converse operation to convex expansion. Hence by induction the only if part is proved. The if part follows by Lemma 5.4. ∎

Let G be a graph. For X, $Y \subseteq V(G)$, $[X, Y]$ denotes the set of edges with one endpoint in X and the other in Y. Now let V_1, $V_2 \subseteq V(G)$ satisfy $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 \neq \emptyset$ and $[V_1 - V_2, V_2 - V_1] = \emptyset$. The *expansion* [40] of G with respect to V_1 and V_2 is the graph G' constructed as follows:

(i) replace each vertex $v \in V_1 \cap V_2$ by two vertices u_v, u'_v, which are joined by an edge;

(ii) join u_v to the neighbours of v in $V_1 - V_2$ and u'_v to those in $V_2 - V_1$;

(iii) if v, $w \in V_1 \cap V_2$ and $[v, w] \in E(G)$, then join u_v to u_w and u'_v to u'_w.

The expansion G' is called *convex* if V_1 and V_2 are convex subsets of $V(G)$.

Lemma 5.6. *In the tope graph G_A of an L^1-system $A=(E, \mathcal{T})$, a set $\mathcal{X}$ of vertices (topes) is convex if and only if $\mathcal{X} = \mathcal{T}(X)$ for some closed acyclon X of A.*

Proof. The if part is clear. To show the only if part, let $\mathcal{X} \subseteq \mathcal{T}$ be convex in G_A and put $X = \cap \mathcal{X}$. Then X is a closed acyclon of A and $\mathcal{X} \subseteq \mathcal{T}(X)$. Now suppose $\mathcal{X} \neq \mathcal{T}(X)$. Then we can choose $Y \in \mathcal{T}(X) - \mathcal{X}$ such that ${}_{\bar{e}}Y \in \mathcal{X}$ for some $e \in E - \underline{X}$. Since $e \in E - \underline{X}$, there is $U \in \mathcal{X}$ such that ${}_{\bar{e}}U \in \mathcal{X}$ and $U_e = Y_e$. Since A is an L^1-system, Y lies on a shortest $(U, {}_{\bar{e}}Y)$-path in G_A, which contradicts the convexity of $\mathcal{X}$. This completes the proof. ∎

By this lemma and Proposition 5.5, we immediately obtain

Theorem 5.7(Mulder [40]). *A graph G is median if and only if G can be obtained from a one-vertex graph K_1 by a sequence of convex expansions.*

Let $A=(E,\mathcal{T})$ be a pair with a finite set E and $\emptyset\neq\mathcal{T}\subseteq\{-,+\}^E$. For $X, Y\in\mathcal{T}$, we define $I_A(X, Y)=\{Z\in\mathcal{T}: D(X, Z)\subseteq D(X, Y)\}$, called the *interval* between X and Y. The index A of $I_A(X, Y)$ is often omitted when it is clear from the context. Note that if $\langle X, Y, Z\rangle$ is an element of $\mathcal{T}$ then it is the unique element of $I_A(X, Y)\cap I_A(Y, Z)\cap I_A(Z, X)$.

Proposition 5.8. *An L^1-system $A=(E,\mathcal{T})$ is median if and only if A satisfies*

(M3) *if $e\in E$ and $X, Y\in\mathcal{T}$ satisfy $X_e=Y_e$ and ${}_{\bar e}X, {}_{\bar e}Y\in\mathcal{T}$, then $Z\in I_A(X, Y)$ implies ${}_{\bar e}Z\in\mathcal{T}$.*

Proof. Let A be a median system. Let $e\in E$ and $X, Y\in\mathcal{T}$ satisfy $X_e=Y_e$ and ${}_{\bar e}X, {}_{\bar e}Y\in\mathcal{T}$, and let $Z\in I(X, Y)$ $(=I_A(X, Y))$. Then ${}_{\bar e}Z=\langle {}_{\bar e}X, {}_{\bar e}Y, Z\rangle\in\mathcal{T}$. Hence A satisfies (M3).

Conversely, let an L^1-system A satisfy (M3). Suppose there are $X, Y, Z\in\mathcal{T}$ such that $\langle X, Y, Z\rangle\notin\mathcal{T}$, and choose such X, Y and Z so that Min $(|D(X, Y)|, |D(Y, Z)|, |D(Z, X)|)$ is minimum. We may assume $|D(X, Y)|$ is minimum. If the set-inclusion relation holds for some two of $D(X, Y)$, $D(Y, Z)$ and $D(Z, X)$, then $\langle X, Y, Z\rangle$ equals one of X, Y or Z, a contradiction. Hence there is $e\in E$ such that $X_e=-Y_e=Z_e$. If there is $P\in I(X, Y)\cap I(Y, Z)-\{Y\}$, $|D(X, P)|<|D(X, Y)|$. By the minimality of $|D(X, Y)|$, $\langle X, P, Z\rangle\in\mathcal{T}$. Since $\langle X, P, Z\rangle\in I(X, P)\cap I(P, Z)\cap I(Z, X)\subseteq I(X, Y)\cap I(Y, Z)\cap I(Z, X)$, $\langle X, Y, Z\rangle=\langle X, P, Z\rangle\in\mathcal{T}$, a contradiction. Hence $I(X, Y)\cap I(Y, Z)=\{Y\}$. Now let $X'\in I(X, Y)$ with $X'_e=X_e$ and ${}_{\bar e}X'\in\mathcal{T}$, and let $Z'\in I(Y, Z)$ with $Z'_e=Z_e$ and ${}_{\bar e}Z'\in\mathcal{T}$. (The existence of X' and Z' is by the condition (L1).) Then by the minimality of $|D(X, Y)|$, $\langle {}_{\bar e}X', {}_{\bar e}Z', Y\rangle\in\mathcal{T}$. Since $I({}_{\bar e}X', Y)\cap I(Y, {}_{\bar e}Z')=\{Y\}$, $\langle {}_{\bar e}X', {}_{\bar e}Z', Y\rangle=Y$ and $Y\in I({}_{\bar e}X', {}_{\bar e}Z')$. By (M3), ${}_{\bar e}Y\in\mathcal{T}$. Hence ${}_{\bar e}Y\in I(X', Y)\cap I(Y, Z')\subseteq I(X, Y)\cap I(Y, Z)$, a contradiction. Thus A satisfies (M1), and clearly (M2). Therefore A is median. ∎

The next theorem is similar to Djoković's theorem that characterizes the graphs isometrically embeddable in a hypercube.

Theorem 5.9. *A graph G is median if and only if G satisfies*

(1) *G is connected bipartite, and*

(2) *$U(a, b)$ is convex for all $[a, b] \in E(G)$, where $U(a, b)$ is the set of all vertices x such that $d_G(a, x) < d_G(b, x)$ and x is adjacent to some vertex y with $d_G(a, y) > d_G(b, y)$.*

Proof. The only if part is proved in [40, 44]. But here we will prove it by another way. By Proposition 5.3, a median graph G is isomorphic to the tope graph G_A of a median system A. Hence G is connected bipartite and, by Proposition 5.8, $U(X, Y)$ is convex for all $[X, Y] \in E(G_A)$. This proves the only if part.

Conversely, let a graph G satisfy the conditions (1) and (2). For $[a, b] \in E(G)$, if $U(a, b)$ is convex, so is $C(a, b)$. Hence, by Djoković's theorem, G is isometrically embeddable in some hypercube. The L^1-system A_G corresponding to G satisfies the condition (M3) by (2). Hence, by Proposition 5.8, A_G is median. This proves the if part. ∎

We proved Theorems 5.7 and 5.9 by using median systems. Such proofs are indeed shorter and easier to understand than direct proofs by the graph language or properties. In the following, we will show another example like this.

The next proposition is useful, which contains the fact that the "face lattice" $\hat{\mathcal{F}} = (\mathcal{F} \cup \{1\}, \leq)$ of a median system satisfies the J-D chain property.

Proposition 5.10. *Let $A = (E, \mathcal{T})$ be an L^1-system with faces $\mathcal{F}$ and closed acyclons $\mathcal{D}$. Then A is median if and only if $\mathcal{F} = \mathcal{D}$ holds.*

Proof. Let A be median, and suppose that there is $X \in \mathcal{D} - \mathcal{F}$. By $X \notin \mathcal{F}$, there is $Y^1 \in \mathcal{T}$ such that $X \circ Y^1 \notin \mathcal{T}$. Put $Z = X \circ Y^1$ and choose $Y^2 \in \mathcal{T}(X)$ so that $D(Z, Y^2)$ is minimal. Since $X \in \mathcal{D}$, for any $e \in D(Z, Y^2)$, there is $Y^3 \in \mathcal{T}(X)$ such that $Y^3_e = -Y^2_e$. Now put $Y = \langle Y^1, Y^2, Y^3 \rangle$, then $Y \in \mathcal{T}(X)$ and $D(Z, Y) \subset D(Z, Y^2)$. This contradicts the minimality of $D(Z, Y^2)$. Thus $\mathcal{F} = \mathcal{D}$ holds. Conversely, if $\mathcal{F} = \mathcal{D}$ holds, then for any $X^1, X^2, X^3 \in \mathcal{T}$, $X^1 \cap X^2 \in \mathcal{D} = \mathcal{F}$, and so $\langle X^1, X^2, X^3 \rangle = (X^1 \cap X^2) \circ X^3 \in \mathcal{T}$. Hence A is median. ∎

Now, extend the definition of $\langle X, Y, Z \rangle$, $X, Y, Z \in \{-, +\}^E$, to the case of any odd numbered elements of $\{-, +\}^E$. That is, for $X^1, X^2, \cdots, X^{2k+1} \in \{-, +\}^E$, $k \geq 0$, $\langle X^1, X^2, \cdots, X^{2k+1} \rangle$ denotes the signed vector U such that, for all $e \in E$, $U_e = i$ if and only if at least $k+1$ of $X^1_e, X^2_e, \cdots, X^{2k+1}_e$ are i. We have

Proposition 5.11. *Let $A = (E, \mathcal{T})$ be a median system and let $X^1, X^2, \cdots,$*

$X^{2k+1}\in\mathcal{T}$. *Then* $\langle X^1, X^2, \cdots, X^{2k+1}\rangle\in\mathcal{T}$.

Proof. The vector $\langle X^1, X^2, \cdots, X^{2k+1}\rangle$ is the composition of all vectors $X^{(K)}$ such that $X^{(K)}=\bigcap_{i\in K} X^i$ for some $(k+1)$-element subset $K\subseteq\{1, 2, \cdots, 2k+1\}$. Here note that this composition does not depend on the order of vectors $X^{(K)}$. By Proposition 5.10, $X^{(K)}\in\mathcal{F}$. In general, in any L^1-system, the set of faces is closed under composition. Hence $\langle X^1, X^2, \cdots, X^{2k+1}\rangle\in\mathcal{T}$. ∎

For vertices $v_1, v_2, \cdots, v_p$ of a connected graph G, any vertex m that minimizes the sum $\sum_{i=1}^{p} d(x, v_i)$, $x\in V(G)$, is called a *median* of $v_1, v_2, \cdots, v_p$. Note that a median graph is a connected one in which any three vertices admit a unique median. It is clear that Proposition 5.11 proves the following result by Bandelt and Barthélemy [3]: a connected graph G is median if and only if each odd numbered family of vertices in G admits a unique median.

§ 6. Perturbation of Acycloids and L^1-Systems

Perturbation of oriented matroids, studied by Edmonds, Fukuda and Mandel [27, 39], is an operation which transfers an oriented matroid on E to another oriented matroid on E. It can be also viewed as a topological operation to locally deform a hypersphere, see [39]. Several different types of perturbations of oriented matroids were studied in [27, 39]. Here we extend the "point perturbation theorem" to acycloids and L^1-systems.

Theorem 6.1(Point perturbation theorem [27, 39]). *Let* $M=(E, \mathcal{F})$ *be an oriented matroid. Let* $f\in E$ *and let* V *be a vertex of* M *satisfying* $V_f=0$ *and* $X+f^+\in\mathcal{F}$ *for all* $X\in\mathcal{F}$ *with* $V<X$ *and* $X_f=0$. *Then* $M'=(E, \mathcal{F}')$ *is an oriented matroid, where*

$$\begin{aligned}\mathcal{F}' &= \mathcal{F}-\{V, -V\}\cup\{\hat{V}, -\hat{V}\}\cup\mathcal{N}\cup-\mathcal{N}\,,\\ \hat{V} &= V+f^+,\quad \mathcal{N}={}^0\mathcal{N}\cup{}^+\mathcal{N}\,,\\ {}^i\mathcal{N} &= \{X+f^i\colon V\lesssim X\in\mathcal{F}\ \text{and}\ X_f=-\}\quad (\text{for } i=0,+)\,.\end{aligned}$$

Corollary 6.2. *Under the same assumptions as in Theorem* 6.1, *let* $\mathcal{T}$ *be the set of topes of* M, *and let*

$$\mathcal{U}=\{{}_{\bar f}X\colon V\lesssim X\in\mathcal{T}\ \text{and}\ X_f=-\}\,.$$

Then the set $\mathcal{T}'=\mathcal{T}\cup\mathcal{U}\cup-\mathcal{U}$ *is the set of topes of the new oriented matroid* M'.

It is not too difficult to see that one can obtain a nonlinear oriented matroid from a linear one using point perturbations. Let's show an example. The "Pappus oriented matroid" is represented by the sphere system such that, in Fig. 2.1 (b), the hypersphere s_3^0 is through the points A'', B'', C'' and their opposite points on the hemisphere s_2^-. Now, take $f=3$ and $V=\sigma(C'')=(-+00-0+-+)$ in the Pappus oriented matroid. Then by Theorem 6.1 we have the Non-Pappus oriented matroid (sphere system) in Fig. 2.1 (b). Topologically speaking, the Non-Pappus sphere system is obtained by slightly pushing down the hyperplane s_3^0 around the point C'' (and pushing it up around the opposite point $-C''$ on s_2^-). Another interesting example is a construction of Non-Bom's, a certain class of nonlinear oriented matroids that are of special importance in oriented matroid programming, see [27]. For the reverse operation of perturbation, see [49].

Now we show that point perturbations of oriented matroids have natural extensions in acycloids.

Theorem 6.3([29]). *Let $A=(E, \mathcal{T})$ be an acycloid, let $f\in E$ and let W be a face of A satisfying $W_f=0$. Let*

$$\mathcal{U} = \{{}_{\bar f}X\colon X\in\mathcal{T}(W), X_f = -\}\ .$$

Then $A'=(E, \mathcal{T}')$, where $\mathcal{T}'=\mathcal{T}\cup\mathcal{U}\cup-\mathcal{U}$, is an acycloid.

Proof. The set of loops of A' is the same as that of A. Denote by $[\cdot]$ a parallel class of A. If f is a loop of A or $|[f]|=1$, then the set of parallel classes of A' is the same as that of A. Otherwise both $[f]-f$ and f are parallel classes of A', and the parallel classes of A' except these are the same as those of A except $[f]$.

If f is a loop of A, then clearly $A'=A$, so assume that f is not a loop of A. Let $\mathcal{T}'=\mathcal{T}\cup\mathcal{U}'\cup-\mathcal{U}'$ be a disjoint union, where $\mathcal{U}'\subseteq\mathcal{U}$. Since the axioms (T1) and (T2) are clearly satisfied, it is enough to show that (T3) is satisfied. Let $X, Y\in\mathcal{T}'$ and $X\neq Y$. Put $D=D(X, Y)$, $X'={}_{\bar f}X$ and $Y'={}_{\bar f}Y$. For convenience, put $\mathcal{T}(W)_{\bar f}=\{X\in\mathcal{T}(W)\colon X_f=-\}$. We first check the following 5 cases (0)-(iv).

(0) $X, Y\in\mathcal{T}$: Trivial.

(i) $X, Y\in\mathcal{U}'$: $X', Y'\in\mathcal{T}(W)_{\bar f}$. Applying (T3) to X' and Y', there is $[g]\subseteq D(X', Y')=D$ such that ${}_{\overline{[g]}}X'\in\mathcal{T}$. Here note that $[g]\neq[f]$. Since ${}_{\overline{[g]}}X'\in\mathcal{T}(W)_{\bar f}$, ${}_{\overline{[g]}}X={}_{\bar f}({}_{\overline{[g]}}X')\in\mathcal{U}$. Thus [g] is our required parallel class of A'.

(ii) $X\in\mathcal{U}'$, $Y\in-\mathcal{U}'$: $f\in D$ and ${}_{\bar f}X\in\mathcal{T}$.

(iii) $X\in\mathcal{T}$, $Y\in\mathcal{U}'$: $Y'\in\mathcal{T}(W)_{\bar f}$.

(iii-1) $X_f=-$: Then note that $f\in D$. If $X=Y'$, ${}_{\bar f}X=Y\in\mathcal{U}'$. Otherwise there is $[g]\subseteq D(X, Y')\subseteq D$ such that ${}_{\overline{[g]}}X\in\mathcal{T}$.

(iii-2) $X_f=+$ and $X\in\mathcal{T}(W)$: If there is $[g]\subseteq D(X, Y')-[f]\subseteq D$ such that ${}_{\overline{[g]}}X\in\mathcal{T}$, there is nothing to do. Otherwise ${}_{\overline{[f]}}X\in\mathcal{T}$ holds. Then if $|[f]|=1$, there is $[h]\subseteq D({}_{\bar f}X, Y')=D$ such that ${}_{\overline{[h]}}({}_{\bar f}X)\in\mathcal{T}(W)_{\bar f}$, i.e. ${}_{\overline{[h]}}X\in\mathcal{U}$, and if $|[f]|\geq 2$, ${}_{\overline{([f]-f)}}X={}_{\bar f}({}_{\overline{[f]}}X)\in\mathcal{U}$.

(iii-3) $X_f=+$ and $X\notin\mathcal{T}(W)$: Put $Z=W\circ X(\in\mathcal{T})$. Applying (T3) to X and Z, there is $[g]\subseteq D(X, Z)\subseteq D$ such that ${}_{\overline{[g]}}X\in\mathcal{T}$.

(iv) $X\in\mathcal{U}'$, $Y\in\mathcal{T}$: $X'\in\mathcal{T}(W)_{\bar f}$.

(iv-1) $Y_f=-$: $f\in D$ and ${}_{\bar f}X=X'\in\mathcal{T}$.

(iv-2) $Y_f=+$ and $Y\in\mathcal{T}(W)$: There is $[g]\subseteq D(X', Y)=D\cup\{f\}$ such that ${}_{\overline{[g]}}X'\in\mathcal{T}$. If $|[f]|=1$, clearly $[g]\neq[f]$, and so ${}_{\overline{[g]}}X'\in\mathcal{T}(W)_{\bar f}$, i.e. ${}_{\overline{[g]}}X={}_{\bar f}({}_{\overline{[g]}}X')\in\mathcal{U}$. When $|[f]|\geq 2$, if $[g]=[f]$, ${}_{\overline{([f]-f)}}X={}_{\overline{[f]}}X'\in\mathcal{T}$, otherwise ${}_{\overline{[g]}}X={}_{\bar f}({}_{\overline{[g]}}X')\in\mathcal{U}$.

(iv-3) $Y_f=+$ and $Y\notin\mathcal{T}(W)$: Put $Z=W\circ Y(\in\mathcal{T})$. Applying case (iv-2) with $Y=Z$, since $D(X, Z)\subseteq D$ this case is proved. The remaining cases are (i') $X, Y\in-\mathcal{U}'$, (ii') $X\in-\mathcal{U}'$, $Y\in\mathcal{U}'$, (iii') $X\in\mathcal{T}$, $Y\in-\mathcal{U}'$, and (iv') $X\in-\mathcal{U}'$, $Y\in\mathcal{T}$. Each case follows immediately from the corresponding case considered above. ∎

Proposition 6.4 ([29]). *Under the same assumptions as in Theorem* 6.3, *if A is matroidal, $0\neq W\in\mathcal{F}-\mathcal{V}$ (where $\mathcal{F}$ and $\mathcal{V}$ are the sets of faces and vertices of A) and $A'\neq A$, then A' is non-matroidal.*

Proof. Since $0\neq W\in\mathcal{F}-\mathcal{V}$, there is $V\in\mathcal{V}$ such that $V<W$. Put $W'=(-V)\circ W$ $(\in\mathcal{F})$. By Proposition 3.4, W' is a coboundary of A. Moreover, by the fact that $W'\not\leq X$ for all $X\in\mathcal{T}'-\mathcal{T}$, W' is also a coboundary of A'. If A' is matroidal, by Proposition 3.4, W' is a face of A'. Then choose any $Y\in\mathcal{U}-\mathcal{T}$ and put $Z=W'\circ Y(\in\mathcal{T}')$. Since $Z_f=+$ and $Z\not\geq W$, $Z\in\mathcal{T}$. Hence we have $Y=W\circ Z\in\mathcal{T}$, a contradiction. Therefore A' is non-matroidal. ∎

For example, let M be the oriented cycle matroid of the graph in Fig. 6.1. Now $E=\{1, 2, 3, 4, 5\}$ and $\mathcal{T}=\{(+++++), (++++-), (-++++), (++-+-), (-+-++), (-+-+-), (+++-+), (+++--), (-++-+), (++---), (-+--+), (-+---)$, and their negatives$\}$. Take $f=3$ and $W=(0+0+0)$ in Theorem 6.3. Then we have $\mathcal{U}=\{(-+++-), (-++++), (++++-)\}$. Here note that W is not a vertex of M because $W>(0+000)\in\mathcal{F}$. By Proposition 6.4, we know that the new acycloid A' is non-matroidal. Indeed, A' equals the acycloid A_3 in Example 7.2. We can see in [29] the topological representation for this example, which is similar to that of an oriented matroid. In general, however, a way to topologically represent acycloids is not known.

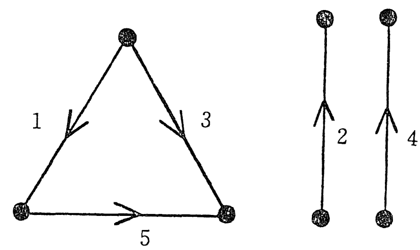


Fig. 6.1

It is easy to see that Theorem 6.3 can be extended to L^1-systems as follows.

Proposition 6.5. *Let $A=(E, \mathcal{T})$ be an L^1-system, let $f \in E$ and let W be a face of A satisfying $W_f=0$. Let*

$$\mathcal{U} = \{{}_{\bar{f}}X\colon X \in \mathcal{T}(W),\ X_f = -\} .$$

Then $A'=(E, \mathcal{T}')$ where $\mathcal{T}'=\mathcal{T} \cup \mathcal{U}$, is an L^1-system.

For example, let A be the L^1-system in Fig. 3.1 (a), and take $f=3$ ($E=\{1, 2, 3, 4\}$) and $W=\mathbf{0}$. Then $\mathcal{T}'-\mathcal{T}=\{(+--+-), (+--++)\}$, and the tope graph of the new L^1-system A' equals the graph in Fig. 6.2.

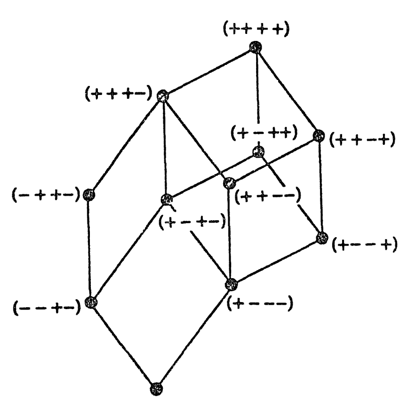


Fig. 6.2

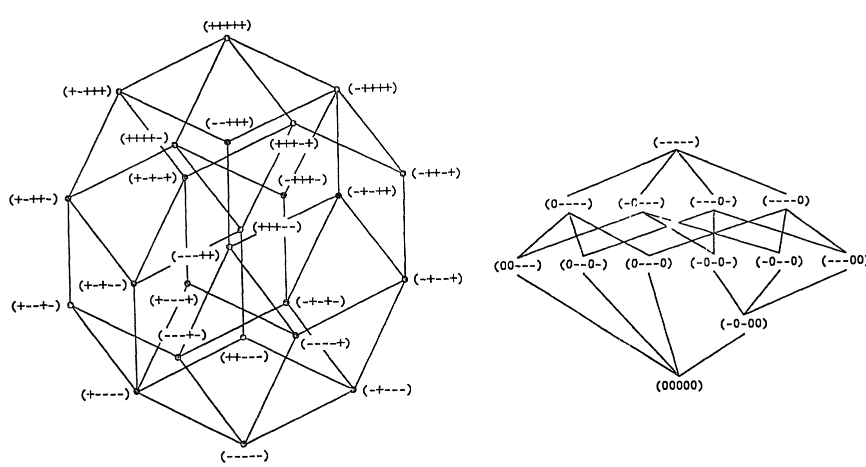
Remark that Proposition 6.5 gives an operation to construct graphs isometrically embeddable in the same hypercube $Q(E)$. (All such graphs cannot be constructed this way, e.g. a cycle of length 6.) It is interesting that this proposition was obtained not from graph theory but from oriented matroid theory.

§ 7. Non-matroidal Acycloids

In this section, we first give some examples of non-matroidal acycloids to show properties contrary to oriented matroids. Then we show that there are exactly 4 non-matroidal acycloids on the 5-element set up to reorientation.

An example of non-matroidal acycloid was first discovered by Fukuda, see [50] and [28, Fig.1 (b)]. We will refer to the Fukuda's example as $A_1=(\{1,2,\cdots,5\},\mathcal{T}_1)$.

Example 7.1. $A_2=(\{1,2,\cdots,5\},\mathcal{T}_2)$, where $\mathcal{T}_2=\mathcal{T}_1\cup\{(-+-++),(+-+--)\}=V(G_{A_2})$. Here G_{A_2} is the tope graph of A_2 and shown in Fig.7.1 (a). The set $\mathcal{B}_2^{+}-\mathcal{F}_2$ of vectors X such that X is not a face but a coboundary is $\{(0+00-),(0+0-0),(000+-)$, and their negatives$\}$. The poset in Fig. 7.1 (b) is an interval of the face lattice $\hat{\mathcal{F}}_2$ of A_2.



(a) (b)

Fig. 7.1

Example 7.2 ([34]). $A_3=(\{1,2,\cdots,5\},\mathcal{T}_3)$, where $\mathcal{T}_3=\mathcal{T}_2\cup\{(++-+-),(--+-+)\}$. Then $\mathcal{T}_3/1/2/3=\{(+0),(-0)\}$ and $\mathcal{T}_3/1/3/2=\{(00)\}$. For the set $\mathcal{F}_3$ of faces of A_3, $\mathrm{Min}\,(\mathcal{F}_3-\{\mathbf{0}\})=\{(+0+00),(00+0+),(+000-),(0++0+0),(0+0+-),(-+0+0)\}$. This set equals the set of cocircuits of the oriented cycle matroid of the graph in Fig. 7.2.

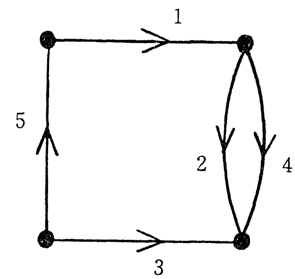


Fig. 7.2

Example 7.3. $A_4=(E, \mathcal{T}_4)$, where $E=\{1, 2, \cdots, 6\}$ and $\mathcal{T}_4=\{-, +\}^E-\{(-+++++), (+---+++)$, and their negatives$\}$. The set of circuits of A_4 is $\{(-+++++), (+---+++)$, and their negatives$\}$, which does not satisfy the axiom (O2').

Next, consider non-matroidal acycloids on the 5-element set.

Proposition 7.4. *If an acycloid $A=(E, \mathcal{T})$ is non-matroidal, then $|E|\geq 5$.*

Proof. Let $\mathcal{C}$ be the set of circuits of A. If A is non-matroidal, there exist $X, Y\in\mathcal{C}$ such that $X\neq -Y$, and $f\in D(X, Y)$, such that the axiom (O3) is not satisfied. First we show $|D(X, Y)|\geq 3$. It is trivial to show $|D(X, Y)|\neq 1$. Now suppose $|D(X, Y)|=2$. Since $(X\circ Y)+f^0$ and $(Y\circ X)+f^0$ are acyclons of A, there are $Z^1, Z^2\in\mathcal{T}$ such that $(X\circ Y)+f^0\lesssim Z^1$ and $(Y\circ X)+f^0\lesssim Z^2$. Here note that we have $Z^1_f=-X_f$ and $Z^2_f=-Y_f$. By repeated applications of (T3) to Z^1 and Z^2, we get eventually $Z\in\mathcal{T}$ such that $X\lesssim Z$ or $Y\lesssim Z$, a contradiction. Hence $|D(X, Y)|\neq 2$. Thus $|D(X, Y)|\geq 3$.

Case (i): $\underline{X}\subseteq\underline{Y}$ or $\underline{X}\supseteq\underline{Y}$. We may assume from symmetry that $\underline{X}\subseteq\underline{Y}$. Then, since $X, -Y\in\mathcal{C}$, $D(X, -Y)\neq\emptyset$. If $D(X, -Y)=\{g\}$ for some $g\in E$, then X conforms to a tope T such that $(-Y)+g^0\lesssim T$, a contradiction. Hence $|D(X, -Y)|\geq 2$, and so $|E|\geq 5$.

Case (ii): $\underline{X}\nsubseteq\underline{Y}$ and $\underline{X}\nsupseteq\underline{Y}$. Trivial. ■

Lemma 7.5. *Let A be an acycloid with circuits $\mathcal{C}$. Then:*

(1) *If $X, Y\in\mathcal{C}$, $X\neq\pm Y$ and $\underline{X}\subseteq\underline{Y}$, then $|D(X, Y)|\geq 3$ and $|D(X, -Y)|\geq 3$.*

(2) *If $X, Y\in\mathcal{C}$ with $X\neq -Y$ do not satisfy the axiom* (O3) *for some $f\in D(X, Y)$, then $|D(X, Y)|\geq 3$. Moreover if X and Y are minimal with respect to $D(X, Y)$, then X, Y do not satisfy* (O3) *for eny $e\in D(X, Y)$.*

Proof. Left to the reader. ■

Lemma 7.6. *If A is a non-matroidal acycloid on the set $E=\{1, 2, 3, 4, 5\}$ with circuits $\mathcal{C}$, then the following statements hold*:

(1) $|\underline{X}|=4$ *for all* $X\in\mathcal{C}$.

(2) *X and Y are not orthogonal for any X, $Y\in\mathcal{C}$.*

Proof. (1) Since $|E|=5$, by Proposition 7.4, A is simple. Hence $|\underline{X}|\geq 3$ for all $X\in\mathcal{C}$. If there is $X\in\mathcal{C}$ such that $|\underline{X}|=5$, by Lemma 7.5 (1), $\mathcal{C}=\{\pm X\}$. This implies A is matroidal. Hence $|\underline{X}|=3$ or 4 for all $X\in\mathcal{C}$. Now by hypothesis, there exist $X^1, X^2\in\mathcal{C}$ (where $X^1\neq -X^2$) for which (O3) does not hold. Then by Lemma 7.5 (2), $|D(X^1, X^2)|=3$, $|\underline{X}^1|=|\underline{X}^2|=4$ and X^1, X^2 does not satisfy (O3) for any $e\in D(X^1, X^2)$. Without loss of generality, let $X^1=(++-0+)$ and $X^2=(-0+--)$. Suppose that there is $X^3\in\mathcal{C}$ such that $|\underline{X}^3|=3$. Then note that $\underline{X}^3$ must contain $\{2,4\}$. We may assume $\underline{X}^3=\{2, 4, 5\}$ without loss of generality. By the second statement of Lemma 7.5 (2), we see that it is enough to check the following 2 cases.

Case (i): $X^3=(0+0+-)$. Since by Lemma 7.5 (2), X^2 and X^3 must satisfy (O3), some circuit Y conforms to $(-++0-)$. The fact of X^1, $Y\in\mathcal{C}$ contradicts the axiom (O2').

Case (ii): $X^3=(0+0++)$. Since $X^2, X^3\in\mathcal{C}$ and $4\in E$ must satisfy (O3), some circuit Z conforms to $(-++0+)$ or $(-++0-)$. The fact of X^1. $Z\in\mathcal{C}$ contradicts the axiom (O2').

(2) Suppose two circuits X^1 and X^2 are orthogonal. By (1) we may assume $X^1=(+++-0)$ and $X^2=(++-0-)$ without loss of generality. Since X^1 and X^2 (resp. X^1 and $-X^2$) must satisfy (O3), there are circuits $X^3=(++0--)$, $X^4=(0i+-+)$ and $X^5=(j0+-+)$. Here we can check that if $i=+$ then $j=-$, and that if $i=-$ then $j=+$. In each case, we have $\mathcal{C}=\{\pm X^1, \pm X^2, \cdots, \pm X^5\}$, and $\mathcal{C}$ satisfies (O3), a contradiction. ∎

Proposition 7.7. *There are exactly* 4 *non-matroidal acycloids on the* 5-*element set up to reorientation.*

Proof. The following 4 acycloids A_i ($i=0, 1, 2, 3$) are known to be non-matroidal, where A_1, A_2 and A_3 are the same acycloids to those in the first half of this section. We will give them by listing the sets $\mathcal{C}_i$ of circuits of A_i:

$\mathcal{C}_0=\{(0+-++), (+0-++), (++0++), (++-0+), (++-+0),$
and their negatives$\}$ (Tamura [48]),

$\mathcal{C}_1=\{(0+-++), (+0-++), (++-0+), (++-+0),$
and their negatives$\}$,

$\mathcal{C}_2 = \{(+0-++), (++-0+), (++--+0)$, and their negatives$\}$,
$\mathcal{C}_3 = \{(+0-++), (++-0+)$, and their negatives$\}$.

By Lemma 7.6, we know that there are not acycloids satisfying the condition of the proposition except these 4 acycloids. ■

§ 8. Conclusion

As we saw in Section 2, topes are very important elements of oriented matroids and they play an essential role in typical examples. We characterized oriented matroids in terms of topes in [35] and tope graphs of oriented matroids of rank at most three in [28]. These results. reviewed in Section 3, were obtained from the study of acycloid and L^1-system, which are generalized notions of oriented matroid. In this paper, we have discussed the recent results on acycloid and L^1-system, and also we have introduced median system. The main contents are as follows.

(1) the close relation between convex geometries and L^1-systems,

(2) applications of median systems to median graphs,

(3) extension of the point perturbation theorem of oriented matroids to acycloids and L^1-systems, and

(4) non-matroidal acycloids.

Finally we will mention some of remaining important problems.

(1) Inductive axiom systems of oriented matroids using deletion and contraction are rather non-practical. Is there a non-inductive axiom system by topes? Cf. Qusetion in Section 3.

(2) Characterize tope graphs of oriented matroids of any rank.

(3) Characterize the set $\mathcal{K}_{om}$. This is an open problem in [22, 24] as we mentioned in Section 4.

The properties of related structures shown in this paper will be probably useful for the characterizations.

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