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Fourier Hyperfunctions as the Boundary Values of Smooth Solutions of Heat Equations

By

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Abstract

We show that if a C^∞-solution $u(x, t)$ of heat equation in $\boldsymbol{R}_+^{n+1}$ does not increase faster than $\exp[\varepsilon(\frac{1}{t}+|x|)]$ then its boundary value determines a unique Fourier hyperfunction. Also, we prove the decomposition theorem for the Fourier hyperfunctions. These results generalize the theorems of T. Kawai and T. Matsuzawa for Fourier hyperfunctions and solve a question given by A. Kaneko.

§ 0. Introduction

T. Kawai and T. Matsuzawa have shown in [10, 15] that the boundary value of a C^∞-solution of heat equation in $\boldsymbol{R}_+^{n+1}$ which does not increase faster than $\exp(\varepsilon/t)$ is a well-defined hyperfunction. However, little is known about the characterization of a solution whose boundary value determines a Fourier hyperfunction near a characteristic boundary point. The purpose of this paper is to discuss this problem, that is, if a C^∞-solution $U(x, t)$ satisfies some growth condition (see (2.2)) then we can assign a unique compactly supported Fourier hyperfunction $u(x)$ to $U(x, t)$. Furthermore we can find such a Fourier tame solution $U(x, t)$ of heat equation for any compactly supported Fourier hyperfunction $u(x)$. To show this, we use the estimate for the heat kernel in [15] and structure theorems of ultradistributions given in [11, 13].

We use the multi-index notations such as $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\partial^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, $\partial_j=\partial/\partial x_j$ for $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)\in\boldsymbol{N}_0^n$ where $\boldsymbol{N}_0$ is the set of nonnegative

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integers, and $\partial_t = \partial/\partial t$.

§ 1. Complex and Real Versions of Fourier Hyperfunctions

First, we are going to introduce the complex and real versions of Fourier hyperfunctions and show their equivalence.

We denote by $\boldsymbol{D}^n$ the compactification $\boldsymbol{R}^n \cup S_\infty^{n-1}$ of $\boldsymbol{R}^n$, where S_∞^{n-1} is an $(n-1)$-dimensional sphere at infinity. When x is a vector in $\boldsymbol{R}^n \setminus \{0\}$, we denote by $x\infty$ the point on S^{n-1} which is represented by x, where we identify S^{n-1} with $\boldsymbol{R}^n \setminus \{0\}/\boldsymbol{R}^+$. The space $\boldsymbol{D}^n$ is given the natural topology, that is: (i) If a point x of $\boldsymbol{D}^n$ belongs to $\boldsymbol{R}^n$, a fundamental system of neighborhoods of x is the set of all open balls containing the point x. (ii) If a point $x \in \boldsymbol{D}^n$ belongs to S_∞^{n-1}, a fundamental system of neighborhoods of x $(=y\infty)$ is given by the following family

$$U_{\tilde{\Delta},A}(y\infty) = \{x \in \boldsymbol{R}^n;\, x/|x| \in \tilde{\Delta},\ |x| > A\} \cup \{a\infty;\, a \in \tilde{\Delta}\}\ ,$$

where $\tilde{\Delta}$ is a neighborhood of y in S^{n-1}.

Definition 1.1. Let K be a compact set in $\boldsymbol{D}^n$. We say that ϕ is in $\mathcal{F}(K)$ if $\phi \in C^\infty(\Omega \cap \boldsymbol{R}^n)$ for any neighborhood Ω of K and if there are positive constants h and k such that

$$|\phi|_{k,h} = \sup_{\substack{x \in \Omega \cap \boldsymbol{R}^n \\ \alpha}} \frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|}\, \alpha!} \exp k|x| < \infty\ .$$

We say that $\phi_j \to 0$ in $\mathcal{F}(K)$ as $j \to \infty$ if there are positive constants h and k such that

$$\sup_{\substack{x \in \Omega \cap \boldsymbol{R}^n \\ \alpha}} \frac{|\partial^\alpha \phi_j(x)|}{h^{|\alpha|}\, \alpha!} \exp k|x| \to 0 \quad \text{as } j \to \infty\ ,$$

where Ω is any neighborhood of K.

We denote by $\mathcal{F}'(K)$ the strong dual space of $\mathcal{F}(K)$ and call its elements Fourier hyperfunctions carried by K.

Definition 1.2. We say that $\phi(z)$ is in $Q(K)$ if $\phi(z)$ is holomorphic in a neighborhood of $\Omega \cap \boldsymbol{R}^n + i\{|y| \le r\}$ for some $r > 0$ and if for some $k > 0$

$$\sup_{z \in \Omega \cap \boldsymbol{R}^n + i\{|y| \le r\}} |\phi(z)| \exp k|z| < \infty\ ,$$

where Ω is a neighborhood of K in $\boldsymbol{D}^n$.

Remark. Let K be a compact subset of $\boldsymbol{D}^n$. Then for any neighborhood Ω of K in $\boldsymbol{D}^n$ there exists a neighborhood V of K in $\boldsymbol{D}^n$ such that for some $\delta>0$

$$(\bar{V}\cap \boldsymbol{R}^n)_\delta \subset \Omega \cap \boldsymbol{R}^n,$$

where $U_\delta=\{x\in \boldsymbol{R}^n \mid |x-y|\leq\delta\}$ for some $y\in U\subset \boldsymbol{R}^n$.

We denote by $E(x, t)$ the n-dimensional heat kernel:

$$E(x, t) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & t>0 \\ 0, & t\leq 0. \end{cases}$$

Theorem 1.3. *For every $\phi\in\mathcal{F}(K)$, let*

$$\phi_t(x) = \int_{\boldsymbol{R}^n} E(x-y, t)\, \phi(y)\, dy, \quad t>0.$$

Then $\phi_t\in Q(\boldsymbol{D}^n)$ and $\phi_t\to\phi$ in $\mathcal{F}(K)$ as $t\to 0_+$.

Proof. Let $\phi\in\mathcal{F}(K)$. Then we can easily show that ϕ_t is in $Q(\boldsymbol{D}^n)$. There are positive constants C, h, k and δ such that

$$\sup_{x\in K_\delta\cap \boldsymbol{R}^n} |\partial^\alpha \phi(x)| \leq C h^{|\alpha|}\alpha! \exp(-k|x|) \tag{1.1}$$

On the other hand we have for any $\delta>0$

$$\begin{aligned} \partial_x^\alpha(\phi_t(x)-\phi(x)) &= \int_{|y|\leq\delta} E(y, t)\, \partial_x^\alpha(\phi(x-y)-\phi(x))\, dy \\ &\quad + \int_{|y|\geq\delta} E(y, t)\, \partial_x^\alpha\, \phi(x-y)\, dy \\ &\quad - \int_{|y|\geq\delta} E(y, t)\, \partial_x^\alpha\, \phi(x)\, dy \\ &= I_1+I_2+I_3. \end{aligned}$$

Making use of (1.1), we have for $|y|\leq\delta$

$$\begin{aligned} &\sup_{x\in K\cap \boldsymbol{R}^n} |\partial^\alpha \phi(x-y)-\partial^\alpha \phi(x)| \\ &\quad \leq C'|y| h^{|\alpha|+1}(|\alpha|+1)! \exp(-k|x|) \\ &\quad \leq C''|y| (Hh)^{|\alpha|}|\alpha|! \exp(-k|x|) \end{aligned}$$

for some C', C'' and $H<1$.

For any $\varepsilon>0$, taking $\delta>0$ so small that $C''\delta<\varepsilon$, we have

$$\frac{|I_1| \exp k|x|}{(Hh)^{|\alpha|}|\alpha|!} < \varepsilon,$$

and it follows from (1.1) that

$$\sup_{\substack{x\in K\cap R^n \\ \alpha}} \frac{|I_3| \exp k|x|}{h^{|\alpha|}\alpha!} \leq C\int_{|y|\geq\delta} E(y, t)\, dy \to 0$$

and

$$\sup_{\substack{x\in K\cap R^n \\ \alpha}} \frac{|I_2| \exp k|x|}{h^{|\alpha|}\alpha!} \leq C\int_{|y|\geq\delta} E(y, t) \exp k|y|\, dy \to 0$$

as $t\to 0_+$. This completes the proof.

Theorem 1.4. *$\mathcal{F}(K)$ is topologically isomorphic to $Q(K)$.*

Proof. Let $\phi\in Q(K)$. Then ϕ is holomorphic in a neighborhood of $\Omega\cap R^n+i\{|y|\leq r\}$ for some $r>0$ and for some $k>0$

$$\sup_{z\in\Omega\cap R^n+i\{|y|\leq r\}} |\phi(z)| \exp k|z| < \infty ,$$

where Ω is any neighborhood of K in D^n. Let $\frac{1}{r}=h>0$. Then for $x\in\Omega\cap R^n$, we have

$$\partial^\alpha \phi(x) = \frac{\alpha!}{(2\pi i)^n}\int_{|z_n-x_n|=r}\cdots\int_{|z_1-x_1|=r} \frac{\phi(z)\, dz}{(z_1-x_1)^{\alpha_1+1}\cdots(z_n-x_n)^{\alpha_n+1}} .$$

Let $z=\xi+i\eta$. If $|x_j|\geq 2r$ then we have

$$|\xi_j|\geq|x_j|/2 .$$

Therefore it follows that

$$\begin{aligned}
|\partial^\alpha \phi(\alpha)| &\leq h^{|\alpha|}\alpha! \sup_{|z_j-x_j|=r} |\phi(z)| \\
&\leq h^{|\alpha|}\alpha! \sup_{|z_j-x_j|=r} |\phi(z)| \exp k(|\xi|-\frac{|x|}{2}) \\
&\leq Ch^{|\alpha|}\alpha! \exp(-k|x|/2) \sup_{z\in\Omega\cap R^n+i\{|\eta|\leq r\}} |\phi(z)| \exp k|z| .
\end{aligned}$$

Hence we have

$$\begin{aligned}
\frac{|\partial^\alpha \phi(x)|}{h^{|\alpha|}\alpha!} \exp k|x|/2 &\leq C \sup_{z\in\Omega\cap R^n+i\{|\eta|\leq r\}} |\phi(z)| \exp k|z| \\
&< \infty
\end{aligned}$$

for any $|x_j|\geq 2r$.

On the other hand, we have for any $|x_j|\leq 2r$

$$|\partial^\alpha \phi(x)| \exp k|x|/2$$

$$\leq \exp(k\sqrt{n}/h)\cdot\alpha!\,h^{|\alpha|}\sup_{z\in\Omega\cap\mathbf{R}^n+i\{|\eta|\leq r\}}|\phi(z)|\exp k|z|\,.$$

Therefore it follows that

$$\begin{aligned}&\sup_{\substack{x\in\Omega\cap\mathbf{R}^n\\ \alpha}}\frac{|\partial^\alpha\phi(x)|}{h^{|\alpha|}\alpha!}\exp k|x|\\ &\leq \exp(2k\sqrt{n}/h)\sup_{z\in\Omega\cap\mathbf{R}^n+i\{|\eta|\leq r\}}|\phi(z)|\exp k|z|\\ &<\infty.\end{aligned}$$

Let $\phi\in\mathcal{F}(K)$. Then it follows from Pringsheim Theorem that ϕ can be analytically continued to a strip $\{z=x+iy\,|\,x\in\Omega\cap\mathbf{R}^n,\ |y|\leq r<\frac{1}{h}\}$

$$\begin{aligned}|\phi(z)|\exp k|x| &= \exp k|x|\,\Big|\sum_\alpha\frac{\partial^\alpha\phi(x)}{\alpha!}(iy)^\alpha\Big|\\ &\leq C\sum_\alpha\frac{|\partial^\alpha\phi(x)|}{h^{|\alpha|}\alpha!}\exp k|x|(h|y|)^{|\alpha|}\\ &\leq C'\sum_\alpha(hr)^{|\alpha|}\,.\end{aligned}$$

Therefore we have

$$\sup_{z\in\Omega\cap\mathbf{R}^n+i\{|y|\leq 1/h\}}|\phi(z)|\exp k|z|<\infty\,,$$

which completes the proof.

Remark. Let K be a compact set in $\mathbf{D}^n$ and let $u\in\mathcal{F}'(K)$. Then for any $h, k>0$ there is a constant C such that

$$|u(\phi)|\leq C\sup_{x\in\Omega\cap\mathbf{R}^n}\frac{|\partial^\alpha\phi(x)|}{h^{|\alpha|}\alpha!}\exp k|x|,\ \phi\in Q(\mathbf{D}^n)\,,$$

where Ω is any neighborhood of K in $\mathbf{D}^n$.

This is equivalent to the condition that for every neighborhood Ω of K and for every $k>0$ there is a constant C such that

$$(1.2)\qquad |u(\phi)|\leq C\sup_{x\in\Omega\cap\mathbf{R}^n+i\{|\eta|+r\}}|\phi(z)|\exp k|z|,\ \phi\in Q(\mathbf{D}^n)\,.$$

Proposition 1.5. *Let $P(\partial)=\sum_{|\alpha|=0}^\infty a_\alpha\partial^\alpha$ be a differential operator of infinite order with constant coefficients satisfying the following: For any $L>0$ there exists a constant $C>0$ such that*

$$|a_\alpha|\leq CL^{|\alpha|}/\alpha!$$

for all α. Then the operators

$$P(\partial)\colon \mathcal{F}(D^n) \to \mathcal{F}(D^n) \tag{1.3}$$

and

$$P(\partial)\colon \mathcal{F}'(D^n) \to \mathcal{F}'(D^n) \tag{1.4}$$

are continuous.

Proof. Let $\phi \in \mathcal{F}(D^n)$ and $h>0$. Then it follows that

$$\begin{aligned}
&|\partial^\beta P(\partial)\, \phi(x)| \exp k|x| \\
&\leq \sum_{|\alpha|=0}^{\infty} |a_\alpha|\, |\partial^{\alpha+\beta} \phi(x)| \exp k|x| \\
&\leq \sum_{|\alpha|=0}^{\infty} |\phi|_{k,h} \frac{CL^{|\alpha|}}{\alpha!} h^{|\alpha+\beta|}(\alpha+\beta)! \\
&\leq C|\phi|_{k,h}(2h)^{|\beta|}\, \beta! \sum_{|\alpha|=0}^{\infty} (2hL)^{|\alpha|}\,.
\end{aligned}$$

Thus if we choose $h>0$ so small that $2Lh<1$ then we obtain

$$|P(\partial)\, \phi(x)|_{k,2h} \leq C|\phi|_{k,h}\,, \quad \phi \in \mathcal{F}(D^n)\,,$$

which proves that (1.3) is continuous. The continuity of (1.4) is easily obtained by this fact.

§ 2. Main Theorems

The following lemma is very useful later. For the details of the proof we refer to Komatsu [13], Lemma 2.9 and Lemma 2.10.

Lemma 2.1. *For any $\varepsilon_1>0$ there exist a function $v(t) \in C_0^\infty(\mathbf{R})$ and an ultradifferential operator $P(d/dt)$ such that*

$$\operatorname{supp} v \subset [0, \varepsilon_1];$$

$$|v(t)| \leq C \exp(-N^*(1/t)),\ t>0\,;$$

$$\textit{for any } h>0\,, \quad P(d/dt) = \sum_{k=0}^{\infty} a_k (d/dt)^k,\ |a_k| \leq C_h h^k/k!^2\,;$$

$$P(d/dt)\, v(t) = \delta + w(t)\,. \tag{2.1}$$

Here $w(t) \in C_0^\infty(\mathbf{R})$, supp $w \subset [\varepsilon_1/2, \varepsilon_1]$ *and δ is a Dirac measure, and*

$$N^*(t) = \sup_p \log \frac{t^p}{h_p\, p!}$$

where $h_p = (l_1 \cdots l_p)^{-1}$ for some sequence l_p decreasing to 0.

In fact. we can construct the above ultradifferential operator $P(d/dt)$ by taking

$$P(\zeta) = (1+\zeta)^2 \prod_{p=1}^{\infty} (1+\frac{l_p \zeta}{p^2}) .$$

Let K be a compact set in $\boldsymbol{D}^n$. Then we denote by $\mathcal{F}_K^{\text{tame}}$ the totality of C^∞-solutions of heat equation $(\partial_t - \Delta)\, u(x, t) = 0$ on $\boldsymbol{R}_+^{n+1}$ which satisfy the following:

For any $\varepsilon > 0$, there is a constant C such that

$$|u(x, t)| \leq C \exp [\varepsilon (\frac{1}{t} + t + |x|) - \frac{\operatorname{dis}(x, K_\delta \cap \boldsymbol{R}^n)^2}{8t}] \tag{2.2}$$

in $\boldsymbol{R}_+^{n+1}$. Then we have:

Theorem 2.2. *Let* $u \in \mathcal{F}'(K)$ *and let*

$$U(x, t) = u_y(E(x-y, t)) , \quad t > 0 . \tag{2.3}$$

Then $U(x, t) \in \mathcal{F}_K^{\text{tame}}$ *and*

$$U(x, t) \to u \quad in \quad \mathcal{F}'(K) \quad as \quad t \to 0_+ . \tag{2.4}$$

Conversely, every element in $\mathcal{F}_K^{\text{tame}}$ *can be expressed in the form* (2.3) *with unique element* $u \in \mathcal{F}'(K)$.

Proof. Let $u \in \mathcal{F}'(K)$. Then it is obvious that the function $U(x, t)$ defined by (2.3) belongs to $C^\infty(\boldsymbol{R}_+^{n+1})$ and satisfies the heat equation on $\boldsymbol{R}_+^{n+1}$;

$$(\partial_t - \Delta)\, U(x, t) = 0 .$$

If follows from (1.2) that for $t > 0$

$$\begin{aligned} |U(x, t)| &\leq C \sup_{\substack{y \in K_\delta \cap \boldsymbol{R}^n \\ |\eta| \leq r}} |E(x-y-i\eta, t)| \exp k|x| \\ &\leq C \sup_{\substack{y \in K_\delta \cap \boldsymbol{R}^n \\ |\eta| \leq r}} (4\pi t)^{-n/2} \exp \left[\frac{-(1/2)|x-y|^2 + 8k^2 t^2 + 4kt\, |x| + \eta^2}{4t} \right] \\ &\leq C' \exp \left[2k^2 t + k|x| + \frac{r^2}{4t} - \frac{\operatorname{dis}(x, K_\delta \cap \boldsymbol{R}^n)^2}{8t} \right]. \end{aligned}$$

Let $\varepsilon = \max \{2k^2, k, r^2/4\}$. Then we obtain (2.2). Hence $U(x, t) \in \mathcal{F}_K^{\text{tame}}$.

Now let

$$G(y, t) = \int_{\boldsymbol{R}^n} E(x-y, t)\, \phi(x)\, dx , \quad \phi \in \mathcal{F}(K) .$$

Then by Theorem 1.3 we can easily see that

$$G(\cdot, t) \to \phi \quad \text{in} \quad \mathcal{F}(K) \quad \text{as} \quad t \to 0_+ . \tag{2.5}$$

Also, we have

$$\int_{\boldsymbol{R}^n} U(x, t)\, \phi(x)\, dx = u_y(G(y, t)) \tag{2.6}$$

by taking the limit of the Riemann sum of the left side. Then applying (2.5) to (2.6) we obtain (2.4).

Now we will prove the converse. Let $U(x, t) \in \mathcal{F}_K^{\text{tame}}$ and let

$$\Omega = \{(x, t) \in \boldsymbol{R}^{n+1};\ t \neq 0 \quad \text{or} \quad x \notin K \cap \boldsymbol{R}^n\} .$$

Since the heat operator is hypoelliptic the condition (2.2) implies

$$\lim_{t \to 0_+} \tilde{P}(\partial_t, \partial_x)\, U(x, t) = 0 , \quad x \notin K \cap \boldsymbol{R}^n$$

for any linear differential operator (with constant coefficients) $\tilde{P}(\partial_t, \partial_x)$ of finite order. It follows that there is a C^∞-function $c(x, t)$ satisfying the following:

$$c(x, t) = U(x, t) \quad \text{in} \quad \boldsymbol{R}_+^{n+1}$$

and

$$c(x, t), \text{ together with all its derivatives vanishes on } \Omega \backslash \boldsymbol{R}_+^{n+1}.$$

The assumption (2.2) implies that $U(x, t)$ does not increase faster than exp $[\varepsilon(\frac{1}{t} + |x|)]$ as $t \to 0_+$. We see that there exists a Fourier hyperfunction $\psi(x, t)$ which satisfies the following:

$$\psi = c \quad \text{on} \quad \Omega$$

and

$$\text{supp}\, \psi \subset \overline{\boldsymbol{R}_+^{n+1}} .$$

In fact, let functions v, w and an ultradifferential operator $P(d/dt)$ be as in Lemma 2.1. Define

$$\tilde{c}(x, t) = \int_0^\infty c(x, t+s)\, v(s)\, ds .$$

Then we have

$$(\partial_t - \Delta)\, \tilde{c}(x, t) = 0 \quad \text{in} \quad \boldsymbol{R}_+^{n+1} .$$

It follows from Lemma 2.1 and (2.2) that

$$|\tilde{c}(x, t)| \leq C' \exp \varepsilon(|x|+t), \quad t \geq 0 .$$

Thus $\tilde{c}(x, t)$ is a continuous function of an infra-exponential type in $\overline{\boldsymbol{R}_+^{n+1}}$. Using (2.1) we obtain for $t>0$.

$$P(-\Delta)\,\tilde{c}(x, t) = P(-d/dt)\,\tilde{c}(x, t) = c(x, t)+\int_0^\infty c(x, t+s)\, w(s)\, ds . \tag{2.7}$$

Since $\tilde{c}(x, t)$ and the second term of the right hand side of (2.7) can be continuously extended beyond the hyperplane $t=0$, we obtain the extension $\psi(x, t)$ of $c(x, t)$.

Since $c(x, t)$ is a C^∞-solution of heat equation on Ω, we have

$$(\partial_t-\Delta)\,\psi(x, t) = 0 \quad \text{on} \quad \Omega .$$

In what follows, a Fourier hyperfunction $\psi(x, t)$ thus obtained shall be called a Fourier tame extension of U for short.

Let $g(x)=\tilde{c}(x, 0)$ and $h(x)=-\int_0^\infty c(x, s)\, w(s)\, ds$. Then g and h are also continuous functions of an infra-exponential type, and hence Fourier hyperfunctions. We define a Fourier hyperfunction u as

$$u(x) = P(-\Delta)\, g(x)+h(x) .$$

Since

$$\lim_{t\to 0_+} U(x, t) = 0,\ x\notin K ,$$

we see that $u\in\mathcal{F}'(K)$.

We define a Fourier hyperfunction $\alpha(x, t)$ by

$$\alpha(x, t) = \begin{cases} \int E(x-y, t)\, u(y)\, dy , & t>0 \\ 0, & t<0 . \end{cases}$$

Let $\alpha_+(x, t)$ be the restriction of α to $\boldsymbol{R}_+^{n+1}$. Then we have $\alpha_+(x, t)\in\mathcal{F}_K^{\text{tame}}$.

Let $\beta(x, t)$ be a Fourier tame extension of $\alpha_+(x, t)$. Note that β does not coincide with α in general. It follows from (2.4) that

$$\lim_{t\to 0_+} \alpha(x, t) = \lim_{t\to 0_+} \beta(x, t) = \lim_{t\to 0_+} \psi(x, t) .$$

Hence we have

$$(\partial_t-\Delta)\,(\psi(x, t)-\alpha(x, t)) = 0 \quad \text{in} \quad \boldsymbol{R}^{n+1} .$$

By the well known uniqueness theorem for the solutions of the Cauchy problem to the heat equation we have $\psi=\alpha$ (See [4]).

Remarks. (i) The estimate (2.2) for $K=\boldsymbol{D}^n$ is the following:

$$|u(x, t)| \leq C \exp \varepsilon\left[\frac{1}{t}+t+|x|\right].$$

(ii) If $K \subset\subset \boldsymbol{R}^n$ then the estimate (2.2) is replaced by

$$|u(x, t)| \leq C \exp \left[\frac{\varepsilon}{t}-\frac{\operatorname{dis}(x, K)^2}{4t}\right].$$

In this case, T. Kawai and T. Matsuzawa have shown in [9, 14] that its boundary value determines a unique hyperfunction with carrier K so that the vanishing of $g(x)$ implies the vanishing of $u(x, t)$.
(iii) Since it suffices to consider the estimate (2.2) for sufficiently small $t>0$, we may omit the term εt in (2.2).

Corollary 2.3. *There exists an isomorphism*

$$b: \mathcal{F}_K^{\text{tame}} \to \mathcal{F}'(K).$$

From the proof of Theorem 2.2, we obtain the following corollaries:

Corollary 2.4. *Each function in $\mathcal{F}_K^{\text{tame}}$ is real analytic.*

Corollary 2.5. *If $u \in \mathcal{F}'(K)$, then there exist an ultradifferential operator $P(d/dt)$ of Gevrey order 2, a continuous function g of an infra-exponential type and a C^∞-function h of an infra-exponential type such that*

$$u(x) = P(-\Delta)\, g(x)+h(x),$$

where $g \in C^\infty(\boldsymbol{R}^n \backslash K)$.

We can consider $\mathcal{F}'(K_1) \subset \mathcal{F}'(K_2)$ if $K_1 \subset K_2 \subset\subset \boldsymbol{D}^n$. Let $\mathcal{F}' = \bigcup_{K \subset\subset \boldsymbol{D}^n} \mathcal{F}'(K)$. Then we have:

Theorem 2.6. *If $u \in \mathcal{F}'$ then there is a smallest compact set $K \subset\subset \boldsymbol{D}^n$ such that $u \in \mathcal{F}'(K)$.*

Proof. Let $u \in \mathcal{F}'$ and let K be the intersection of all compact set K' in $\boldsymbol{D}^n$ such that $u \in \mathcal{F}'(K')$. By Theorem 2.2 a defining function

$$U(x, t) = u_y(E(x-y, t)), \quad t>0$$

is uniquely defined and satisfies the heat equation in $\boldsymbol{R}^{n+1} \backslash (K \times \{0\})$. Noting that $u=\lim_{t\to 0+} U(\cdot, t)$, we see that $u\in\mathcal{F}'(K)$.

Theorem 2.7. *Let $K_1, \cdots, K_r$ be compact subsets of $\boldsymbol{D}^n$ and $u\in\mathcal{F}'(K_1\cup\cdots\cup K_r)$. Then we can find $u_j\in\mathcal{F}'(K_1)$ so that $u=u_1+\cdots+u_r$.*

Proof. It is sufficient to prove the statement when $r=2$. Let $U(x, t)$ be the function defined by (2.3). The theorem will be proved if we can split U into a sum $U=U_1+U_2$ where $U_j\in\mathcal{F}^{\text{tame}}_{K_j}$, $j=1, 2$. Let $\tilde{U}$ be a Fourier tame extension of U. Then $\tilde{U}$ satisfies the heat equation in $\boldsymbol{R}^{n+1}\backslash(\tilde{K}_1\cup\tilde{K}_2)$ where $\tilde{K}_j=K_j\times\{0\}\cap\boldsymbol{R}^{n+1}$, $j=1, 2$. We take a function $\psi\in C^\infty(\boldsymbol{R}^{n+1}\backslash\tilde{K}_1\cap\tilde{K}_2)$ constructed in [7, Corollary 1.4.11] such that $\psi=0$ for large $|x|+t$ and near $(\tilde{K}_2\backslash(\tilde{K}_1\cap\tilde{K}_2))$, $\psi=1$ near $(\tilde{K}_1\backslash\tilde{K}_1\cap\tilde{K}_2)$ and $\psi\in L^\infty(\boldsymbol{R}^{n+1})$. Here "near" means in the sense of the slowly varying metric defined in [6, Chap. 1]. We will split $\tilde{U}$ as follows:

$$\tilde{U}_1 = \psi\tilde{U}-\tilde{V}\,, \quad \tilde{U}_2 = (1-\psi)\,\tilde{U}+\tilde{V}\,.$$

We define $\psi\tilde{U}\in\mathcal{F}'(\boldsymbol{D}^{n+1})$ such that $\psi\tilde{U}=0$ near $(\tilde{K}_2\backslash(\tilde{K}_1\cap\tilde{K}_2))$ and $(1-\psi)\,\tilde{U}=0$ near $(\tilde{K}_1\backslash(\tilde{K}_1\cap\tilde{K}_2))$. We can write

$$(\partial_t-\Delta)\,\psi\tilde{U}=\tilde{F}+f$$

where $\tilde{F}$ and f are in $\mathcal{F}'(\boldsymbol{D}^{n+1})$ such that

$$\tilde{F}=\begin{cases}(\partial_t-\Delta)\,(\psi U)\,, & t>0\\ 0\,, & t<0\end{cases}$$

and $f\in\mathcal{F}'(\boldsymbol{D}^{n+1})$, $\operatorname{supp} f\subset K_1\times\{0\}$. Now we define

$$\tilde{V}(x, t) = E*\tilde{F}(x, t)\in\mathcal{F}'(\boldsymbol{D}^{n+1})$$

and $V(x, t)\equiv\tilde{V}(x, t)$ for $t>0$. Then we have

$$\tilde{V}\in C^\infty(\boldsymbol{R}^{n+1}\backslash\tilde{K}_1\cap\tilde{K}_2)\,, \quad \operatorname{supp}\tilde{V}\subset\overline{\boldsymbol{R}^{n+1}_+}$$

and

$$V(\cdot,t)\to 0 \ \text{ uniformly in } \ \{x;\, \operatorname{dis}(x, K_1\cap K_2)\geq\delta\}$$

for every $\delta>0$ as $t\to 0_+$. Since we have

$$V(x, t) = \psi U-E*f(x. t)\,,$$

we have for any $\varepsilon>0$

$$V(x, t) = O(\exp\varepsilon[1/t+t+|x|]) \ \text{ as } \ t\to 0_+\,.$$

Thus we have the desired property that

$$U_1 = \psi U - V \in \mathscr{F}_{K_1}^{\text{tame}}$$

and

$$U_2 = (1-\psi)\, U + V \in \mathscr{F}_{K_2}^{\text{tame}}.$$

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