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Further Generalization of Generalized Verma Modules

By

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§0. Introduction

0.1. Let G be a complex semisimple Lie group, B a Borel subgroup, P a parabolic subgroup containing B, $\mathfrak{g}=\mathrm{Lie}(G)$, $\mathfrak{b}=\mathrm{Lie}(B)$, $\mathfrak{p}=\mathrm{Lie}(P)$, and E a finite dimensional irreducible $U(\mathfrak{p})$-module, where $U(-)$ denotes the enveloping algebra. A $U(\mathfrak{g})$-module of the form $U(\mathfrak{g})\otimes_{U(\mathfrak{p})}E$ is called *a generalized Verma module* [24] and, in the special case where $\mathfrak{p}=\mathfrak{b}$, it is called *a Verma module* (cf. [9] and its references).

In the course of proving the Kazhdan-Lusztig conjecture [21], it was shown [1], [8] that the Verma modules correspond to the local cohomologies at the B-orbits on G/B via the localization functor. Thus it is natural to ask what are the $U(\mathfrak{g})$-modules corresponding to the local cohomologies at the B-orbits on G/P.

In this paper, we shall give an answer to this problem. It turns out that here appears a further generalization of the generalized Verma modules. We shall construct these $U(\mathfrak{g})$-modules in a purely algebraic way as follows. Let $\mathfrak{p}^{\vee}$ be the set of linear characters of the Lie algebra $\mathfrak{p}$, A the ring of polynomial functions on $\mathfrak{p}^{\vee}$, and $c: \mathfrak{p}\to A$ the canonical homomorphism, which we shall consider as an A-valued character of a Cartan subalgebra, say $\mathfrak{t}$, contained in $\mathfrak{b}=\mathrm{Lie}(B)$. Let λ be the lowest weight of a finite dimensional irreducible $\mathfrak{p}$-module, W the Weyl group, W_I the Weyl subgroup of W corresponding to P, and w an element of W which is longest in the coset wW_I. Let $U_A(-)=U(-)\otimes_{C}A$ and define the 'universal' Verma module $M_A(w(c+\lambda-\rho)-\rho)$ by $M_A(w(c+\lambda-\rho)-\rho)=U_A(\mathfrak{g})\otimes_{U_A(\mathfrak{b}),\,w(c+\lambda-\rho)-\rho}A$, where ρ is the half of the sum of the positive roots. Note that $w(c+\lambda-\rho)-\rho$ is not fully universal as a character of $\mathfrak{t}$ but it is universal among the characters lying on a certain facet with respect to the reflection group W (translated by $w(\lambda-\rho)-\rho$). Hence $M_A(w(c+\lambda-\rho)-\rho)$ resembles to reducible Verma modules and we can construct its quotient $V_A(w, c+\lambda, \mathfrak{p})$ in the same way as the construction of the simple quotient

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of the usual Verma module. The $U(\mathfrak{g})$-module investigated in this paper is the specialization of $V_A(w, c+\lambda, \mathfrak{p})$

$$V(w, \lambda, \mathfrak{p})=V_A(w, c+\lambda, \mathfrak{p})\otimes_A \boldsymbol{C},$$

where $\boldsymbol{C}$ is considered as a trivial A-module. (In other words, $V(w, \lambda, \mathfrak{p})$ is obtained from $V_A(w, c+\lambda, \mathfrak{p})$ by the specialization $c\to 0$.)

In contrast with the case of the usual generalized Verma modules, the most difficult point in the study of our $\mathfrak{g}$-modules is the character formula, which will be proved in (6.3). Once we get the character formula, we can deduce several consequences from it. For instance, we show in (6.8) that our $\mathfrak{g}$-modules are a generalization of the generalized Verma modules, and, we construct in §7 a resolution of our $\mathfrak{g}$-module by Verma modules which is a generalization of the resolution of a finite dimensional representation constructed by Bernstein-Gelfand-Gelfand [2].

0.2. Using the character formula, we can also show that our $U(\mathfrak{g})$-module $V(w, \lambda, \mathfrak{p})$ actually corresponds to the local cohomology at a B-orbit of G/P. See (6.6). This fact enables us a $\mathcal{D}$-module theoretic study of our $U(\mathfrak{g})$-modules, by which we get an irreducibility criterion (9.13) for generalized Verma modules in terms of the b-functions of the semi-invariants. (See (9.2) for the semi-invariants.) Our irreducibility criterion is far different from, and unfortunately, less complete than the Jantzen's one [16], for we need to assume the anti-dominancy in order to use the generalities concerning the localization functor [1]. In this regard, see (9.14).

0.3. Let us explain our motivation. Assume that $\mathfrak{g}$ is simple, the nilpotent radical $\mathfrak{u}$ of $\mathfrak{p}$ is commutative, and a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ is normalized by the longest element of the Weyl group. Let L be the Levi subgroup of P corresponding to $\mathfrak{l}$. Then it is known that (L, adjoint action, $\mathfrak{u}$) is an irreducible regular prehomogeneous vector space, that there is an irreducible polynomial f on $\mathfrak{u}$ which is relatively L-invariant, and that there is a unique fundamental weight ϖ which can be extended to a character of the Lie algebra $\mathfrak{p}$. (See [30], [12] for prehomogeneous vector spaces, and [25], [27] for there special kind of prehomogeneous vector spaces.) Let $b(s)$ be the Bernstein-Sato polynomial ($=b$-function) of f. In [31], S. Suga observed a relation between the simplicity of the generalized Verma module $M(\lambda)=U(\mathfrak{g})\otimes_{U(\mathfrak{p}), \lambda\varpi}\boldsymbol{C}$ and the zeros of $b(s)$. (More precisely, $M(\lambda)$ is simple if and only if $b(\lambda-j)\neq 0$ for $j=1, 2, \cdots$.) The original motivation of the present work was to explain and generalize this observation.

Roughly speaking

(0.3.1) $$(L, \text{ adjoint action, } \mathfrak{u}) \doteqdot (L, \text{ left action, } G/P).$$

At one hand, we have the $\mathscr{D}$-module $\mathscr{D}f^{\lambda}$, which is related to the left hand side of (0.3.1). We can show that $\mathscr{D}f^{\lambda}$ is simple if and only if $b(\lambda-j)\neq 0$ for any $j\in\boldsymbol{Z}$. On the other hand, we can expect that we get a $\mathscr{D}$-module, say $\mathscr{M}(\lambda)$, on G/P by "localizing" the generalized Verma module $M(\lambda)$ as in [1]. Then $\mathscr{M}(\lambda)$, which is related to the right hand side of (0.3.1), would be simple if and only if $M(\lambda)$ is simple. Hence, by showing that $\mathscr{M}(\lambda)\doteqdot\mathscr{D}f^{\lambda}$, we would be able to explain the observation of Suga to some extent.

In this paper, we have tried to realize this idea and get (9.13), which is our first result in this direction, although it is still unsatisfactory.

0.4. A deeply related problem is studied by M. Kashiwara [20]. The relation between the present work and [20] will become clear in [13].

0.5. This paper consists of 9 sections. In Section 1, we define a new generalization $V(w, \lambda, \mathfrak{p})$ of Verma modules in (1.3) (cf. (4.1.1)) and give the basic lemma (1.12), which is used to prove the character formula. In Sections 2 and 3, we review some known facts about the twisted $\mathscr{D}$-modules and the localization functor, respectively. In Section 4, we construct a certain $\mathfrak{g}$-module, which is used to deduce the character formula from (1.12). In Section 5, we prove Proposition 5.2, which is used in (9.4). In Section 6, we prove the character formula in (6.3). Using it, we prove in (6.6) that the dual $\mathfrak{g}$-module of $V(w, \lambda, \mathfrak{p})$ corresponds to the local cohomology at BwP/P. In (6.7)-(6.9), we study the relation between our $\mathfrak{g}$-modules $V(w, \lambda, \mathfrak{p})$ and the usual generalized Verma modules. In Section 7, we construct a resolution of $V(w, \lambda, \mathfrak{p})$ by the Verma modules, which is a generalization of the resolution of a finite dimensional representation constructed by Bernstein-Gelfand-Gelfand. In Section 8, we give a simplicity criterion (8.4) for a certain type of $\mathscr{D}$-modules, which is used in Section 9 to obtain an irreducibility criterion (9.13) for the generalized Verma modules.

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Convention. We denote the complex (resp. rational) number field by $\boldsymbol{C}$ (resp. $\boldsymbol{Q}$), the rational integer ring by $\boldsymbol{Z}$, and we put $\boldsymbol{N}=\{0, 1, 2, \cdots\}$. If two objects, say X and Y, are *naturally* isomorphic, we often write $X=Y$.

§1. A Generalization of Verma Modules

1.0. In this section, we define a new generalization of Verma modules, and prove a basic lemma, which will be used to prove the character formula (6.3). First we review basic facts concerning Lie algebras in order to fix notations. We define our generalization of Verma modules in (1.3). After studying elementary properties of our modules, we give a basic lemma in (1.12). The remainder of the section is devoted to the proof of this lemma.

1.1. Let G be a connected reductive group over the complex number field C, B a Borel subgroup of G, T a maximal torus contained in B, $W=N_G(T)/T$, and $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{t}$ the Lie algebras of G, B and T, respectively. Let $\mathfrak{t}^\vee=\mathrm{Hom}_C(\mathfrak{t}, C)$, $R(\subset\mathfrak{t}^\vee)$ be the root system of $(\mathfrak{g}, \mathfrak{t})$, $\mathfrak{g}(\alpha)$ the root subspace of $\mathfrak{g}$ corresponding to $\alpha\in R$, R_+ the set of $\alpha\in R$ such that $\mathfrak{g}(\alpha)\subset\mathfrak{b}$, $R_-=-R_+$, and $\mathfrak{n}_\pm=\sum_{\alpha\in R_\pm}\mathfrak{g}(\alpha)$. Let $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ be the simple roots, $\Pi^\vee=\{\alpha_1^\vee, \cdots, \alpha_l^\vee\}$ the simple coroots, $\{\varpi_1, \cdots, \varpi_l\}$ the fundamental weights, and $\{\varpi_1^\vee, \cdots, \varpi_l^\vee\}$ the fundamental coweights. For $\alpha\in R$, let $\alpha^\vee\in\mathfrak{t}$ be the corresponding coroot. Let $Q=\sum_{i=1}^l \boldsymbol{Z}\alpha_i$ and $Q_+=\sum_{i=1}^l \boldsymbol{N}\alpha_i$. Define $\mu\leqq\lambda$ $(\mu, \lambda\in\mathfrak{t}^\vee)$ if $\lambda-\mu\in Q_+$. Let r_α be the reflection with respect to α, and $S=\{r_\alpha|\alpha\in\Pi\}$. For $w\in W$, denote its length by $l(w)$. Let $\leqq$ be the Bruhat order in W, where the identity element is minimal.

Let I be a subset of S, W_I the subgroup of W generated by I, w_I the longest element of W_I, $\Pi_I=\{\alpha\in\Pi|r_\alpha\in I\}$, R_I the root subsystem of R generated by Π_I, $\mathfrak{l}=\mathfrak{l}(I)=\mathfrak{t}+\sum_{\alpha\in R_I}\mathfrak{g}(\alpha)$, $\mathfrak{u}_\pm=\mathfrak{u}_\pm(I)=\sum_{\alpha\in R_\pm\setminus R_I}\mathfrak{g}(\alpha)$, $\mathfrak{p}=\mathfrak{p}(I)=\mathfrak{l}+\mathfrak{u}_+$, and $\mathfrak{p}_-=\mathfrak{p}_-(I)=\mathfrak{l}+\mathfrak{u}_-$. We denote the connected subgroups of G corresponding to $\mathfrak{l}$, $\mathfrak{n}_\pm$, $\mathfrak{p}$ and $\mathfrak{p}_-$ by $L=L(I)$, $U_\pm=U_\pm(I)$, $P=P(I)$ and $P_-=P(I)$, respectively. For J, $K\subset S$, let $(W_J\backslash W/W_K)_s$ (resp. $(W_J\backslash W/W_K)_l$) be the shortest (resp. longest) representatives of the double cosets in $W_J\backslash W/W_K$. For a subset K of S, let $K'=w_SKw_S(\subset S)$. Then $(W_J\backslash W/W_K)_l=\{ww_S|w\in(W_J\backslash W/W_{K'})_s\}$. We write $(W/W_I)_l$ etc. for $(W_\phi\backslash W/W_I)_l$ etc.

Take a $\boldsymbol{Q}$-subspace $\mathfrak{z}_Q$ of the center $\mathfrak{z}$ such that $\mathfrak{z}_Q\otimes C=\mathfrak{z}$. Let $\mathfrak{g}_Q$ be the $\boldsymbol{Q}$-linear span of a Chevalley basis of $[\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}_Q$. Put $\mathfrak{t}_Q=\mathfrak{z}_Q+\sum_{\alpha\in R}\boldsymbol{Q}\alpha^\vee$ and $\mathfrak{t}_{I,Q}=\mathfrak{t}_Q/\sum_{\alpha\in\Pi_I}\boldsymbol{Q}\alpha^\vee$. Let $\mathfrak{t}_Q^\vee$ and $\mathfrak{t}_{I,Q}^\vee$ be their dual spaces. For a $\boldsymbol{Q}$-algebra A, put $\mathfrak{g}_A=\mathfrak{g}_Q\otimes A$ etc. If $A=\boldsymbol{C}$, we omit the suffix $A(=\boldsymbol{C})$. We identify $\mathfrak{t}_{I,A}^\vee$ with the W_I-invariant elements in $\mathrm{Hom}_Q(\mathfrak{t}_Q, A)=\mathrm{Hom}_A(\mathfrak{t}_A, A)$. We say that $\lambda\in\mathfrak{t}^\vee$ is *anti-dominant* (resp. *regular*) if $\langle\lambda, \alpha^\vee\rangle\notin\boldsymbol{N}\setminus\{0\}$ (resp. $\neq 0$) for any $\alpha\in R_+$. Let $\mathfrak{t}_{ad}^\vee$ (resp. $\mathfrak{t}_{rad}^\vee$) be the set of $\lambda\in\mathfrak{t}^\vee$ such that $\lambda-\rho$ is anti-dominant (resp. regular and anti-dominant). Put $\mathfrak{t}_{I,ad}^\vee=\mathfrak{t}_I^\vee\cap\mathfrak{t}_{ad}^\vee$ and $\mathfrak{t}_{I,rad}^\vee=\mathfrak{t}_I^\vee\cap\mathfrak{t}_{rad}^\vee$.

Let k be a field of characteristic zero. For a Lie algebra $\mathfrak{a}$ over k, let $U(\mathfrak{a})=U_k(\mathfrak{a})$ be the enveloping algebra, and $Z(\mathfrak{a})$ the center of $U(\mathfrak{a})$. Express $z\in Z(\mathfrak{g})$ as $z=\varphi(z)+z'$ with $\varphi(z)\in U(\mathfrak{t})$ and $z'\in U(\mathfrak{g})\mathfrak{n}_+$. Consider $\lambda\in\mathfrak{t}^\vee$ as a character of $U(\mathfrak{t})$. Let $\chi_\lambda=\lambda\circ\varphi$ and $U(\lambda, \mathfrak{g})=U(\mathfrak{g})/U(\mathfrak{g})\ker\chi_\lambda$. Let $\rho=(1/2)\sum_{\alpha\in R_+}\alpha$ and $\gamma: U(\mathfrak{t})\to U(\mathfrak{t})$ be the isomorphism defined by $\gamma(H)=H-\rho(H)$ for $H\in\mathfrak{t}$. Then $\gamma\circ\varphi$ gives an isomporphism $Z(\mathfrak{g})\to U(\mathfrak{t})^W$ (cf. [9, 7.4.5]), which is called the *Harish-Chandra homomorphism*. Here $U(\mathfrak{t})^W$ denotes the totality of the W-invariant elements of $U(\mathfrak{t})$. Note that $\lambda\circ\gamma\circ\varphi=\chi_{\lambda-\rho}$. Hence $\chi_{w\lambda-\rho}=\chi_{\lambda-\rho}$ and $U(w\lambda-\rho, \mathfrak{g})=U(\lambda-\rho, \mathfrak{g})$ for any $w\in W$.

For a $U_k(\mathfrak{t})$-module V and $\mu\in\mathfrak{t}_k^\vee$, let V_μ be the set of $v\in V$ such that for any $H\in\mathfrak{t}_k$, there exists an integer n such that $(H-\mu(H))^n v=0$. We call V_μ

the weight space of weight μ. If $\dim V_\mu<\infty$ for any $\mu\in\mathfrak{t}_k^\vee$, we define the character $\mathrm{ch}(V)$ of V as the formal sum $\mathrm{ch}(V)=\sum_{\mu\in t_k^\vee}(\dim V_\mu)e^\mu$.

Let τ be an automorphism of $\mathfrak{g}$ which normalizes $\mathfrak{t}$ and induces -1 on $\mathfrak{t}$. Then $\tau(\mathfrak{g}(\alpha))=\mathfrak{g}(-\alpha)$. For any $U(\mathfrak{g})$-module M, define a $\mathfrak{g}$-module structure in $\mathrm{Hom}_{\boldsymbol{C}}(M, \boldsymbol{C})$ by $\langle Af, u\rangle=-\langle f, \tau(A)u\rangle$ for $A\in\mathfrak{g}$, $f\in\mathrm{Hom}_{\boldsymbol{C}}(M, \boldsymbol{C})$ and $u\in M$. If M is a direct sum of weight spaces, and each weight space is of finite dimension, then put $M^*=\{f\in\mathrm{Hom}_{\boldsymbol{C}}(M, \boldsymbol{C})\mid f(M_\mu)=0$ except for finitely many $\mu\in\mathfrak{t}^\vee\}=\bigoplus_{\mu\in t^\vee}\mathrm{Hom}(M_\mu, \boldsymbol{C})$. Then $(M^*)^*=M$, and $\mathrm{ch}(M^*)=\mathrm{ch}(M)$.

1.2. Verma module. Extend $\lambda\in\mathfrak{t}_k^\vee=\mathrm{Hom}_k(\mathfrak{t}_k, k)$ to a linear character of the Lie algebra $\mathfrak{b}_k$ by putting $\lambda|\mathfrak{n}_k\equiv 0$. Let $k(\lambda)$ be the corresponding $U(\mathfrak{b}_k)$-module, 1_λ its basis element, and

$$M(\lambda)=M_k(\lambda)=M_k(\lambda, \mathfrak{b}_k)=U(\mathfrak{g}_k)\otimes_{U(\mathfrak{b}_k)}k(\lambda),$$

which is called a *Verma module* [33], [3] (cf. [9]). Denote its simple quotient by $V(\lambda)=V_k(\lambda)$.

1.3. A generalization of Verma modules. For a k-algebra A, put $U_A(\mathfrak{a})=U_k(\mathfrak{a})\otimes_k A$. For $a\in\mathfrak{t}_A^\vee$, let $A(a)=U_A(\mathfrak{b}_k)/(U_A(\mathfrak{b}_k)\mathfrak{n}_{+,k}+\sum_{H\in t_k}U_A(\mathfrak{b}_k)(H-a(H)))$, 1_a be the element of $A(a)$ corresponding to $1\in U_A(\mathfrak{b}_k)$, and $M_A(a)=U_A(\mathfrak{g}_k)\otimes_{U_A(\mathfrak{b}_k)}A(a)$, which we consider as a family of Verma modules. We can show that $M_A(a)$ is a free $U_A(\mathfrak{n}_{-,k})$-module generated by $u(a)=u_A(a):=1\otimes 1_a$. Let $M_A(a)_+=U_A(\mathfrak{n}_{-,k})\mathfrak{n}_{-,k}u_A(a)$, $J_A(a)$ be the (unique) maximal $U_A(\mathfrak{g}_k)$-submodule of $M_A(a)$ contained in $M_A(a)_+$, $V_A(a)=M_A(a)/J_A(a)$, and $v(a)=v_A(a):=(u_A(a) \bmod J_A(a))$.

Let L be an affine subspace of $\mathfrak{t}_k^\vee$, $A=A(L)=A_k(L)$ the algebra $k[L]$ of polynomial functions on L, $c=c_L$ the natural homomorphism $\mathfrak{t}_k\to A$, $M(L)=M_A(L):=M_{A(L)}(c_L)=U_{A(L)}(\mathfrak{g}_k)\otimes_{U_{A(L)}(\mathfrak{b}_k)}A(c_L)$, $M(L)_+=M_A(L)_+:=M_{A(L)}(c_L)_+$, $J(L)=J_A(L):=J_{A(L)}(c_L)$, and $V(L)=V_A(L):=V_{A(L)}(c_L)$. Let p be a prime ideal of $A(L)$, $k(p)$ the residue field at p, and $M(L, p)=M(L)\otimes_{A(L)}k(p)$, $V(L, p)=V(L)\otimes_{A(L)}k(p)$ etc. If $p(\lambda)$ is the maximal ideal of $A(L)$ consisting of polynomial functions vanishing at $\lambda\in L$, then $A(L)\to k(p(\lambda))=k$ is the evaluation at λ, which we shall denote by the same letter λ. Put

$$V(L, \lambda)=V_k(L, \lambda):=V(L, p(\lambda))=V(L)\otimes_{A(L),\lambda}k.$$

Let $p(\eta):=\{0\}$, $K=K(L):=k(p(\eta))$, $M_K(L):=M(L, p(\eta))=M(L)\otimes_{A(L)}K$, $V_K(L):=V(L, p(\eta))=V(L)\otimes_{A(L)}K$, etc. (specializations at the generic point η of L). Note that $K(L)$ is the quotient field of $A(L)$. Thus we can consider the composition $\eta=\eta_L(\in\mathfrak{t}_K^\vee)$ of $\mathfrak{t}_k\xrightarrow{c_L}A(L)\to K(L)$, the Verma module $M_K(\eta_L)$, and its simple quotient $V_K(\eta_L)$.

Lemma 1.4. (1) $M_K(\eta_L)=M_K(L)$. (2) $V_K(\eta_L)=V_K(L)$.

Proof. (1) is trivial. Consider $M_A(L)$ as a submodule of $M_K(L)$. Since $V_K(\eta)=M_K(\eta)/J_K(\eta)$ and $V_K(L)=K\otimes(M_A(L)/J_A(L))=M_K(\eta)/KJ_A(L)$, it remains to prove that

$$J_K(\eta)=KJ_A(L). \tag{1.4.1}$$

Since $J_K(\eta)\cap M_A(L)$ is a $U_A(\mathfrak{g}_k)$-submodule of $M_A(L)$ contained in $M_A(L)_+$, we have $KJ_A(L)\supset K(J_K(\eta)\cap M_A(L))=J_K(\eta)$. On the other hand, $KJ_A(L)$ is a $U_K(\mathfrak{g}_k)$-submodule of $M_K(\eta)$ contained in $M_K(\eta)_+$. Hence $KJ_A(L)\subset J_K(\eta)$.

Lemma 1.5. (1) *Let* $\lambda\in L$. *If* $V_k(L, \lambda)$ *is simple as a* $U(\mathfrak{g}_k)$*-module, then it is absolutely simple.* (2) $V_K(L)$ *is absolutely simple.*

Proof. These assertions follow from (1.4, (2)), and [9, 2.6.5 and 7.1.8, (iv)].

1.6. Field extension. Let k' be a field containing k, $L'=L\otimes_k k'$, $A'=A_{k'}(L')$, and K' the quotient field of A'. We can naturally consider $\mathfrak{t}_k$ (resp. $\mathfrak{t}_k^\vee$) as a subspace of $\mathfrak{t}_{k'}$ (resp. $\mathfrak{t}_{k'}^\vee$). Then $\mathfrak{t}_k^\vee=\{\lambda\in \mathrm{Hom}_{k'}(\mathfrak{t}_{k'}, k')\,|\,\lambda(\mathfrak{t}_k)\subset k\}$ and $L=\{\lambda\in L'\,|\,\lambda(\mathfrak{t}_k)\subset k\}$.

Lemma 1.7. (1) $V_A(L)\otimes_A A'=V_{A'}(L')$. (2) $V_K(\eta_L)\otimes_K K'=V_K(L)\otimes_K K'=V_{K'}(L')=V_{K'}(\eta_{L'})$. (3) $V_k(L, \lambda)\otimes_k k'=V_{k'}(L', \lambda)$ *for* $\lambda\in L(\subset L')$.

Proof. (1) Since $M_A(L)\otimes_A A'=M_{A'}(L')$, it suffices to show that $A'J_A(L)=J_{A'}(L')$. Since $M_{A'}(L')=U_{A'}(\mathfrak{n}_{-,k})u_{A'}(c)\cong U_{A'}(\mathfrak{n}_{-,k})$, we can take a free A'-basis $\{u_1, u_2, \cdots\}$ of $M_{A'}(L')$ in $U_k(\mathfrak{n}_{-,k})u_{A'}(c)$. An element u of $M_{A'}(L')$ belongs to $J_{A'}(L')$ if and only if $U(\mathfrak{g}_k)u\subset M_{A'}(L')_+$. For $n\in \boldsymbol{N}$ and $u=\sum a_i'u_i\in\sum_{i\leq n}A'u_i$, this condition can be written as a system of homogeneous A-linear equations in $(a_i')_{i\leq n}\in A'^n$. Since A' is flat over A, every solution (a_i') in A'^n can be expressed as an A'-linear combination of solutions in A^n [7, Chap. 1, § 2, Corollary 2 of Proposition 13]. Hence $(\sum_{i\leq n}A'u_i)\cap J_{A'}(L')=A'\cdot(\sum_{i\leq n}Au_i\cap J_A(L))$. Letting $n\to\infty$, we get $J_{A'}(L')=A'J_A(L)$. (2) By (1.4, (2)), it suffices to prove the equality $V_K(L)\otimes K'=V_{K'}(L')$, which can be proved in the same way as above. (3) By (1), we have $V_{k'}(L', \lambda)=V_{A'}(L')\otimes_{A',\lambda}k'=(V_A(L)\otimes_A A')\otimes_{A',\lambda}k'=V_A(L)\otimes_{A,\lambda}k'=(V_A(L)\otimes_{A,\lambda}k)\otimes_k k'=V_k(L, k)\otimes_k k'$.

1.8. Let $r=\dim L$. If $0\in L$, we can find a k'-linear basis $\lambda_1, \cdots, \lambda_r$ of L' contained in L. In this case, put $\lambda_0=0$. If $0\notin L$, we can find linearly independent elements $\lambda_0, \lambda_1, \cdots, \lambda_r\in L$ such that $L'=\{\sum_{i=0}^r a_i\lambda_i\,|\,a_i\in k', \sum_{i=0}^r a_i=1\}$. This equality also holds in the case where $0\in L$. Take an element $\lambda=\sum_{i=0}^r a_i\lambda_i$ of L'. Let $k\langle\lambda\rangle$ be the subfield of K' generated by k and $\lambda(\mathfrak{t}_k)$. Then $k\langle\lambda\rangle=k(a_1, \cdots, a_r)$. Hence the transcendental degree $\mathrm{tr.deg}_k k\langle\lambda\rangle$ of $k\langle\lambda\rangle$ over k is at most $r=\dim L$.

1.9. If $k'=\boldsymbol{C}$ and $\mathrm{tr.deg}_{\boldsymbol{Q}}k<\infty$, then $\{\lambda\in L' \mid \mathrm{tr.deg}_k k\langle\lambda\rangle=\dim L\}\neq\phi$ is stable with respect to the translations by elements of L and hence everywhere dense in $L'(\subset \mathfrak{t}_{\boldsymbol{C}}^{\vee}=\mathfrak{t}^{\vee})$.

Lemma 1.10. *If* $\mathrm{tr.deg}_k k\langle\lambda\rangle=\dim_k L$, *then* $V_{k'}(L', \lambda)$ *is simple.*

Proof. By (1.5, (1)) and (1.7, (3)), we may assume that $k'=k\langle\lambda\rangle$. Let $I_L=\{\varphi\in k[\mathfrak{t}_k^{\vee}]\,;\,\varphi|L\equiv 0\}$, $H_1, \cdots, H_r$ be elements of $\mathfrak{t}_k$ such that $\lambda_i(H_j)=\delta_{ij}$ $(0\leqq i\leqq r, 1\leqq j\leqq r)$, and $\bar{H}_i$ the corresponding elements of $k[\mathfrak{t}_k^{\vee}]/I_L$. Then A is the symmetric algebra $k[\bar{H}_1, \cdots, \bar{H}_r]$ and K is its quotient field $k(\bar{H}_1, \cdots, \bar{H}_r)$. Since $a_i=\lambda(H_i)$ $(1\leqq i\leqq r)$ are algebraically independent over k, the homomorphism λ: $A=k[\bar{H}_1, \cdots, \bar{H}_r]\to k[\lambda(H_1), \cdots, \lambda(H_r)]$ can be extended to an isomorphism λ: $K=k(\bar{H}_1, \cdots, \bar{H}_r)\xrightarrow{\sim} k(\lambda(H_1), \cdots, \lambda(H_r))=k'$. Since $V_K(L)\otimes_{K,\lambda}k'=V_A(L)\otimes_{A,\lambda}k'=(V_A(L)\otimes_A A')\otimes_{A',\lambda}k'=V_{k'}(L', \lambda)$, we get the assertion by (1.5, (2)).

1.11. From now on, the base field is always $\boldsymbol{C}$. Later in (4.2), (4.6) and (4.7), for a special $\boldsymbol{C}$-subspace L of $\mathfrak{t}^{\vee}$ defined over $\boldsymbol{Q}$, we shall construct using a family of twisted $\mathcal{D}$-modules, a certain $U_A(\mathfrak{g})$-module $M_A(w)^*$ $(A=A(L))$ and a $U_A(\mathfrak{g})$-homomorphism φ: $M_A(L)\to M_A(w)^*$ satisfying the following conditions:
(1.11.1) Let j: $M_A(w)^*\to M_A(w)^*\otimes_A K$ be the canonical morphism and $M_A(w)^*(\mu)=j^{-1}((M_A(w)^*\otimes_A K)_{\eta-\mu})$ for $\mu\in Q_+$. Then each $M_A(w)^*(\mu)$ is a free A-module of finite type and $M_A(w)^*=\bigoplus_{\mu\in Q_+}M_A(w)^*(\mu)$.
(1.11.2) Let φ_λ: $M(\lambda)=M_A(L)\otimes_{A,\lambda}\boldsymbol{C}\to M_A(w)^*\otimes_{A,\lambda}\boldsymbol{C}(\lambda\in L)$ be the homomorphism induced by φ. Then $\varphi_\lambda\not\equiv 0$ for any $\lambda\in L$.
(1.11.3) There exists an open dense subset L^0 of L with respect to the classical topology such that $M_A(w)^*\otimes_{A,\lambda}\boldsymbol{C}$ is a simple $U(\mathfrak{g})$-module for $\lambda\in L^0$.

The remainder of this section is devoted to the proof of the following lemma.

Lemma 1.12. *Assume that* $L(\subset\mathfrak{t}^{\vee})$ *is defined over* $\boldsymbol{Q}$. *If a* $U_A(\mathfrak{g})$-*module* $M_A(w)^*$ *and a* $U_A(\mathfrak{g})$-*homomorphism* φ: $M_A(L)\to M_A(w)^*$ *satisfy* (1.11.1)-(1.11.3), *then* $\mathrm{ch}V(L, \lambda)=\mathrm{ch}(M_A(w)^*\otimes_{A,\lambda}\boldsymbol{C})$ $(\lambda\in L)$.

1.13. Let us fix $\lambda\in L$, and an affine line L_1 of L containing λ. Let $A_1=A(L_1)$, and $K_1=K(L_1)$ be the quotient field of A_1. Then there are natural morphisms $A\to A_1\xrightarrow{\lambda}\boldsymbol{C}$, whose composition is λ: $A\to\boldsymbol{C}$. Let $M_{A_1}(w)^*=M_A(w)^*\otimes_A A_1$, $M_{A_1}(w)^*(\mu)=M_A(w)^*(\mu)\otimes_A A_1$, $\varphi_1=\varphi\otimes_A A_1$, etc. As a first step of the proof of (1.12), let us show that the natural homomorphism $M_{A_1}(L_1)\to V_{A_1}(L_1)$ induces

$$\varphi_1(M_{A_1}(L_1))=M_{A_1}(L_1)/\ker\varphi_1 \longrightarrow V_{A_1}(L_1), \tag{1.13.1}$$

i.e., that $\ker\varphi_1\subset M_{A_1}(L_1)_+$. Assume that $\ker\varphi_1\not\subset M_{A_1}(L_1)_+$. Take an element

$u \in \ker\varphi_1$ such that $u=u_+ + u_0$ with $u_+ \in M_{A_1}(L_1)_+$ and $0 \neq u_0 \in A_1 u(c)$. Then

$$\varphi_1(u_0) = -\varphi_1(u_+) \in M_{A_1}(w)^*(0) \cap \bigoplus_{\mu \neq 0} M_{A_1}(w)^*(\mu) = 0 \tag{1.13.2}$$

by (1.11.1). Since $u_0 \neq 0$, there exists $\lambda' \in L_1$ such that the image $\bar{u}_0$ of u_0 by the natural homomorphism $M_{A_1}(L_1) \to M_{A_1}(L_1) \otimes_{A_1, \lambda'} \boldsymbol{C} = M(\lambda')$ is non-zero. Let $\varphi' := \varphi_1 \otimes_{A_1, \lambda'} \boldsymbol{C} = \varphi \otimes_{A, \lambda'} \boldsymbol{C}$. By (1.13.2), $\varphi'(\bar{u}_0)=0$. Since $\bar{u}_0$ is a generator of $M(\lambda')$, this contradicts (1.11.2). Thus we get (1.13.1). Similarly, letting $\varphi_{K_1} = \varphi_1 \otimes K_1$, we get a surjective $U_{K_1}(\mathfrak{g})$-homomorphism

$$M_{K_1}(L_1)/\ker\varphi_{K_1} \longrightarrow V_{K_1}(L_1). \tag{1.13.3}$$

Lemma 1.14. *If* $L^0 \cap L_1 \neq \phi$, *then* $M_{K_1}(w)^* = M_{A_1}(w)^* \otimes_{A_1} K_1$ *is a simple* $U_{K_1}(\mathfrak{g})$*-module. (See* (1.11.3) *for* L^0*.)*

Proof. Let N be a $U_{K_1}(\mathfrak{g})$-submodule of $M_{K_1}(w)^*$. By (1.11.1), we can regard $M_{A_1}(w)^*$ as a submodule of $M_{K_1}(w)^*$. Let $N_{A_1} = M_{A_1}(w)^* \cap N$ and $N_{A_1}(\mu) = M_{A_1}(w)^*(\mu) \cap N$. For any $\mu \in Q_+$, there is a finite subset $L_1(\mu)$ of L_1 such that the quasi-coherent sheaf on L_1 obtained by localizing the A_1-module $M_{A_1}(w)^*(\mu)/N_{A_1}(\mu)$ is locally free on $L_1 \backslash L_1(\mu)$. (Note that a finitely generated module over the principal ideal domain A_1 is a direct sum of a free module and a torsion module.) For $\mu \in Q_+$, take $a(\mu) \in A_1$ so that $L_1 \backslash L_1(\mu) = \operatorname{Spec} A_1[a(\mu)^{-1}]$. Take $\lambda' \in (L^0 \cap L_1) \backslash \bigcup_{\mu \in Q_+} L_1(\mu) \neq \phi$. Then λ' gives an algebra homomorphism $\lambda' \colon A_1[a(\mu)^{-1}] \to \boldsymbol{C}$ for any $\mu \in Q_+$, and

$$\begin{aligned} 0 = &\operatorname{Tor}_1^{A_1[a(\mu)^{-1}]}(((M_{A_1}(w)^*(\mu)/N_{A_1}(\mu))[a(\mu)^{-1}], \boldsymbol{C}(\lambda')) \\ &\longrightarrow N_{A_1}(\mu) \otimes_{A_1, \lambda'} \boldsymbol{C} \longrightarrow M_{A_1}(w)^*(\mu) \otimes_{A_1, \lambda'} \boldsymbol{C} \end{aligned} \tag{1.14.1}$$

is exact for any $\mu \in Q_+$. Any $u \in N_{A_1}$ can be uniquely expressed as $u = \sum_{\mu \in Q_+} u(\mu)$ with $u(\mu) \in N \cap M_{K_1}(w)^*(\mu)$. On the other hand, by (1.11.1), $u \in N_{A_1} \subset M_{A_1}(w)^*$ can be uniquely expressed as $\sum u(\mu)$ with $u(\mu) \in M_{A_1}(w)^*(\mu)$. Hence $u(\mu) \in N \cap M_{A_1}(w)^*(\mu) = N_{A_1}(\mu)$ and

$$N_{A_1} = \bigoplus_{\mu \in Q_+} N_{A_1}(\mu). \tag{1.14.2}$$

By (1.14.1) and (1.14.2),

$$0 \longrightarrow N_{A_1} \otimes_{A_1, \lambda'} \boldsymbol{C} \longrightarrow M_{A_1}(w)^* \otimes_{A_1, \lambda'} \boldsymbol{C} \longrightarrow (M_{A_1}(w)^*/N_{A_1}) \otimes_{A_1, \lambda'} \boldsymbol{C} \longrightarrow 0$$

is exact. Since $\lambda' \in L^0$, $M_{A_1}(w)^* \otimes_{A_1, \lambda'} \boldsymbol{C} = M_A(w)^* \otimes_{A, \lambda'} \boldsymbol{C}$ is a simple $U(\mathfrak{g})$-module. Hence $N_{A_1} \otimes_{A_1, \lambda'} \boldsymbol{C} = 0$ or $(M_{A_1}(w)^*/N_{A_1}) \otimes_{A_1, \lambda'} \boldsymbol{C} = 0$. Assume that $N_{A_1} \otimes_{A_1, \lambda'} \boldsymbol{C} = 0$. Then $N_{A_1}(\mu) \otimes_{A_1, \lambda'} \boldsymbol{C} = 0$ for any $\mu \in Q_+$ by (1.14.2). Since the submodule $N_{A_1}(\mu)$ of the free module $M_{A_1}(w)^*(\mu)$ is also A_1-free, $N_{A_1}(\mu) \otimes_{A_1, \lambda'} \boldsymbol{C} = 0$ implies $N_{A_1}(\mu) = 0$, $N_{A_1} = 0$ and hence $N=0$. Next assume that $(M_{A_1}(w)^*/N_{A_1}) \otimes_{A_1, \lambda'} \boldsymbol{C} = 0$. Then $(M_{A_1}(w)^*(\mu)/N_{A_1}(\mu)) \otimes_{A_1, \lambda'} \boldsymbol{C} = 0$ by (1.11.1) and (1.14.2). Since $(M_{A_1}(w)^*(\mu)/N_{A_1}(\mu))[a(\mu)^{-1}]$ is a free $A_1[a(\mu)^{-1}]$-module, it implies $(M_{A_1}(w)^*(\mu)/N_{A_1}(\mu))$

$[a(\mu)^{-1}]=0$, i.e., $M_{A_1}(w)^*(\mu)[a(\mu)^{-1}]=N_{A_1}(\mu)[a(\mu)^{-1}]$. Considering the K_1-subspaces of $M_{K_1}(w)^*(\mu)$ generated by the both members, we get $M_{K_1}(w)^*(\mu)=M_{K_1}(w)^*(\mu)\cap N$, and hence $N=M_{K_1}(w)^*$. Therefore, $M_{K_1}(w)^*$ is a simple $U_{K_1}(\mathfrak{g})$-module.

Lemma 1.15. *If* $L^0\cap L_1\neq\phi$, *then* $\varphi_{K_1}=\varphi_1\otimes K_1: M_{K_1}(L_1)\to M_{K_1}(w)^*$ *is surjective.*

Proof. By (1.11.1) and (1.11.2), $\varphi_{K_1}\not\equiv 0$. Since $M_{K_1}(w)^*$ is a simple $U_{K_1}(\mathfrak{g})$-module by (1.14), we get the assertion.

Lemma 1.16. *If* $L^0\cap L_1\neq\phi$, *then* $M_{K_1}(L_1)/\mathrm{Ker}\varphi_{K_1}\xrightarrow{\sim}V_{K_1}(L_1)$.

Proof. By (1.14) and (1.15), $M_{K_1}(L_1)/\ker\varphi_{K_1}$ is a simple $U_{K_1}(\mathfrak{g})$-module. Hence (1.13.3) is an isomorphism. (Note that $V_{A_1}(L_1)=(M_{A_1}(L_1)_+/J_{A_1}(L_1))\oplus A_1v(c)$ and hence $V_{K_1}(L_1)\neq 0$.)

1.17. Let $M_K(L)(\mu)=M_K(L)_{\eta-\mu}$, and $M_A(L)(\mu)=M_A(L)\cap M_K(L)(\mu)$ for $\mu\in Q_+$. Let $u\in J_A(L)$ and decompose it as $u=\sum_{\mu\in Q_+}u(\mu)$ with $u(\mu)\in M_A(L)(\mu)$. Since $J_A(L)$ is a $U_A(\mathfrak{t})$-stable and $u(\mu)$'s belong to different weight spaces, $u(\mu)\in KJ_A(L)$. Hence

(1.17.1) $$J_A(L)\subset\bigoplus_{\mu\in Q_+}KJ_A(L)\cap M_A(L)(\mu).$$

The right side is contained in $M_A(L)_+$ and stable under the actions of A, $\mathfrak{t}$ and $\mathfrak{g}(\alpha)$ $(\alpha\in R)$. Thus the right side of (1.17.1) is also a $U_A(\mathfrak{g})$-submodule contained in $M_A(L)_+$ and hence (1.17.1) is an equality. Put $J_A(L)(\mu)=KJ_A(L)\cap M_A(L)(\mu)$, $V_A(L)(\mu)=M_A(L)(\mu)/J_A(L)(\mu)$, and $V(L,\lambda)(\mu)=V(L,\lambda)_{\lambda-\mu}$ for $\lambda\in L$. Define $J_{A_1}(L_1)(\mu)$ and $V_{A_1}(L_1)(\mu)$ in the same way. Then

(1.17.2) $$J_A(L)=\bigoplus_{\mu\in Q_+}J_A(L)(\mu),$$

(1.17.3) $$V_A(L)=\bigoplus_{\mu\in Q_+}V_A(L)(\mu),$$

(1.17.4) $$V(L,\lambda)=V_A(L)\otimes_{A,\lambda}\boldsymbol{C}=\bigoplus_{\mu\in Q_+}V_A(L)(\mu)\otimes_{A,\lambda}\boldsymbol{C},\quad\text{and}$$

(1.17.5) $$V(L,\lambda)(\mu)=V_A(L)(\mu)\otimes_{A,\lambda}\boldsymbol{C}=V_{A_1}(L_1)(\mu)\otimes_{A_1,\lambda}\boldsymbol{C}.$$

We can also see that (1.13.1) induces a surjective $U_{A_1}(\mathfrak{g})$-homomorphism

(1.17.6) $$\begin{aligned}\varphi_1(M_{A_1}(L_1)(\mu))&=\mathrm{image}(M_{A_1}(L_1)(\mu)\longrightarrow M_{A_1}(L_1)/\ker\varphi_1)\\&\longrightarrow\mathrm{image}(M_{A_1}(L_1)(\mu)\longrightarrow V_{A_1}(L_1))=V_{A_1}(L_1)(\mu).\end{aligned}$$

Since $V_A(L)(\mu)$ is an A-module of finite type, we can prove by the "Nakayama's lemma" that for any $\mu\in Q_+$ and $\lambda\in L$, there is a Zariski open neighbourhood $U(\mu)$ of λ in L such that

(1.17.7) $$\dim V(L, \lambda')(\mu) \leqq \dim V(L, \lambda)(\mu)$$

for any $\lambda' \in U(\mu)$. By (1.7, (3)), we may assume that $U(\mu)$ are defined over $\boldsymbol{Q}\langle\lambda\rangle$.

1.18. *Proof of* (1.12). Let L be a $\boldsymbol{C}$-subspace of $\mathfrak{t}^{\vee}$ defined over $\boldsymbol{Q}$, λ an element of L, and $U(\mu)$ as above. Take $\lambda' \in L^0$ so that $\mathrm{tr.deg}_{\boldsymbol{Q}\langle\lambda\rangle}\boldsymbol{Q}\langle\lambda\rangle\langle\lambda'\rangle = \dim L$ (cf. (1.9)). Since $U(\mu)$ are defined over $\boldsymbol{Q}\langle\lambda\rangle$, $\lambda' \in U(\mu)$ for any μ. Let L_1 be an affine line containing λ and λ'. By (1.11.2), the image of $\varphi_{\lambda'}$ is a non-zero $U(\mathfrak{g})$-submodule of $M_A(w)^* \otimes_{A, \lambda'} \boldsymbol{C}$. Since $\lambda' \in L^0$, $M_A(w)^* \otimes_{A, \lambda'} \boldsymbol{C}$ is simple by (1.11.3). Hence $\varphi_{\lambda'}$ is surjective, and $M_A(w)^* \otimes_{A, \lambda'} \boldsymbol{C}$ is a simple quotient of the Verma module $M(\lambda')$. By (1.10), $V(L, \lambda')$ is also a simple quotient of the same Verma module. Hence

(1.18.1) $$M_A(w)^* \otimes_{A, \lambda'} \boldsymbol{C} \cong V(L, \lambda').$$

By (1.15), $\varphi_{K_1} = \varphi_1 \otimes_{A_1} K_1 : M_{K_1}(L_1) \to M_{K_1}(w)^*$ is surjective. Since the homomorphisms induced by $\varphi_1 \otimes_{A_1} K_1$ between the weight spaces are also surjective, $\varphi_1(M_{A_1}(L_1)(\mu))$ is an A_1-lattice of the free A_1-module $M_{A_1}(w)^*(\mu)$. Since A_1 is a principal ideal domain,

(1.18.2) $$\varphi_1(M_{A_1}(L_1)(\mu)) \cong M_{A_1}(w)^*(\mu)$$

as A_1-modules. By (1.17.5), the surjection (1.17.6) induces a surjection $\varphi_1(M_{A_1}(L_1)(\mu)) \otimes_{A_1, \lambda} \boldsymbol{C} \to V(L, \lambda)(\mu)$ for any $\mu \in Q_+$. Thus by (1.18.2), we get the inequality

$$\dim(M_{A_1}(w)^*(\mu) \otimes_{A_1, \lambda} \boldsymbol{C}) \geqq \dim V(L, \lambda)(\mu).$$

On the other hand,

$$\begin{aligned} &\dim V(L, \lambda)(\mu) \\ &\geqq \dim V(L, \lambda')(\mu) && \text{by (1.17.7)} \\ &= \dim(M_A(w)^*(\mu) \otimes_{A, \lambda'} \boldsymbol{C}) && \text{by (1.18.1)} \\ &= \dim(M_A(w)^*(\mu) \otimes_{A, \lambda} \boldsymbol{C}) && \text{by (1.11.1)} \\ &= \dim(M_{A_1}(w)^*(\mu) \otimes_{A_1, \lambda} \boldsymbol{C}). \end{aligned}$$

Thus we get the desired equality.

Remark 1.19. (1) If $L = \{\lambda\}$, then $V(L, \lambda) = V_k(\lambda)$. (2) If $L = \mathfrak{t}^{\vee}$, then $V(L, \lambda) = M_k(\lambda)$ for any λ. Thus our module is a generalization of Verma modules and also of their simple quotients.

Let us prove (2). By [9, 7.6.24], $M_K(\eta)$ is simple in this case. Hence $J_K(\eta) = 0$. By (1.4.1), it follows that $J_A(L) = 0$, $V_A(L) = M_A(L)$, and $V(L, \lambda) = M_k(\lambda)$.

§ 2. Twisted $\mathcal{D}$-Modules on Homogeneous Spaces

2.0. The purpose of this section is to review the concept of twisted ring of differential operators $\mathcal{D}_X(\lambda)$ due to Beilinson-Bernstein [1] (cf. (2.2)), and to define the $\mathcal{D}_X(\lambda)$-modules $\mathcal{O}(V_0, \lambda)$ (cf. (2.5)) and $H^k_T(X, \mathcal{O}_X(\lambda))$ (cf. (2.9)). We also consider their 'relative versions' (cf. (2.10)-(2.13)).

2.1. First we fix some notations used in this section. For a smooth algebraic variety X over the complex number field $\boldsymbol{C}$, denote the underlying complex manifold by X^{an}. A morphism $f: X \to Y$ of smooth algebraic varieties is denoted by f^{an} if it is considered as a morphism between the underlying complex manifolds. For a complex manifold $\tilde{X}$, we denote the sheaf of holomorphic functions by $\mathcal{O}^{an} = \mathcal{O}^{an}_{\tilde{X}}$. We write $\mathcal{O}^{an}_X$ for $\mathcal{O}^{an}_{X^{an}}$. Let $\iota = \iota_X : (X^{an}, \mathcal{O}^{an}) \to (X, \mathcal{O})$ be the morphism of ringed spaces induced by the identity mapping, where $\mathcal{O} = \mathcal{O}_X$ is the sheaf of regular functions. Let G be a complex algebraic group and $\mathfrak{g}$ its Lie algebra. For an algebraic action $\sigma: G \times X \to X$, let $\sigma(g)x = \sigma(g, x)$, and $(\sigma(A)f)(x) = (d/dt)f(\sigma(e^{-tA})x)|_{t=0}$ for $A \in \mathfrak{g}$ and a smooth function f on X. Define G-actions R and L on G itself by $R(g)x = xg^{-1}$ and $L(g)x = gx$ for any $g, x \in G$.

2.2. Twisted ring of differential operators $\mathcal{D}_X(\lambda)$. Let H be a connected algebraic subgroup of G, $\mathfrak{h} = \mathrm{Lie}(H)$ and $\lambda: \mathfrak{h} \to \boldsymbol{C}$ a character of the Lie algebra $\mathfrak{h}$. Let $F = F(\lambda) = F(\lambda, H)$ be the sheaf on G^{an} of local holomorphic functions f such that $R(A)f = -\lambda(A)f$ for any $A \in \mathfrak{h}$. Let $X = G/H$ and $p = p_X: G \to X = G/H$ be the natural projection. Since F has a $(p^{an})^{-1}\mathcal{O}^{an}_X$-module structure, it also has an $\iota_G^{-1} p^{-1}\mathcal{O}_X$-module structure. Since $L(A)$ $(A \in \mathfrak{g})$ preserves F, F has a structure of $\mathfrak{g}$-module. Let $\mathcal{D}_X(\lambda)$ be the subring of $\iota_* p^{an}_* \underline{End}_C(F)$ generated by the endomorphisms induced by $\mathcal{O}_X$ and $\mathfrak{g}$. Here $\underline{End}$ is the sheaf of local endomorphisms. Then $\mathcal{D}_X(\lambda)$ is a twisted ring of (algebraic) differential operators (cf. [19, 2.3.3]). Since $\mathcal{D}_X(\lambda)$ is locally isomorphic to $\mathcal{D}_X$ [19, 4.16], we can naturally generalize definitions concerning $\mathcal{D}_X$-modules to general $\mathcal{D}_X(\lambda)$-modules. We shall use the concept of characteristic variety $SS(\mathcal{M})$ of a $\mathcal{D}_X(\lambda)$-module $\mathcal{M}$, holonomicity, etc. without further explanation.

2.3. $\mathcal{D}_X(\lambda)$**-module** $\mathcal{O}^{an}(V_0, \lambda)$. Assume that there exists an algebraic subvariety V_0 of G such that $p|V_0: V_0 \to X(=G/H)$ is an open immersion, and let V be an open neighbourhood of V_0 in G with respect to the classical topology such that $p(V_0) = p(V)$ and each fibre of $p|V$ is connected and simply connected. Then the restriction $f \to f|V_0$ defines an isomorphism

$$r = r_{V_0}: (p^{an}|V)_*(F|V) \longrightarrow (p^{an}|V_0)_*\mathcal{O}^{an}_{V_0}(\cong \mathcal{O}^{an}_{p(V_0)}).$$

Since $\mathcal{D}_X(\lambda)|p(V_0)$ acts on $\iota_*(p^{an}|V)_*(F|V)$, $\iota_*\mathcal{O}^{an}_{p(V_0)}$ has a $\mathcal{D}_X(\lambda)$-module structure.

(Here and below, we say that $\iota_*\mathcal{O}^{an}_{p(V_0)}$ etc. is a $\mathcal{D}_X(\lambda)$-module instead of a $(\mathcal{D}_X(\lambda)|p(V_0))$-module etc., if there is no fear of confusion.) This $\mathcal{D}_X(\lambda)$-module structure of $\iota_*\mathcal{O}^{an}_{p(V_0)}$ does not depend on the choice of a neighbourhood V of a given V_0, but as we shall see later in (2.7), it depends on the choice of V_0. If we need to specify this dependence, we write $\mathcal{O}^{an}(V_0, \lambda)$ for $\iota_*\mathcal{O}^{an}_{p(V_0)}$.

2.4. Let H_0 be a connected, simply connected, open neighbourhood of the identity element e of H, and $\mathcal{H}$ the set of such open neighbourhoods H_0. For any $H_0 \in \mathcal{H}$, a linear character λ of the Lie algebra $\mathfrak{h}$ determines a unique holomorphic function $\lambda' : H_0 \to C^{\times}$ such that $\lambda'(e)=1$ and $\lambda'(xy)=\lambda'(x)\lambda'(y)$ whenever x, y and xy are contained in H_0. In the sequel, until the end of (2.9), we fix a character λ of $\mathfrak{h}$, and write λ for λ'. Let $i: V_0 \times H \to V_0 H$ be the natural isomorphism. Then for $H_0 \in \mathcal{H}$, $V=V_0H_0$ satisfies the condition of (2.3) and

(2.4.1) $$(i^{an}|V_0\times H_0)^{-1}(F(\lambda, H)|V)=\mathcal{O}^{an}_{V_0}\otimes_C C\lambda^{-1}.$$

Lemma 2.5. *The twisted ring of (algebraic) differential operators $\mathcal{D}_X(\lambda)$ preserves the subsheaf $\mathcal{O}_{p(V_0)}$ of $\iota_*\mathcal{O}^{an}_{p(V_0)}$. We denote this $\mathcal{D}_X(\lambda)$-module $\mathcal{O}_{p(V_0)}$ by $\mathcal{O}(V_0, \lambda)$ if we need to specify the dependence on V_0.*

We omit the proof, since it is essentially contained in the proof of (2.12).

2.6. Dependence of $\mathcal{O}(V_0, \lambda)$ on V_0. If V_0 and V_0' are algebraic subvarieties of G such that $p(V_0)=p(V_0')$ and, $p: V_0 \to X$ $(=G/H)$ and $p: V_0' \to X$ are both open immersions. Then there exists a unique morphism $s: V_0 \to H$ such that $V_0'=\{gs(g)|g\in V_0\}$.

Lemma 2.7. *Let V_0, V_0' and s be as above. The $\mathcal{D}_X(\lambda)$-module structure of $\mathcal{O}(V_0, \lambda)$ (resp. $\mathcal{O}^{an}(V_0, \lambda)$) and $\mathcal{O}(V_0', \lambda)$ (resp. $\mathcal{O}^{an}(V_0', \lambda)$) are the same if and only if $\lambda\circ s$ is locally constant on $s^{-1}(H_0)\cap V_0$ for any $H_0\in\mathcal{H}$. (Note that $\mathcal{O}(V_0, \lambda)=\mathcal{O}(V_0', \lambda)=\mathcal{O}_{p(V_0)}$ and $\mathcal{O}^{an}(V_0, \lambda)=\mathcal{O}^{an}(V_0', \lambda)=\iota_*\mathcal{O}^{an}_{p(V_0)}$ as sheaves on $p(V_0)$.)*

Proof. It is enough to consider the $\mathfrak{g}$-module structures on $\mathcal{O}^{an}(V_0, \lambda)$ and $\mathcal{O}^{an}(V_0', \lambda)$. Although we have assumed V_0 and V_0' to be algebraic subvarieties, we may assume them to be analytic subvarieties as far as we are dealing with the analytic case. Since the problem is local with respect to the classical topology, we may shrink V_0 arbitrarily. For a given $v_0\in V_0$, we can find $H_0\in\mathcal{H}$ which contains $s(v_0)$ and $s(v_0)^{-1}$. Since in a small neighbourhood of v_0, the value of s is always contained in H_0, we may assume from the beginning that $s(V_0)\subset H_0$. Then $V:=V_0H_0$ is an open neighbourhood of V_0 and V_0'. Let $A\in\mathfrak{g}$, $f\in\mathcal{O}^{an}_{p(V_0)}=\mathcal{O}^{an}_{p(V_0')}$, $r_{V_0}^{-1}(fp|V_0)=f_0$ and $r_{V_0'}^{-1}(fp|V_0')=f_0'$. (See (2.3) for r_{V_0} and $r_{V_0'}$.) Then $f_0|V_0=fp|V_0$ and $f_0(vh)=f_0(v)\lambda(h)^{-1}$ for any $v\in V_0$ and $h\in H_0$. Analogous equalities hold for f'_0 and V'_0. For any $v_0 \in V_0$ and $t \in \boldsymbol{C}$, $e^{-tA}v_0$ can be uniquely expressed as $e^{-tA}v_0 = vh$ with $v = v(t) \in V_0$ and $h = h(t) \in H_0$, if $|t|$ is sufficiently small. Then $f_0(e^{-tA}v_0) = (fp)(v)\lambda(h)^{-1}$ and $f'_0(e^{-tA}v_0 s(v_0)) = f'_0(vs(v)s(v)^{-1}hs(v_0)) = (fp)(v)\lambda(s(v)^{-1}hs(v_0))^{-1}$. Note that, if $|t|$ is sufficiently small, $v(t) \doteqdot v_0$, $h(t) \doteqdot e$ and $s(v(t))^{-1}h(t)s(v_0) \doteqdot e$. Hence $s(v(t))^{-1}h(t)s(v_0) \in H_0$ and $\lambda(s(v)^{-1}hs(v_0))$ is defined. Moreover $s(v(t))^{\pm 1} \doteqdot s(v_0)^{\pm 1} \in H_0$ and $h(t)s(v_0) \doteqdot s(v_0) \in H_0$ etc. Thus $\lambda(s(v)^{-1}hs(v_0)) = \lambda(s(v)^{-1})\lambda(hs(v_0)) = \lambda(s(v)^{-1})\lambda(h)\lambda(s(v_0)) = \lambda(s(v)s(v_0)^{-1})^{-1}\lambda(h)$ and $f'_0(e^{-tA}v_0 s(v_0)) = (fp)(v)\lambda(h(t))^{-1}\lambda(s(v(t))s(v_0)^{-1})$. Hence

$$\frac{d}{dt} f_0(e^{-tA}v_0)|_{t=0} = \frac{d}{dt} fp(v(t))\lambda(h(t))^{-1}|_{t=0} \tag{2.7.1}$$

and

$$\begin{aligned} &\frac{d}{dt} f'_0(e^{-tA}v_0 s(v_0))|_{t=0} \\ &= \frac{d}{dt}(fp)(v(t))\lambda(h(t))^{-1}|_{t=0} + (fp)(v_0)\frac{d}{dt}\lambda(s(v(t))s(v_0)^{-1})|_{t=0}. \end{aligned} \tag{2.7.2}$$

Thus (2.7.1) coincides with (2.7.2) if and only if

$$\frac{d}{dt}\lambda(s(v(t))s(v_0)^{-1})|_{t=0} = 0. \tag{2.7.3}$$

Since $\lambda(s(v(t))s(v_0)^{-1}) = \lambda(s(v(t)))\lambda(s(v_0)^{-1})$ if $|t|$ is sufficiently small, the condition (2.7.3) holds for any $v_0 \in V_0$ and $A \in \mathfrak{g}$ if and only if $\lambda \circ s$ is locally constant on V_0.

2.8. Let S be a smooth algebraic variety, T a closed subvariety of S, and $I = \{f \in \mathcal{O}_S \mid f \equiv 0 \text{ on } T\}$. For an $\mathcal{O}_S$-module M, let $\Gamma_T(M) = \varinjlim_m \underline{Hom}_{\mathcal{O}_S}(\mathcal{O}_S/I^m, M)$. Here $\underline{Hom}$ denotes the sheaf of local homomorphisms. Let $\bar{S}$ be another smooth algebraic variety containing S as an open dense subset, and $j: S \to \bar{S}$ the inclusion mapping. For an $\mathcal{O}_{\bar{S}}$-module M, put $\Gamma_T(M) := j_*\Gamma_T(M|S)$, $\Gamma_T(\bar{S}, M) := \Gamma(\bar{S}, \Gamma_T(M))$, $H^i_T(M) := H^i(R\Gamma_T(M))$, and $H^i_T(\bar{S}, M) := H^i(R\Gamma_T(\bar{S}, M))$. Note that $\Gamma_T(M)$ etc. depend only on $j^{-1}M$, and hence they can be defined also for $\mathcal{O}_S$-modules.

2.9. Let V_0 etc. be as before. Let $j: p(V_0) \to X$ be the inclusion mapping of the open subvariety $p(V_0)$ of X, and T a closed subvariety of $p(V_0)$. Then the $\mathcal{D}_X(\lambda)$-module structure of $\mathcal{O}_{p(V_0)} = \mathcal{O}(V_0, \lambda)$ induces $\mathcal{D}_X(\lambda)$-module structures in $H^i_T(\mathcal{O}_{p(V_0)})$ and $j_*H^i_T(\mathcal{O}_{p(V_0)})$. We shall (abusively) denote the latter sheaf by $H^i_T(\mathcal{O}_X(\lambda))$, if there is no fear of confusion. Similarly, we sometimes denote the $\Gamma(X, \mathcal{D}_X(\lambda))$-module $H^i_T(X, \mathcal{O}(V_0, \lambda))$ by $H^i_T(X, \mathcal{O}(\lambda))$.

2.10. In the remainder of this section, we shall consider 'a relative version' of what we have considered. Let E be a subvariety of $\mathfrak{h}^\vee = \mathrm{Hom}_{Lie\ algebra}(\mathfrak{h}, \boldsymbol{C})$. We define G-actions $R = R_E$ and $L = L_E$ on $G \times E$ by $R(g)(g', \lambda) = (g'g^{-1}, \lambda)$ and $L(g)(g', \lambda) = (gg', \lambda)$ for $g, g' \in G$ and $\lambda \in E$. Let $F(c) = F(c, H)$ be the sheaf on $(G \times E)^{an}$ of local holomorphic functions f such that $(R(A)f)(g, \lambda) = -\lambda(A)f(g, \lambda)$ for $A \in \mathfrak{h}$ and $(g, \lambda) \in G \times E$. Let $p_E: G \times E \to X \times E (=(G/H) \times E)$ be the natural projection. Since $F(c)$ has a $(p_E^{an})^{-1}\mathcal{O}_{X\times E}^{an}$-module structure, it also has an $\iota_{G\times E}^{-1}p_E^{-1}\mathcal{O}_{X\times E}$-module structure. Since $L(A)$ $(A \in \mathfrak{g})$ preserves $F(c)$, $F(c)$ has a structure of $\mathfrak{g}$-module. Let $\mathcal{D}_{X,E}$ be the subring of $\iota_* p_{E*}^{an}\underline{End}_C(F(c))$ generated by the endomorphisms induced by $\mathcal{O}_{X\times E}$ and $\mathfrak{g}$. We want to consider $\mathcal{D}_{X,E}$ as a family of twisted rings of differential operators $\mathcal{D}_X(\lambda)$ parametrized by $\lambda \in E$.

2.11. Let V_0 and V be as in (2.3). Then the restriction $f \to f|V_0 \times E$ defines an isomorphism

$$r = r_{V_0}: (p_E^{an}|V \times E)_*(F(c)|V \times E) \longrightarrow (p_E^{an}|V_0 \times E)_* \mathcal{O}_{V_0\times E}^{an} (\cong \mathcal{O}_{p(V_0)\times E}^{an}).$$

Using this isomorphism, we can define a $\mathcal{D}_{X,E}$-module stucture of $\iota_*\mathcal{O}_{p(V_0)\times E}^{an}$ in a similar way as in (2.3). Let $H_0 \in \mathcal{H}$ (cf. (2.4) for $\mathcal{H}$) and c be the function on $H_0 \times E$ such that $c(e, \lambda) = 1$ for any $\lambda \in E$, where e is the identity element of H, and $(d/dt)c(e^{tA}h, \lambda)|_{t=0} = \lambda(A)c(h, \lambda)$ for $A \in \mathfrak{h}$, $h \in H_0$ and $\lambda \in E$. (In the notation of (2.4), $c(h, \lambda) = \lambda(h)$.)

Lemma 2.12. *The family of twisted rings of differential operators $\mathcal{D}_{X,E}$ preserves the subsheaf $\mathcal{O}_{p(V_0)\times E}$ of $\iota_*\mathcal{O}_{p(V_0)\times E}^{an}$. We denote this $\mathcal{D}_{X,E}$-module $\mathcal{O}_{p(V_0)\times E}$ by $\mathcal{O}(V_0, c)$ if we need to specify the dependence on V_0.*

Proof. Let $f(v_0, \lambda)$ be a regular function on $V_0 \times E$, and $g: \boldsymbol{C}^\times \to G$ be an algebraic homomorphism. If $t \doteq 1$, $g(t)v_0$ can be expressed as $g(t)v_0 = v(t)h(t)$ with rational morphisms $v: \boldsymbol{C}^\times \to V_0$ and $h: \boldsymbol{C}^\times \to H$ which are regular at $t=1$ and satisfy $v(1) = v_0$ and $h(1) = e$. Since $(d/dt)f(g(t)v_0, \lambda)|_{t=1} = (d/dt)f(v(t), \lambda)$ $\lambda(h(t))^{-1}|_{t=1} = (d/dt)f(v(t), \lambda)|_{t=1} + f(v_0, \lambda) \cdot (d/dt)\lambda(h(t))^{-1}|_{t=1}$, it is enough to show that $(d/dt)\lambda(h(t))^{-1}|_{t=1}$ is a rational function of (v_0, λ). More generally, let us show that $(d/dt)\lambda(h(t, v_0))|_{t=1}$ is a rational function of (v_0, λ) if $h(t, v_0)$ is regular in a neighbourhood of $\{(1, v_0)|v_0 \in V_0\}$ and $h(1, v_0) = e$. Take an (algebraic) local coordinate system $\{x_1, \cdots, x_n\}$ in a neighbourhood of $e \in H$. Let $(d/dt)(x_i h)(t, v_0)|_{t=1} =: h_i(v_0)$ and $(\partial\lambda/\partial x_i)(e) =: \lambda_i$. Then $(d/dt)\lambda(h(t, v_0))|_{t=1} = \sum_{i=1}^n \lambda_i h_i(v_0)$ is a rational function of $(v_0, \lambda) = (v_0, (\lambda_1, \cdots, \lambda_n)) \in V_0 \times E$.

2.13. Take a closed subvariety T of $p(V_0)$ as in (2.9), and let $j_E: p(V_0) \times E \to X \times E$ be the inclusion mapping. The $\mathcal{D}_{X,E}$-module structure of $\mathcal{O}_{p(V_0)\times E} = \mathcal{O}(V_0, c)$ induces $\mathcal{D}_{X,E}$-module structures in $H_{T\times E}^i(\mathcal{O}_{p(V_0)\times E})$ and $j_{E*}H_{T\times E}^i(\mathcal{O}_{p(V_0)\times E})$. It also induces a $\Gamma(X \times E, \mathcal{D}_{X,E})$-module structure in $H_{T\times E}^i(X \times E, \mathcal{O}(V_0, c))$. The evaluation of $f \in \Gamma(E, \mathcal{O}_E)$ at $\lambda \in E$ gives a $\boldsymbol{C}$-algebra homomorphism $\Gamma(E, \mathcal{O}_E) \to \boldsymbol{C}$, which we shall denote by λ. Then, as $\Gamma(X, \mathcal{D}_X(\lambda))$-modules.

$$H_{T\times E}^i(X \times E, \mathcal{O}(V_0, c)) \otimes_{\Gamma(E, \mathcal{O}_E), \lambda} \boldsymbol{C} \cong H_T^i(X, \mathcal{O}_X(\lambda)). \tag{2.13.1}$$

§ 3. Localization of $\mathfrak{g}$-modules

3.0. In this section, we recollect results of Beilinson and Bernstein [1] concerning the localization of $\mathfrak{g}$-modules, which give a correspondence between a category of $U(\mathfrak{g})$-modules and a category of 'twisted $\mathcal{D}$-modules' on the flag manifold $X=G/B$. (Here and below, G denotes again a complex reductive group as in § 1.) Here we follow the exposition of Kashiwara [19], but we keep the notations of the previous sections. Thus our notations here become slightly different from those given in [19]. The twisted ring of differential operators $A_X(\lambda)=D_{\lambda-\rho}$ in the notation of [19] is denoted by $\mathcal{D}_X(\lambda)$ (cf. (2.2)) here. Also $U_{\lambda+\rho}(\mathfrak{g})$ in [19] is denoted by $U(\lambda, \mathfrak{g})$ (cf. (1.1)). See also [14]. Henceforth, a $\mathcal{D}$-module means a $\mathcal{D}$-module which is quasi-coherent over $\mathcal{O}$.

Lemma. 3.1. (Cf. [19, 6.2.3].) *For any $\lambda\in\mathfrak{t}^\vee$, the natural ring homomorphism $U(\mathfrak{g})\to\Gamma(X, \mathcal{D}_X(\lambda))$ induces an isomorphism $U(\lambda-2\rho, \mathfrak{g})\to\Gamma(X, \mathcal{D}_X(\lambda))$. Hence $U(w(\lambda-\rho)-\rho, \mathfrak{g})=U(\lambda-2\rho, \mathfrak{g})=\Gamma(X, \mathcal{D}_X(\lambda))$ for any $w\in W$.*

3.2. Let $Mod_{qc}(\lambda)$ be the category of $\mathcal{D}_X(\lambda)$-modules $\mathcal{M}$ which are quasi-coherent over $\mathcal{O}_X$, and $Mod_{qc,0}(\lambda)$ the subcategory of $Mod_{qc}(\lambda)$ consisting of $\mathcal{M}$ satisfying the following conditions: (a) $\mathcal{M}$ is generated by global sections. (b) If a $\mathcal{D}_X(\lambda)$-submodule $\mathcal{N}$ of $\mathcal{M}$ is quasi-coherent over $\mathcal{O}_X$ and $\Gamma(X, \mathcal{N})=0$, then $\mathcal{N}=0$.

Let $Mod(\lambda, \mathfrak{g})$ be the category of $U(\lambda, \mathfrak{g})$-modules. Note that $Mod(w(\lambda-\rho)-\rho, \mathfrak{g})=Mod(\lambda-2\rho, \mathfrak{g})$ for any $w\in W$. Define the functors Γ: $Mod_{qc}(\lambda)\to Mod(\lambda-2\rho, \mathfrak{g})$ and $\otimes$: $Mod(\lambda-2\rho, \mathfrak{g})\to Mod_{qc}(\lambda)$ by $\Gamma(\mathcal{M})=\Gamma(X, \mathcal{M})$ and $\otimes(M)=\mathcal{D}_X(\lambda)\otimes_{U(\lambda-2\rho,\mathfrak{g})}M$.

Lemma 3.3. ([19, 1.5 and 6.4.2]) *If $\lambda-\rho$ is anti-dominant, i.e., $\langle\alpha^\vee, \lambda-\rho\rangle\neq 1, 2, \cdots$ for any $\alpha\in R_+$, then Γ is an exact functor and $\Gamma\circ\otimes=id$. By the functors Γ and $\otimes$, $Mod_{qc,0}(\lambda)$ is equivalent to $Mod(\lambda-2\rho, \mathfrak{g})$.*

Lemma 3.4. ([1]. Cf. [19, 6.4.1].) *If $\lambda-\rho$ is regular and anti-dominant, i.e., $\langle\alpha^\vee, \lambda-\rho\rangle\neq 0, 1, 2, \cdots$ for any $\alpha\in R_+$, then $Mod_{qc}(\lambda)$ is equivalent to $Mod(\lambda-2\rho, \mathfrak{g})$ by Γ and $\otimes$.*

Lemma 3.5. *If $\lambda-\rho$ is anti-dominant and $\mathcal{M}\in Mod_{qc}(\lambda)$ is holonomic, then* $\mathrm{supp}(\otimes\Gamma(\mathcal{M}))\subset\mathrm{supp}(\mathcal{M})$ *and $SS(\otimes\Gamma(\mathcal{M}))\subset SS(\mathcal{M})$. Here* supp *(resp. SS) denotes the support (resp. the characteristic variety).*

Proof. Assume first that $\mathcal{M}\neq 0$ does not have a proper $\mathcal{D}_X(\lambda)$-submodule. If $\mathcal{M}$ is not generated by the global sections, then $\Gamma(X, \mathcal{M})=0$ and we get the desired inclusion. If $\mathcal{M}$ is generated by the global sections, then $\Gamma(X, \mathcal{M})\neq 0$. Hence if a $\mathcal{D}_X$-submodule $\mathcal{N}$ of $\mathcal{M}$ does not have a non-zero global section, then $\mathcal{N}\subsetneqq\mathcal{M}$ and, consequently, $\mathcal{N}=0$. Hence $\mathcal{M}\in Mod_{qc,0}(\lambda)$ and $\otimes\Gamma(\mathcal{M})=\mathcal{M}$ by (3.3). Thus we get the inclusion also in this case. In general, we prove the assertion by the induction on the length of $\mathcal{M}$ in $Mod_{qc}(\lambda)$. Let $\mathcal{N}\in Mod_{qc}(\lambda)$ be a proper submodule of $\mathcal{M}$. Since Γ is exact by (3.3), $\otimes\Gamma(\mathcal{N})\to\otimes\Gamma(\mathcal{M})\to\otimes\Gamma(\mathcal{M}/\mathcal{N})\to 0$ is exact. Hence $\mathrm{supp}(\otimes\Gamma(\mathcal{M}))\subset\mathrm{supp}(\otimes\Gamma(\mathcal{N}))\cup\mathrm{supp}(\otimes\Gamma(\mathcal{M}/\mathcal{N}))\subset\mathrm{supp}(\mathcal{N})\cup\mathrm{supp}(\mathcal{M}/\mathcal{N})=\mathrm{supp}(\mathcal{M})$. The assertion concerning the characteristic varieties can be proved in the same way.

Lemma 3.6. *Assume that $\lambda\in\mathfrak{t}^\vee$ is anti-dominant.* (1) *We have $JH(M(w\lambda-\rho))\subset\{[V(w'\lambda-\rho)]\,|\,w'\leqq w\}$, where $JH(-)$ denotes the set of composition factors, and $[-]$ the isomorphism class. (See* (1.2) *for $V(-)$.)* (2) *Let Z be a subset of $JH(M(w\lambda-\rho))$ which does not contain $[V(w\lambda-\rho)]$. Then $Z\subset\bigcup_{w'\lneqq w}JH(M(w'\lambda-\rho))$.*

Proof. (2) follows from (1). Let us prove (1). By [3] (cf. [9, 7.6.23]), $JH(M(w\lambda-\rho))$ consists of $[V(x\lambda-\rho)]$ with the following property: There exist $\gamma_1,\cdots,\gamma_n\in R_+$ such that $w\lambda\gneqq r_{\gamma_1}w\lambda\gneqq\cdots\gneqq r_{\gamma_n}\cdots r_{\gamma_1}w\lambda=x\lambda$. Put $w_i=r_{\gamma_i}\cdots r_{\gamma_1}w_1$. Since $r_{\gamma_i}w_{i-1}\lambda=w_{i-1}\lambda-\langle w_{i-1}\lambda,\gamma_i^\vee\rangle\gamma_i\lneqq w_{i-1}\lambda$, $\langle\lambda, w_{i-1}^{-1}\gamma_i^\vee\rangle\in\{1,2,\cdots\}$. Since λ is anti-dominant, $w_{i-1}^{-1}\gamma_i$ is a negative root. Hence $w_{i-1}>r_{\gamma_i}w_{i-1}(=w_i)$ by [4, 2.3]. Thus $w=w_0>w_1>\cdots>w_n$ and $V(x\lambda-\rho)=V(w_n\lambda-\rho)$.

3.7. Let B and $\mathfrak{n}_\pm$ be as in (1.1), and $N_\pm$ the connected subgroup of G corresponding to the Lie subalgebra $\mathfrak{n}_\pm$ of $\mathfrak{g}$. Let w be an element of W. Let $p=p_X: G\to G/B=X$ be the natural projection, $x_0=p_X(e)$, and $X(w)=Bwx_0$. (Here and below, we denote a representative element of $w\subset N_G(T)/T$ by the same letter.) Then $p|wN_-: wN_-\to X=G/B$ is an open immersion, and $X(w)$ is a closed subvariety of $p(wN_-)$ which is of pure codimension $cd(w):=l(w_S)-l(w)$. Hence, as in (2.9), for any character λ of $\mathfrak{b}$, we can consider the $\mathcal{D}_X(\lambda)$-module

$$\mathcal{X}(w)=\mathcal{X}(w,\lambda):=j_*H^{cd(w)}_{X(w)}(\mathcal{O}(wN_-,\lambda))=H^{cd(w)}_{X(w)}(\mathcal{O}_X(\lambda)),$$

which is holonomic. Here $j: p(wN_-)\to X$ is the inclusion mapping. By (2.7), the $\mathcal{D}_X(\lambda)$-module structure of $\mathcal{X}(w)=H^{cd(w)}_{X(w)}(\mathcal{O}_X(\lambda))$ does not depend on the choice of the representative element of w. We can also consider the $\Gamma(X,\mathcal{D}_X(\lambda))$-module $\Gamma(X,\mathcal{X}(w,\lambda))=H^{cd(w)}_{X(w)}(X,\mathcal{O}_X(\lambda))$. Recall that $\Gamma(X,\mathcal{D}_X(\lambda))=U(\lambda-2\rho,\mathfrak{g})=U(w(\lambda-\rho)-\rho,\mathfrak{g})$.

Lemma 3.8. *If $\lambda-\rho$ is anti-dominant, $\Gamma(X,\mathcal{X}(w,\lambda))=M(w(\lambda-\rho)-\rho,\mathfrak{b})^*$.*

Proof. As in [22], we can calculate the character of $\Gamma(X,\mathcal{X}(w,\lambda))$, and we get (without the assumption of anti-dominancy)

(3.8.1) $\quad\mathrm{ch}\Gamma(X,\mathcal{X}(w,\lambda))=\mathrm{ch}M(w(\lambda-\rho)-\rho,\mathfrak{b})=\mathrm{ch}M(w(\lambda-\rho)-\rho,\mathfrak{b})^*.$

(Cf. (3.9).) Since $w(\lambda-\rho)-\rho$ is the highest among the weights of $\Gamma(X, \mathcal{X}(w, \lambda))^*$, we get a non-zero $U(\mathfrak{g})$-homomorphism $\sigma: M(w(\lambda-\rho)-\rho, \mathfrak{b}) \to \Gamma(X, \mathcal{X}(w, \lambda))^*$ and its dual $\sigma^*: \Gamma(X, \mathcal{X}(w, \lambda)) \to M(w(\lambda-\rho)-\rho, \mathfrak{b})^*$. Let $K=\ker\sigma^*$. From the diagram

$$\begin{array}{ccccc} \otimes K & \xrightarrow{\delta} & \otimes\Gamma(X, \mathcal{X}(w, \lambda)) & \xrightarrow{\otimes(\sigma^*)} & \otimes M(w(\lambda-\rho)-\rho, \mathfrak{b})^* \\ & & \downarrow{\scriptstyle \varepsilon} & & \\ & & \mathcal{X}(w, \lambda) & & \end{array}$$

we get the following diagram.

$$\begin{array}{ccccc} K & \xrightarrow{\Gamma(\delta)} & \Gamma(X, \mathcal{X}(w, \lambda)) & \xrightarrow{\sigma^*} & M(w(\lambda-\rho)-\rho, \mathfrak{b})^* \\ & & \downarrow{\scriptstyle id=\Gamma(\varepsilon)} & & \\ & & \Gamma(X, \mathcal{X}(w, \lambda)). & & \end{array}$$

By (3.6, (2)) and (3.8.1), we can show that

$$JH(K) \subset \bigcup_{w' \lneq w} JH(M(w'(\lambda-\rho)-\rho, \mathfrak{b})) = \bigcup_{w' \lneq w} JH(\Gamma(X, \mathcal{X}(w', \lambda))).$$

By (3.5),

$$\begin{aligned} \operatorname{supp}(\otimes K) &\subset \bigcup_{w' \lneq w} \operatorname{supp}(\otimes\Gamma(X, \mathcal{X}(w', \lambda))) \\ &\subset \bigcup_{w' \lneq w} \operatorname{supp}(\mathcal{X}(w', \lambda)) = \bigcup_{w' \lneq w} \overline{X(w')}. \end{aligned} \tag{3.8.2}$$

Since $\mathcal{X}(w, \lambda)=H^{\operatorname{codim}(w)}_{X(w)}(\mathcal{O}_X(\lambda))$ does not have non-zero $\mathcal{O}_X$-submodules whose supports are contained in $\overline{X(w)}\backslash X(w)$, (3.8.2) implies that $\varepsilon\delta=0$. Hence $\Gamma(\delta)=\Gamma(\varepsilon\delta)=0$ and σ^* is injective. Comparing the characters (3.8.1), we can show that σ^* is an isomorphism.

Remark 3.9. In (3.8), we have assumed that $\lambda-\rho$ is anti-dominant. Here, we shall show that the assertion becomes false without this assumption.

Let $G=SL_2$, B (resp. T) be the subgroup of G consisting of the upper-triangular matrices (resp. the diagonal matrices), and $w=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The mapping $G \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (a\infty+b)/(c\infty+d) \in \boldsymbol{P}^1$ induces an isomorphism $X=G/B \to \boldsymbol{P}^1$, by which we shall identify G/B with $\boldsymbol{P}^1$. Then G acts on $\boldsymbol{P}^1$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = (ax+b)/(cx+d)$, and $x_0=p_X(e)$ is identified with ∞. Note that $N_-=\left\{\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \middle| t \in \boldsymbol{C}\right\}$.

First, let us consider $H^0_{X(w)}(X, \mathcal{O}_X(\lambda))$. Denote the function $w\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}x_0 \to t$ by t. Then as vector spaces, $H^0_{X(w)}(X, \mathcal{O}_X(\lambda))=H^0_{X(w)}(X, \mathcal{O}_X)=\boldsymbol{C}[t]$. For $\lambda \in \boldsymbol{C}$,

$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \to a^{\lambda}$ gives a 'multi-valued character' of T and induces an element of $\mathfrak{t}^{\vee}$, which we shall denote by the same letter λ. Then $\lambda=1$ corresponds to ρ. The function $p_X\left(w\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) \to t^n$ on $X(w)=\boldsymbol{C}$, which we shall denote by t^n, is identified via $r=r_{wN_-}$ (cf. (2.3)) with the following function f_n in $F(\lambda, B)$ (cf. (2.2));

$$SL_2 \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -ac^{-1} & 1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \longrightarrow (-ac^{-1})^n \cdot c^{-\lambda}.$$

Hence $\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f_n\right)\left(w\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = (d/du) f_n\left(\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right)|_{u=0} = nt^{n-1}$, and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t^n = nt^{n-1}$. In the same way, we get $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t^n = (-\lambda-2n)t^n$, and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} t^n = (-\lambda-n)t^{n+1}$. Let $\tau(x)=-{}^t x$, $\boldsymbol{C}[t]^*$ be the dual sl_2-module (cf. (1.1)), and $\{e_n\}_{n\geq 0}$ the dual basis of $\{t^n\}_{n\geq 0}$. Then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e_n = (-\lambda-n+1)e_{n-1}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e_n = (-\lambda-2n)e_n$, and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} e_n = (n+1)e_{n+1}$. (Here we put $e_{-1}=0$.) Thus $H^0_{X(w)}(X, \mathcal{O}_X(\lambda))^*$ is isomorphic to the Verma module for any λ, whose highest weight vector is e_0 and the highest weight is $-\lambda=-\lambda\rho$.

Let us consider $H^1_{X(e)}(X, \mathcal{O}_X(\lambda))$. Denote the function $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} x_0 \to t$ by t. Then $H^0_{N_-\cdot x_0}(X, \mathcal{O}_X) = \boldsymbol{C}[t]$ and $H^1_{\{x_0\}}(X, \mathcal{O}_X) = \bigoplus_{j\geq 1} \boldsymbol{C}\delta_j(t)$, where $\delta_j = (t^{-j} \bmod \boldsymbol{C}[t]) \in \boldsymbol{C}[t, t^{-1}]/\boldsymbol{C}[t]$. The rational function $p_X\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) \to t^n$ $(n \in \boldsymbol{Z})$ on $N_-\cdot x_0$, which we shall denote by t^{-n}, is identified via $r=r_{N_-}$ with the following function f_n in $F(\lambda, B)$;

$$SL_2 \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-1}c & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \longrightarrow (ac^{-1})^n a^{-\lambda}.$$

Hence $\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f_n\right)\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = (d/du) f_n\left(\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right)|_{u=0} = (\lambda-n)t^{-n+1}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \delta_n = (\lambda-n)\delta_{n-1}$. In the same way, we get $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \delta_n = (\lambda-2n)\delta_n$, and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \delta_n = n\delta_{n+1}$. Let $\{e_n\}_{n\geq 1}$ be the dual basis of $\{\delta_n\}_{n\geq 1}$. Then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e_n = (n-1)e_{n-1}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e_n = (\lambda-2n)e_n$, and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} e_n = (\lambda-n-1)e_{n+1}$. (Here we put $e_0=0$.) Thus $H^1_{X(e)}(X, \mathcal{O}_X(\lambda))^*$ is isomorphic to the Verma module if and only if $\lambda-j\neq 0$ for $j=2, 3, 4, \cdots$, i.e., $\lambda-\rho$ is anti-dominant. (For any λ, $H^1_{X(e)}(X, \mathcal{O}_X(\lambda))$ is the Verma module.)

§ 4. The $U_A(\mathfrak{g})$-module $M_A(w)^*$

4.0. Let λ_c be a W_I-invariant character of $\mathfrak{t}$, λ_d the lowest weight of a finite dimensional irreducible P-module, $\lambda=\lambda_c+\lambda_d$, and $w\in(W/W_I)_l$. In this section, we construct a certain $U_A(\mathfrak{g})$-module $M_A(w)^*$ and a $U_A(\mathfrak{g})$-homomorphism $\varphi: M_A(L)\to M_A(w)^*$, which satisfy the conditions (1.11.1) and (1.11.2). Later, in §6, we shall show that $M_A(w)^*$ also satisfies the condition (1.11.3), and shall calculate the character of $V(w, \lambda, \mathfrak{p})$ using (1.12). In this section, we fix a subset I of S. In order to make the account easier to read, first we consider the case where $\lambda_d=0$, and at the end of this section, indicate how to generalize it.

4.1. Naturally identify $\mathfrak{t}_I^\vee$ with the set $\mathfrak{p}^\vee$ of characters of the Lie algebra $\mathfrak{p}$. Let $\lambda_c\in\mathfrak{t}_I^\vee=\mathfrak{p}^\vee=: E$, λ_d be as in (4.0) and $\lambda=\lambda_c+\lambda_d$. Put $L=L(w, \lambda_d, \mathfrak{p})=\{w(\lambda_c'+\lambda_d-\rho)-\rho \mid \lambda_c'\in E\}$, $A=A(L)$, $A'=A(E)$, and K' the quotient field of A'. Since for our argument here, it is more convenient to consider E instead of L, we construct a $U_{A'}(\mathfrak{g})$-module $M_{A'}(w)^*$ and a $U_{A'}(\mathfrak{g})$-homomorphism φ': $M_{A'}(w(c'+\lambda_d-\rho)-\rho)\to M_{A'}(w)^*$ such that $M_{A'}(w)^*\otimes_{A'}A$ and $\varphi'\otimes_{A'}A$ satisfy the conditions (1.11.1)-(1.11.3). Here $c'\in\mathrm{Hom}_C(\mathfrak{t}, A')=\mathrm{Hom}_{A'}(\mathfrak{t}_{A'}, A')$ is the natural character and A is considered as an A'-algebra by the isomorphism induced by $E\ni\lambda_c'\to w(\lambda_c'+\lambda_d-\rho)-\rho\in L$. Let us write out the conditions corresponding to (1.11.1)-(1.11.3), which $M_{A'}(w)^*$ and φ' should satisfy.

(1.11.1′) Let $j: M_{A'}(w)^*\to M_{A'}(w)^*\otimes_{A'}K'$ be the canonical morphism and $M_{A'}(w)^*(\mu)=j^{-1}((M_{A'}(w)^*\otimes_{A'}K')_{w(\eta'+\lambda_d-\rho)-\rho-\mu})$ for $\mu\in Q_+$. Then each $M_{A'}(w)^*(\mu)$ is a free A'-module of finite type and $M_{A'}(w)^*=\bigoplus_{\mu\in Q_+}M_{A'}(w)^*(\mu)$. Here $\eta'=c'\otimes_{A'}K'$.

(1.11.2′) Let $\varphi'_{\lambda_c'}: M(w(\lambda_c'+\lambda_d-\rho)-\rho, \mathfrak{b})\to M_{A'}(w)^*\otimes_{A,\lambda_c'}C$ $(\lambda_c'\in\mathfrak{t}_I^\vee)$ be the homomorhism induced by φ'. Then $\varphi'_{\lambda_c'}\neq0$ for any $\lambda_c'\in\mathfrak{t}_I^\vee$.

(1.11.3′) There exists an open dense subset $\mathfrak{t}_0^\vee$ of $\mathfrak{t}_I^\vee$ with respect to the classical topology such that $M_{A'}(w)^*\otimes_{A',\lambda_c'}C$ is a simple $U(\mathfrak{g})$-module for $\lambda_c'\in\mathfrak{t}_0^\vee$.

Henceforth, we shall exclusively consider E as the parameter space, and we write A, K, φ, c, η, etc. for A', K', φ', c', η', etc. for the simplicity of notation. For $L=L(w, \lambda_d, \mathfrak{p})=w(\mathfrak{p}^\vee+\lambda_d-\rho)-\rho$, we put $M_A(w(c+\lambda_d-\rho)-\rho):=M(L)$, $J_A(w, c+\lambda_d, \mathfrak{p}):=J(L)$, $V_A(w, c+\lambda_d, \mathfrak{p}):=V(L)$,

$$V(w, \lambda, \mathfrak{p}):=V(L, w(\lambda-\rho)-\rho) \tag{4.1.1}$$

$\eta:=c\otimes K$, $V_K(w, \eta+\lambda_d, \mathfrak{p}):=V_K(L)$, etc. (Cf. (1.3).)

4.2. The $U_A(\mathfrak{g})$-module $M_A(w)$. Let $X=G/B$ and $Y=G/P$. Let $p_X: G\to X$, $p_Y: G\to Y$ and $q: X\to Y$ be the natural projections, $x_0=p_X(e)$, $y_0=p_Y(e)$, where e is the identity element of G, $X(w)=Bwx_0$ and $Y(w)=Bwy_0$ for $w\in W$. Let $w\in(W/W_I)_l$. Then $Y(w)$ is a closed subvariety of $p_Y(wU_-)=wU_-\cdot y_0$ of pure codimension $cd(w)=l(w_S)-l(w)$. (Note that $X(w)$ is an open subset of $q^{-1}Y(w)$, and is of pure codimension $cd(w)$ in X.) Let $\mathcal{D}_{Y,E}$ be as in (2.10). Then as in (2.13), we can consider the $\Gamma(Y\times E, \mathcal{D}_{Y,E})$-module

$$M_A(w):=H^{cd(w)}_{Y(w)\times E}(Y\times E, \mathcal{O}(wU_-, c))=H^{cd(w)}_{Y(w)}(Y, \mathcal{O}_Y)\otimes_C A. \tag{4.2.1}$$

Since $\mathcal{D}_{Y,E}$ is generated by the operators induced by $\mathfrak{g}$ and $\mathcal{O}_{Y\times E}$ (cf. (2.10)), we have natural morphisms $U(\mathfrak{g})\to\Gamma(Y\times E,\ \mathcal{D}_{Y,E})$ and $A=\Gamma(E,\ \mathcal{O}_E)\to\Gamma(Y\times E,\ \mathcal{O}_{Y\times E})\to\Gamma(Y\times E,\ \mathcal{D}_{Y,E})$, by which $M_A(w)$ becomes a $U_A(\mathfrak{g})$-module.

4.3. A free A-basis of $M_A(w)$. The natural G-action on Y induces actions of the maximal torus T on $Y(w)=Bwy_0$ and $wU_-\cdot y_0$. Let $\mathfrak{u}_w=\bigoplus_{\alpha\in R_+\cap w(R_-\setminus R_I)}\mathfrak{g}(\alpha)$ and $w(\mathfrak{u}_-)=\bigoplus_{\alpha\in w(R_-\setminus R_I)}\mathfrak{g}(\alpha)$. By the isomorphism $x\to(\exp x)wy_0$, the pair $(w(\mathfrak{u}_-),\ \mathfrak{u}_w)$ is isomorphic to $(wU_-\cdot y_0,\ Bwy_0)$ including the natural T-actions. Hence

$$H^{cd(w)}_{Y(w)}(Y,\ \mathcal{O}_Y)\cong\Big(\bigotimes_{\alpha\in R_+\cap w(R_-\setminus R_I)}\Gamma(\mathfrak{g}(\alpha),\ \mathcal{O})\Big)\otimes\Big(\bigotimes_{\alpha\in w(R_-\setminus R_I)\setminus R_+}H^1_{\{0\}}(\mathfrak{g}(\alpha),\ \mathcal{O})\Big). \tag{4.3.1}$$

We can get a $\boldsymbol{C}$-basis of the left hand side of (4.3.1) by using the expression of the right hand side as follows. The exact sequence

$$0=\Gamma_{\{0\}}(\boldsymbol{C},\ \mathcal{O})\longrightarrow\Gamma(\boldsymbol{C},\ \mathcal{O})\longrightarrow\Gamma(\boldsymbol{C}-\{0\},\ \mathcal{O})\longrightarrow H^1_{\{0\}}(\boldsymbol{C},\ \mathcal{O})\longrightarrow 0$$

can be identified with the exact sequence

$$0\longrightarrow\boldsymbol{C}[x]\longrightarrow\boldsymbol{C}[x,\ x^{-1}]\longrightarrow H^1_{\{0\}}(\boldsymbol{C},\ \mathcal{O})\longrightarrow 0.$$

Let $((d/dx)^n x^{-1} \bmod \boldsymbol{C}[x])=:\ \delta^{(n)}(x)$. Then

$$\Gamma(\boldsymbol{C},\ \mathcal{O})=\bigoplus_{n\geqq 0}\boldsymbol{C}x^n \quad\text{and}\quad H^1_{\{0\}}(\boldsymbol{C},\ \mathcal{O})=\bigoplus_{n\geqq 0}\boldsymbol{C}\delta^{(n)}. \tag{4.3.2}$$

Let u_α be a linear coordinate function on $\mathfrak{g}(\alpha)$ and $\delta^{(n)}_\alpha=\delta^{(n)}(u_\alpha)$. As is seen from (4.3.1) and (4.3.2), the set of the elements of the form

$$\Big(\prod_{\alpha\in R_+\cap w(R_-\setminus R_I)}u_\alpha^{n(\alpha)}\Big)\times\Big(\prod_{-\alpha\in w(R_-\setminus R_I)\setminus R_+}\delta_{-\alpha}^{(n(\alpha))}\Big) \tag{4.3.3}$$

with $n(\alpha)\geqq 0$ gives a $\boldsymbol{C}$-basis of $H^{cd(w)}_{Y(w)}(Y,\ \mathcal{O}_Y)$. It also gives a free A-basis of $M_A(w)$.

Lemma 4.4. *Let v be the element* (4.3.3). *Then*

$$Hv=\langle w(c-\rho)-\rho-\sum_{\alpha\in R_+\setminus wR_I}n(\alpha)\alpha,\ H\rangle v \qquad \textit{for } H\in\mathfrak{t}.$$

Proof. The element (4.3.3) is a weight vector of the weight

$$\begin{aligned} &wc-\sum_{\alpha\in R_+\cap w(R_-\setminus R_I)}n(\alpha)\alpha+\sum_{\alpha\in R_+\cap w(R_+\setminus R_I)}(n(\alpha)+1)(-\alpha)\\ &=wc-\sum_{\alpha\in R_+\cap w(R_-\setminus R_I)}\alpha-\sum_{\alpha\in R_+\setminus wR_I}n(\alpha)\alpha. \end{aligned} \tag{4.4.1}$$

Since $w\in(W/W_I)_l$, we have $w(R_+\cap R_I)\subset R_-$, and hence

$$\sum_{\alpha\in R_+\cap w(R_+\setminus R_I)}\alpha=\sum_{\alpha\in R_+\cap wR_+}\alpha=\rho+w\rho. \tag{4.4.2}$$

By (4.4.1) and (4.4.2), we get the assertion.

Corollary 4.5. *For $\lambda\in\mathfrak{t}_I^\vee$ and $w\in(W/W_I)_l$,*

$$\mathrm{ch}(M_A(w)\otimes_{A,\lambda}\boldsymbol{C})=e^{w(\lambda-\rho)-\rho}\prod_{\alpha\in R_+\setminus wR_I}(1-e^{-\alpha})^{-1}.$$

4.6. For $w\in(W/W_I)_l$ and $\mu\in Q_+$, let $M_A(w)(\mu)=\{v\in M_A(w)\mid Hv=\langle w(c-\rho)-\rho-\mu, H\rangle v$ for $H\in\mathfrak{t}^\vee\}$. By (4.3) and (4.4), $M_A(w)=\bigoplus_{\mu\in Q_+}M_A(w)(\mu)$ and each $M_A(w)(\mu)$ is a free A-module of finite type. Let $M_A(w)^*(\mu)=\mathrm{Hom}_A(M_A(w)(\mu), A)$, and $M_A(w)^*=\bigoplus_{\mu\in Q_+}M_A(w)^*(\mu)$. Then $M_A(w)^*$ has a natural $U_A(\mathfrak{g})$-module structure and

$$\mathrm{ch}(M_A(w)^*\otimes_{A,\lambda}\boldsymbol{C})=\mathrm{ch}(M_A(w)\otimes_{A,\lambda}\boldsymbol{C})=e^{w(\lambda-\rho)-\rho}\prod_{\alpha\in R_+\setminus wR_I}(1-e^{-\alpha})^{-1}$$

for $\lambda\in\mathfrak{t}_I^\vee$. Obviously $M_A(w)^*$ satisfies the condition (1.11.1′) of (4.1).

4.7. We have a $U_A(\mathfrak{b})$-homomorphism $A(w(c-\rho)-\rho)\to M_A(w)^*$ whose image is the weight space $M_A(w)^*(0)=M_A(w)^*_{w(c-\rho)-\rho}(\cong A)$. This homomorphism induces a $U_A(\mathfrak{g})$-homomorphism $\varphi\colon M_A(w(c-\rho)-\rho, \mathfrak{b})\to M_A(w)^*$. Obviously, φ satisfies the condition (1.11.2′) of (4.1). The condition (1.11.3′) will be proved in §6.

4.8. Let us explain how to generalize the argument of this section to the case where λ_d is not necessarily zero. We keep the notations of (4.2). Let λ_d be the lowest weight of a finite dimensional irreducible P-module, and $\mathcal{O}_X(\lambda_d)$ the line bundle on X consisting of (local) regular functions f such that $f(gb)=f(g)\lambda_d(b)^{-1}$ for $g\in G$ and $b\in B$. Let $\lambda_c\in\mathfrak{t}_I^\vee$ and $\lambda=\lambda_c+\lambda_d$. (Note that the lowest weight λ of a finite dimensional irreducible $\mathfrak{p}$-module can be always expressed in this way.) Then $q^*\mathcal{O}(wU_-, \lambda_c)\otimes_{\mathcal{O}_X}\mathcal{O}_X(\lambda_d)$ has a natural $\mathcal{D}_X(\lambda)$-module structure, which induces a $\Gamma(X, \mathcal{D}_X(\lambda))$-module structure on

$$\begin{aligned}(4.8.1)\qquad &H^{cd(w)}_{q^{-1}Y(w)}(q^{-1}(wU_-\cdot y_0), q^*\mathcal{O}(wU_-, \lambda_c)\otimes_{\mathcal{O}_X}\mathcal{O}_X(\lambda_d))\\ &=H^{cd(w)}_{Y(w)}(wU_-\cdot y_0, \mathcal{O}(wU_-, \lambda_c)\otimes_{\mathcal{O}_Y}q_*\mathcal{O}_X(\lambda_d))\\ &=H^{cd(w)}_{Y(w)}(Y, q_*\mathcal{O}_X(\lambda_d)),\qquad \text{(as a vector space)}.\end{aligned}$$

(Note that $Rq_*\mathcal{O}(\lambda_d)=q_*\mathcal{O}(\lambda_d)$ by the Borel-Weil theory [5].) "Varying $\lambda_c\in\mathfrak{t}_I^\vee$ in (4.8.1)", we get a $\Gamma(X\times E, \mathcal{D}_{X,E})$-module

$$M_A(w):=H^{cd(w)}_{Y(w)}(Y, q_*\mathcal{O}_X(\lambda_d))\otimes_{\boldsymbol{C}}A,$$

which has also a $U_A(\mathfrak{g})$-module structure as in (4.2). Let $\{f_\iota\}$ be a basis of the P-module $\Gamma(P/B, \mathcal{O}_X(\lambda_d))$ consisting of weight vectors, and μ_i the weight of f_ι. Then the functions $wU_-\cdot P\ni wup\to f_\iota(p)$, which we shall denote by f_ι, give a $\Gamma(wU_-\cdot y_0, \mathcal{O}_Y)$-basis of $\Gamma(wU_-\cdot y_0, q_*\mathcal{O}_X(\lambda_d))$. Let $V_I(\lambda')$ be the irreducible $\mathfrak{p}$-module with the lowest weight λ'. Since $\Gamma(P/B, \mathcal{O}_X(\lambda_d))\cong V_I(\lambda_d)$ (cf. (9.2) below), we have

$$(4.8.2)\qquad \mathrm{ch}(M_A(w)\otimes_{A,\lambda_c}\boldsymbol{C})=e^{-w\rho-\rho}\cdot w(\mathrm{ch}V_I(\lambda))\prod_{\alpha\in R_+\setminus wR_I}(1-e^{-\alpha})^{-1},$$

where $w(e^{\lambda'})=e^{w\lambda'}$ etc. (As is indicated by the calculation

$$(u_\alpha^n f_i)(t^{-1}x_\alpha(u)wp,\ \lambda_c)=(u_\alpha^n f_i)(x_\alpha(\alpha(t)^{-1}u)w\cdot w^{-1}t^{-1}wp,\ \lambda_c)$$
$$=(\alpha(t)^{-n}u^n\cdot\lambda_c(w^{-1}tw)\cdot\lambda_c(p)^{-1})\cdot(\mu_i(w^{-1}tw)f_i(p))$$
$$=(w\mu_i)(t)\cdot(w\lambda_c-n\alpha)(t)\cdot(u_\alpha^n f_i)(x_\alpha(u)wp,\ \lambda_c),$$

the character (4.8.2) is the same as the case $\lambda_d=0$ except for the contribution from the factor $(w\mu_i)(t)$, which amounts to $w(\mathrm{ch}V_I(\lambda))$.) Since $w\in(W/W_I)_l$, the highest weight of $M_A(w)\otimes_{A,\lambda_c}C$ is $w(\lambda-\rho)-\rho$. As in (4.6) and (4.7), we can construct a $U_A(\mathfrak{g})$-module $M_A(w)^*$ and a $U_A(\mathfrak{g})$-homomorphism φ: $M_A(w(c+\lambda_d-\rho)-\rho,\ \mathfrak{b})\to M_A(w)^*$ which satisfy (1.11.1′) and (1.11.2′).

§ 5. Stability of the Submodule Lattice by a Smooth Pull-back

5.1. Let X be a smooth algebraic variety over the complex number field C, and $\mathcal{O}=\mathcal{O}_X$ the sheaf of regular functions on X. Let Y be another smooth algebraic variety and $f: X\to Y$ a smooth morphism. Let $\mathcal{D}_X$ be the sheaf of algebraic differential operators on X and $\mathcal{D}_f$ the subring of $\mathcal{D}_X$ generated by $\mathcal{O}_X$ and the tangent vector fields which are tangent to the fibres of f. Let M' be an $\mathcal{O}_Y$-module and $M=f^*M'=\mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}M'$. Then M has a natural $\mathcal{D}_f$-module structure. The purpose of this section is to prove the following proposition.

Proposition 5.2. (1) *Let N be a $\mathcal{D}_f$-submodule of $M=f^*M'$ which is quasi-coherent as an $\mathcal{O}_X$-module. For any $p\in X$, there exists an open neighbourhood U of p and a uniquely determined $\mathcal{O}_{f(U)}$-submodule $N'(U)$ of $M'|f(U)$ such that $N|U=(f|U)^*N'(U)$. (Note that f is an open mapping.)*

(2) *Assume that f is surjective and the fibres of f are connected. Then there is a uniquely determined $\mathcal{O}_Y$-submodule N' of M' such that $N=f^*N'$.*

Remark 5.3. Since f is smooth, $\mathcal{O}_{X,p}$ is faithfully flat over $(f^{-1}\mathcal{O}_Y)_p=\mathcal{O}_{Y,f(p)}$. Hence for any $\mathcal{O}_Y$-submodule N' of M', f^*N' is an $\mathcal{O}_X$-submodule of f^*M', and, by [7, Chap. 1, § 3, Prop. 9], $f^{-1}N'$ is a subsheaf of f^*N'.

Corollary 5.4. *Assume that $f: X\to Y$ is a smooth surjective morphism whose fibres are connected. Let M' be a quasi-coherent $\mathcal{O}_Y$-module, $M=f^*M'$, $S(M)$ the set of $\mathcal{D}_f$-submodules of M which are quasi-coherent as $\mathcal{O}_X$-modules, and $S(M')$ the set of quasi-coherent $\mathcal{O}_Y$-submodules of M'. Then $N'\to f^*N'$ defines a bijection $S(M')\to S(M)$.*

5.5. Let $y=(y_1, \cdots, y_r)$ be a local coordinate system of Y at $q=f(p)$. If there is no fear of confusion, we regard $\mathcal{O}_{Y,q}$ as a subring of $\mathcal{O}_{X,p}$ by f^*. Especially, we identify $y_i\in\mathcal{O}_{Y,q}$ with $f^*y_i=y_i\circ f\in\mathcal{O}_{X,p}$. Choose $z_1, \cdots, z_s\in$

$\mathcal{O}_{X,p}$ so that $x=(y_1, \cdots, y_r; z_1, \cdots, z_s)$ gives a local coordianate system of X at p. For $v=(v_1, \cdots, v_s)\in \boldsymbol{Z}^s$, we write $v\geqq 0$ if $v_i\geqq 0$ for any i, and $v>0$ if $v\geqq 0$ and $v\neq 0$. If $v\in \boldsymbol{Z}^s$ and $v\geqq 0$, we set $z(v)=\prod_{i=1}^s z_i^{v_i}/v_i!$ and $\partial^v=\partial_z^v=(\partial/\partial z_1)^{v_1}\cdots(\partial/\partial z_s)^{v_s}$. If $v\ngeqq 0$, we set $z(v)=0$. Then $\partial^v z(w)=z(w-v)$.

5.6. Assume that we are given an n-tuple $A(x)=(a_1(x), \cdots, a_n(x))\in \mathcal{O}_{X,p}^n$. Consider its power series expansion with respect to z; $A(x)=\sum_{v\geqq 0}A_v(y)z(v)$ with $A_v(y)\in \mathcal{O}_{Y,q}^n$. Set $J=\sum_{v\geqq 0}\mathcal{O}_{Y,q}A_v(y)\,(\subset \mathcal{O}_{Y,q}^n)$, and let $\{B_1(y), \cdots, B_g(y)\}$ be a minimal generating system of the $\mathcal{O}_{Y,q}$-module J. Then $A_v(y)$'s can be expressed as

$$A_v(y)=\sum_{i=1}^g c_{v,i}(y)B_i(y) \tag{5.6.1}$$

with some $c_{v,i}(y)\in \mathcal{O}_{Y,q}$. Set

$$c_i(x)=\sum_{v\geqq 0} c_{v,i}(y)z(v)\,. \tag{5.6.2}$$

Then $c_i(x)\in \hat{\mathcal{O}}_{X,p}$, where $\hat{\mathcal{O}}_{X,p}$ is the completion of the local ring $\mathcal{O}_{X,p}$ with respect to its maximal ideal $m_{X,p}$. Moreover $c_i=c_i(x)$ $(1\leqq i\leqq g)$ satisfy the system of linear equations with coefficients in $\mathcal{O}_{X,p}$;

$$A(x)=\sum_{i=1}^g c_iB_i(y)\,. \tag{5.6.3}$$

Since (5.6.3) has a solution in $\hat{\mathcal{O}}_{X,p}$, and since $\hat{\mathcal{O}}_{X,p}$ is faithfully flat over $\mathcal{O}_{X,p}$, (5.6.3) has a solution in $\mathcal{O}_{X,p}$ [7, Chap. 1, § 3, Prop. 13]. Hence we may take $c_i(x)$ in $\mathcal{O}_{X,p}$. (Note that, if we define $c_{v,i}(y)$'s as the coefficients of the power series expansion (5.6.2) of $c_i(x)$, the equality (5.6.1) holds.) Since $\{A_v(y)\,|\,v\geqq 0\}$ and $\{B_i(y)\,|\,1\leqq i\leqq g\}$ are both generators of J, $B_i(y)$'s can be expressed as $B_i(y)=\sum_{v\geqq 0}d_{i,v}(y)A_v(y)$ (a finite sum) with some $d_{i,v}(y)\in \mathcal{O}_{Y,q}$. Set $K=\mathcal{O}_{X,p}J$ and $P_i=P_i(y, \partial_z)=\sum_{v\geqq 0}d_{i,v}(y)\partial_z^v$ $(1\leqq i\leqq g)$. Then

$$\begin{aligned} P_iA(x)&=\Big(\sum_{v\geqq 0} d_{i,v}(y)\partial_z^v\Big)\Big(\sum_{w\geqq 0} A_w(y)z(w)\Big) \\ &\equiv \sum_{v\geqq 0} d_{i,v}(y)A_v(y)=B_i(y) \qquad (\mathrm{mod}\ Km_{X,p})\,. \end{aligned} \tag{5.6.4}$$

Set $L=\sum_{i=1}^g\mathcal{O}_{X,p}(P_iA(x))$. By (5.6.4), $B_i(y)\in L+Km_{X,p}$. Since $\{B_1(y), \cdots, B_g(y)\}$ generates the $\mathcal{O}_{X,p}$-module K,

$$K\subset L+Km_{X,p}\,. \tag{5.6.5}$$

On the other hand, by (5.6.3), $P_jA(x)=P_j(y, \partial_z)\sum_{i=1}^g B_i(y)c_i(x)=\sum_{i=1}^g B_i(y)\cdot P_jc_i(x)\in K$. Hence

$$L\subset K\,. \tag{5.6.6}$$

By (5.6.5), (5.6.6) and the "Nakayama's lemma", we get $K=L$. Hence there exist $e_{ij}(x)\in \mathcal{O}_{X,p}$ $(1\leqq i,\ j\leqq g)$ such that $B_i(y)=\sum_{j=1}^g e_{ij}(x)\cdot P_j(y, \partial_z)A(x)$. By setting $Q_\iota(x, \partial_z)=\sum_{j=1}^{g} e_{\iota j}(x)P_j(y, \partial_z)$, we get the following lemma.

Lemma 5.7. *Given* $A(x)=(a_1(x), \cdots, a_n(x))\in\mathcal{O}_{X,p}^n$, *there exist n-tuples* $B_\iota(y)=(b_{\iota 1}(y), \cdots, b_{\iota n}(y))\in\mathcal{O}_{Y,f(p)}^n$, $Q_\iota\in\mathcal{D}_{f,p}$ *and* $c_\iota(x)\in\mathcal{O}_{X,p}$ $(1\leqq i\leqq g)$ *such that* $B_\iota(y)=Q_\iota A(x)$ $(1\leqq i\leqq g)$, *and* $A(x)=\sum_{i=1}^{g} c_\iota(x)B_\iota(y)$.

5.8. *Proof of* (5.2). Let u be a section of $M=f^*M'=\mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y}M'$ on an open neighbourhood U of p. By shrinking U, if necessary, we may assume that u can be expressed as $u=\sum_{j=1}^{n} a_j(x)\otimes u_j'$ with $a_j\in\Gamma(U, \mathcal{O}_x)$ and $u_j'\in\Gamma(f(U), M')$. (Note that f is an open mapping.) Let $A(x)=(a_1(x), \cdots, a_n(x))\in\mathcal{O}_{X,p}^n$, and take $B_\iota(y)=(b_{\iota 1}(y), \cdots, b_{\iota n}(y))$, Q_ι and $c_\iota(x)$ as in (5.7). Set $v_\iota'=\sum_{j=1}^{n} b_{\iota j}(y)u_j'$. Then $Q_\iota u=\sum_{j=1}^{n} Q_\iota a_j(x)\otimes u_j'=\sum_{j=1}^{n} b_{\iota j}(y)\otimes u_j'=\sum_{j=1}^{n} 1\otimes b_{\iota j}(y)u_j'=1\otimes v_i'$ and $\sum_{i=1}^{g} c_\iota(x)(1\otimes v_i')=\sum_{i=1}^{g}\sum_{j=1}^{n} c_\iota(x)b_{ij}(y)\otimes u_j'=\sum_{j=1}^{n} a_j(x)\otimes u_j'=u$. If we set $\sum_{i=1}^{g}\mathcal{O}_{Y,f(p)}v_\iota'=T(u, p)$, then $\mathcal{D}_{f,p}u=\sum_{i=1}^{g}\mathcal{O}_{X,p}(1\otimes v_\iota')=\mathcal{O}_{X,p}\otimes_{\mathcal{O}_{Y,f(p)}}T(u, p)$. Since $\mathcal{O}_{X,p}$ is faithfully flat over $\mathcal{O}_{Y,f(p)}$, the $\mathcal{O}_{Y,f(p)}$-submodule $T(u, p)$ of $M_{f(p)}'$ is uniquely determined by u (cf. [7, Chap. 1, §3, Prop. 10]). By shrinking U, if necessary, we may assume that c_ι's and all the coefficients of Q_ι's are regular on U. By the same argument as above, we get $\mathcal{D}_{f,p'}u=\mathcal{O}_{X,p'}\otimes_{\mathcal{O}_{Y,f(p')}}(\sum_{i=1}^{g}\mathcal{O}_{Y,f(p')}v_i')$ for any $p'\in U$. In other words, if we define a subsheaf $T(u,U)$ of $M'|f(U)$ by $T(u, U)=\sum_{i=1}^{g}\mathcal{O}_{f(U)}v_i'$, then $(\mathcal{D}_f|U)u=(f|U)^*T(u, U)$. By shrinking U, we may assume that U is an affine open subset of X. Since we are assuming N to be a quasi-coherent $\mathcal{O}_X$-module, there exist sections $u_\alpha\in\Gamma(U,N)$ $(\alpha\in A)$ such that $N|U=\sum_{\alpha\in A}\mathcal{O}_U u_\alpha$. Define an $\mathcal{O}_{f(U)}$-submodule $N'(U)$ of $M'|f(U)$ by $N'(U)=\sum_{\alpha\in A}T(u_\alpha, U)$. Then $N|U=\sum_{\alpha\in A}(\mathcal{D}_f|U)u_\alpha=\sum_{\alpha\in A}(f|U)^*T(u_\alpha, U)=(f|U)^*N'(U)$. Moreover, if an $\mathcal{O}_{f(U)}$-submodule $N'(U)$ of $M'|f(U)$ satisfies $N|U=(f|U)^*N'(U)$, then $N_{p'}=\mathcal{O}_{X,p'}\otimes_{\mathcal{O}_{Y,f(p')}}N'(U)_{f(p')}$ for any $p'\in U$. Since $\mathcal{O}_{X,p'}$ is faithfully flat over $\mathcal{O}_{Y,f(p')}$, $N'(U)_{f(p')}$ is uniquely determined by $N_{p'}$. Hence $N'(U)$ is unique. Thus we get the first assertion.

Let p and p' be two points of X. Take an open neighbourhood U (resp. U') of p (resp. p'), and an $\mathcal{O}_{f(U)}$-module $N(U)$ (resp. an $\mathcal{O}_{f(U')}$-module $N(U')$) as in the first part. If $p''\in U\cap U'$, $N_{p''}=\mathcal{O}_{X,p''}\otimes_{\mathcal{O}_{Y,f(p'')}}N'(U)_{f(p'')}=\mathcal{O}_{X,p''}\otimes_{\mathcal{O}_{Y,f(p'')}}N'(U')_{f(p'')}$. Hence $N'(U)=N'(U')$ on $f(U\cap U')$. In order to get the second assertion, it is enough to prove that $N'(U)=N'(U')$ on $f(U)\cap f(U')$, assuming the connectedness of the fibres of f. Hence it is enough to prove that $f(U\cap U')=f(U)\cap f(U')$. Let $q\in f(U)\cap f(U')$. Then $f^{-1}(q)\cap U\neq\phi$ and $f^{-1}(q)\cap U'\neq\phi$. Since $f^{-1}(q)$ is connected, we can find an element $p''\in f^{-1}(q)\cap U\cap U'$. Then $q=f(p'')\in f(U\cap U')$. Thus we have completed the proof.

§6. Character of $V(w, \lambda, \mathfrak{p})$

6.0. In this section, first we determine the character of $V(w, \lambda, \mathfrak{p})$. Once the character formula is obtained, we can deduce several consequences from it. See (6.6) and (6.8).

6.1. Let λ_d be the lowest weight of a finite dimensional irreducible P-module, and

$$\mathfrak{t}_{I,rad}^{\vee}(\lambda_d)=\{\lambda_c\in\mathfrak{t}_I^{\vee}\,|\,\langle\lambda_c+\lambda_d-\rho,\ \alpha^{\vee}\rangle\neq 0,\ 1,\ 2,\ \cdots\quad\text{for any }\alpha\in R_+\}.$$

An easy calculation using fundamental weights shows that $\mathfrak{t}_{I,rad}^{\vee}(\lambda_d)$ is an open dense subset of $\mathfrak{t}_I^{\vee}$. Let $\lambda_c\in\mathfrak{t}_I^{\vee}$, $\lambda=\lambda_c+\lambda_d$, and assume that $\lambda_c\in\mathfrak{t}_{I,rad}^{\vee}(\lambda_d)$. Let us consider the simplicity of the $U(\mathfrak{g})$-module

$$M_A(w)^*\otimes_{A,\lambda_c}C=H^{cd(w)}_{q^{-1}Y(w)}(X,\ \mathcal{O}_X(\lambda))$$

for $w\in(W/W_I)_l$, where $cd(w)=l(w_S)-l(w)=\mathrm{codim}_X X(w)=\mathrm{codim}_Y Y(w)$ and $\mathcal{O}_X(\lambda)$ denotes (abusively) the $\mathcal{D}_X(\lambda)$-module $q^*\mathcal{O}(wU_-,\ \lambda_c)\otimes\mathcal{O}_X(\lambda_d)$. (Cf. (2.9).) Note that the inclusion $q^{-1}Y(w)\to X$ is an affine morphism and hence, $H^{cd(w)}_{q^{-1}Y(w)}(\mathcal{O}_X(\lambda))=R\Gamma_{q^{-1}Y(w)}(\mathcal{O}_X(\lambda))[cd(w)]$ and $H^{cd(w)}_{q^{-1}Y(w)}(X,\ \mathcal{O}_X(\lambda))=R\Gamma_{q^{-1}Y(w)}(X,\ \mathcal{O}_X(\lambda))[cd(w)]$. (To see this, it suffices to show that the inclusion morphism $Y(w)\to Y$ is affine, whose proof we do not give here since a similar argument appears later in (7.2).) Since

$$H^{cd(w)}_{q^{-1}Y(w)}(X,\ \mathcal{O}_X(\lambda))=\Gamma(X,\ H^{cd(w)}_{q^{-1}Y(w)}(\mathcal{O}_X(\lambda_c))\otimes\mathcal{O}_X(\lambda_d)),$$

the simplicity of the $U(\mathfrak{g})$-module $M_A(w)^*\otimes_{A,\lambda_c}C$ is equivalent to the simplicity of the $\mathcal{D}_X(\lambda)$-module

$$H^{cd(w)}_{q^{-1}Y(w)}(\mathcal{O}_X(\lambda_c))\otimes\mathcal{O}_X(\lambda_d)=q^*H^{cd(w)}_{Y(w)}(\mathcal{O}_Y(\lambda_c))\otimes\mathcal{O}_X(\lambda_d),$$

which is equivalent to the simplicity of the $\mathcal{D}_X(\lambda_c)$-module $H^{cd(w)}_{q^{-1}Y(w)}(\mathcal{O}_X(\lambda_c))=q^*H^{cd(w)}_{Y(w)}(\mathcal{O}_Y(\lambda_c))$. (Note that $\lambda-\rho$ is regular anti-dominant, and use (3.4).) Furthermore, by (5.4), the simplicity of $q^*H^{cd(w)}_{Y(w)}(\mathcal{O}_Y(\lambda_c))$ is equivalent to that of the $\mathcal{D}_Y(\lambda_c)$-module $H^{cd(w)}_{Y(w)}(\mathcal{O}_Y(\lambda_c))$.

6.2. Let $\lambda_c\in\mathfrak{t}_{I,rad}^{\vee}(\lambda_d)\cap\mathfrak{t}_{I,rad}^{\vee}(0)$, and assume that $H^{cd(w)}_{q^{-1}Y(w)}(X,\ \mathcal{O}_X(\lambda))$ is not simple. By (6.1), $N:=H^{cd(w)}_{q^{-1}Y(w)}(X,\ \mathcal{O}_X(\lambda_c))=\Gamma(H^{cd(w)}_{q^{-1}Y(w)}(\mathcal{O}_X(\lambda_c)))$ is not simple. (See (3.2) for Γ.) Let L be a simple submodule of N. Put $M_w:=M(w(\lambda_c-\rho)-\rho,\ \mathfrak{b})$. Since $N\subset H^{cd(w)}_{X(w)}(X,\ \mathcal{O}_X(\lambda_c))=M_w^*$ by (3.8), L^* ($\simeq L$) is the simple quotient of M_w. (The injectivity of $N=H^{cd(w)}_{q^{-1}Y(w)}\to H^{cd(w)}_{X(w)}$ follows from the vanishing of $H^{cd(w)-1}_{\overline{X(w)}-q^{-1}Y(w)}(X,\ \mathcal{O}_X(\lambda))$, which can be proved by the usual 'dévissage'.) Let $[Q]\in JH(N/L)\subset JH(M_w^*/L)=JH(M_w)\setminus\{[L]\}$. By (3.6), $Q=V(w'(\lambda_c-\rho)-\rho)$ with some $w'\lneqq w$. By (3.4), $\otimes Q$ is a composition factor of $\otimes N=H^{cd(w)}_{q^{-1}Y(w)}(\mathcal{O}_X(\lambda_c))$. Hence

$$\text{(6.2.1)}\quad \mathrm{supp}(\otimes Q)=\bigcup_{w''\in W_0}\overline{X(w'')}\quad\text{for some }W_0\subset\{w''\in(W/W_I)_l\,|\,w''\leqq w\}.$$

(Consider the characteristic variety.) On the other hand $\otimes V(w'(\lambda_c-\rho)-\rho)^*$ is *the* simple submodule of $\otimes M_{w'}^*=H^{cd(w')}_{X(w')}(\mathcal{O}_X(\lambda_c))$. Since $H^{cd(w')}_{X(w')}(\mathcal{O}_X(\lambda_c))$ does not have a non-zero submodule supported by $\overline{X(w')}\setminus X(w')$,

$$\text{(6.2.2)}\qquad \mathrm{supp}(\otimes Q)=\mathrm{supp}(\otimes Q^*)=\mathrm{supp}(\otimes V(w'(\lambda_c-\rho)-\rho)^*)=\overline{X(w')}.$$

By (6.2.1) and (6.2.2), $w' \in (W/W_I)_l$. Since the highest weight $w'(\lambda_c - \rho) - \rho$ of Q is also a weight of $M_w = M(w(\lambda_c - \rho) - \rho, \mathfrak{b})$,

(6.2.3) $(w(\lambda_c - \rho) - \rho) - (w'(\lambda_c - \rho) - \rho) \in Q_+$, $w, w' \in (W/W_I)_l$, and $w > w'$.

Theorem 6.3. *Let λ_d be the lowest weight of a finite dimensional irreducible P-module, $\lambda_c \in \mathfrak{t}_I^\vee$, $\lambda = \lambda_c + \lambda_d$, and $w \in (W/W_I)_l$. Then*

$$\mathrm{ch}V(w, \lambda, \mathfrak{p}) = e^{-w\rho - \rho} w(\mathrm{ch}V_I(\lambda)) \prod_{\alpha \in R_+ \setminus wR_I} (1 - e^{-\alpha})^{-1}.$$

($V_I(\lambda)$ is the simple $\mathfrak{p}$-module with the lowest weight λ.)

Proof. We calculate the character of $V(w, \lambda, \mathfrak{p})$ using (1.12). Since we have already proved that $(M_A(w)^*, \varphi)$ defined in § 4 satisfies (1.11.1′) and (1.11.2′), it remains only to prove (1.11.3′). (Cf. (4.1).) Let

$$\mathfrak{t}_0^\vee = (\mathfrak{t}_{I,rad}^\vee(\lambda_d) \cap \mathfrak{t}_{I,rad}^\vee(0)) - \bigcup_{\substack{w_1, w_2 \in (W/W_I)_l \\ w_1 \neq w_2 \\ \mu \in Q}} \{\lambda' \in \mathfrak{t}_I^\vee \mid w_1\lambda' - w_2\lambda' = \mu\}.$$

Then (6.2.3) does not hold for any $\lambda_c' \in \mathfrak{t}_0^\vee$. Hence $M_A(w)^* \otimes_{A, \lambda_c'} C = H^{cd(w)}_{q^{-1}Y(w)}(X, \mathcal{O}_X(\lambda_c' + \lambda_d))$ is simple for $\lambda_c' \in \mathfrak{t}_0^\vee$. Moreover $\mathfrak{t}_0^\vee$ is an open dense subset of $\mathfrak{t}_I^\vee$ with respect to the classical topology. Thus $M_A(w)^*$ satisfies (1.11.3′). Hence

$$\begin{aligned} \mathrm{ch}V(w, \lambda, \mathfrak{p}) &= \mathrm{ch}M_A(w)^* \otimes_{A, \lambda_c} C && \text{by (1.12)} \\ &= \mathrm{ch}M_A(w) \otimes_{A, \lambda_c} C && \text{by (1.1)} \\ &= e^{-w\rho - \rho} \cdot w(\mathrm{ch}V_I(\lambda)) \prod_{\alpha \in R_+ \setminus wR_I} (1 - e^{-\alpha})^{-1} && \text{by (4.8.2)}. \end{aligned}$$

Corollary 6.4. *Let $w \in (W/W_I)_l$, λ_d be as in (6.3), and $\mu \in Q_+$.*

(1) *$V_A(w, c + \lambda_d, \mathfrak{p})(\mu)$ is a free A-module of finite type.*

(2) $\mathrm{ch}V_K(w, \eta + \lambda_d, \mathfrak{p}) = \mathrm{ch}M_A(w)^* \otimes_A K = e^{-w\rho - \rho} \cdot w(e^\eta \mathrm{ch}V_I(\lambda_d)) \prod_{\alpha \in R_+ \setminus wR_I} (1 - e^{-\alpha})^{-1}$. (See (4.1) for $V_A(w, c + \lambda_d, \mathfrak{p})$ etc.)

Proof. Since $V_A(w, c + \lambda_d, \mathfrak{p})(\mu)$ is a quotient of the finitely generated A-module $M_A(w(c + \lambda_d - \rho) - \rho)(\mu)$, (cf. (1.17) and (4.1)) it is enough to prove that $V_A(w, c + \lambda_d, \mathfrak{p})(\mu)$ is a projective A-module [26], [32]. The projectivity follows from the following lemma together with (6.3).

Lemma 6.5. *Let C be a polynomial ring over a field and M a C-module of finite type. Assume that the dimension of the (C/m)-vector space M/mM does not depend on the maximal ideal m of C. Then M is a projective C-module.*

Proof. It is enough to prove that the quasi-coherent sheaf $\tilde{M}$ on Spec$C$ corresponding to M is $\tilde{C}$-free in a neighbourhood of any closed point $m \in \mathrm{Spec}C$, where $\tilde{C}$ is the structure sheaf. Let C_m be the local ring at m, and let

$u_1, \cdots, u_n$ be elements of M which give a basis of M/mM. Let $e_1, \cdots, e_n$ be the natural basis of C^n and define a C-homomorphism $\varphi: C^n \to M$ by $\varphi(e_i)=u_i$. Let $\{v_j\}$ be a generator system of M. Since $\varphi \otimes_C C_m$ is surjective by the "Nakayama's lemma", $v_j = \sum_i u_i c_{ij}$ with some $c_{ij} \in C_m$. Take an element $f \in C$ such that all the c_{ij}'s come from $C_f = C[f^{-1}]$. Then $\varphi \otimes C_f$ is surjective. Let $M_f = M \otimes_C C_f$ and $K = \ker\varphi \otimes C_f$. From the exact sequence $0 \to K \to C_f^n \to M_f \to 0$, we get the exact sequence $(C/m') \otimes K \to (C/m')^n \to M/m'M \to 0$ for any maximal ideal m' contained in $\mathrm{Spec}\, C_f$. Since the dimension of the (C/m')-vector space $M/m'M$ is equal to n, the image of $(C/m') \otimes K$ in $(C/m')^n$ equals 0, i.e., $K \subset m'C_f^n$. Since the intersection of $m'C_f^n$ for maximal ideals m' of C_f is 0, we get $K=0$ and $M_f = C_f^n$.

Theorem 6.6. *Let λ_c, λ_d, λ and w be as in* (6.3). *If $\lambda-\rho$ is anti-dominant, then*

$$\Gamma(X, q^* H^{cd(w)}_{Y(w)}(\mathcal{O}_Y(\lambda_c)) \otimes \mathcal{O}_X(\lambda_d)) = H^{cd(w)}_{q^{-1}Y(w)}(X, \mathcal{O}_X(\lambda)) \cong V(w, \lambda, \mathfrak{p})^*$$

as $U(\mathfrak{g})$-modules. (See (6.1) for $\mathcal{O}_X(\lambda)$.)

Proof. Using the notation of (4.1), put $M_A = M_A(w(c+\lambda_d-\rho)-\rho)$, $J_A = J_A(w, c+\lambda_d, \mathfrak{p})$, $V_A = M_A/J_A$, $M_K = M_A \otimes_A K$, $J_K = J_A \otimes_A K$, $V_K = V_A \otimes_A K$, and $M_K(w)^* = M_A(w)^* \otimes_A K$. First, let us show that the kernel of the homomorphism $\varphi: M_A(w(c+\lambda_d-\rho)-\rho) \to M_A(w)^*$ defined in (4.7) and (4.8) is J_A. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_A & \longrightarrow & M_A & \longrightarrow & V_A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J_K & \longrightarrow & M_K & \longrightarrow & V_K & \longrightarrow & 0 \end{array}$$

consisting of the natural morphisms. Since V_A is a free A-module by (6.4, (1)), the first horizontal sequence is a split exact sequence, and hence the second one is also exact. Since M_A and V_A are A-free, the second and the third vertical arrows are injections, and hence the remaining one is also an injection. (In fact, we can show that J_A is a free A-module together with its 'weight spaces' using [26], [32] and (1.17.2).) By (1.4, (2)), V_K is *the* simple quotient of the Verma module M_K and J_K *the* maximal submodule of M_K. Since $\mathrm{ch} V_K = \mathrm{ch} M_K(w)^*$ by (6.4, (2)), $V_K \cong M_K(w)^*$. Since $M_A(w)^*$ is A-free, the natural morphism $M_A(w)^* \to M_K(w)^*$ is injective, $\varphi_K := \varphi \otimes K: M_K \to M_K(w)^*$ is not identically zero, and hence surjective. Thus we can identify $\varphi_K: M_K \to M_K(w)^*$ with the projection of the Verma module to its simple quotient. Especially its kernel is J_K. Hence $J_A \subset M_A \cap J_K = M_A \cap \ker\varphi_K = \ker\varphi$. Since $\ker\varphi$ is contained in $M_A(w(c+\lambda_d-\rho)-\rho)_+$, $\ker\varphi = J_A$. Thus φ induces homomorphisms $V_A(w, c+\lambda_d, \mathfrak{p}) \to M_A(w)^*$, $V(w, \lambda, \mathfrak{p}) \to M_A(w)^* \otimes_{A, \lambda_c} C = H^{cd(w)}_{q^{-1}Y(w)}(X, \mathcal{O}_X(\lambda))^*$ and σ^*: $H^{cd(w)}_{q^{-1}Y(w)}(X, \mathcal{O}_X(\lambda)) \to V(w, \lambda, \mathfrak{p})^*$. Note that the image of σ^* contains the weight space of the highest weight. Let $K=\ker\sigma^*$. By (3.8), $M(w(\lambda-\rho)-\rho, \mathfrak{b})^*=H_{X(w)}^{cd(w)}(X, \mathcal{O}_X(\lambda))\supset H_{q^{-1}Y(w)}^{cd(w)}(X, \mathcal{O}_X(\lambda))\supset K$, and K^* is a quotient of $M(w(\lambda-\rho)-\rho, \mathfrak{b})$. If $K\neq 0$, then K contains the weight space of weight $w(\lambda-\rho)-\rho$. This contradicts the fact that the image of σ^* contains the weight space of weight $w(\lambda-\rho)-\rho$. Hence $K=0$ and σ^* is injective. Since

$$\begin{aligned}\mathrm{ch}H_{q^{-1}Y(w)}^{cd(w)}(X, \mathcal{O}_X(\lambda))&=\mathrm{ch}M_A(w)\otimes_{A,\lambda_c}\boldsymbol{C} && \text{(cf. (4.8))}\\ &=e^{-w\rho-\rho}\cdot w(\mathrm{ch}V_I(\lambda))\prod_{\alpha\in R_+\setminus wR_I}(1-e^{-\alpha})^{-1} && \text{by (4.8.2)}\\ &=\mathrm{ch}V(w, \lambda, \mathfrak{p})^* && \text{by (6.3)},\end{aligned}$$

σ^* is an isomorphism.

Remark 6.6.1. In the above theorem, we can not omit the assumption that $\lambda-\rho$ is anti-dominant. See (3.9) and (1.19).

6.7. Generalized Verma modules. Let $V_I'(\lambda')$ be the finite dimensional irreducible $\mathfrak{p}(I)$-module with the highest weight λ', and

$$M(\lambda', \mathfrak{p})=U(\mathfrak{g})\otimes_{U(\mathfrak{p})}V_I'(\lambda').$$

Such a $U(\mathfrak{g})$-module is called a *generalized Verma module*. Obviously

$$\text{(6.7.1)}\qquad \mathrm{ch}M(\lambda', \mathfrak{p})=\mathrm{ch}V_I'(\lambda')\prod_{\alpha\in R_+\setminus R_I}(1-e^{-\alpha})^{-1}.$$

Theorem 6.8. *Let $I\subset S$, $w\in(W/W_I)_l$ and λ be a character of $\mathfrak{t}$ such that $\langle\lambda, \alpha^\vee\rangle$ $(\alpha\in\Pi_I)$ are non-positive integers. If*

$$\text{(6.8.1)}\qquad J:=wIw^{-1}\subset S,$$

then $w(\lambda-\rho)-\rho$ is the highest weight of a finite dimensional irreducible $\mathfrak{p}(J)$-module, and

$$M(w(\lambda-\rho)-\rho, \mathfrak{p}(J))=V(w, \lambda, \mathfrak{p}(I)).$$

Proof. Since $w\in(W/W_I)_l$, $w(\Pi_I)=-\Pi_J$. Let $\alpha\in\Pi_I$ and $w(\alpha)=-\beta$. Then $\beta\in\Pi_J$ and $\langle w(\lambda-\rho)-\rho, \beta^\vee\rangle=\langle\lambda, w^{-1}\beta^\vee\rangle-\langle\rho, w^{-1}\beta^\vee\rangle-\langle\rho, \beta^\vee\rangle=-\langle\lambda, \alpha^\vee\rangle+\langle\rho, \alpha^\vee\rangle-\langle\rho, \beta^\vee\rangle=-\langle\lambda, \alpha^\vee\rangle$. (Note that $\langle\rho, \gamma^\vee\rangle=1$ for $\gamma\in\Pi$.) Thus $\langle w(\lambda-\rho)-\rho, \beta^\vee\rangle$ $(\beta\in\Pi_J)$ are non-negative integers, and we can consider the generalized Verma module $M(w(\lambda-\rho)-\rho, \mathfrak{p}(J))$. Consider the $U_A(\mathfrak{b})$-module $A(w(c-\rho)-\rho)$ as in (1.3). Extend this $U_A(\mathfrak{b})$-module to a $U_A(\mathfrak{p}(J))$-module by putting $\mathfrak{g}(\alpha)1_{w(c-\rho)-\rho}=0$ for $\alpha\in R_-\cap R_J$. (See (1.3) for $1_{w(c-\rho)-\rho}$. Note that the W_I-invariance of c implies the W_J-invariance of $w(c-\rho)-\rho$.) Let $wV_I(\lambda_d)$ be the irreducible $P(J)$-module with the highest weight $w\lambda_d$. Then $A(w(c-\rho)-\rho)\otimes_C wV_I(\lambda_d)$ has a $U_A(\mathfrak{p}(J))$-module structure. Let

$$M_A(w(c+\lambda_d-\rho)-\rho, \mathfrak{p}(J)):=U_A(\mathfrak{g})\otimes_{U_A(\mathfrak{p}(J))}(A(w(c-\rho)-\rho)\otimes wV_I(\lambda_d)).$$

It is enough to prove that

$$M_A(w(c+\lambda_d-\rho)-\rho,\ \mathfrak{p}(J))\cong V_A(w,\ c+\lambda_d,\ \mathfrak{p}(I)). \tag{6.8.2}$$

Denote the left (resp. right) hand side of (6.8.2) by M_A (resp. V_A). As is easily seen, M_A is isomorphic to $U_A(\mathfrak{u}_-(J))\otimes_C wV_I(\lambda_d)$ as an A-module, and the kernel of the natural homomorphism $M_A(w(c+\lambda_d-\rho)-\rho,\ \mathfrak{b})\to M_A(w(c+\lambda_d-\rho)-\rho,\ \mathfrak{p}(J))$ is contained in $M_A(w(c+\lambda_d-\rho)-\rho,\ \mathfrak{b})_+$. Hence we get a surjective $U_A(\mathfrak{g})$-homomorphism $\psi\colon M_A\to V_A$. For $\mu\in Q_+$, let $M_A(\mu)$ and $V_A(\mu)$ be the images of $M_A(w(c+\lambda_d-\rho)-\rho,\ \mathfrak{b})(\mu)$ (i.e., 'the weight space' of weight $w(c+\lambda_d-\rho)-\rho-\mu$) by the natural projections. As is easily seen, M_A is a direct sum of $M_A(\mu)$'s. On the other hand, V_A is also a direct sum of $V_A(\mu)$'s, by (1.17.3). Since ψ induces a surjective A-homomorphism $\psi(\mu)\colon M_A(\mu)\to V_A(\mu)$ for each μ, it is enough to show that $\psi(\mu)$ is bijective. As is easily seen, $M_A(\mu)$ is a free A-module of finite type. On the other hand, $V_A(\mu)$ is also a free A-module of finite type by (6.4, (1)). Since $\psi(\mu)$ is a surjection between free A-modules of finite type, it is enough to show that $\dim M_A(\mu)\otimes_{A,\lambda_c}\boldsymbol{C}=\dim V_A(\mu)\otimes_{A,\lambda_c}\boldsymbol{C}$ for $\lambda_c\in\mathfrak{t}_I^\vee$, i.e.,

$$\mathrm{ch}M(w(\lambda-\rho)-\rho,\ \mathfrak{p}(J))=\mathrm{ch}V(w,\ \lambda,\ \mathfrak{p}(I)), \tag{6.8.3}$$

where $\lambda=\lambda_c+\lambda_d$. By (6.7.1) and the Weyl's character formula, the left hand side of (6.8.3) is equal to

$$\sum_{w''\in W_J}\varepsilon(w'')e^{w''(w(\lambda-\rho)-\rho+\rho(J))-\rho(J)}\prod_{\alpha\in R_+}(1-e^{-\alpha})^{-1}, \tag{6.8.4}$$

where $\varepsilon(w'')=(-1)^{l(w'')}$ and $\rho(J)=(1/2)\sum_{\alpha\in R_+\cap R_J}\alpha$. Let $\beta\in\Pi_J$ and $\alpha=-w^{-1}(\beta)$ $(\in\Pi_I)$. Then $\langle w\rho+\rho,\ \beta^\vee\rangle=-\langle\rho,\ \alpha^\vee\rangle+\langle\rho,\ \beta^\vee\rangle=0$, and hence $w\rho+\rho$ is W_J-invariant. Since $wW_Iw^{-1}=W_J$ and $w\rho(I)=-\rho(J)$, (6.8.4) is equal to

$$e^{-w\rho-\rho}\sum_{w'\in W_I}\varepsilon(w')e^{ww'(\lambda-\rho(I))+w\rho(I)}\prod_{\alpha\in R_+}(1-e^{-\alpha})^{-1}. \tag{6.8.5}$$

On the other hand, by (6.3) and the Weyl's character formula again, the right hand side of (6.8.3) is equal to

$$e^{-w\rho-\rho}w\Big(\sum_{w'\in W_I}\varepsilon(w')e^{w'(w_I\lambda+\rho(I))-\rho(I)}\prod_{\alpha\in R_+\cap R_I}(1-e^{-\alpha})^{-1}\Big)\cdot\prod_{\alpha\in R_+\setminus wR_I}(1-e^{-\alpha})^{-1}. \tag{6.8.6}$$

(Note that the highest weight of $V_I(\lambda)$ is $w_I\lambda$.) Since

$$\begin{aligned} w\prod_{\alpha\in R_+\cap R_I}(1-e^{-\alpha})^{-1}&=w\Big(\varepsilon(w_I)e^{2\rho(I)}\prod_{\alpha\in R_+\cap R_I}(1-e^{\alpha})^{-1}\Big)\\ &=\varepsilon(w_I)e^{2w\rho(I)}\prod_{\alpha\in R_+\cap R_J}(1-e^{-\alpha})^{-1}, \end{aligned}$$

replacing w' with $w'w_I$ in (6.8.6), we can see that (6.8.5) is equal to (6.8.6). Thus we get (6.8.3) and complete the proof.

Remark 6.9. Let us show that the set

(6.9.1) $\{V(w, \lambda, \mathfrak{p}(I)) | I \subset S,\ w \in (W/W_I)_l,\ \langle \lambda, \alpha^\vee \rangle \in \boldsymbol{Z}_{\leq 0} \text{ for } \alpha \in \Pi_I\}/\cong$

is strictly larger than the set

(6.9.2) $\{M(\lambda', \mathfrak{p}(J)) | J \subset S,\ \langle \lambda', \beta^\vee \rangle \in \boldsymbol{Z}_{\geq 0} \text{ for } \beta \in \Pi_J\}/\cong.$

Let $M(\lambda', \mathfrak{p}(J))$ be a module which belongs to (6.9.2). Then by (6.8), $M(\lambda', \mathfrak{p}(J)) = V(w_S, w_S\lambda', \mathfrak{p}(w_S J w_S))$ belongs to (6.9.1). Next let us find a module which belongs to (6.9.1) but not to (6.9.2). For this purpose, it suffices to show that $V(w, \lambda, \mathfrak{p}(I))$ in (6.9.1) with $\lambda \in \mathfrak{t}_I^\vee$ belongs to (6.9.2) if and only if $w(\Pi_I) \subset -\Pi$. Assume that $V(w, \lambda, \mathfrak{p}(I))$ is isomorphic to some module $M(\lambda', \mathfrak{p}(J))$ in the set (6.9.2). By the Weyl's character formula and by (6.7.1),

(6.9.3) $$\mathrm{ch} M(\lambda', \mathfrak{p}(J)) = \sum_{w' \in W_J} \varepsilon(w') e^{w'(\lambda' + \rho(J)) - \rho(J)} \prod_{\alpha \in R_+} (1 - e^{-\alpha})^{-1}.$$

Note that the numerator of the right hand side of (6.9.3) can be expressed as $e^{\lambda'} \cdot f$, where f is a Laurent polynomial in $\{e^\beta | \beta \in \Pi_J\}$. Hence among the factors of the denominator, $(1 - e^{-\alpha})$ for $\alpha \in R_+ \backslash R_J$ can not be canceled. (Note that the group ring of the root lattice is a unique factorization domain.) On the other hand, for $\lambda \in \mathfrak{t}_I^\vee$,

(6.9.4) $$\mathrm{ch} V(w, \lambda, \mathfrak{p}(I)) = e^{w(\lambda - \rho) - \rho} \prod_{\alpha \in R_+ \backslash w R_I} (1 - e^{-\alpha})^{-1},$$

by (6.3). Hence

(6.9.5) $$R_+ \backslash R_J \subset R_+ \backslash w R_I.$$

By (6.9.3) and (6.9.4), we also get

(6.9.6) $$e^{w(\lambda - \rho) - \rho + \rho(J)} \prod_{\alpha \in R_+ \cap w R_I} (1 - e^{-\alpha}) = \sum_{w' \in W_J} \varepsilon(w') e^{w'(\lambda' + \rho(J))}.$$

Since the right hand side is W_J-antisymmetric, the left hand side is divisible by $\prod_{\alpha \in R_+ \cap R_J} (1 - e^{-\alpha})$ [6, Chap. 6, no. 3.3, Prop. 2]. Thus (6.9.5) becomes an equality. Since $w \in (W/W_I)_l$, $w(R_I \cap R_+)$ should be equal to $-(R_J \cap R_+)$, and

(6.9.7) $$w(\Pi_I) = -\Pi_J.$$

(Moreover, comparing the highest terms of (6.9.6), we get

(6.9.8) $$w(\lambda - \rho) - \rho = \lambda').$$

Conversely, assume $V(w, \lambda, \mathfrak{p}(I))$ in (6.9.1) is given and $w(\Pi_I) \subset -\Pi$. Then, by (6.8), $V(w, \lambda, \mathfrak{p}(I)) = M(w(\lambda - \rho) - \rho, \mathfrak{p}(w I w^{-1}))$ belongs to (6.9.2). Hence, if λ is W_I-invariant, $V(w, \lambda, \mathfrak{p}(I))$ in (6.9.1) belongs to (6.9.2) if and only if $w(\Pi_I) \subset -\Pi$. Thus (6.9.1) is strictly larger than (6.9.2). In other words, our $U(\mathfrak{g})$-module $V(w, \lambda, \mathfrak{p})$ is a further generalization of generalized Verma modules.

§7. Resolutions of $V(w, \lambda, \mathfrak{p})$

7.0. Resolutions of a finite dimensional $U(\mathfrak{g})$-modules by the Verma modules were constructed in [2] and [22] in several ways. In this section, we shall construct resolutions of $V(w, \lambda, \mathfrak{p})$ by the Verma modules. First we construct a resolution of $V(w, \lambda, \mathfrak{p})$ using the Grothendieck-Cousin complex [22]. Then using this resolution instead of [2, 9.9], and following the argument of [2], we construct a second resolution, which is a generalization of the resolution of a finite dimensional representation constructed by Bernstein-Gelfand-Gelfand [2, 10.1 and 10.1′]. In order to reduce our task, we consider only the case where λ is W_I-invariant.

7.1. Grothendieck-Cousin complex. We fix an element $w\in(W/W_I)_l$ until the end of (7.6). Let $W_I^{(i)}=\{x\in W_I \mid l(x)=i\}$, $W(w, i)=\{w'\in W \mid w\geqq w',\ l(w')=l(w)-i\}$, $Z_i=\bigcup_{j\geqq i}\bigcup_{x\in W(w,j)}X(x)$ for $i\geqq 0$, $Z_i'=Z_i$ for $i>0$ and $Z_i'=X$ for $i\leqq 0$. (See (3.7) and (4.2) for $X(w)$ etc. Note that $W(w, i)\cap wW_I=wW_I^{(i)}$ and $(Z_i'-Z_{i+1}')\cap q^{-1}Y(w)=\bigcup_{x\in W_I^{(i)}}X(wx)$.) For any sheaf $\mathcal{F}'$ on X, $X=Z_0'\supset Z_1'\supset\cdots\supset Z_{l'}'\supset Z_{l'+1}'=\phi$ gives the *global Cousin complex* of $\mathcal{F}'$ with respect to the filtration $\{Z_i'\}$

$$0\longrightarrow \Gamma(X, \mathcal{F}')\longrightarrow H^0_{Z_0'/Z_1'}(X, \mathcal{F}')\longrightarrow H^1_{Z_1'/Z_2'}(X, \mathcal{F}')\longrightarrow\cdots. \tag{7.1.1}$$

See [22, 7.8]. We have also the *local Cousin complex* [22, 8.6]

$$0\longrightarrow \mathcal{F}'\longrightarrow H^0_{Z_0'/Z_1'}(\mathcal{F}')\longrightarrow H^1_{Z_1'/Z_2'}(\mathcal{F}')\longrightarrow\cdots. \tag{7.1.2}$$

7.2. Let C be the full subcategory of the category of quasi-coherent $\mathcal{O}$-modules consisting of sheaves whose supports are contained in $\overline{q^{-1}Y(w)}$. Then C is closed under kernels, cokernels and extensions. Since $H^j_{X/Z_i'}(\mathcal{F}')\in C$ for any $\mathcal{F}'\in C$ and $i, j\in \boldsymbol{Z}$, all the basic assumptions of [22, 9.4 and several lines preceding it] are satisfied with our category C and the filtration $\{Z_i'\}$. Let us show that the following conditions are satisfied for $i\geqq 0$.

$(L.V.)_i$ $\qquad R\Gamma_{X/Z_{i+1}'}(\mathcal{F}')=\Gamma_{X/Z_{i+1}'}(\mathcal{F}')$ if $\mathcal{F}'\in C$ and $\mathrm{supp}\mathcal{F}'\subset Z_i'$.

Let ι_i': $Z_i'-Z_{i+1}'\to X$ be the inclusion mappings. If $i>0$, then the connected components of $Z_i'-Z_{i+1}'$ are affine spaces $X(x)$ ($x\in W(w, i)$), and hence for any $w'\in W$, $\iota_i'^{-1}(w'X(w_S))$ is a disjoint union of $X(x)\cap w'X(w_S)$ ($x\in W(w, i)$). Each $X(x)\cap w'X(w_S)$ is empty or a complement of a hypersurface in the affine space $X(x)$, and hence $\iota_i'^{-1}(w'X(w_S))$ ($i>0$) are affine varieties. Since $\{w'X(w_S)\mid w'\in W\}$ is an affine open covering of X, ι_i' ($i>0$) are affine morphisms. Hence by the argument of the proof of [22, 9.6], we can prove $(L.V.)_i$ for $i>0$. Since $R\Gamma_{X/Z_1'}(\mathcal{F}')=R\Gamma_{Z_0/Z_1}(\mathcal{F}')$ and $\Gamma_{X/Z_1'}(\mathcal{F}')=\Gamma_{Z_0/Z_1}(\mathcal{F}')$ for $\mathcal{F}'\in C$, we get $(L.V.)_0$ by the same argument. Thus the condition $(L.V.)$ of [22, p 362] is satisfied. In a similar way, using the fact that Z_i-Z_{i+1} are affine varieties, we can show that the condition $(G.V.)$ of [22, p 362] is also satisfied.

7.3. First construction of a resolution. Let $m:=cd(w)=\mathrm{codim}_Y Y(w)=\mathrm{codim}_X q^{-1}Y(w)=l(w_S)-l(w)$, and $\mathcal{F}:=H^m_{q^{-1}Y(w)}(\mathcal{O}_X)=R\Gamma_{q^{-1}Y(w)}(\mathcal{O}_X)[m]$. (Recall that $w\in(W/W_I)_l$.) Then $\mathcal{F}\in\mathcal{C}$. Note that, for $i\geqq 0$,

$$R\Gamma_{Z'_i/Z'_{i+1}}R\Gamma_{q^{-1}Y(w)}=R\Gamma_{(Z'_i-Z'_{i+1})\cap q^{-1}Y(w)}=\bigoplus_{w\in W_I^{(i)}}R\Gamma_{X(wx)}$$

in $\mathcal{C}$, and $\mathrm{codim}_X X(wx)=m+l(x)$ for $x\in W_I$. Hence, for $i\geqq 0$,

$$H^i_{Z'_i/Z'_{i+1}}(\mathcal{F})=H^i_{Z'_i/Z'_{i+1}}(H^m_{q^{-1}Y(w)}(\mathcal{O}_X))=\bigoplus_{x\in W_I^{(i)}}H^{m+i}_{X(wx)}(\mathcal{O}_X) \tag{7.3.1}$$

and

$$H^j_{Z'_i/Z'_{i+1}}(\mathcal{F})=0, \qquad \text{if } j\neq i. \tag{7.3.2}$$

By [22, 10.5], $\mathcal{F}$ is locally Cohen-Macaulay with respect to $\{Z'_i\}$. In other words, the complex (7.1.2) is exact for $\mathcal{F}'=\mathcal{F}$. By [22, 9.5.(e)], $H^i_{Z'_i/Z'_{i+1}}(\mathcal{F})$ are $\Gamma(X,-)$-acyclic. Hence $H^i(X,\mathcal{F})$ are the cohomologies of the complex

$$\Gamma(X, H^0_{Z'_0/Z'_1}(\mathcal{F}))\longrightarrow\Gamma(X, H^1_{Z'_1/Z'_2}(\mathcal{F}))\longrightarrow\cdots. \tag{7.3.3}$$

Again by [22, 9.5.(e)], the complex (7.3.3) can be naturally identified with the complex

$$H^0_{Z'_0/Z'_1}(X,\mathcal{F})\longrightarrow H^1_{Z'_1/Z'_2}(X,\mathcal{F})\longrightarrow\cdots. \tag{7.3.4}$$

By the same calculation as (7.3.1), we get

$$H^i_{Z'_i/Z'_{i+1}}(X,\mathcal{F})=\bigoplus_{x\in W_I^{(i)}}H^{m+i}_{X(wx)}(X,\mathcal{O}_X). \tag{7.3.5}$$

Since

$$\begin{aligned}R\Gamma(X,\mathcal{F})&=R\Gamma(X, R\Gamma_{q^{-1}Y(w)}(\mathcal{O}_X))[m]=R\Gamma(Y, Rq_*R\Gamma_{q^{-1}Y(w)}(\mathcal{O}_X))[m]\\&=R\Gamma(Y, R\Gamma_{Y(w)}Rq_*(\mathcal{O}_X))[m]=R\Gamma(Y, R\Gamma_{Y(w)}(\mathcal{O}_Y))[m]=R\Gamma_{Y(w)}(Y,\mathcal{O}_Y)[m],\end{aligned}$$

we have

$$H^i(X,\mathcal{F})=\begin{cases}H^m_{Y(w)}(Y,\mathcal{O}), & \text{if } i=0\\ 0, & \text{if } i\neq 0.\end{cases} \tag{7.3.6}$$

Since the cohomologies of the complex (7.3.4) can be identified with (7.3.6), and each term of this complex are given by (7.3.5), we get the exact sequence

$$0\longrightarrow H^m_{Y(w)}(Y,\mathcal{O}_Y)\longrightarrow A^0\longrightarrow A^1\longrightarrow\cdots\longrightarrow A^{l'}\longrightarrow 0, \tag{7.3.7}$$

where

$$A^i=\bigoplus_{x\in W_I^{(i)}}H^{m+i}_{X(wx)}(X,\mathcal{O}_X).$$

Lemma 7.4. *For $\lambda\in\mathfrak{t}^{\vee}_{I,ad}$, we have an exact sequence*

(7.4.1) $$0\longleftarrow H^m_{Y(w)}(Y,\mathcal{O}_Y(\lambda))^*\longleftarrow B_0\longleftarrow B_1\longleftarrow\cdots\longleftarrow B_{l'}\longleftarrow 0,$$

where $l'=l(w_I)$ *and*

$$B_i=\bigoplus_{x\in W_I^{(i)}}M(wx(\lambda-\rho)-\rho).$$

Proof. As a dual of (7.3.7), we get the exact sequence

$$0\longleftarrow H^m_{Y(w)}(Y,\mathcal{O}_Y(\lambda))^*\longleftarrow(A^0)^*\longleftarrow\cdots\longleftarrow(A^{l'})^*\longleftarrow 0.$$

Thus the assertion follows from (3.8).

Using (6.6), the above assertion can be also stated as follows.

Lemma 7.5. *If $\lambda\in\mathfrak{t}^{\vee}_{I,ad}$, we have an exact sequence of $U(\mathfrak{g})$-modules*

(7.5.1) $$0\longleftarrow V(w,\lambda,\mathfrak{p})\longleftarrow B_0\longleftarrow B_1\longleftarrow\cdots\longleftarrow B_{l'}\longleftarrow 0$$

where l' and B_i are as in (7.4).

Lemma 7.6. *For $\lambda\in\mathfrak{t}^{\vee}_{I,ad}$,* $\dim\mathrm{Tor}_i^{\mathfrak{n}_-}(\boldsymbol{C},V(w,\lambda,\mathfrak{p}))=\mathrm{card}\,W_I^{(i)}$, *where $\boldsymbol{C}$ is considered as a trivial right $\mathfrak{n}_-$-module.*

Proof. Since (7.5.1) gives a free $U(\mathfrak{n}_-)$-resolution of $V(w,\lambda,\mathfrak{p})$, the torsion groups are the homology groups of the complex

$$0\longleftarrow B_0/\mathfrak{n}_-B_0\overset{\bar{d}_1}{\longleftarrow}B_1/\mathfrak{n}_-B_1\overset{\bar{d}_2}{\longleftarrow}\cdots\overset{\bar{d}_{l'}}{\longleftarrow}B_{l'}/\mathfrak{n}_-B_{l'}\longleftarrow 0.$$

For $x\in W_I^{(i)}$ and $y\in W_I^{(i-1)}$, we have $(wy(\lambda-\rho)-\rho)-(wx(\lambda-\rho)-\rho)=w((\rho-y\rho)-(\rho-x\rho))$, which can not be equal to zero by [2, 9.8]. Hence $\bar{d}_i=0$. (Note that $B_i/\mathfrak{n}_-B_i\cong\bigoplus_{x\in W_I^{(i)}}\boldsymbol{C}u(wx(\lambda-\rho)-\rho)$, where $u(-)$ is the highest weight vector of the Verma module $M(-)$.) Since $\dim B_i/\mathfrak{n}_-B_i=\mathrm{card}\,W_I^{(i)}$, we get the assertion.

Lemma 7.7. ([3]. Cf. [9].) *For any field k of characteristic zero, and λ, $\mu\in\mathfrak{t}_k^{\vee}$, the following conditions are equivalent*: (1) $M_k(\lambda-\rho,\mathfrak{b}_k)\supset M_k(\mu-\rho,\mathfrak{b}_k)$. (2) *There is a sequence $\gamma_1,\cdots,\gamma_n$ of roots such that* $\lambda\geqq r_{\gamma_1}(\lambda)\geqq r_{\gamma_2}r_{\gamma_1}(\lambda)\geqq\cdots\geqq r_{\gamma_n}\cdots r_{\gamma_2}r_{\gamma_1}(\lambda)=\mu$.

7.8. For x, $y\in W$ and $\gamma\in R$, we write $x\overset{\gamma}{\rightarrow}y$ if $xr_\gamma=y$ and $l(x)+1=l(y)$. Sometimes we shall omit the symbol γ on the arrow. For x, $y\in W$, the following conditions are known to be equivalent (cf. [4]): (1) There exists a sequence $\gamma_1,\cdots,\gamma_n$ in R such that $x=x_0\overset{\gamma_1}{\rightarrow}x_1\overset{\gamma_2}{\rightarrow}\cdots\overset{\gamma_n}{\rightarrow}x_n=y$. (2) $x\leqq y$. (3) $\overline{BxB}\subset\overline{ByB}$.

Lemma 7.9. *Let $\gamma\in R$, $w\in W$ and $w'=wr_\gamma$. Then $w\overset{\gamma}{\rightarrow}w'$ if and only if*

$w(-\rho)\leqq w'(-\rho)$.

Proof. Let $n=\langle\gamma^\vee, \rho\rangle$. Then $0\neq n\in \boldsymbol{Z}$ and $w\rho-w'\rho=nw(\gamma)$. Hence $w\rho>w'\rho$ if and only if $nw(\gamma)>0$. Let $\gamma'=w(\gamma)$. We may assume that $\gamma'>0$ by replacing γ with $-\gamma$ if necessary. Since $w'=r_{\gamma'}w$ and $w\rho-r_\gamma w\rho=n\gamma'$, the condition $n\gamma'>0$, i.e., $n>0$ is equivalent to $r_{\gamma'}w>w$, i.e., $w\overset{\gamma}{\to}w'$ by [2, 8.10].

Lemma 7.10. *For $w, w'\in W$, the following conditions are equivalent*: (1) $M_K(w(\eta-\rho)-\rho, \mathfrak{b})\supset M_K(w'(\eta-\rho)-\rho, \mathfrak{b})$. (*Recall that* $\eta=c\otimes_A K$.) (2) $M(-w\rho-\rho, \mathfrak{b})\supset M(-w'\rho-\rho, \mathfrak{b})$ *and* $wW_I=w'W_I$. (3) $w\geqq w'$ *and* $wW_I=w'W_I$. (4) *There is a sequence* $\gamma_1, \cdots, \gamma_n$ *in* R_I *such that* $w=w_0\overset{\gamma_1}{\leftarrow}w_1\overset{\gamma_2}{\leftarrow}\cdots\overset{\gamma_n}{\leftarrow}w_n=w'$.

Proof. Assume that $M_K(w(\eta-\rho)-\rho, \mathfrak{b})\supset M_K(w'(\eta-\rho)-\rho, \mathfrak{b})$. By (7.7), there exists a sequence $\gamma_1', \cdots, \gamma_n'$ in R such that $w(\eta-\rho)\geqq r_{\gamma_1'}w(\eta-\rho)\geqq\cdots\geqq r_{\gamma_n'}\cdots r_{\gamma_2'}r_{\gamma_1'}w(\eta-\rho)=w'(\eta-\rho)$. For any $x, y\in W$, $x(\eta-\rho)\geqq y(\eta-\rho)$ implies that $xW_I=yW_I$ and $x(-\rho)\geqq y(-\rho)$. Hence $wW_I=w'W_I$ and

$$-w\rho\geqq -r_{\gamma_1'}w\rho\geqq\cdots\geqq -r_{\gamma_n'}\cdots r_{\gamma_2'}r_{\gamma_1'}w\rho=-w'\rho\,. \tag{7.10.1}$$

Let $w^{-1}(\gamma_i')=\gamma_i$. Since $wW_I=r_{\gamma_i'}\cdots r_{\gamma_1'}wW_I=wr_{\gamma_i}\cdots r_{\gamma_1}W_I$, the reflections r_{γ_i} belong to W_I. Hence $\gamma_i\in R_I$ and $-w\rho\geqq -wr_{\gamma_1}\rho\geqq\cdots\geqq -wr_{\gamma_n}\cdots r_{\gamma_2}r_{\gamma_1}\rho=-w'\rho$. By (7.9), this relation is equivalent to the assertion (4). Thus we have proved the implication (1)$\Rightarrow$(4). The implications (4)$\Rightarrow$(1) and (2)$\Leftrightarrow$(4) can be proved in a similar way. Let us prove (3)$\Rightarrow$(1). Take $x\in(W/W_I)_s$ so that $xW_I=wW_I=w'W_I$. If $w\geqq w'$, then x, w and w' can be expressed as $x=t_1\cdots t_a$, where $a=l(x)$ and $t_i\in S$, $w=t_1\cdots t_a r_1\cdots r_b$, where $a+b=l(w)$ and $r_j\in I$, and $w'=t_{i_1}\cdots t_{i_{a'}}r_{j_1}\cdots r_{j_{b'}}$, where $1\leqq i_1<\cdots<i_{a'}\leqq a$, $1\leqq j_1<\cdots<j_{b'}\leqq b$ and $a'+b'=l(w')$. Since $x\in(W/W_I)_s$ and $t_{i_1}\cdots t_{i_{a'}}W_I=w'W_I=xW_I=t_1\cdots t_aW_I$, the expression of w' should be $w'=t_1\cdots t_a r_{j_1}\cdots r_{j_{b'}}$. Let $y_0=r_1\cdots r_b$. Since $r_1\cdots r_b\geqq r_{j_1}\cdots r_{j_{b'}}$, we can find a sequence $\gamma_1, \cdots, \gamma_n\in R_I$ such that

$$r_1\cdots r_b=y_0\overset{\gamma_1}{\longleftarrow}y_1\overset{\gamma_2}{\longleftarrow}\cdots\overset{\gamma_n}{\longleftarrow}y_n=r_{j_1}\cdots r_{j_{b'}}\,.$$

(Apply (7.8) to the Weyl group W_I with the set of simple reflections I.) Then

$$w=xy_0\overset{\gamma_1}{\longleftarrow}xy_1\overset{\gamma_2}{\longleftarrow}\cdots\overset{\gamma_n}{\longleftarrow}xy_n=w', \tag{7.10.2}$$

and $y_i\in W_I$. Since η is W_I-invariant,

$$xy_{i-1}(\eta-\rho)-xy_i(\eta-\rho)=xy_{i-1}(-\rho)-xy_i(-\rho)\,. \tag{7.10.3}$$

By (7.9) and (7.10.2), $xy_{i-1}(-\rho)\geqq xy_i(-\rho)$. By (7.10.3), $w(\eta-\rho)=xy_0(\eta-\rho)\geqq\cdots\geqq xy_n(\eta-\rho)=w'(\eta-\rho)$. Thus, by (7.7), we get the implication (3)$\Rightarrow$(1). The implication (4)$\Rightarrow$(3) follows from (7.8).

Lemma 7.11. *Let $R=R(x)$ be an $n\times m$-matrix with components in $\boldsymbol{C}[x]=\boldsymbol{C}[x_1, \cdots, x_l]$, where $x_1, \cdots, x_l$ are indeterminates. Assume that the rank of $R(\lambda)$ is $m-1$ for any $\lambda\in\boldsymbol{C}^l$. Then there exists a unique vector $u=u(x)\in\boldsymbol{C}[x]^m$ up to a non-zero constant multiple such that $R(x)u(x)=0$ and $u(x)$ is not divisible by any element of $\boldsymbol{C}[x]\backslash\boldsymbol{C}$. Moreover, $u(\lambda)\neq 0$ for any $\lambda\in\boldsymbol{C}^l$.*

Proof. The assertion concerning the existence and the uniqueness is obvious. Let $R_j(x)$ $(1\leqq j\leqq m)$ be the column vectors of $R(x)$. For any $\lambda\in\boldsymbol{C}^l$, there exists a unique nontrivial linear relation $\sum_{j=1}^m c_j R_j(\lambda)=0$ up to a constant multiple. Let $U_k=\{\lambda\in\boldsymbol{C}^l\,|\,c_k\neq 0\}$. For any $\lambda\in U_k$, there exist unique $u'_j(\lambda)\in\boldsymbol{C}$ $(j\neq k)$ such that $R_k(\lambda)=\sum_{j\neq k}u'_j(\lambda)R_j(\lambda)$. Using the formula of Cramer for a system of linear equations, we can express $u'_j(x)$'s as regular functions in $x\in U_k$. By multiplying the denominators of $u'_j(x)$'s, we get a relation of the form $\sum_{j=1}^m u''_j(x)R_j(x)=0$ with $u''_j(x)\in\boldsymbol{C}[x]$ such that $u''_k(\lambda)\neq 0$ for any $\lambda\in U_k$. Dividing by the greatest common divisor, we may assume that $u''_j(x)$'s are relatively prime. Since u is a non-zero constant multiple of u'', $u(\lambda)\neq 0$ for any $\lambda\in\boldsymbol{C}^l=\bigcup_k U_k$.

Lemma 7.12. *If the equivalent conditions of* (7.10) *are satisfied with w and w', then there exists a $U_A(\mathfrak{g})$-homomorphism φ: $M_A(w'(c-\rho)-\rho, \mathfrak{b})\to M_A(w(c-\rho)-\rho, \mathfrak{b})$ such that $\varphi\otimes_{A,\lambda}\boldsymbol{C}\neq 0$ for any $\lambda\in\mathfrak{t}_I^\vee$. Such a homomorphism φ is unique up to multiplication of a non-zero complex number.*

Proof. It suffices to consider the case where $w=w'r_\alpha$ with $\alpha\in R_I\cap R_+$ and $l(w)=l(w')+1$. Then $w'(\alpha)\in R_+$,

$$w(c-\rho)-\rho=w'(c-\rho)-\rho+nw'\alpha\geqq w'(c-\rho)-\rho \tag{7.12.1}$$

with $n:=\langle\rho, \alpha^\vee\rangle$, and $M_K(w'(\eta-\rho)-\rho, \mathfrak{b})\subset M_K(w(\eta-\rho)-\rho, \mathfrak{b})$. Hence there is a non-zero element u of $U_K(\mathfrak{n}_-)$ such that

$$[H, u]=-\langle nw'\alpha, H\rangle \quad \text{for } H\in\mathfrak{t}, \text{ and} \tag{7.12.2}$$

$$\mathfrak{n}_+uv=0, \tag{7.12.3}$$

where v denotes the canonical generator of $M_A(w(c-\rho)-\rho, \mathfrak{b})$. Multiplying and/or dividing an element of A if necessary, we may assume that $u\in U_A(\mathfrak{n}_-)$ and $a^{-1}u\notin U_A(\mathfrak{n}_-)$ for any $a\in A\backslash\boldsymbol{C}$. Identify A with a polynomial ring $\boldsymbol{C}[x_1, \cdots, x_l]$, let $U(\mathfrak{n}_-)(-nw'\alpha)$ be the set of $u\in U(\mathfrak{n}_-)$ satisfying (7.12.2), and fix a $\boldsymbol{C}$-basis of $U(\mathfrak{n}_-)(-nw'\alpha)$. Then $U(\mathfrak{n}_-)(-nw'\alpha)\otimes A$ can be identified with A^m, where $m=\dim U(\mathfrak{n}_-)(-nw'\alpha)$. Let $u=u(x)={}^t(u_1(x), \cdots, u_m(x))$. As is seen from the proof of [9, 7.6.12], the condition (7.12.3) on u can be written as a system of linear equations in $u_i(x)$ $(1\leqq i\leqq m)$ with coefficients in $A=\boldsymbol{C}[x_1, \cdots, x_l]$, say, (7.12.4) $$R(x)u(x)=0.$$

Since $w(\lambda-\rho)-\rho=w'(\lambda-\rho)-\rho+nw'\alpha\geqq w'(\lambda-\rho)-\rho$ for any $\lambda\in\mathfrak{t}_I^{\vee}$, dim Hom $(M(w'(\lambda-\rho)-\rho, \mathfrak{b}), M(w(\lambda-\rho)-\rho, \mathfrak{b}))=1$ by [9, 7.6.6 and 7.6.23]. Hence the solution space of (7.12.4) is one-dimensional for any $(\lambda_1, \cdots, \lambda_l)\in \boldsymbol{C}^l$, i. e., the rank of $R(\lambda)$ is equal to $m-1$. Thus applying (7.11) to our situation, we get an element $v'=uv$ of $M_A(w(c-\rho)-\rho, \mathfrak{b})$ such that $v'\otimes_{A,\lambda}1\neq 0$ for any $\lambda\in\mathfrak{t}_I^{\vee}$. Then the $U_A(\mathfrak{g})$-homomorphism which sends the canonical generator of $M_A(w'(c-\rho)-\rho, \mathfrak{b})$ to v' satisfies the condition.

7.13. Fix a reduced expression of each $w\in W_I\backslash\{e\}$ and let $\sigma(w)\in I$ be the last factor of the fixed expression. For any arrow $w\rightarrow w'$, define the function $s(w, w')$ by the induction on $l(w)$ as follows. If $w\sigma(w')>w$, let $s(w, w')=1$. (Especially $s(e, w')=1$.) If $w\sigma(w')<w$, let $s(w, w')=-s(w\sigma(w'), w'\sigma(w'))$.

Lemma 7.14. ([2, 10.3 and 10.4]) (1) *If $w_1\rightarrow w_3\rightarrow w_2$ with $w_i\in W_I$, there exists exactly one $w_4\in W_I-\{w_3\}$ such that $w_1\rightarrow w_4\rightarrow w_2$.* (2) *For any quadruple (w_1, w_2, w_3, w_4) as in* (1), $s(w_1, w_3)s(w_3, w_2)+s(w_1, w_4)s(w_4, w_2)=0$.

7.15. Let $w\in(W/W_I)_l$. For any $x\in W_I$, $M_K(wx(\eta-\rho)-\rho, \mathfrak{b})$ can be embedded in $M_K(w(\eta-\rho)-\rho, \mathfrak{b})$ by (7.10). Let $c'(e, x)$ $(x\in W_I)$ be such embeddings. By (7.12), we may assume that $c'(e, x)$ induces an embedding $c'_A(e, x)$ of $M_A(wx(c-\rho)-\rho, \mathfrak{b})$ into $M_A(w(c-p)-\rho, \mathfrak{b})$, and that $c'_A(e, x)\otimes_{A,\lambda}\boldsymbol{C}\not\equiv 0$ for any $\lambda\in\mathfrak{t}_I^{\vee}$. Fix such an embedding for each $x\in W_I$. Then, if $x_1, x_2\in W_I$ and $x_1\rightarrow x_2$, there is a unique embedding $c'(x_1, x_2)$: $M_K(wx_2(\eta-\rho)-\rho, \mathfrak{b})\rightarrow M_K(wx_1(\eta-\rho)-\rho, \mathfrak{b})$, which is compatible with the fixed embeddings $c'(e, x)$. Put $x_0=e$, $M_i=M_A(wx_i(c-\rho)-\rho, \mathfrak{b})$ $(i=0, 1, 2)$, and let u_i be the canonical generator of M_i. By (7.12), we can take $t\in K^{\times}$ so that $u'_2:=tc'(x_1, x_2)(u_2)\in M_1$ and $u'_2\otimes_{A,\lambda}1\neq 0\,(\in M_1\otimes_{A,\lambda}\boldsymbol{C})$ for any $\lambda\in\mathfrak{t}_I^{\vee}$. Then $tc'(e, x_2)$ sends u_2 to $c'_A(e, x_1)(u'_2)\in M_0$. Hence $tc'(e, x_2)$ sends M_2 into M_0, and for any $\lambda\in\mathfrak{t}_I^{\vee}$, $(tc'(e, x_2)\otimes_{A,\lambda}\boldsymbol{C})(u_2\otimes 1)=(c'_A(e, x_1)\otimes_{A,\lambda}\boldsymbol{C})(u'_2\otimes 1)\neq 0$, since $c'_A(e, x_1)\otimes\boldsymbol{C}$ is an embedding of a Verma module. Then by the uniqueness part of (7.12), $t\in\boldsymbol{C}^{\times}$. Hence $c'(x_1, x_2)$ sends $M_A(wx_2(c-\rho)-\rho, \mathfrak{b})$ into $M_A(wx_1(c-\rho)-\rho, \mathfrak{b})$ and $c'(x_1, x_2)\otimes_{A,\lambda}\boldsymbol{C}\not\equiv 0$ for any $\lambda\in\mathfrak{t}_I^{\vee}$. Let

$$c(x_1, x_2)=\begin{cases} s(x_1, x_2)c'(x_1, x_2), & \text{if } x_1\rightarrow x_2, \\ 0, & \text{otherwise.}\end{cases}$$

Let $W_I^{(i)}=\{x\in W_I\,|\,l(x)=i\}$, $l'=l(x_I)$ as before, and

$$C_{A,i}=\bigoplus_{x\in W_I^{(i)}} M_A(wx(c-\rho)-\rho, \mathfrak{b}).$$

Then $d_i=(c(x, y))_{x\in W_I^{(i)}, y\in W_I^{(i+1)}}$ and the natural projection $\varepsilon: \boldsymbol{C}_{A,0}=M_A(w(c-\rho)-\rho, \mathfrak{b})\rightarrow V_A(w, c, \mathfrak{p})$ defines a sequence

(7.15.1) $0 \longleftarrow V_A(w, c, \mathfrak{p}) \longleftarrow C_{A,0} \longleftarrow C_{A,1} \longleftarrow \cdots \longleftarrow C_{A,l'} \longleftarrow 0$

which is a complex by (7.14). (Note that $C_{A,1} \subset J_A(w(c-\rho)-\rho) \subset M_A(w(c-\rho)-\rho, \mathfrak{b}) = C_{A,0}$.)

Theorem 7.16. *For* $w \in (W/W_I)_l$, *the complex*

$$0 \longleftarrow V_K(w, \eta, \mathfrak{p}) \longleftarrow C_{K,0} \longleftarrow C_{K,1} \longleftarrow \cdots \longleftarrow C_{K,l'} \longleftarrow 0$$

obtained as (7.15.1)$\otimes_A K$ *is exact.*

Theorem 7.17. *For* $w \in (W/W_I)_l$ *and* $\lambda \in \mathfrak{t}_I^\vee$, *let*

$$C_i(\lambda) = \bigoplus_{x \in W_I^{(i)}} M(wx(\lambda-\rho)-\rho, \mathfrak{b}) = C_{A,i} \otimes_{A,\lambda} C .$$

Then the complex

(7.17.1) $0 \longleftarrow V(w, \lambda, \mathfrak{p}) \overset{\varepsilon}{\longleftarrow} C_0(\lambda) \overset{d_1}{\longleftarrow} C_1(\lambda) \longleftarrow \cdots \overset{d_{l'}}{\longleftarrow} C_{l'}(\lambda) \longleftarrow 0$

obtained as (7.15.1)$\otimes_{A,\lambda} C$ *is exact, if* $\lambda \in \mathfrak{t}_{I,ad}^\vee$.

Proof. Since (7.16) and (7.17) can be proved in the same way, we shall prove only (7.17). The surjectivity of ε is obvious. Assume that we have already proved the exactness at $C_0(\lambda), \cdots, C_{i-1}(\lambda)$, and let us prove the exactness at $C_i(\lambda)$. (If $i=0$, we do not assume anything.) Let $C_j = C_j(\lambda)$ and $K_j = \ker d_j$. The desired equality $d_{i+1}(C_{i+1}) = K_i$ is obtained by modifying the proof of [2, 10.1′]. Here, we provisionally use notations close to those in [2]. Also in our case, it is enough to prove the same assertions as Lemmas 10.5, 10.6 and 10.7 of [2]. We do not need any modification concerning Lemma 10.5 of [2]. As in [2], the proof of "Lemma 10.6" is divided into following two steps.

Lemma a. $JH(K_i) \subset JH(C_{i+1})$.

The proof is the same as in [2] except that we use the exact sequenece (7.5.1) instead of the one constructed in [2, 9.9].

Lemma b. *Let* $\lambda \in \mathfrak{t}^\vee$, $M \in O$, *and* $L(\lambda-\rho)$ *be the simple quotient of the Verma module* $M(\lambda-\rho) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C(\lambda-\rho)$. (See [2, § 8] for O). *Assume that* λ *is maximal in* $\{\psi \in \mathfrak{t}^\vee \mid L(\psi-\rho)$ *occurs in* $JH(M)\}$. *Let* $\tau\colon M(\lambda-\rho) \to M$ *be a homomorphism such that the image* $\tau(f_{\lambda-\rho})$ *of the canonical generator* $f_{\lambda-\rho}$ *of* $M(\lambda-\rho)$ *is not zero. Then the image of* $\tau(f_{\lambda-\rho})$ *in* $M/\mathfrak{n}_- \cdot M$ *is also not zero.*

Proof. We shall prove the assertion by the induction on the length of M. Let $f_{\psi-\rho} \in M$ be a weight vector whose weight $\psi-\rho$ is maximal among the weights of M and $N \subset M$ the submodule generated by $f_{\psi-\rho}$. Concerning the case where $\tau(f_{\lambda-\rho}) \notin N$, we do not need any modification of the proof given in [2]. Assume that $\tau(f_{\lambda-\rho}) \in N$. Then $L(\lambda-\rho) \in JH(N) \subset JH(M(\psi-\rho))$. Hence $\lambda \leqq \psi$. On the other hand, $L(\psi-\rho) \in JH(N) \subset JH(M)$. According to the condition of the lemma, we get $\lambda = \psi$. Since $\psi-\rho$ is a maximal weight of

M, $\tau(f_{\lambda-\rho})\notin\mathfrak{n}_-\cdot M$.

To complete the proof of "Lemma 10.6", i. e., the injectivity of the mapping $C_{i+1}/\mathfrak{n}_-\cdot C_{i+1}\to K_i/\mathfrak{n}_-\cdot K_i$, it suffices to apply Lemma b to the module $M=K_i$. Cf. the proof of Lemma 10.6 of [2].

Lastly, we modify the proof of Lemma 10.7 of [2], namely, we replace the proof of the identity $\dim_C C_{i+1}/\mathfrak{n}_- C_{i+1}=\dim_C K_i/\mathfrak{n}_- K_i<\infty$ with the following argument. Define the modules C and D, and the morphisms η, ϑ, $\bar{\eta}$ and $\bar{\vartheta}$ in the same way as in the proof of Lemma 10.7 of [2]. Then we get exact sequences $D\xrightarrow{\eta}C\xrightarrow{\tau}K_i\to 0$, and $C\xrightarrow{\vartheta}C_i\xrightarrow{d_i}K_{i-1}\to 0$. (The latter sequence should be replaced with $C\xrightarrow{\vartheta}C_0\xrightarrow{\varepsilon}V(w,\lambda,\mathfrak{p})\to 0$, if $i=0$.) As in [2], using these sequences, we can show that $\dim K_i/\mathfrak{n}_- K_i=\dim \mathrm{Tor}_{i+1}^{\mathfrak{n}_-}(C, V(w,\lambda,\mathfrak{p}))$. On the other hand, by (7.6), we get $\dim C_{i+1}/\mathfrak{n}_- C_{i+1}=\mathrm{card}\ W_I^{(i+1)}=\dim \mathrm{Tor}_{i+1}^{\mathfrak{n}_-}(C, V(w,\lambda,\mathfrak{p}))$.

Remark 7.18. We assumed that $\lambda-\rho$ is anti-dominant in (7.17). The author does not know whether this condition is necessary or not. A deeply related result is obtained by O. Gabber and A. Joseph [10].

§ 8. $\mathscr{D}$-modules Associated to Complex Powers of Functions

8.0. The purpose of this section is to prove (8.4), which will be used in the next section.

8.1. Let X be a connected smooth variety over C, $\mathcal{O}=\mathcal{O}_X$ the sheaf of regular functions, $\mathscr{D}_X$ the sheaf of algebraic differential operators, $f_1\cdots, f_l\in\Gamma(X,\mathcal{O}_X)$ which are not identically zero, $I=\{1,2,\cdots,l\}$, $e_k=(0,\cdots,0,1,0,\cdots,0)\in C^l$, where 1 appears as the k-th component, $\underline{s}=(s_1,\cdots,s_l)$ the linear coordinate functions of C^l, $g=\prod_{i=1}^l f_i$, B an open ball contained in $\Omega=X\backslash g^{-1}(0)$, and $\underline{f}^{\underline{s}}=f_1^{s_1}\cdots f_l^{s_l}$ a single valued branch of $B\times C^l\ni(x,\underline{s})\to f_1(x)^{s_1}\cdots f_l(x)^{s_l}$. Let $\mathscr{D}[\underline{s}]=\mathscr{D}\otimes_C C[s_1,\cdots,s_l]$, $\mathfrak{N}'=\mathfrak{N}'_{f_1,\cdots,f_l}=\mathscr{D}[\underline{s}]\underline{f}^{\underline{s}}$, $\mathfrak{N}'(\underline{\alpha})=\mathfrak{N}'_{f_1,\cdots,f_l}(\underline{\alpha})=\mathfrak{N}'/\sum_{i=1}^l(s_i-\alpha_i)\mathfrak{N}'$, $\mathfrak{N}(\underline{\alpha})=\mathfrak{N}'(\underline{\alpha})[f_1^{-1},\cdots,f_l^{-1}]$, for $\underline{\alpha}=(\alpha_1,\cdots,\alpha_l)\in C^l$ and $u(\underline{\alpha})$ (resp. $u'(\underline{\alpha})$) the section of the $\mathscr{D}$-module $\mathfrak{N}(\underline{\alpha})$ (resp. $\mathfrak{N}'(\underline{\alpha})$) corresponding to $\underline{f}^{\underline{s}}$.

Lemma 8.2. *For any $k\in I$, there exist $P_k=P_k(\underline{s})\in\mathscr{D}[\underline{s}]$ and $b_k(\underline{s})\in C[\underline{s}]$ such that $P_k\underline{f}^{\underline{s}+e_k}=b_k(\underline{s})\underline{f}^{\underline{s}}$ and b_k is a product of polynomials of degree at most* 1.

Proof. This lemma is essentially due to Sabbah [28]. Since Sabbah works in the analytic category, we need to deduce from it the corresponding assertion in the algebraic category. The necessary argument is the same as the last part of the proof of [12, 2.5.4].

Lemma 8.3. *Let $g=\prod f_i$ as above,*

$$W'=\{(x, s\ \mathrm{grad}\log g(x))\in T^*\Omega\mid s\in C^\times\},$$

$$W = \text{the Zariski closure of } W' \text{ in } T^*X, \text{ and}$$

$$W_0 = \{(x, y) \in W \mid g(x)y_1 = \cdots = g(x)y_n = 0\}.$$

For any $\underline{\alpha} \in C^l$, *the characteristic variety of* $\mathcal{N}(\underline{\alpha})$ *is* W_0. *Especially* $\mathcal{N}(\alpha)$ *is holonomic. (Here* T^*X *denotes the cotangent bundle of* X.)

Proof. If $g^{-1}(0)$ is normal crossing, the assertion can be easily verified. Moreover, we can show that the characteristic cycle of $\mathcal{N}(\alpha)$ does not depend on α in this case. In general, by the Hironaka's desingularization theorem [15], there exist a smooth algebraic variety $\tilde{X}$ and a projective morphism p: $\tilde{X} \to X$ such that $(g \circ p)^{-1}(0)$ is normal crossing and p induces an isomorphism $p: \tilde{\Omega} := \tilde{X} \setminus (g \circ p)^{-1}(0) \to X \setminus g^{-1}(0) = \Omega$. Let $\tilde{\mathcal{N}}(\underline{\alpha})$ be the $\mathcal{D}_{\tilde{X}}$-module defined in the same way as $\mathcal{N}(\alpha)$ using $f_1 \circ p, \cdots, f_l \circ p$ instead of $f_1, \cdots, f_l$. Then $\mathcal{N}(\underline{\alpha}) = j_*(\mathcal{N}(\underline{\alpha}) \mid \Omega) = j_*\left(\int_{p \mid \tilde{\Omega}} \tilde{\mathcal{N}}(\underline{\alpha}) \mid \tilde{\Omega}\right) = \int_p \tilde{j}_*(\tilde{\mathcal{N}}(\underline{\alpha}) \mid \tilde{\Omega}) = \int_p \tilde{\mathcal{N}}(\underline{\alpha})$, where $j: \Omega \to X$ and $\tilde{j}: \tilde{\Omega} \to \tilde{X}$ are the inclusion mappings, and $\int_p$ is the integration along fibres. (See [17].) Since $\tilde{\mathcal{N}}(\underline{\alpha})$ is holonomic, the characteristic cycle of $\mathcal{N}(\underline{\alpha}) = \int_p \tilde{\mathcal{N}}(\underline{\alpha})$ depends only on the characteristic cycle of $\tilde{\mathcal{N}}(\underline{\alpha})$ [23]. Hence the characteristic cycle of $\mathcal{N}(\underline{\alpha}) = \mathcal{N}_{f_1, \cdots, f_l}(\alpha_1, \cdots, \alpha_l)$ coincides with that of $\mathcal{N}_g(0)$, whose support is known to be W_0 [29] (cf. [12, 2.4.6, (2)]).

The purpose of this section is to prove the following assertion.

Proposition 8.4. *Let* p *be a point of* W_0, *and assume that there exist invertible micro-differential operators* Q_k *in a neighbourhood of* p *such that*

$$Q_k \underline{f}^{\underline{s}+e_k} = b_k(\underline{s}) \underline{f}^{\underline{s}}, \tag{8.4.1}$$

where b_k*'s are the polynomials appeared in* (8.2). *Then* $\mathcal{N}(\underline{\alpha})$ *is a simple* $\mathcal{D}$-*module (i.e., it does not have a non-trivial coherent* $\mathcal{D}$-*submodule) if and only if* $b_k(\underline{\alpha} - \underline{\nu}) \neq 0$ *for any* $k \in I$ *and* $\underline{\nu} \in Z^l$.

8.5. *Proof of the "if part".* Let $\mathcal{M}$ be a coherent non-zero $\mathcal{D}$-submodule of $\mathcal{N}(\underline{\alpha})$. Since $\mathcal{N}(\underline{\alpha})$ does not have a non-zero submodule supported by $g^{-1}(0)$ and since $\mathcal{N}(\underline{\alpha}) \mid (X \setminus g^{-1}(0))$ is a simple $\mathcal{D}$-module, $\mathcal{M} = \mathcal{N}(\underline{\alpha})$ on $X \setminus g^{-1}(0)$. Let $u''(\underline{\alpha})$ be the element of $\mathcal{N}(\underline{\alpha})/\mathcal{M}$ corresponding to $u(\underline{\alpha})$. Then $\mathcal{N}(\underline{\alpha})/\mathcal{M} = \bigcup_{m \in Z} \mathcal{D}(g^m u''(\underline{\alpha}))$. Fix an integer m arbitrarily. Since the support of $\mathcal{N}(\underline{\alpha})/\mathcal{M}$ is contained in $g^{-1}(0)$, $g^{m+m'} u''(\underline{\alpha}) = 0$ for a sufficiently large integer m'. Applying P_k's several times to this relation, we get a relation of the form $(\prod_{(k, \nu) \in S} b_k(\underline{\alpha} + \underline{\nu})) g^m u''(\underline{\alpha}) = 0$ with some finite subset S of $I \times Z^l$. It follows from our assumption that $g^m u''(\underline{\alpha}) = 0$ and $\mathcal{N}(\underline{\alpha}) = \mathcal{M}$.

Remark 8.5.1. The assumptions on Q_k's are not used in the "if part".

8.6. In order to prove the "only if part", we need some preliminaries.

8.6.1. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module, u a section of $\mathcal{M}$, and $f\in\mathcal{O}_X$. Consider the left ideal $\mathcal{I}$ of $\mathcal{D}[s]$ consisting of differential operators $P(s)\in\mathcal{D}[s]$ such that $(f^{m-s}P(s)f^s)u=0$ holds in $\boldsymbol{C}[s]\otimes_{C}\mathcal{M}$ for a sufficiently large integer m. (Note that $f^{m-s}P(s)f^s\in\mathcal{D}[s]$ if m is sufficiently large.) Let $f^s u$ be the section of $\mathcal{L}:=\mathcal{D}[s]/\mathcal{I}$ corresponding to the identity element of $\mathcal{D}[s]$. Then $\mathcal{L}=\mathcal{D}[s]/\mathcal{I}=\mathcal{D}[s](f^s u)$. For a complex number α, let $f^\alpha u$ be the section of $\mathcal{L}/(s-\alpha)\mathcal{L}$ corresponding to $f^s u$. Then $\mathcal{L}/(s-\alpha)\mathcal{L}=\mathcal{D}(f^\alpha u)$.

8.6.2. Define an endomorphism t of the $\mathcal{D}$-module $\mathcal{L}$ by $t: P(s)(f^s u)\to P(s+1)(f\cdot f^s u)$. Then t is well-defined and injective.

Proof. Let m be a sufficiently large integer, and $f^{m-s}\cdot P(s)\cdot f^s=\sum_{j\geq 0}s^j P_j$. The following conditions are equivalent: $P(s)(f^s u)=0$. $\sum_{j\geq 0}s^j P_j u=0$. $\sum_{j\geq 0}(s+1)^j P_j u=0$. $P(s+1)(f\cdot f^s u)=0$. Thus we get the assertion.

8.6.3. If $\mathcal{D}u$ is holonomic, then $\mathcal{L}$ is a subholonomic $\mathcal{D}$-module, and $\mathcal{L}/(s-\alpha)\mathcal{L}$ $(\alpha\in\boldsymbol{C})$ and $\mathcal{L}/t\mathcal{L}$ are holonomic.

Proof. The first assertion is due to Kashiwara [18, Theorem 2.5]. The remaining assertions follow from it.

8.6.4. If $\mathcal{D}u$ is holonomic, the composition factors of $\mathcal{L}/(s-\alpha)\mathcal{L}$ (including multiplicities) depend only on (α mod$\boldsymbol{Z}$).

Proof. Let $\boldsymbol{C}[s, t]$ be the $\boldsymbol{C}$-algebra defined by the relation $ts=(s+1)t$, and $\mathcal{D}[s, t]=\mathcal{D}\otimes_{C}\boldsymbol{C}[s, t]$. The multiplication by s and the endomorphism t defined in (8.6.2) give a $\mathcal{D}[s, t]$-module structure in $\mathcal{L}$. The assertion follows from (8.6.2), (8.6.3) and [12, 2.8.5].

8.6.5. Assume that there exist a differential operator $P(s)$ and a polynomial $c(s)\in\boldsymbol{C}[s]$ such that $P(s)(f\cdot f^s u)=c(s)f^s u$. If $c(\alpha-j)\neq 0$ for $j=1, 2, \cdots$, then $(\mathcal{L}/(s-\alpha)\mathcal{L}=)\mathcal{D}(f^\alpha u)=\mathcal{D}(f^\alpha u)[f^{-1}]$ (cf. [12, 2.3.8]).

8.6.6. Let p be a point of the conormal bundle T^*X, and assume that p is contained in the characteristic variety of $\mathcal{L}/(s-\alpha)\mathcal{L}$ for any $\alpha\in\boldsymbol{C}$. Let $\mathcal{E}_p$ be the ring of germs of micro-differential operators at p. If there exists a micro-differential operator $Q\in\mathcal{E}_p$ which is invertible and satisfies $Q(f\cdot f^s u)=c(s)f^s u$ with the same $c(s)$ as in (8.6.5), then $c(s)$ is a minimal polynomial of $s\in End_{\mathcal{D}}(\mathcal{L}/t\mathcal{L})$.

Proof. Let $\mathcal{L}_p=\mathcal{E}_p\otimes_{\mathcal{D}}\mathcal{L}$. Since p lies in the characteristic variety of $\mathcal{L}/(s-\alpha)\mathcal{L}$, $0\neq\mathcal{E}_p\otimes_{\mathcal{D}}\mathcal{L}/(s-\alpha)\mathcal{L}=\mathcal{L}_p/(s-\alpha)\mathcal{L}_p$ for any $\alpha\in\boldsymbol{C}$. Hence $(s-\alpha)\mathcal{L}_p$

$\subsetneq \mathcal{L}_p$ for any $\alpha \in \boldsymbol{C}$. Since $a(s)\mathcal{L}_p \neq 0$ for any $a(s) \in \boldsymbol{C}[s]-\{0\}$, the above relation implies the faithful flatness of $\mathcal{L}_p$ over $\boldsymbol{C}[s]$. Let $c_1(s)$ be a minimal polynomial of $s \in End(\mathcal{L}/t\mathcal{L})$. Then $c(s)=c_1(s)d(s)$ with some $d(s) \in \boldsymbol{C}[s]$. If $d(s) \notin \boldsymbol{C}'$, then $c(s)\boldsymbol{C}[s] \subsetneq c_1(s)\boldsymbol{C}[s]$ and $c(s)\mathcal{L}_p \subsetneq c_1(s)\mathcal{L}_p \subset t\mathcal{L}_p = \mathcal{E}_p[s](f^{s+1}u)$. On the other hand, since Q is invertible in $\mathcal{E}_p$, $c(s)\mathcal{L}_p = c(s)\mathcal{E}_p[s](f^s u) = \mathcal{E}_p[s] Q(f^{s+1}u) = \mathcal{E}_p[s](f^{s+1}u)$. Thus we get a contradiction. Hence $d(s) \in \boldsymbol{C}^\times$, i.e., $c(s)$ is a minimal polynomial.

8.7. *Proof of the "only if part"*. Assume that $b_1(\underline{\alpha}-\underline{\nu})=0$ for some $\underline{\nu} \in \boldsymbol{Z}^l$, and let us prove that $\mathcal{N}(\underline{\alpha})$ is not simple. Since $\mathcal{N}(\underline{\alpha})=j_*(\mathcal{N}(\underline{\alpha})|\Omega)$ and $\mathcal{N}(\underline{\alpha})|\Omega$ depends only on ($\underline{\alpha}$ mod$\boldsymbol{Z}^l$), we may freely replace $\underline{\alpha}$ with other element in the same residue class modulo $\boldsymbol{Z}^l$. Especially, we may assume from the beginning that $b_1(\underline{\alpha})=0$. We have $(P_1P_2 \cdots P_l)g^{s+1}u(\underline{\alpha})=c(s)g^s u(\underline{\alpha})$, where $c(s)=\prod_{k=1}^l b_k(s+\alpha_1+1, \cdots, s+\alpha_{k-1}+1, s+\alpha_k, \cdots, s+\alpha_l)$. Put $\mathcal{L}=\mathcal{D}[s](g^s u(\underline{\alpha}))$ and let us show that the conditions of (8.6.6) are satisfied. For any $\beta \in \boldsymbol{C}$, there exists an integer m such that

$$\mathcal{L}/(s-\beta+m)\mathcal{L}=\mathcal{D}(g^{\beta-m}u(\underline{\alpha}))=\mathcal{D}(g^\beta u(\underline{\alpha}))[g^{-1}]=\mathcal{N}(\underline{\alpha}+\beta\underline{\delta}), \tag{8.7.1}$$

where $\underline{\delta}=(1, \cdots, 1)$. (Cf. (8.6.5).) By (8.6.4), the characteristic variety of $\mathcal{L}/(s-\beta)\mathcal{L}$ coincides with that of $\mathcal{L}/(s-\beta+m)\mathcal{L}=\mathcal{N}(\underline{\alpha}+\beta\underline{\delta})$, which is $\boldsymbol{W}_0$ by (8.3). Moreover, $(Q_1Q_2 \cdots Q_l)g^{s+1}u(\underline{s})=c(s)g^s u(\underline{s})$ and $Q_1Q_2 \cdots Q_l$ is invertible at $p \in \boldsymbol{W}_0$. Thus the conditions of (8.6.6) are satisfied. Hence $c(s)$ is a minimal polynomial of $s \in End_{\mathcal{D}}(\mathcal{L}/t\mathcal{L})$. Since $b_1(\underline{\alpha})=0$, $c(0)=0$. Let $c(s)=c_1(s)s$. If s is surjective, then s is an automorphism of $\mathcal{L}/t\mathcal{L}$, for $\mathcal{L}/t\mathcal{L}$ is holonomic. But, then $c_1(s)=0$ as an endomorphim of $\mathcal{L}/t\mathcal{L}$, which contradicts the minimality of $c(s)$. Hence $s \in End(\mathcal{L}/t\mathcal{L})$ is not surjective, i.e., $s\mathcal{L}+t\mathcal{L} \subsetneq \mathcal{L}$. Then $\mathcal{T}:=(s\mathcal{L}+t\mathcal{L})/s\mathcal{L} \subsetneq \mathcal{L}/s\mathcal{L}$. Since $t\mathcal{L}=\mathcal{L}$ on Ω, $\mathcal{T} \neq 0$. Thus we get a proper submodule of $\mathcal{L}/s\mathcal{L}$. Since $\mathcal{L}/s\mathcal{L}$ and $\mathcal{L}/(s+m)\mathcal{L}=\mathcal{N}(\underline{\alpha})$ (cf. (8.7.1)) have the same composition factors, $\mathcal{N}(\underline{\alpha})$ is not simple.

§ 9. Submodules of $V(w, \lambda, \mathfrak{p})$ and b-functions

9.0. The purpose of this section is to prove (9.4) and its corollary (9.13). In (9.4), we describe the submodule lattice of $V(w, \lambda, \mathfrak{p})$. In order to state (9.4), we need some definitions, which are given in (9.1)-(9.3). The proof of (9.4) is given in (9.5)-(9.11). In (9.13), we give a criterion for the simplicity of generalized Verma modules, by combining (8.4) and (9.4).

9.1. Let λ be a W_I-invariant character. If we forget the $\mathcal{D}_Y(\lambda)$-module structure, then $\mathcal{O}(wU_-, \lambda)=\mathcal{O}_Y|wU_- \cdot y_0$. (See (2.5) for $\mathcal{O}(wU_-, \lambda)$.) Hence we can regard $1 \in \Gamma(wU_- \cdot y_0, \mathcal{O}_Y)$ as a section of $\mathcal{O}(wU_-, \lambda)$, which we shall denote by 1_w^λ. We write 1^λ for 1_e^λ. Then $\mathcal{D}_Y|U_- \cdot y_0$ is isomorphic to $\mathcal{D}_Y(\lambda)|U_- \cdot y_0$ by

$\mathcal{D}_Y \ni P \to 1^\lambda \otimes P \otimes 1^{-\lambda}$. Thus from any $(\mathcal{D}_Y(\lambda)|U_-\cdot y_0)$-module $\mathcal{M}$, we get a $(\mathcal{D}_Y|U_-\cdot y_0)$-module $1^{-\lambda}\otimes\mathcal{M}$.

9.2. Semi-invariants. From now on, we assume that G is semi-simple and simply connected. Note that these assumptions are not restrictive for the study of (generalized) flag manifolds, $U(\mathfrak{g})$-modules, etc. Then each $\varpi=\sum_{i=1}^{l} n_i\varpi_i$ $(n_i\in \boldsymbol{Z})$ determines rational characters of T, B and $B_-:=w_S B w_S$ (via the projection $B_-\to T$), which we shall denote by the same letter ϖ. If every n_i $(1\leq i\leq l)$ is non-negative, then there exists a regular function f^{ϖ} on G such that $f^{\varpi}(e)=1$ and $f^{\varpi}(b'xb)=\varpi(b')\varpi(b)f^{\varpi}(x)$ for $x\in G$, $b'\in B_-$ and $b\in B$. We call such a polynomial *a semi-invariant*. These functions f^{ϖ} can be constructed as follows. Let V_{ϖ} be a finite dimensional irreducible representation of G with highest weight ϖ, v_{ϖ} its highest weight vector, and $v_{-\varpi}$ the lowest weight vector of the contragradient representation $V_{\varpi}^{\vee}$ of V_{ϖ} such that $\langle v_{-\varpi}, v_{\varpi}\rangle=1$. Then the regular function f^{ϖ} is given by

$$f^{\varpi}(g)=\langle v_{-\varpi}, gv_{\varpi}\rangle. \tag{9.2.1}$$

Let $f_i=f^{\varpi_i}$. Then $f^{\varpi}=\prod_{i=1}^{l} f_i^{n_i}$.

9.3. Assume that $\Pi_I=\{\alpha_{k+1},\cdots,\alpha_l\}$. Then $\mathfrak{t}_I^{\vee}=\{\lambda=\sum_{i=1}^{k}\lambda_i\varpi_i\,|\,\lambda_i\in \boldsymbol{C}\}$. Let $s=(s_i)_{1\leq i\leq k}$ be independent complex variables, $f^s(x)=\prod_{i=1}^{k} f_i^{s_i}(x)$, $\mathcal{N}'=\mathcal{D}_G[s_1,\cdots,s_k]f^s$, $\mathcal{N}'(\lambda)=\mathcal{N}'/\sum_{i=1}^{k}(s_i-\lambda_i)\mathcal{N}'$, and $u(\lambda)$ the generator of $\mathcal{N}'(\lambda)$ corresponding to f^s. For any $\mathcal{D}_G$-module $\mathcal{M}$, let $g\mathcal{M}:=L_{g^{-1}}^*\mathcal{M}(g\in G)$, where $L_{g^{-1}}$ is the left translation by g^{-1}.

Theorem 9.4. *Assume that $\lambda=\lambda_c+\lambda_d\in\mathfrak{t}_{rad}^{\vee}$, where λ_c is W_I-invariant and λ_d is the lowest weight of a finite dimensional irreducible P-module. For $w\in(W/W_I)_l$, put*

$L_1=\{U(\mathfrak{g})$*-submodules of* $V(w,\lambda,\mathfrak{p})\}$,

$L_1^*=\{U(\mathfrak{g})$*-submodules of* $V(w,\lambda,\mathfrak{p})^*\}$,

$L_2=\{$*coherent* $\mathcal{D}_X(\lambda)$*-submodules of* $H_{X(w)}^{cd(w)}(\mathcal{O}_X(\lambda_c))\otimes\mathcal{O}_X(\lambda_d)\}$

$L_3=\{$*coherent* $\mathcal{D}_X(\lambda_c)$*-submodules of* $H_{X(w)}^{cd(w)}(\mathcal{O}_X(\lambda_c))\}$,

$L_4=\{$*coherent* $\mathcal{D}_Y(\lambda_c)$*-submodules of* $H_{Y(w)}^{cd(w)}(\mathcal{O}_Y(\lambda_c))\}$,

$L_5=\{$*coherent* $(\mathcal{D}_Y(\lambda_c)|U_-\cdot y_0)$*-submodules of* $H_{Y(w)}^{cd(w)}(\mathcal{O}_Y(\lambda_c))|U_-\cdot y_0\}$,

$L_6=\{$*coherent* $(\mathcal{D}_Y|U_-\cdot y_0)$*-submodules of* $1^{-\lambda_c}\otimes(H_{Y(w)}^{cd(w)}(\mathcal{O}_Y(\lambda_c))|U_-\cdot y_0)\}$, *and*

$L_7=\{$*coherent* $\mathcal{D}_G$*-submodules of* $H_{BwP}^{cd(w)}(w\mathcal{N}'(-\lambda_c))\}$.

Then as lattice-ordered sets,

$$L_1^{opp}\cong L_1^*\cong L_2\cong L_3\cong L_4\cong L_5\cong L_6\cong L_7.$$

(See (1.1) *for* $\check{\mathfrak{t}}_{rad}$, $(W/W_I)_l$, U_-, *and* $*$, (4.8) *for* $\mathcal{O}_X(\lambda_d)$, *and* (4.2) *for* X, Y, $X(w)$, $Y(w)$, $cd(w)$ *and* y_0. *We denote the dual of* L *by* L^{opp}.)

Proof. Obviously, $L_1^{opp}\cong L_1^*$, $L_2\cong L_3$ and $L_5\cong L_6$. By (3.4) and (6.6), $L_1^*\cong L_2$. By (5.4), $L_3\cong L_4$. In order to prove $L_4\cong L_5$ and $L_4\cong L_7$, we need some preliminaries. Henceforth until the end of (9.11), we write λ for λ_c, since we exclusively consider the W_I-invariant characters.

Lemma 9.5. *For any* $w\in W$, $Y(w)\cap U_-\cdot y_0\neq\phi$

Proof. Since Bw_SB is a Zariski open subset of G, $Bw_SBg\cap Bw_SB\neq\phi$ for any $g\in G$. Hence $Bg\cap w_SBw_SB\neq\phi$. Especially, $Y(w)\cap U_-\cdot y_0=Bwy_0\cap w_SBw_S\cdot By_0\neq\phi$ for any $w\in W$.

Lemma 9.6. *Let* S *be a smooth algebraic variety over* $\mathbf{C}$, U *a Zariski open subset of* S, A_S *a twisted ring of algebraic differential operators on* S ([19, 2.3.3]), $A_U=A_S|U$, M *a coherent* A_S*-module, and* N' *a coherent* A_U*-submodule of* $M|U$. *Then there exists a coherent* A_S*-submodule* N *of* M *such that* $N|U=N'$.

Proof. Since M and N' are quasi-coherent over $\mathcal{O}_S$ and $\mathcal{O}_U$, respectively, we can find a quasi-coherent $\mathcal{O}_S$-submodule N_1 of M such that $N_1|U=N'$, by [11, (5.9.2)]. Let N be the A_S-submodule of M generated by N_1. Then N is a coherent A_S-submodule of M such that $N|U=N'$.

9.7. Let us prove that $L_4\cong L_5$. Define a mapping $L_4\to L_5$ by the restriction to $U_-\cdot y_0$. By (9.6), this morphism is surjective. Assume that two modules M and N in L_4 restricts to the same module in L_5. Then $M/(M\cap N)$ is supported by the complement of $U_-\cdot y_0$. Hence every irreducible component of the characteristic variety of $M/(M\cap N)$ is the conormal bundle of some subvariety of $Y\backslash U_-\cdot y_0$. On the other hand, each irreducible component of the characteristic variety of $H^{cd(w)}_{Y(w)}(\mathcal{O}_Y(\lambda))$ is the conormal bundle of $Y(w')$ with some $w'\in W/W_I$, which is not a conormal bundle of a subvariety of $Y\backslash U_-\cdot y_0$ by (9.5). Hence $M/(M\cap N)=0$, i.e., $M=N$.

Thus it remains to prove that $L_4\cong L_7$.

Lemma 9.8. *A defining equation of the hypersurface* $\overline{B_-\cdot r_{\alpha_i}B}$ *of* G *is given by* $f_i=0$.

Proof. Since $r_{\alpha_j}(\varpi_i)=\varpi_i-\delta_{ij}\alpha_j$, we can show that $f_i(r_{\alpha_j})\neq0$ (resp. $=0$) if $i\neq j$ (resp. $i=j$) by (9.2.1). Since $f_i^{-1}(0)$ is a union of cosets in $B_-\backslash G/B$, and is a hypersurface of G, we have $f_i^{-1}(0)=\overline{B_-\cdot r_{\alpha_i}B}$. For $A\in\mathfrak{g}(\alpha_i)$ and $A'\in\mathfrak{g}(-\alpha_i)$

such that $[A, A']=\alpha_i^\vee$, we have

$$f_i(\exp(tA)r_{\alpha_i})=\langle v_{-\varpi_i}, r_{\alpha_i}v_{\varpi_i}\rangle+t\langle v_{-\varpi_i}, Ar_{\alpha_i}v_{\varpi_i}\rangle+O(t^2)$$
$$=ct\langle v_{-\varpi_i}, AA'v_{\varpi_i}\rangle+O(t^2)$$

with some $c\in C\setminus\{0\}$. (By the representation theory of sl_2, we can show that $A'v_{\varpi_i}\neq 0$. Considering the weight of $r_{\alpha_i}v_{\varpi_i}$, we get $r_{\alpha_i}v_{\varpi_i}=cA'v_{\varpi_i}$ $(c\neq 0)$.) But $AA'v_{\varpi_i}=[A, A']v_{\varpi_i}=\alpha_i^\vee v_{\varpi_i}=v_{\varpi_i}$. Hence $f_i(\exp(tA)r_{\alpha_i})=ct+O(t^2)$ $(c\neq 0)$. Thus $f_i=0$ is a defining equation of $\overline{B_-\cdot r_{\alpha_i}B}$.

Lemma 9.9. $G\setminus B_-\cdot P=\bigcup_{r\in S\setminus I}\overline{B_-\cdot rB}$.

Proof. The minimal elements of $W\setminus W_I$ are $S\setminus I$. Hence the maximal elements of $W\setminus w_SW_I$ are $\{w_Sr\mid r\in S\setminus I\}$. By the equivalence $w\geqq w'\Leftrightarrow\overline{BwB}\supset\overline{Bw'B}$, we get $G\setminus Bw_SP=\bigcup_{w\in W\setminus w_SW_I}BwB=\bigcup_{r\in S\setminus I}\overline{Bw_SrB}$. Multiplying w_S from the left, we get the assertion.

Lemma 9.10. *The rational characters ϖ_i $(1\leqq i\leqq k)$ of B can be extended to those of P and P_-. Denoting them by the same letter ϖ_i, we have $f_i(p'gp)=\varpi_i(p')\varpi_i(p)f_i(g)$ for $g\in G$, $p'\in P_-$ and $p\in P$.*

Proof. Let $1\leqq i\leqq k$, $\alpha\in\prod_I=\{\alpha_{k+1},\cdots,\alpha_l\}$, $A\in\mathfrak{g}(\alpha)$ and $A'\in\mathfrak{g}(\alpha)$. Then we have $Av_{\varpi_i}=0$ and $\alpha^\vee v_{\varpi_i}=0$. Hence by the representation theory of sl_2, $A'v_{\varpi_i}=0$ and $f_i(g\exp A')=\langle v_{-\varpi_i}, g\exp A'\cdot v_{\varpi_i}\rangle=\langle v_{-\omega_i}, gv_{\varpi_i}\rangle=f_i(g)$. Thus we get the relative invariance with respect to P. The relative invariance with respect to P_- can be proved in the same way.

9.11. Let us prove that $L_4\cong L_7$. It is enough to prove that

$$p_Y^*H_{Y(w)}^{cd(w)}(\mathcal{O}_Y(\lambda))=H_{BwP}^{cd(w)}(w\mathcal{N}'(-\lambda)) \tag{9.11.1}$$

(cf. (5.4)). Let $j_G\colon B_-\cdot P\to G$ and $j_Y\colon B_-\cdot y_0\to Y$ be inclusion mappings. Then $w^{-1}H_{BwP}^{cd(w)}(w\mathcal{N}'(-\lambda))=(j_G)_*H_{w^{-1}BwP}^{cd(w)}(\mathcal{N}'(-\lambda)\mid B_-\cdot P)$. Since $w^{-1}H_{Y(w)}^{cd(w)}(\mathcal{O}_Y(\lambda))=(j_Y)_*H_{w^{-1}Y(w)}^{cd(w)}(\mathcal{O}(U_-,\lambda)\mid B_-\cdot y_0)$, it is enough to show that $p_Y^*\mathcal{O}(U_-,\lambda)=\mathcal{N}'(-\lambda)$ on $B_-\cdot P$. Let $\lambda=\sum_{i=1}^k\lambda_i\varpi_i$ and $f^{-\lambda}=\prod_{i=1}^kf_i^{-\lambda_i}$. Then $f^{-\lambda}$ gives a multi-valued holomorphic function on $G\setminus\bigcup_{i=1}^kf_i^{-1}(0)=B_-\cdot P$. (Cf. (9.8) and (9.9).) Denote by $\mathcal{O}f^{-\lambda}$ the $\mathcal{O}$-module on $B_-\cdot P$ generated by $f^{-\lambda}$. Then $\mathcal{O}f^{-\lambda}=\mathcal{D}f^{-\lambda}=\mathcal{N}'(-\lambda)\mid B_-\cdot P$. By (9.10), $f^{-\lambda}$ gives a section of $F(\lambda, P)$ on $U_-\cdot P_0$, where P_0 is a connected, simply connected open neighbourhood of the identity element of P. (See (2.2) for $F(\lambda, P)$.) Thus $f^{-\lambda}$ determines a section of $\mathcal{O}^{an}(U_-,\lambda)=\iota_*(p_Y^{an}\mid U_-\cdot P_0)_*(F(\lambda, P)\mid U_-\cdot P_0)$. Take a single-valued branch of $f^{-\lambda}$ on $U_-\cdot P_0$ so that $f^{-\lambda}(e)=1$. Then $f^{-\lambda}\equiv 1$ on U_-, i.e., $f^{-\lambda}$ determines $1\in\Gamma(U_-\cdot y_0,\mathcal{O}_Y)=\Gamma(U_-\cdot y_0,\mathcal{O}(U_-,\lambda))$, which we have denoted by 1^λ in (9.1). For $g\in G$, we have $(L(g)f^{-\lambda})(x)=f^{-\lambda}(g^{-1}x)=(p_Y^*\varphi_g)(x)f^{-\lambda}(x)$ with a locally defined analytic function φ_g on Y.

For $A\in\mathfrak{g}$, let $\varphi_A=(d/dt)\varphi_{\exp tA}|_{t=0}$. Then φ_A is a regular function, $A\cdot f^{-\lambda}=(p_Y^*\varphi_A)\cdot f^{-\lambda}$, and $A\cdot 1^\lambda=\varphi_A\cdot 1^\lambda$. Hence $1\otimes 1^\lambda\in p_Y^*\mathcal{D}_Y(\lambda)\otimes_{p_Y^{-1}\mathcal{D}_Y(\lambda)}p_Y^{-1}\mathcal{O}(U_-,\lambda)=p_Y^*\mathcal{O}(U_-,\lambda)$ satisfies $A(1\otimes 1^\lambda)=(p_Y^*\varphi_A)(1\otimes 1^\lambda)$. Thus all the linear differential equations satisfied by $f^{-\lambda}$ are also satisfied by $1\otimes 1^\lambda$. Since $\mathcal{O}_{B_-\cdot P}f^{-\lambda}$ and $p_Y^*(\mathcal{O}_{B_-\cdot y_0}1^\lambda)$ are integrable connections of rank one, $\mathcal{N}'(-\lambda)|B_-\cdot P=\mathcal{O}_{B_-\cdot P}f^{-\lambda}=p_Y^*(\mathcal{O}_{B_-\cdot y_0}1^\lambda)=p_Y^*\mathcal{O}(U_-,\lambda)$. Thus we have completed the proof of (9.4).

9.12. As an application of (9.4), we get a criterion for the simplicity of $V(w,\lambda,\mathfrak{p})$ in terms of $\mathcal{D}$-modules under certain assumptions. Here we restrict ourselves to the generalized Verma modules.

Let $\lambda=\lambda_c+\lambda_d$ be a character of $\mathfrak{t}$, where λ_c is W_I-invariant and λ_d is a highest weight of a finite dimensional P-module. First, assume that

(9.12.1) $\Pi_I=\{\alpha_{k+1},\cdots,\alpha_l\}$ with $k>0$, and

(9.12.2) $\langle\lambda+\rho,\alpha^\vee\rangle\neq 0,-1,-2,\cdots$ for any $\alpha\in R_+$.

Let W_0 be the characteristic variety of $\mathcal{N}'(\lambda_c)\,[f_1^{-1},\cdots,f_k^{-1}]$. Consider two more assumptions that

(9.12.3) for any $i\leqq k$, there exists $P_i\in\mathcal{D}_G$ and $b_i(\underline{s})\in\boldsymbol{C}[\underline{s}]$ such that $P_i\underline{f}^{\underline{s}+e_i}=b_i(\underline{s})\underline{f}^{\underline{s}}$, where $\underline{f}^{\underline{s}}=f_1^{s_1}\cdots f_k^{s_k}$ and $e_i=(0,\cdots,0,1,0,\cdots,0)$ (1 appears as the i-th component), and that

(9.12.4) there exists a point $p\in W_0$ and for any $i\leqq k$, there exists an invertible micro-differential operator Q_i in a neighbourhood of p such that $Q_i\underline{f}^{\underline{s}+e_i}=b_i(\underline{s})\underline{f}^{\underline{s}}$ with the same $b_i(\underline{s})$'s as in (9.12.3).

Let us identify $\underline{\lambda}_c=(\lambda_1,\cdots,\lambda_k)\in\boldsymbol{C}^k$ with $\lambda_c=\sum_{i=1}^k\lambda_i\varpi_i$.

Theorem 9.13. *Under the above four assumptions, the following conditions are equivalent:*

(1) *The generalized Verma module $M(\lambda,\mathfrak{p}(I))$ is simple as a $\mathfrak{g}$-module.*

(2) *The generalized Verma module $M(-w_S\lambda,\mathfrak{p}(w_SIw_S))$ is simple as a $\mathfrak{g}$-module.*

(3) *The coherent $\mathcal{D}_G$-module $H^0_{Bw_SP}(w_S\mathcal{N}'(\lambda_c))$ is simple, i.e., it does not have non-trivial coherent $\mathcal{D}_G$-submodules.*

(4) $b_i(\underline{\lambda}_c-\nu)\neq 0$ *for any* $1\leqq i\leqq k$ *and* $\nu\in\sum_{i=1}^k\boldsymbol{Z}\varpi_i$.

Proof. Since the action of $-w_S$ on $\mathfrak{t}$ extends to an automorphism of $\mathfrak{g}$ preserving $\mathfrak{b}$, we get (1)$\Leftrightarrow$(2). By (6.8) and (9.4), we get (2)$\Leftrightarrow$(3). Since $w_S^{-1}Bw_SP=G\backslash\bigcup_{i=1}^k f_i^{-1}(0)$ by (9.8) and (9.9), $w_S^{-1}H^0_{Bw_SP}(w_S\mathcal{N}'(\lambda_c))=\mathcal{N}'(\lambda_c)[f_1^{-1},\cdots,f_k^{-1}]$. Hence we get (3)$\Leftrightarrow$(4) by (8.4).

Remark 9.14. We assumed (9.12.1) only to exclude the trivial case. The author conjectures that the assumptions (9.12.3) and (9.12.4) are always satisfied. Thus the assumptions except (9.12.2) would be harmless. But (9.12.2) is essential and, because of this assumption, our irreducibility criterion is less complete than the one given by Jantzen [16]. In our forthcoming paper [13], we shall start to study the simplicity of generalized Verma modules and the b-functions of the semi-invariants without such assumptions. The relation between (9.13) and the result of Suga [31] will also become clear in [13].

References

[1] Beilinson, A. and Bernstein, J., Localisation de $\mathfrak{g}$-modules, *C. R. Acad. Sci. Paris*, **292** (1981), 15-18.

[2] Bernstein, I. N., Gelfand, I. M. and Gelfand, S. I., Differential operators on the base affine space and a study of $\mathfrak{g}$-modules, *Lie groups and their representations, Proc. summer school in group representations*, (1975), 21-64, Halsted, New York.

[3] ———, Structure of representations generated by vectors of highest weight, *Funct. Anal. Appl.*, **5** (1971), 1-8.

[4] ———, Schubert cells and cohomology of the spaces G/P, *Russian Math. Surveys*, **28** (1973), 1-26.

[5] Bott, R., Homogeneous vector bundles, *Ann. of Math.*, **66** (1957), 203-248.

[6] Bourbaki, N., *Groupes et algèbres de Lie, chapitres* 4, 5 *et* 6, Masson, 1981.

[7] ———, *Algèbre commutative, chapitres* 1 *à* 4, Masson, 1985.

[8] Brylinski, J. L. and Kashiwara, M., Kazhdan-Lusztig conjecture and holonomic systems, *Invent. math.*, **64** (1981), 387-410.

[9] Dixmier, J., *Algèbres enveloppantes*, Gauthier-Villars, Paris, 1974.

[10] Gabber, O. and Joseph, A., On the Bernstein-Gelfand-Gelfand resolution and the Duflo sum formula, *Compositio Math.*, **43** (1981), 107-131.

[11] Grothendieck, A. and Dieundonné, J. A., *Eléments de géométrie algébrique*, Springer, Berlin-Heidelberg-New York, 1971.

[12] Gyoja, A., Theory of prehomogeneous vector spaces without regularity condition, *Publ. RIMS. Kyoto Univ.*, **27** (1991), 861-922.

[13] ———, Highest weight modules and b-functions of semi-invariants, *preprint*.

[14] Hecht, H., Miličić, D., Schmid, W. and Wolf, J. A., Localization and standard modules for real semisimple Lie groups I: The duality theorem, *Invent. math.*, **90** (1987), 297-332.

[15] Hironaka, H., Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. of Math.*, **79** (1964), 109-326.

[16] Jantzen, J. C., Kontravariante Formen auf induzierten Darstellungen halbeinfacher Lie-Algebren, *Math. Ann.*, **226** (1977), 53-65.

[17] Kashiwara, M., b-Functions and holonomic systems, *Invent. math.*, **38** (1976), 33-53.

[18] ———, On the holonomic systems of linear differential equations, II, *Invent. math.*, **49** (1978), 121-135.

[19] ———, Representation theory and D-modules on flag varieties, *preprint*.

[20] ———, The universal Verma module and the b-function, *Advanced studies in pure math.*, **6** (1985), 67-81.

[21] Kazhdan, D. and Lusztig, G., Representations of Coxeter groups and Hecke algebras, *Invent. math.*, **53** (1979), 165-184.

[22] Kempf, G., The Grothendieck-Cousin complex of an induced representation, *Adv. in Math.*, **29** (1978). 310-396.

[23] Laumon, G., Sur la catégorie dérivée des D-modules filtrés, *Springer Lecture Note*, **1016** (1983), 151-237.

[24] Lepowsky, J., Generalized Verma modules, the Cartan-Helgason theorem, and the

Harish-Chandra homomorphism, *J. Algebra*, **49** (1977), 470-495.

[25] Muller, I., Rubenthaler, H. and Schiffmann, G., Structures des espaces préhomogènes associés à certaines algèbres de Lie graduées, *Math. Ann.*, **274** (1986), 95-123.

[26] Quillen, D., Projective modules over polynomial rings, *Invent. math.*, **36** (1976), 167-171.

[27] Rubenthaler, H., Espaces préhomogènes de type parabolique, *Lec. in Math., Kyoto Univ.*, **14** (1982), 189-221.

[28] Sabbah, C., Proximité évanescente II, *Compositio Math.*, **64** (1987), 213-241.

[29] Sato, M., Kashiwara, M., Kimura, T. and Oshima, T., Micro-local analysis of prehomogeneous vector spaces, *Invent. math.*, **62** (1980), 117-179.

[30] Sato, M. and Kimura, T., A classification of irreducible prehomogeneous vector spaces and their relative invariants, *Nagoya Math. J.*, **65** (1977), 1-155.

[31] Suga, S., Highest weight modules associated with classical irreducible regular prehomogeneous vector space of commutative parabolic type, *Osaka J. Math.*, **28** (1991), 323-346.

[32] Suslin, A., Projective modules over a polynomial ring are free, *Soviet Math. Dokl.*, **17** (1976), 1160-1164.

[33] Verma, D. N., Structure of certain induced representations of complex semisimple Lie algebras, *Bull. Amer. Math. Soc.*, **74** (1968), 160-166.