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Phantom Maps and Monoids of Endomorphisms of $K(\boldsymbol{Z}, m)\times S^n$

By

Yoshimi SHITANDA*

Abstract

The compositions hf, hg of phantom pair f, g and a phantom map h are homotopic. The compositions hf, kf of phantom pair h, k and a phantom map f are homotopic. We determine the homotopy set $[K(\boldsymbol{Z},m)\times S^n, K(\boldsymbol{Z},m)\times S^n]$ and its monoid structure given by the composition of maps for all $m, n\geqq 1$.

Introduction

Two continuous maps f, g: $X\to Y$ are called a phantom pair, if the restriction maps $f|X^n$, $g|X^n$ on the n-skeleton X^n are homotopic for all $n\geqq 0$ or equivalently q_nf and q_ng are homotopic for all $n\geqq 0$ where $q_n: Y\to Y_n$ is the Postnikov n-stage of Y. A characterization of a phantom pair is given by Theorem 3.6 of [5] which is a generalization of [11]. When g is the constant map, f is called a phantom map. When g is an identity map, f is called a weak identity map. Many topologists studied properties of phantom maps [3, 11]. We also studied properties of phantom pairs [5]. J. Roitberg [6] studied the set $WI(X)$ of weak identities of X. In this paper, we shall work in the category of nilpotent CW-complexes with base point and base point preserving continuous maps except for some special cases. We use the notations and terminologies of [5, 11].

C. A. McGibbon and B. Gray [3] proved that the composition of phantom maps is homotopic to the constant map. In this paper, we shall generalize this result for more general cases.

Theorem 0.1. *Let X, Y and Z be nilpotent CW-complexes of finite type with $\pi_1(Y)$, $\pi_1(Z)$ finite groups. Let f, $g: X\to Y$ be a phantom pair and $h: Y\to Z$ a phantom map. Then hf and hg are homotopic. Let $f: X\to Y$ be a phantom map and h, $k: Y\to Z$ a phantom pair. Then hf and kf are homotopic.*

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* Meiji University, Izumi Campus 9-1 Eifuku 1-chome, Suginami-ku, Tokyo 168 Japan.

J. Roitberg [7] calculated the group $\mathrm{Aut}(K(\boldsymbol{Z}, 2)\times S^3)$ of the self-homotopy equivalences of $K(\boldsymbol{Z}, 2)\times S^3$. That is, there is an exact sequence:

$$0 \longrightarrow \boldsymbol{Z}^{\wedge}/\boldsymbol{Z} \longrightarrow \mathrm{Aut}(K(\boldsymbol{Z}, 2)\times S^3) \longrightarrow \boldsymbol{Z}/2\boldsymbol{Z}\times\boldsymbol{Z}/2\boldsymbol{Z} \longrightarrow 0$$

C. A. McGibbon and J. M. Møller [4] determined also $\mathrm{Aut}(K(Z, m)\times S^{m+1})$.

In this paper, we shall calculate the homotopy set $\mathrm{End}(K(Z, m)\times S^n)=[K(\boldsymbol{Z}, m)\times S^n, K(\boldsymbol{Z}, m)\times S^n]$ for all $m, n\geqq 1$. We determine also the monoid structure of $\mathrm{End}(K(\boldsymbol{Z}, m)\times S^n)$ given by the composition of maps, $\mathrm{Aut}(K(\boldsymbol{Z}, m)\times S^n)$ and the group $WI(K(\boldsymbol{Z}, m)\times S^n)$ of weak identities. We get the next result.

Theorem 0.2. *If $m>1$, the elements of* $\mathrm{End}(K(\boldsymbol{Z}, m)\times S^n)$ *are in one to one correspondence with 2×2 matrices of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a\in[K(\boldsymbol{Z}, m), K(\boldsymbol{Z}, m)]$, $b\in[S^n, K(\boldsymbol{Z}, m)]$, $d\in[S^n, S^n]$, and c is the class of a map $K(\boldsymbol{Z}, m)\to S^n$ if $hm\neq n-2$ for even m, n, any $h>0$ or m or n is odd, and a phantom map $c: K(\boldsymbol{Z}, m)\to Map\ (S^n, S^n)$ otherwise.

For $m=1$, $\mathrm{End}(K(\boldsymbol{Z}, m)\times S^n)$ is given in section 4 with a little different form. We can determine the multiplication of matrices corresponding to the composition of maps which is described in Theorem 3.2, 3.3 and 4.3. From these results, we can easily determine $\mathrm{Aut}(K(\boldsymbol{Z}, m)\times S^n)$ and $WI(K(\boldsymbol{Z}, m)\times S^n)$.

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§ 1. Phantom Maps and Weak Identities

In this section, we generalize the results of J. Roitberg [6] and B. Gray and C. A. McGibbon [3]. They proved that the composition of phantom maps is homotopic to the constant map. This statement is generalized to Theorem 1.1 and 1.4. J. Roitberg proved also that weak identities of X are described by phantom maps of X for homotopy associative H-spaces X, and the group $WI(X)$ is a divisible abelian group which does not depend on the multiplication of X. We generalize the results for rational H-spaces (Theorem 1.7).

Theorem 1.1. *Let X, Y and Z be nilpotent CW-complexes of finite type and $\pi_1(Y)$ finite group. Let $f, g: X\to Y$ be a phantom pair and $h: Y\to Z$ a phantom map. Then hf and hg are homotopic.*

Proof. If f, $g: X\to Y$ are a phantom pair, $r_Y f$ and $r_Y g$ are homotopic where $r_Y: Y\to Y_Q$ is the rationalization. Since $q_n f$ and $q_n g$ are homotopic by the definition, $r_Y^n q_n f$ and $r_Y^n q_n g$ are homotopic where $r_Y^n: Y_n\to(Y_n)_Q$ is the rationalization. By using the equality of the dual form of Theorem 3.3 of [9],

$$[X, Y_Q]=\varprojlim_k [X, (Y_k)_Q]$$

we see that $r_Y f$ and $r_Y g$ are homotopic.

By Theorem B of [11], h can be decomposed as $h=kr_Y$ where $k: Y_Q\to Z$. By using the above result, we obtain the result by the following,

$$hf\sim kr_Y f\sim kr_Y g\sim hg\,.$$

By using this result, we obtain the next corollary (cf. [3, 6])

Corollary 1.2. *Let X, Y and Z be nilpotent CW-complexes of finite type and $\pi_1(Y)$ finite group. Let $f: X\to Y$ and $h: Y\to Z$ be phantom maps. Then hf is homotopic to the constant map.*

Let $\rho: X_\rho\to X$ be a homotopy fiber of Sullivan completion $e^\wedge: X\to X^\wedge$. The homotopy group of X_ρ is a finite direct sum of $Z^\wedge/Z$. A map $f: X\to Y$ is uniquely lifted to $(f)_\rho: X_\rho\to Y_\rho$ because of $[X_\rho, \Omega^i Y^\wedge]=0$. By the method similar to Theorem 1.1, we can obtain the next result by using Theorem 2.1 of [9].

Proposition 1.3. *Let X, Y be finite type CW-complexes with finite fundamental groups. If f, $g: X\to Y$ are a phantom pair, $(f)_\rho$ and $(g)_\rho$ are homotopic.*

Proof. By the assumption and the universality of the completion, $f^\wedge$ and $g^\wedge$ are homotopic and also $q_n f\sim q_n g$. Hence we have $(q_n f)_\rho\sim(q_n g)_\rho$ by the above remark. Since the homotopy groups of X_ρ and Y_ρ are finitely generated over $\boldsymbol{Z}^\wedge/\boldsymbol{Z}$, $(f)_\rho$ and $(g)_\rho$ are homotopic by Theorem 2.1 of [9].

The dual version of Theorem 1.1 is obtained by the similar method.

Theorem 1.4. *Let X, Y and Z be nilpotent CW-complexes of finite type and $\pi_1(Y)$, $\pi_1(Z)$ finite groups. Let $f: X\to Y$ be a phantom map and g, $h: Y\to Z$ a phantom pair. Then gf and hf are homotopic.*

A map $f: X\to X$ is called a weak identity, if f and identity map id_X are a phantom pair. Let $WI(X)$ be the set of weak identities of X. We prepare some elementary results. Let X be a rational H-space. We consider the arithmetic square (cf. [2, 5]):

$$
(1.5)\qquad
\begin{array}{ccccccc}
X_\rho & \xrightarrow{\rho} & X & \xrightarrow{e^{\wedge}} & X^{\wedge} & \longrightarrow & BX_\rho \\
 & & \downarrow & & \downarrow & & \\
X_\rho & \longrightarrow & X_Q & \longrightarrow & X_{\hat{Q}} & \longrightarrow & BX_\rho
\end{array}
$$

where $\rho: X_\rho \to X$ is the homotopy fiber of $e^{\wedge}: X \to X^{\wedge}$ and BX_ρ is the classifying space of X_ρ. Note that X_ρ, X_Q, $X_{\hat{Q}}$ and BX_ρ are products of Eilenberg-MacLane spaces and the upper sequence of (1.5) does not depend on the structure of a rational H-space.

Let $\mu: X_\rho \times X \to X$ be the action of the principal fibration $e^{\wedge}: X \to X^{\wedge}$ which satisfies $e^{\wedge}\mu \sim e^{\wedge}\pi_X$. Hence the homotopy set $[Z, X_\rho]$ acts on $[Z, X]$ by the formula

$$\phi * f = \mu\langle \phi, f \rangle$$

where $\phi: Z \to X_\rho$, $f: Z \to X$. This action satisfies the next formulas where 0 is the constant map, $f: Z \to X$ and $\phi, \psi: Z \to X_\rho$.

$$
(1.6)\qquad
\begin{gathered}
0 * f \sim f \\
(\phi + \psi) * f \sim \phi * (\psi * f) \\
\phi * 0 \sim \rho\phi\,.
\end{gathered}
$$

Two maps $\phi * f$ and f are a phantom pair for $\phi: Z \to X_\rho$ and $f: Z \to X$. Conversely if f and g are a phantom pair, $\phi * g$ and f are homotopic for some $\phi: Z \to X_\rho$. Now, there is a map,

$$\Theta: [X, X_\rho] \longrightarrow WI(X)$$

defined by $\Theta(\phi) = \phi * \mathrm{Id}_X$. $[X, X_\rho]$ has an abelian group structure induced by the H-space structure of X_ρ and $WI(X)$ has a group structure defined by the composition.

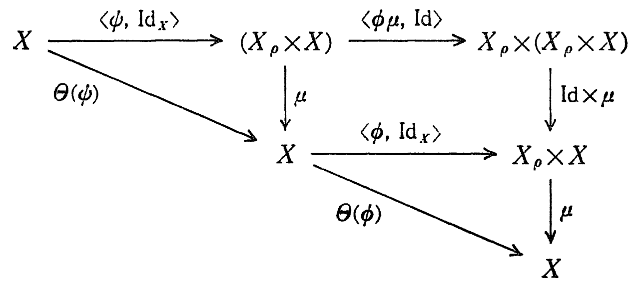
Theorem 1.7. *Let X be a rational H-space of finite type with a finite fundamental group. There is a group isomorphism induced by the above Θ*

$$\Theta: [X, X_\rho]/S\Theta \longrightarrow WI(X)$$

where $S\Theta$ is a stationary subgroup.

Proof. It sufficies to show that $\Theta(\phi + \psi) = \Theta(\phi)\Theta(\psi)$. Here the right-hand side means the composition of maps.

Now we consider the next commutative diagram.



By the above diagram, the next claim and the definition of the action, we have the result.

Claim.

$$\psi=\phi\mu\langle\psi, \mathrm{Id}_X\rangle : X \longrightarrow X_\rho\times X \longrightarrow X \longrightarrow X_\rho .$$

Now we shall prove the claim. Since X is a rational H-space, ϕ can be decomposed by the composition

$$\phi=\beta\alpha : X \longrightarrow K \longrightarrow X_\rho$$

where $K=\prod_j K(\pi_j(X), j)$. Since X_ρ is a rational space, a map

$$(\mu\langle\psi, \mathrm{Id}_X\rangle)^* : [X, K] \longrightarrow [X_\rho\times X, K] \longrightarrow [X, K]$$

is the identity map modulo torsion group. Hence we obtain the result.

§2. Endomorphisms of $K(\boldsymbol{Z}, m)\times S^n$ for $m, n>1$

In this section, we determine the homotopy set $[K(\boldsymbol{Z}, m)\times S^n, K(\boldsymbol{Z}, m)\times S^n]$ for all $m, n>1$. For $m=1$, $\mathrm{Aut}(S^1\times S^n)$ was calculated by N. Sawashita [8] and P. I. Booth and P. R. Heath [1]. We shall also calculate $\mathrm{End}(S^1\times S^n)$ and its monoid structure in section 4. It is elementary for $\mathrm{End}(K(\boldsymbol{Z}, m)\times S^1)$.

We use the notations for mapping spaces as follows:

$\mathrm{Map}(X, Y)$; a space of continuous maps from X to Y.
$\mathrm{Map}_*(X, Y)$; a space of based continuous maps form X to Y.
$\mathrm{Map}(X, Y; f)$; a connected component of $\mathrm{Map}(X, Y)$ which contains f.
$\mathrm{Map}_*(X, Y; f)$; a connected component of $\mathrm{Map}_*(X, Y)$ which contains f.

Lemma 2.1. *Let $k : S^n\to S^n$ be a map of degree k. The free parts of homotopy groups of* $\mathrm{Map}(S^n, S^n; k)$ *are as follows:*

$$\pi_j(\mathrm{Map}(S^n, S^n; k)\otimes\boldsymbol{Q}=\boldsymbol{Q}\begin{cases} j=n & \text{for odd } n, \\ j=2n-1 & \text{for even } n \text{ and } k\neq 0, \\ j=n-1,\ n,\ 2n-1 & \text{for even } n \text{ and } k=0. \end{cases}$$

$=0$ *otherwise*

Proof. Consider the fibration

$$\text{(2.2)} \qquad \mathrm{Map}_*(S^n, S^n; k) \longrightarrow \mathrm{Map}(S^n, S^n; k) \xrightarrow{ev} S^n$$

where ev is the evaluation map $f \mapsto f(*)$. The free part of $\pi_j(\mathrm{Map}_*(S^n, S^n; k))$ vanishes unless n is even, $j=n-1$. Thus we have the result for odd n. The result for the case n even and $k=0$ follows from the fact that (2.2) has a cross section. Finally since the Whitehead products $[k, h]$ are not 0-homotopic for all $k \neq 0$, $h \neq 0$ and even n, a map $h: S^n \to S^n$ can not be lifted to $\mathrm{Map}(S^n, S^n; k)$. Hence $\pi_n(ev)$ is a 0-map and we get the result. We can prove also this lemma by [10].

Lemma 2.3. *The homotopy set* $[K(\boldsymbol{Z}, m), \mathrm{Map}(S^n, S^n; k)\hat{\ }]$ *is* 0 *for* $m>1$.

Proof. We remark that $\mathrm{Map}(S^n, S^n; k)$ is a nilpotent space by using the fibration (2.2). Since $[K(\boldsymbol{Z}, m), \mathrm{Map}(S^n, S^n; k)\hat{\ }] = [K(\boldsymbol{Q}/\boldsymbol{Z}, m-1), \mathrm{Map}(S^n, S^n; k)]$ by Theorem 2.3 of [5], the latter set is equal to 0 by Theorem C of [11].

Now we calculate the homotopy set $[K(\boldsymbol{Z}, m) \times S^n, S^n]$ for $m, n>1$. Since S^n is simply connected, it is sufficient to calculate the free homotopy set $[K(\boldsymbol{Z}, m) \times S^n, S^n]_{\mathrm{free}}$. The free homotopy set $[K(\boldsymbol{Z}, m), \mathrm{Map}(S^n, S^n; k)]_{\mathrm{free}}$ is equal to the based homotopy set $[K(\boldsymbol{Z}, m), \mathrm{Map}(S^n, S^n; k)]$, because the fundamental group of $\mathrm{Map}(S^n, S^n; k)$ acts trivially on the free part of the homotopy group at least modulo torsion group by using the fibration (2.2). We can also prove it by the naturality of the forgetful functor from the based homotopy set to the free homotopy set. Hence we get the next result by Lemma 2.3 and Theorem D of [11].

Proposition 2.4. *The homotopy sets* $E(m, n, k) = [K(\boldsymbol{Z}, m), \mathrm{Map}(S^n, S^n; k)]$ *are given as follows for* $m, n>1$:

(1) m, n; *odd,*
  $E(m, n, k)=0$

(2) m; *even and* n; *odd,*
  (a) $E(m, n, k)=\boldsymbol{Z}\hat{\ }/\boldsymbol{Z}$ *if* $im=n-1$ *for some* $i>0$
  (b) $E(m, n, k)=0$ *if* $im \neq n-1$ *for any* $i>0$

(3) m; *odd and* n; *even,*
  (a) $E(m, n, k)=\boldsymbol{Z}\hat{\ }/\boldsymbol{Z}$ *if* $m=n-1$ *and* $k=0$
  (b) $E(m, n, k)=0$ *otherwise*

(4) m, n; *even,*
  (a) $E(m, n, k)=\boldsymbol{Z}\hat{\ }/\boldsymbol{Z}$ *if* $im=2n-2$ *for some* $i>0$ *and* $k\neq 0$ *or*
  $im=n-2$ *for some* $i>0$ *and* $jm \neq 2n-2$

for any $j>0$ *and* $k=0$, *or*
$im\neq n-2$ *for any* $i>0$ *and* $jm=2n-2$
for some $j>0$ *and* $k=0$

(b) $E(m, n, 0)=\boldsymbol{Z}\hat{}/\boldsymbol{Z}\times\boldsymbol{Z}\hat{}/\boldsymbol{Z}$
if $im=n-2$ *for some* $i>0$ *and* $jm=2n-2$
for some $j>0$ (*i.e.* $m=2$)

(c) $E(m, n, k)=0$ *otherwise*

Remark. If $im=n-2$ for some $i>0$ and $jm=2n-2$ for some $j>0$, we get $m(j-2i)=2$ and hence $m=2$.

The above result is summarized as follows. For odd m or odd n or $k\neq 0$ or $im\neq n-2$ for any $i>0$, a map $x: K(\boldsymbol{Z}, m)\times S^n\to S^n$ is evaluated by the restriction map $x': K(\boldsymbol{Z}, m)\vee S^n\to S^n$. For even m, n, $im=n-2$ for some i and $k=0$, a map $x: K(\boldsymbol{Z}, m)\times S^n\to S^n$ is a phantom map and is evaluated by elements of $H^*(K(\boldsymbol{Z}, m)\times S^n; \boldsymbol{Z}\hat{}/\boldsymbol{Z})$.

We associate a 2×2 matrix $X(x)$ with a self-map $x: K(\boldsymbol{Z}, m)\times S^n\to K(\boldsymbol{Z}, m)\times S^n$. $X(x)$ has (h, k)-components as follows. Suppose that m or n is odd or $im\neq n-2$ for any $i>0$ and even m. The next maps a, b, c and d are defined by the restrictions and projections of x.

$$X(x)=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\quad \begin{matrix} a\in[K(\boldsymbol{Z}, m), K(\boldsymbol{Z}, m)]\boldsymbol{Z}, & b\in[S^n, K(\boldsymbol{Z}, m)]=0 \text{ or } \boldsymbol{Z} \\ c\in[K(\boldsymbol{Z}, m), S^n]=0 \text{ or } \boldsymbol{Z}\hat{}/\boldsymbol{Z}, & d\in[S^n, S^n]=\boldsymbol{Z}.\end{matrix}$$

In the case $im=n-2$ for some $i>0$ and even m, we associate a 2×2 matrix $X(x)$ where maps a, b and d are defined as above. An element c is a phantom map $c: K(\boldsymbol{Z}, m)\times S^n\to S^n$ for $d=0$, and a phantom map $K(\boldsymbol{Z}, m)\times\{*\}\to S^n$ for $d\neq 0$. A map c corresponds U^iV for the case (4) (a) in Theorem 2.5 where U and V are the generators of $H^m(K(\boldsymbol{Z}, m); \boldsymbol{Z}\hat{}/\boldsymbol{Z})$, $H^n(S^n; \boldsymbol{Z}\hat{}/\boldsymbol{Z})$ respectively. For the case (4) (e), $c: K(\boldsymbol{Z}, 2)\times S^n\to S^n$ is represented by the elements $c_1, c_2\in H^{2n-2}(K(\boldsymbol{Z}, 2)\times S^n; \boldsymbol{Z}\hat{}/\boldsymbol{Z})$. Elements c_1, c_2 correspond U^{n-1}, $U^{n/2-1}V$ respectively. The (2, 1)-component in the case (4) (e) of Theorem 2.5 is represented as vector $(c_1, c_2)\in\boldsymbol{Z}\hat{}/\boldsymbol{Z}\times\boldsymbol{Z}\hat{}/\boldsymbol{Z}$ for $d=0$ and $c_1=(c_1, 0)\in\boldsymbol{Z}\hat{}/\boldsymbol{Z}$ for $d\neq 0$. See also the section 3. From these results, we get the following theorem.

Theorem 2.5. *The set* End $(K(\boldsymbol{Z}, m)\times S^n)$ *consists of* 2×2 *matrices*

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

where

(1) m, n; *odd* (a) $a, b, d\in\boldsymbol{Z}$ *and* $c=0$ *for* $m=n$
(b) $a, d\in\boldsymbol{Z}$ *and* $b=c=0$ *for* $m\neq n$

(2) m; *even*, n; *odd* (a) $a, d\in\boldsymbol{Z}$, $c\in\boldsymbol{Z}\hat{}/\boldsymbol{Z}$ *and* $b=0$ *for* $im=n-1$ *for some* $i>0$

(b) $a, d \in \boldsymbol{Z}$ and $b=c=0$ for $im \neq n-1$ for any $i>0$

(3) m; odd, n; even (a) $a, d \in \boldsymbol{Z}$, $b=0$ and $c \in \boldsymbol{Z}^{\wedge}/\boldsymbol{Z}$ with $cd=0$ for $m=n-1$

(b) $a, d \in \boldsymbol{Z}$ and $b=c=0$ for $m \neq n-1$

(4) m, n; even (a) $a, d \in \boldsymbol{Z}$, $b=0$ and $c \in \boldsymbol{Z}^{\wedge}/\boldsymbol{Z}$ with $cd=0$ for $im=n-2$ for some $i>0$ and $jm \neq 2n-2$ for any $j>0$

(b) $a, d \in \boldsymbol{Z}$, $b=0$ and $c \in \boldsymbol{Z}^{\wedge}/\boldsymbol{Z}$ for $im \neq n-2$ for any $i>0$ and $jm=2n-2$ for some $j>0$, $m \neq n$

(c) $a, d \in \boldsymbol{Z}$ and $b=c=0$ for $im \neq n-2$ for any $i>0$ and $jm \neq 2n-2$ for any $j>0$, $m \neq n$

(d) $a, b, d \in \boldsymbol{Z}$ and $c=0$ for $m=n>2$

(e) $a, d \in \boldsymbol{Z}$, $b=0$ and $c=(c_1, c_2) \in \boldsymbol{Z}^{\wedge}/\boldsymbol{Z} \times \boldsymbol{Z}^{\wedge}/\boldsymbol{Z}$ with $c_2 d=0$ for $m=2$, $n>2$

(f) $a, b, d \in \boldsymbol{Z}$, $c \in \boldsymbol{Z}^{\wedge}/\boldsymbol{Z}$ for $m=n=2$

§3. Monoid Structure of End$(K(\boldsymbol{Z}, m) \times S^n)$ for $m, n>1$

In this section, we determine the monoid structure of End$(K(\boldsymbol{Z}, m) \times S^n)$ for $m, n>1$.

Let $(S^{2n})_\rho$ be the homotopy fiber of Sullivan completion $e^{\wedge}: S^{2n} \to S^{2n\wedge}$. Since $(S^{2n})_Q$ is the homotopy fiber of the cup square map $K(\boldsymbol{Q}, 2n) \to K(\boldsymbol{Q}, 4n)$, we get the following diagram in which every horizontal and vertical sequences are fiber sequences:

$$\begin{array}{ccccc}
(S^{2n})_\rho & \longrightarrow & (S^{2n})_Q & \longrightarrow & (S^{2n})^{\wedge}_0 \\
\downarrow & & \downarrow & & \downarrow \\
K(\boldsymbol{Z}^{\wedge}/\boldsymbol{Z}, 2n-1) & \longrightarrow & K(\boldsymbol{Q}, 2n) & \longrightarrow & K(\boldsymbol{Z}^{\wedge} \otimes \boldsymbol{Q}, 2n) \\
\downarrow & & \downarrow & & \downarrow \\
K(\boldsymbol{Z}^{\wedge}/\boldsymbol{Z}, 4n-1) & \longrightarrow & K(\boldsymbol{Q}, 4n) & \longrightarrow & K(\boldsymbol{Z}^{\wedge} \otimes \boldsymbol{Q}, 4n)
\end{array}$$

Then, $(S^{2n})_\rho$ is a homotopy fiber of $K(\boldsymbol{Z}^{\wedge}/\boldsymbol{Z}, 2n-1) \to K(\boldsymbol{Z}^{\wedge}/\boldsymbol{Z}, 4n-1)$. If a map $S^{2n} \to S^{2n}$ has degree d, the induced homomorphism $\pi_j(S^{2n}) \to \pi_j(S^{2n})$ has degrees d for $j=2n$, d^2 for $j=4n-1$, and the induced homomorphism $\pi_j((S^{2n})_\rho) \to \pi_j((S^{2n})_\rho)$ has degree d for $j=2n-1$, d^2 for $j=4n-2$. Hence a phantom map $f: K(\boldsymbol{Z}, m) \times S^n \to S^n$ is represented by an element of $H^*(K(\boldsymbol{Z}, m) \times S^n; \boldsymbol{Z}^{\wedge}/\boldsymbol{Z})$ and the set $\theta[K(\boldsymbol{Z}, m) \times S^n, S^n]$ of phantom maps is equal to $[K(\boldsymbol{Z}, m) \times S^n, K(\boldsymbol{Z}^{\wedge}/\boldsymbol{Z}, 2n-2)]$ for $im=n-2$ for some $i>0$ and even m.

Since a map $g: K(\boldsymbol{Z}, m) \times S^n \to S^n$ is evaluated by the restriction $g': K(\boldsymbol{Z}, m) \vee S^n \to S^n$ except for the cases (4) (a, e) in Theorem 2.5, we can calculate the multiplication of matrices corresponding to the composition of maps. Let

$y: K(\boldsymbol{Z}, m)\to K(\boldsymbol{Z}, m)\times S^n$ and $x: K(\boldsymbol{Z}, m)\times S^n\to S^n$ be maps represented as ${}^t[e, g]$ and $[c, d]$ respectively. By using the result $[K(\boldsymbol{Z}, m)\times(S^n)_\rho, (S^n)^\wedge]=[(S^n)_\rho, \mathrm{Map}(K(\boldsymbol{Z}, m), (S^n)^\wedge)]=[(S^n)_\rho, (S^n)^\wedge]=0$, we get $[K(\boldsymbol{Z}, m)\times(S^n)_\rho, S^n]=[K(\boldsymbol{Z}, m)\times(S^n)_\rho, (S^n)_\rho]$. Hence we get the next commutative diagram where $c=\rho c'$, $g=\rho g'$ and $d^\sim: (S^n)_\rho\to(S^n)_\rho$.

$$\begin{array}{ccccc} K(\boldsymbol{Z}, m) & \xrightarrow{{}^t[e, g]} & K(\boldsymbol{Z}, m)\times S^n & \xrightarrow{[c, d]} & S^n \\ & {\scriptstyle {}^t[e, g']}\searrow & \uparrow{\scriptstyle \mathrm{Id}\times\rho} & & \uparrow{\scriptstyle \rho} \\ & & K(\boldsymbol{Z}, m)\times(S^n)_\rho & \xrightarrow[{[c', d^\sim]}]{} & (S^n)_\rho \end{array} .$$

Hence we can get the next lemma, which is the case except for (4) (a, e).

Lemma 3.1. *Suppose that m or n is odd, or $im\neq n-2$ for even m, n, any $i>0$. Let $y: K(\boldsymbol{Z}, m)\to K(\boldsymbol{Z}, m)\times S^n$ and $x: K(\boldsymbol{Z}, m)\times S^n\to S^n$ be maps represented as ${}^t[e, g]$ and $[c, d]$ respectively. Then, the composition $xy=[c, d]{}^t[e, g]$ is equal to ce^h+d^kg where h is i or j according as $im=n-1$ or $jm=2n-2$ respectively and k is* 2 *or* 1 *according as the cases* (4) *(b, f) or otherwise.*

Proof. Maps c and g are represented as $c=\rho c'$ and $g=\rho g'$ respectively by Theorem 2.3 of [5]. A map d induces a map $d^\sim: (S^n)_\rho\to(S^n)_\rho$ and the maps c, g are determined by $c', g'\in H^{2n-2}(K(\boldsymbol{Z}, m); \boldsymbol{Z}^\wedge/\boldsymbol{Z})$ or $c', g'\in H^{n-1}(K(\boldsymbol{Z}, m); \boldsymbol{Z}^\wedge/\boldsymbol{Z})$. Hence we get the following equalities by using the cohomology expression for phantom maps.

$$\begin{aligned} xy=[c, d]{}^t[e, g]&=\rho[c', d^\sim]{}^t[e, g']=\rho(c'e+d^\sim g') \\ &=ce^h+d^2g \qquad \text{for the case (4) (b, f)} \\ &=ce^h+dg \qquad \text{otherwise.} \end{aligned}$$

We describe the multiplication of matrices by using Lemma 3.1 and properties of maps $K(\boldsymbol{Z}, m)\to K(\boldsymbol{Z}, m)\times S^n\to K(\boldsymbol{Z}, m)$. $S^n\to K(\boldsymbol{Z}, m)\times S^n\to K(\boldsymbol{Z}, m)$ and $S^n\to K(\boldsymbol{Z}, m)\times S^n\to S^n$. Note that $K(\boldsymbol{Z}, m)\to S^n\to K(\boldsymbol{Z}, m)$ and $S^n\to K(\boldsymbol{Z}, m)\to S^n$ are 0-homotopic for $m, n>1$.

Theorem 3.2. *Except for the cases* (4) (a) *and* (4) (e), *the multiplication of matrices which corresponds the composition of maps is given as follows*:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} ae & af+bh \\ ce^h+d^kg & dh \end{pmatrix}$$

where h is i or j according as $im=n-1$ or $jm=2n-2$ respectively and k is 2 *or* 1 *according as* (4) (b, f) *or otherwise.*

Example. In the cases (2, a), (4, f), we get the next formulas respectively.

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\begin{pmatrix} e & 0 \\ g & h \end{pmatrix}=\begin{pmatrix} ae & 0 \\ ce^i+dg & dh \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} ae & af+bh \\ ce+d^2g & dh \end{pmatrix}.$$

A map $x: K(\boldsymbol{Z}, 2)\times S^n\to S^n$ was classified by $[c, d]=[(c_1, c_2), d]$ with $c_2d=0$ for even n where $d=x\mid\{*\}\times S^n$, and c_1, c_2 are elements of the cohomology group. For $im=n-2$, even $m>2$, x is represented by $c=[0, c]$. Hence we get the next theorem by the same method of Theorem 3.2.

Theorem 3.3. *For the cases* (4) (a, e), *the composition of endomorphisms of* $K(\boldsymbol{Z}, m)\times S^n$ *is given by the following formula of matrices,*

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\begin{pmatrix} e & 0 \\ g & h \end{pmatrix}=\begin{pmatrix} ae & 0 \\ c*(e, h)+d^2g & dh \end{pmatrix}$$

where $c*(e, h)+d^2g$ *is defined by* $(c_1e^j+d^2g_1, c_2e^ih+d^2g_2)$ *for* $c=(c_1, c_2)$, $g=(g_1, g_2)$, $im=n-2$. *Here the first factor is* 0 *in the case* (4) (a), j *is given by* $jm=2n-2$ *in the case* (4) (e).

Proof. We shall prove only the case (4, e). The case (4, a) is similar. It is sufficient to determine the (2, 1)-component. If $d=0$, we factor $c=\rho'c'$: $K(\boldsymbol{Z}, 2)\times S^n\to K(\boldsymbol{Z}^{\wedge}/\boldsymbol{Z}, 2n-2)\to S^n$. Hence we can get the result for $d=0$ by using the cohomology expression of phantom maps. If $h=0$, $d\neq 0$, we also factor $h=\rho'h'$ as above. By lifting a map c to $(c_i', d^2): K(\boldsymbol{Z}, 2)\times K(\boldsymbol{Z}^{\wedge}/\boldsymbol{Z}, 2n-2)\to K(\boldsymbol{Z}^{\wedge}/\boldsymbol{Z}, 2n-2)$ with $\rho'(c_i', d^2)=c(\mathrm{Id}\times\rho')$, we can get the result as the proof of Lemma 3.1. For $hd\neq 0$, it is proved by the same way as Theorem 3.2. Note that $c_2=0$, $g_2=0$ in this case.

Remark. By using Theorem 3.2 and 3.3, we can easily determine the group of self-equivalences and the group of weak identities for $K(\boldsymbol{Z}, m)\times S^n$. For example, we get $\mathrm{Aut}(K(\boldsymbol{Z}, m)\times S^n)=\boldsymbol{Z}/2\boldsymbol{Z}\times\boldsymbol{Z}/2\boldsymbol{Z}$, $WI(K(\boldsymbol{Z}, m)\times S^n)=0$ in the case (4) (a), and the next exact sequence (*) and $WI(K(\boldsymbol{Z}, m)\times S^n)=\boldsymbol{Z}^{\wedge}/\boldsymbol{Z}$ in the case (2) (a) which contains the result of J. Roitberg [7]. The group extension of the next exact sequence is seen by the multiplication of matrices and the action of $\boldsymbol{Z}/2\boldsymbol{Z}\times\boldsymbol{Z}/2\boldsymbol{Z}$ on $\boldsymbol{Z}^{\wedge}/\boldsymbol{Z}$ is given by the multiplication $(i, j)c=(-1)^{i+j}c$ where $(i, j)\in\boldsymbol{Z}/2\boldsymbol{Z}\times\boldsymbol{Z}/2\boldsymbol{Z}$, $i, j\in\{0, 1\}=\boldsymbol{Z}/2\boldsymbol{Z}$ and $c\in\boldsymbol{Z}^{\wedge}/\boldsymbol{Z}$.

$$(*)\qquad 0\longrightarrow WI(K(\boldsymbol{Z}, m)\times S^n)\longrightarrow \mathrm{Aut}(K(\boldsymbol{Z}, m)\times S^n)\longrightarrow \boldsymbol{Z}/2\boldsymbol{Z}\times\boldsymbol{Z}/2\boldsymbol{Z}\longrightarrow 0$$

§4. Monoid Structure of Endomorphisms of $S^1\times S^n$

In this section, we determine $\mathrm{End}(S^1\times S^n)$ for $n>1$. For $n=1$, it is equal to the set of 2×2 matrices of integer components which has the usual multiplication. Now by using the fibration (2.2) and the section 3 of [10], we have the boundary operator $\partial: \pi_n(S^n)\to\pi_{n-1}(\mathrm{Map}_*(S^n, S^n; k))=\pi_{2n-1}(S^n)$, $\partial(h)=[h, k]$.

Lemma 4.1. *The fundamental group of* $\mathrm{Map}(S^n, S^n; k)$ *is* $\boldsymbol{Z}/2\boldsymbol{Z}$ *for* $n>2$ *and* $\boldsymbol{Z}/2k\boldsymbol{Z}$ *for* $n=2$.

Proof. For $n>2$, $\pi_1(\mathrm{Map}(S^n, S^n; k))=\pi_1(\mathrm{Map}_*(S^n, S^n; k))=\pi_{n+1}(S^n)=Z/2Z$. For $n=2$, $\partial(\mathrm{id})=[\mathrm{id}, k]=2k\eta$ where η is the Hopf map, we get the result.

A map $z: S^1\times S^n\to S^n$ is classified by $z|\{*\}\times S^n: S^n\to S^n$ and adjoint map $z^\sim: S^1\to\mathrm{Map}(S^n, S^n; k)$ where k is the degree of the restriction map. By the above lemma, z is represented by (k, q) where $k\in\boldsymbol{Z}$, $q\in\boldsymbol{Z}/2\boldsymbol{Z}$ for $n>2$ or $q\in\boldsymbol{Z}/2k\boldsymbol{Z}$ for $n=2$. To each map $x: S^1\times S^n\to S^1\times S^n$, we attach a 2×2 matrix $X(x)$.

$$X(x)=\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}\quad \begin{array}{l} a\in[S^n, S^n]=\boldsymbol{Z},\quad d\in[S^1, S^1]=\boldsymbol{Z} \\ b\in[S^1, \mathrm{Map}(S^n, S^n; a)]=\boldsymbol{Z}/2\boldsymbol{Z} \text{ for } n>2,\ \boldsymbol{Z}/2a\boldsymbol{Z} \text{ for } n=2\end{array}$$

Maps a, d are the restriction maps of x and b is the adjoint map of $S^1\times S^n\to S^n$. To determine the composition of two maps, we prepare the next lemma.

Lemma 4.2. *Let* $h: S^n\to S^n$ *be a map of degree* h. *Then the homomorphism* $\pi_1(\mathrm{Map}(S^n, S^n; k))\to\pi_1(\mathrm{Map}(S^n, S^n; hk))$ *induced by the map*

$$h_*: \mathrm{Map}(S^n, S^n; k)\longrightarrow \mathrm{Map}(S^n, S^n; hk),\qquad x\mapsto hx$$

is the multiplication $\times h: \boldsymbol{Z}/2\boldsymbol{Z}\to\boldsymbol{Z}/2\boldsymbol{Z}$ *for* $n>2$ *or* $\times h^2: \boldsymbol{Z}/2k\boldsymbol{Z}\to\boldsymbol{Z}/2kh\boldsymbol{Z}$ *according as* $n>2$ *or* $n=2$. *Moreover, the map* $h^*: \mathrm{Map}(S^n, S^n; k)\to \mathrm{Map}(S^n, S^n; kh)$, $x\mapsto xh$ *induces the multiplication by* h *on fundamental groups.*

Proof. At first, we remark the next diagram is commutative where the vertical maps are defined by $k^-(x)=x-k$, $(hk)^-(x)=x-hk$ respectively.

$$\begin{array}{ccc} \mathrm{Map}_*(S^n, S^n; k) & \xrightarrow{h_*} & \mathrm{Map}_*(S^n, S^n; hk) \\ \downarrow k^- & & \downarrow (hk)^- \\ \mathrm{Map}_*(S^n, S^n; 0) & \xrightarrow[h_*]{} & \mathrm{Map}_*(S^n, S^n; 0)\end{array}$$

By the above diagram, we can see $\pi_1(h_*): \pi_1(\mathrm{Map}_*(S^n, S^n; k))\to \pi_1(\mathrm{Map}_*(S^n, S^n; hk))$ is equal to the multiplication $\times h: \boldsymbol{Z}/2\boldsymbol{Z}\to\boldsymbol{Z}/2\boldsymbol{Z}$ for $n>2$

and $\times h^2: \boldsymbol{Z}\to\boldsymbol{Z}$ for $n=2$ respectively. Hence by considering $\pi_1(\mathrm{Map}_*(S^n, S^n; k))\to\pi_1(\mathrm{Map}(S^n, S^n; k))$, we get the result. For $h^*(x)=xh$, the proof is similar.

Theorem 4.3. *Let us identify maps y, $x: S^n\times S^1\to S^n\times S^1$ with 2×2 matrices of the forms $X(y)$, $X(x)$ respectively. Then the composition yx is given by the next formula for $n>2$ (resp. $n=2$).*

$$\begin{pmatrix} e & f \\ 0 & h\end{pmatrix}\begin{pmatrix} a & b \\ 0 & d\end{pmatrix}=\begin{pmatrix} ea & eb+fad \\ 0 & hd\end{pmatrix}\left(resp.\begin{pmatrix} ea & e^2b+fad \\ 0 & hd\end{pmatrix}\right).$$

Proof. It is sufficient to determine (1, 2)-component. By taking the adjoint map of the composition $yx: S^n\times S^1\to S^n\times S^1\to S^n\times S^1$, we have $(yx)^\sim=\mathrm{Map}(y, S^n)x^\sim: S^1\to\mathrm{Map}(S^n, S^n\times S^1)\to\mathrm{Map}(S^n, S^n\times S^1)$. $x^\sim: S^1\to\mathrm{Map}(S^n, S^n\times S^1)$ is determined by $a: S^n\to S^n$, $b: S^1\to\mathrm{Map}(S^n, S^n; a)$ and $d: S^1\to S^1$. To determine $\mathrm{Map}(S^n, y): \mathrm{Map}(S^n, S^n\times S^1)\to\mathrm{Map}(S^n, S^n\times S^1)$, it is sufficient to determine $\mathrm{Map}(S^n, S^n\times S^1; 1)\to\mathrm{Map}(S^n, S^n\times S^1; e)$ where 1 induces $\mathrm{id}: S^n\to S^n$. By calculating $y^\sim=\mathrm{Map}(S^n, y)(\mathrm{id})^\sim: S^1\to\mathrm{Map}(S^n, S^n\times S^1: 1)\to\mathrm{Map}(S^n, S^n; e)\times S^1$, the induced map of fundamental groups is equal to $[f, h]: Z\to Z/2Z\times Z$ for $n>2$ where f, h mean the multiplications by f, h. Hence, $(\mathrm{Map}(a, \mathrm{id})\times\mathrm{id})(f, h): S^1\to\mathrm{Map}(S^n, S^n: e)\times S^1\to\mathrm{Map}(S^n, S^n; ea)\times S^1$ induces $[fa, h]: \boldsymbol{Z}\to\boldsymbol{Z}/2\boldsymbol{Z}\times\boldsymbol{Z}$ on fundamental groups for $n>2$, by Lemma 4.2. Hence we get the induced map of fundamental groups for $\mathrm{Map}(S^n, y): \mathrm{Map}(S^n, S^n; a)\times S^1\to\mathrm{Map}(S^n, S^n; ea)\times S^1$ which is equal to a map $[i, j]\mapsto[ei+faj, hj]$. $\pi_1(x^\sim)$ is equal to a map $[i]\mapsto[bi, di]$. By composing $\pi_1(x^\sim)$ and this map, we have the (1, 2)-component $eb+afd$ for $n>2$. Similarly we get the result for $n=2$.

By Theorem 4.3, 2-power x^2 of any element x of $\mathrm{Aut}(S^1\times S^n)$ is the identity. Hence it is abelian and we have the next result which is the rediscovery of the result of [1, 8].

Theorem 4.4. $\mathrm{Aut}(S^1\times S^n)$ *is the group* $(\boldsymbol{Z}/2\boldsymbol{Z})\times(\boldsymbol{Z}/2\boldsymbol{Z})\times(\boldsymbol{Z}/2\boldsymbol{Z})$ *for* $n>1$.

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