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Root Systems and Periods on Hirzebruch Surfaces

By

Jun-ichi MATSUZAWA*

§ I. Introduction

Cayley classified all nonsingular cubic surfaces in three-dimensional complex projective space by using the configuration of the 27 lines on the surface [3]. The symmetry of these 27 lines can be described by the Weyl group and root system of type E_6 ([4]). Furthermore these objects, namely the 27 lines, the Weyl group of type E_6, and root system of type E_6, have natural realization in the Picard group of cubic surface ([8]).

In [10] a fine moduli space M of marked cubic surfaces was constructed explicitly in such a manner that the relation between the geometrical structure of M and the structure of root system became clear. On the other hand the fine moduli spaces for certain classes of rational surfaces were constructed in terms of root system and periods, which are integrals of a meromorphic 2-form over 2-cycles on the surface corresponding to roots ([7]). The moduli space M was reconstructed in terms of the root system and the periods in the same way (Appendix in [10]).

In this paper, we discuss the moduli problem for certain class of rational surfaces in terms of the root system of type A. Let X be the rational surface obtained from the n-th Hirzebruch surface or rational ruled surface with invariant n by blowing up n points. The relation between the Hirzebruch surfaces with n points blown up and the root system of type A_{n-1} is similar to the relation between the cubic surface and the root system of type E_6. We prove a Torelli theorem for the pairs of X and a certain anticanonical divisor on X by using the structure of the root system of type A_{n-1} in the Picard group of X.

We shall construct a family $p: \mathfrak{X} \to S$ of the Hirzebruch surfaces with n points blown up and study a period mapping for the fibration $p: \mathfrak{X} \to S$, where the base space S is the quotient space of a maximal torus of the simple Lie group of type A_{n-1} by its Weyl group. The fiber $\mathfrak{X}_t$ of p can be regarded as a compactification of the fiber of semi-universal deformation of the simple sur-

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* Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606, Japan.

face singularity of type A_{n-1}. This relation between the Hirzebruch surfaces with n points blown up and simple surface singularities of type A_{n-1} is similar to the relation between Del Pezzo surfaces and simple surface singularities of type E (see Remark 5.4). In order to define a period mapping, we fix a meromorphic 2-form ω on $\mathfrak{X}$. Denote by $\Delta\subset S$ the discriminant variety of p and by $\mathfrak{D}_t$ ($t\in S\setminus\Delta$) the anticanonical divisor on the fiber $\mathfrak{X}_t$ such that the restriction of ω to the fiber $\mathfrak{X}_t$ has poles only along $\mathfrak{D}_t$. The fundamental group $\pi_1(S\setminus\Delta)$ of the space $S\setminus\Delta$ is isomorphic to the Artin group associated to the extended Dynkin diagram of type A_{n-1} ([11]). The monodromy group or the image of the monodromy representation of $\pi_1(S\setminus\Delta)$ on the second homology group of $\mathfrak{X}_t\setminus\mathfrak{D}_t$ is isomorphic to the affine Weyl group $\widetilde{W}$ of the root system of type A_{n-1}. The group $\pi_1(S\setminus\Delta)$ acts on the period domain as an affine transformation group which is isomorphic to $\widetilde{W}$.

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§ 2. Hirzebruch Surfaces with Several Points Blown up

We denote by Σ_n, $n\geqq 0$, the n-th Hirzebruch surface or the rational ruled surface with invariant n. The surface Σ_n is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$. For $n\geqq 1$, Σ_n are obtained by desingularising the projective cone Y in $\boldsymbol{P}^{n+1}$ over a nonsingular rational curve of degree n which lies in a hyperplane of $\boldsymbol{P}^{n+1}$. Especially Σ_0 is $\boldsymbol{P}^1\times\boldsymbol{P}^1$. The Hirzebruch surfaces have only one ruling except for Σ_0 (see e.g. [1, Chap. V. 4], [6, Chap. V]). For $n\geqq 1$, let $\pi:\Sigma_n\to\boldsymbol{P}^1$ be the ruling and S the unique section with $S\cdot S=-n$.

For later use, we shall give another realization of Σ_n. Σ_n is isomorphic to the variety in $\boldsymbol{P}^2\times\boldsymbol{P}^1$

$$\Sigma_n=\{(\zeta_0:\zeta_1:\zeta_2)(s:t)\in\boldsymbol{P}^2\times\boldsymbol{P}^1\,|\,s^n\zeta_0=t^n\zeta_1\}. \tag{2.1}$$

The second projection gives the ruling and

$$S=\{(\zeta_0:\zeta_1:\zeta_2)(s:t)\in\Sigma_n\,|\,\zeta_0=\zeta_1=0\}. \tag{2.2}$$

Put

$$\begin{cases} F=\{(\zeta_0:\zeta_1:\zeta_2)(s:t)\in\Sigma_n\,|\,s=0\}, \\ F'=\{(\zeta_0:\zeta_1:\zeta_2)(s:t)\in\Sigma_n\,|\,t=0\}, \\ C_0=\{(\zeta_0:\zeta_1:\zeta_2)(s:t)\in\Sigma_n\,|\,\zeta_2=0\}. \end{cases} \tag{2.3}$$

Then F and F' are two fibers and C_0 is a section with $C_0\cdot C_0=n$. Σ_n is covered by 4 copies U_i, $1\leqq i\leqq 4$, of C^2 with coordinates $(z_1^{(i)}, z_2^{(i)})$, which are defined by

$$(2.4)\qquad \begin{cases} U_1=\Sigma_n\setminus(F'\cup S), & (z_1^{(1)}, z_2^{(1)})=(s/t, \zeta_2/\zeta_0), \\ U_2=\Sigma_n\setminus(F\cup S), & (z_1^{(2)}, z_2^{(2)})=(t/s, \zeta_2/\zeta_1), \\ U_3=\Sigma_n\setminus(F\cup C_0), & (z_1^{(3)}, z_2^{(3)})=(t/s, \zeta_1/\zeta_2), \\ U_4=\Sigma_n\setminus(F'\cup C_0), & (z_1^{(4)}, z_2^{(4)})=(s/t, \zeta_0/\zeta_2). \end{cases}$$

The transition functions among these coordinates are given by

$$(2.5)\qquad \begin{cases} z_1^{(1)}z_1^{(2)}=z_1^{(2)}z_1^{(4)}=1, \quad z_1^{(2)}=z_1^{(3)} \\ z_2^{(2)}=(z_1^{(1)})^{-n}z_2^{(1)}, \quad z_2^{(1)}z_2^{(4)}=1, \quad z_2^{(2)}z_2^{(3)}=1. \end{cases}$$

Definition 2.1. We say that n points $P_1, \cdots, P_n$ of Σ_n are 'in general position' if no two of them lie on a fiber and no one lies on the section S.

Remark 2.2. For $n\geqq 1$, let $\Phi:\Sigma_n\to Y$ be the morphism obtained by desingularising the projective cone Y in P^{n+1}. If n points $P_1, \cdots, P_n$ of Σ_n are in general position, the n points $Q_i=\Phi(P_i)$, $1\leqq i\leqq n$, are contained in a unique hyperplane H such that H does not pass the vertex of Y and the intersection $H\cap Y$ is an irreducible nonsingular curve. By Bertini's theorem, the set of hypersurfaces H such that $H\cap Y$ is an irreducible nonsingular curve is an open dense subset of the complete linear system $|H|$, considered as a projective space. Therefore the set of the points $(P_1, \cdots, P_n)$ in general position is an open dense subset of the variety $\Sigma_n\times\cdots\times\Sigma_n$ (n times).

From now on, we assume that $n\geqq 1$. Let $P_1, \cdots, P_n$ be n points of Σ_n in general position and

$$p: X_n \longrightarrow \Sigma_n$$

the morphism obtained by blowing up these n points.

Let f and s be the linear equivalence classes of the total transforms of a fiber of π and the section S respectively. Let $E_1, \cdots, E_n(E_i=\pi^{-1}(P_i))$ be the exceptional curves and $e_1, \cdots, e_n$ the linear equivalence classes of them. Then we have (see e.g. [6, Chap. V])

Proposition 2.3.

(1) *The Picard group $Pic(X_n)$ is generated by $f, s, e_1, \cdots, e_n$.*

(2) *The intersection pairing on X_n is given by $f^2=0$, $s^2=-n$, $e_i^2=-1$ $(1\leqq i\leqq n)$, $f\cdot s=1$, $f\cdot e_i=0$, $s\cdot e_i=0$, $e_i\cdot e_j=0$ $(i\neq j)$.*

(3) *The canonical class is $k=-(n+2)f-2s+e_1+\cdots+e_n$.*

(4) *Let C be an irreducible curve on X_n, other than $E_1, \cdots, E_n$, and $c=xf+ys-\sum_{i=1}^n b_ie_i$ the linear equivalence class of C. Then we have either* (i) $x=1$,

$y=0,\ b_i\geqq 0\ (1\leqq i\leqq n)$, (ii) $x=0,\ y=1,\ b_i=0\ (1\leqq i\leqq n)$, *or* (iii) $x\geqq ny,\ y>0,\ b_i\geqq 0$ $(1\leqq i\leqq n)$.

Notation 2.4. Let D be an anticanonical divisor on X_n whose irreducible components are the proper transforms $\bar{F}$, $\bar{F}'$ of two distinct fibers F, F' of the projection $\pi: \Sigma_n \to \boldsymbol{P}^1$ not containing $P_1, \cdots, P_n$, the proper transform $\bar{S}$ of the section S, and the proper transform $\bar{C}$ of the section C of π with $C^2=n$ which passes through $P_1, \cdots, P_n$. We shall give an order $\bar{F}_1$, $\bar{F}_2$ to these two components $\bar{F}$, $\bar{F}'$ and call it an orientation of the divisor

$$D=\bar{F}_1+\bar{F}_2+\bar{S}+\bar{C}\ .$$

By $\mathfrak{d}(X_n)$ we denote the set of such anticanonical divisors. Since the linear equivalence classes of $\bar{F}_1$, $\bar{F}_2$, $\bar{S}$, $\bar{C}$ are f, f, s, and $nf+s-e_1-\cdots-e_n$ respectively, the linear equivalence class of D is $(n+2)f+2s-e_1-\cdots-e_n$. Thus D is an anticanonical divisor on X_n.

Definition 2.5. Let X_n and X'_n be surfaces obtained by blowing up n points in general position of Σ_n. For $D=\bar{F}_1+\bar{F}_2+\bar{S}+\bar{C}\in\mathfrak{d}(X_n)$ and $D'=\bar{F}'_1+\bar{F}'_2+\bar{S}'+\bar{C}'\in\mathfrak{d}(X'_n)$, if there exists an isomorphism $\phi: X_n\to X'_n$ such that

$$\phi(\bar{F}_i)=\bar{F}'_i\ (i=1,2),\qquad \phi(\bar{S})=\bar{S}',\qquad \phi(\bar{C})=\bar{C}'\ ,$$

then we say that the pairs (X_n, D) and (X'_n, D') are isomorphic.

We next consider the isomorphic classes of the pairs (X_n, D).

Lemma 2.6. *Let $F_0=\pi^{-1}(0)$ and $F_\infty=\pi^{-1}(\infty)$ be fibers of Σ_n, S the $(-n)$-section, and C an n-section of $\pi: \Sigma_n\to\boldsymbol{P}^1$. Let $p': X'_n\to\Sigma_n$ be a blowing-up at n points in general position and $D'=\bar{F}'_1+\bar{F}'_2+\bar{S}'+\bar{C}'\in\mathfrak{d}(X'_n)$. Then there exists n points $P_1, \cdots, P_n$ of Σ_n in general position which have the following properties*:

(1) $P_1, \cdots, P_n\in C\setminus(F_0\cup F_\infty)$,

(2) *Let X_n be the surface obtained by blowing up $P_1, \cdots, P_n$. Then there exists an isomorphism $\Phi: X_n\to X'_n$ such that*

$$\Phi(\bar{F}_0)=\bar{F}'_1,\quad \Phi(\bar{F}_\infty)=\bar{F}'_2,\quad \Phi(\bar{S})=\bar{S}',\quad \Phi(\bar{C})=\bar{C}',$$

where $\bar{F}_0$, $\bar{F}_\infty$, $\bar{S}$, $\bar{C}$ are the proper transforms of F_0, F_∞, S, C respectively.

Proof. The Hirzebruch surface Σ_n can be obtained by blowing up the vertex Q of a projective cone Y in $\boldsymbol{P}^{n+1}$ over a nonsingular rational curve of degree n lying in a hyperplane of $\boldsymbol{P}^{n+1}$. The n-sections of Σ_n are the strict transforms of the hyperplane sections of Y not containing the vertex Q. Let H and H' be the hyperplanes in $\boldsymbol{P}^{n+1}$ corresponding to C and $C'=p'(\bar{C}')$. There exists a projective automorphism ψ of $\boldsymbol{P}^{n+1}$ which sends H to H' and preserves the cone Y. Let φ_1 be the automorphism of Σ_n induced by ψ, which sends C to C' and preserves the fibers. Let φ_2 be the automorphism of Σ_n induced by the automorphism of $\pi(\Sigma_n)\cong \boldsymbol{P}^1$ which sends F_0 to $F_1'=p'(\bar{F}_1')$ and F_∞ to $F_2'=p'(\bar{F}_2')$. Put $\varphi=\varphi_1\circ\varphi_2$. Then we have

$$\varphi(F_0)=F_1',\quad \varphi(F_\infty)=F_2',\quad \varphi(S)=S',\quad \varphi(C)=C'.$$

Let $Q_1, \cdots, Q_n$ be the centers of the blowing-up $p': X_n'\to\Sigma_n$. Put $P_i=\varphi^{-1}(Q_i)$, $1\leqq i\leqq n$, then $P_1, \cdots, P_n\in C\setminus(F_0\cup F_\infty)$. Let X_n be the surface obtained by blowing up $P_1, \cdots, P_n$ and $\Phi: X_n\to X_n'$ the induced isomorphism by φ. Then Φ satisfies the second condition.

Proposition 2.7. *Let $p: X_n\to\Sigma_n$ (resp. $p': X_n'\to\Sigma_n$) be the morphism obtained by blowing up n points $P_1, \cdots, P_n$ (resp. $P_1', \cdots, P_n'$) in general position and $D=\bar{F}_1+\bar{F}_2+\bar{S}+\bar{C}\in\mathfrak{d}(X_n)$ (resp. $D'=\bar{F}_1'+\bar{F}_2'+\bar{S}'+\bar{C}'\in\mathfrak{d}(X_n')$). Let*

$$t_0=\pi(p(\bar{F}_1)),\quad t_\infty=\pi(p(\bar{F}_2)),\quad t_i=\pi(P_i)\in\boldsymbol{P}^1,\quad 1\leqq i\leqq n,$$

$$t_0'=\pi(p'(\bar{F}_1')),\quad t_\infty'=\pi(p'(\bar{F}_2')),\quad t_i'=\pi(P_i')\in\boldsymbol{P}^1,\quad 1\leqq i\leqq n.$$

Then the pairs (X_n, D) and (X_n', D') are isomorphic if and only if there exists an automorphism g of $\boldsymbol{P}^1$ such that

$$g(t_0)=t_0',\qquad g(t_\infty)=t_\infty',$$

$$\{g(t_1), \cdots, g(t_n)\}=\{t_1', \cdots, t_n'\}.$$

Proof. This follows from Lemma 2.6.

§ 3. Homology and Root System

Throughout this section, we assume that $n\geqq 2$. We shall study the homology groups of the surfaces X_n and $X_n\setminus D$ ($D=\bar{F}_1+\bar{F}_2+\bar{S}+\bar{C}\in\mathfrak{d}(X_n)$) with integral coefficients. The root systems of type A_{n-1} can be realized in the homology groups of X_n. The realization is similar to that of the root systems in the homology groups of Del Pezzo surfaces and certain rational surfaces ([8], [7]).

We consider the homology exact sequence:

$$\begin{array}{l} \cdots \qquad\qquad \longrightarrow H_3(X_n;\boldsymbol{Z}) \longrightarrow H_3(X_n, X_n\setminus D;\boldsymbol{Z}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \\ \xrightarrow{\partial_*} H_2(X_n\setminus D;\boldsymbol{Z}) \xrightarrow{i_*} H_2(X_n;\boldsymbol{Z}) \xrightarrow{j_*} H_2(X_n, X_n\setminus D;\boldsymbol{Z}) \\ \longrightarrow \qquad \cdots \end{array}$$

We extend the intersection form in $H_2(X_n;\boldsymbol{Z})$ to $H_2(X_n;\boldsymbol{Z})\otimes_{\boldsymbol{Z}}\boldsymbol{R}$. Let

$$Q=\ker j_*\subset H_2(X_n;\boldsymbol{Z})$$

and

$$R=\{\alpha\in Q \mid \alpha\cdot\alpha=-2\}.$$

Proposition 3.1. *Let Q and R be as above. Then Q is given by*

$$Q=\{\alpha\in H_2(X_n\,;\boldsymbol{Z}) \mid \alpha\cdot f=\alpha\cdot s=\alpha\cdot(nf+s-e_1-\cdots-e_n)=0\} \tag{3.1}$$

and R is a root system of type A_{n-1} in $Q\otimes_{\boldsymbol{Z}}\boldsymbol{R}$ and Q is generated by R. The set $\Pi=\{e_i-e_{i+1} \mid 1\leqq i\leqq n\}$ is a basis of R, where e_i is the class of the exceptional curve $E_i=p^{-1}(P_i)$.

Proof. We have the following duality:

$$H_2(X_n,\ X_n\setminus D\,;\boldsymbol{Z})\cong H^2(D\,;\boldsymbol{Z})$$

Thus $\ker j_*$ is the lattice whose elements are orthogonal to the classes of the components of D. Since the classes of the components of D are f, s, and $nf+s-e_1-\cdots-e_n$ (see Notation 2.4), we have (3.1).

Let $\alpha=xf+ys+\sum_{i=1}^n b_ie_i$ be an element of Q. It follows from

$$\alpha\cdot f=\alpha\cdot s=\alpha\cdot(nf+s-e_1-\cdots-e_n)=0$$

that

$$x=y=0\,,\qquad \sum_{i=1}^n b_i=0\,.$$

Thus

$$Q=\left\{\sum_{i=1}^n b_ie_i\in H_2(X_n\,;\boldsymbol{Z}) \mid \sum_{i=1}^n b_i=0\right\}. \tag{3.2}$$

Let $\alpha=\sum_{i=1}^n b_ie_i\in R$. Since $\alpha\cdot\alpha=-2$, we have $\sum_{i=1}^n b_i^2=2$. Thus $b_i=\pm 1$, $b_j=\pm 1$ for some i, j $(i\neq j)$ and the rest are 0. By (3.2), α must be $\pm(e_i-e_j)$ and we have $R=\{e_i-e_j \mid i\neq j\}$. Therefore Q is generated by R. Furthermore R is a root system of type A_{n-1} in $Q\otimes_{\boldsymbol{Z}}\boldsymbol{R}$ and Π is a basis of R. ■

We next consider the second homology group $H_2(X_n\setminus D\,;\boldsymbol{Z})$. By Proposition 3.1, we have the short exact sequence

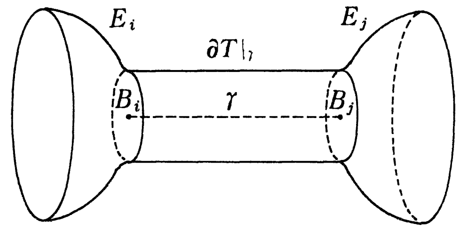
$$0\longrightarrow H_3(X_n,\ X_n\setminus D\,;\boldsymbol{Z})\overset{\partial_*}{\longrightarrow} H_2(X_n\setminus D\,;\boldsymbol{Z})\overset{i_*}{\longrightarrow} Q\longrightarrow 0\,. \tag{3.3}$$

Let $E_i=p^{-1}(P_i)$ and $E_j=p^{-1}(P_j)$ be the exceptional curves and $B_i=E_i\cap\bar{C}$. Let T be a closed tubular neighborhood of $\bar{C}$ in X_n such that $T\cap E_i$ and $T\cap E_j$ are fibers. Let γ be an injective path in $\bar{C}$ from B_i to B_j and let

$$\Gamma_{i,j}=(E_i\setminus(E_i\cap T))\cup\partial T\,|_\gamma\cup(E_j\setminus(E_j\cap T))\,. \tag{3.4}$$

We can take the orientation such that $\Gamma_{i,j}$ is homologous to E_i-E_j in X_n. Hence we have

$$i_*([\Gamma_{i,j}])=e_i-e_j\,.$$



We shall write

$$D=D_1+D_2+D_3+D_4\,,$$

where $D_1=\bar{F}_1$, $D_2=\bar{C}$, $D_3=\bar{F}_2$, and $D_4=\bar{S}$.

By Lemma 2.6 and (2.4), we can take a local coordinate $(z_1^{(i)}, z_2^{(i)})$ $(1\leqq i\leqq 4)$ around the intersection $D_i\cap D_{i+1}$ on X_n $(D_5=D_1)$. For $r_1^{(i)}, r_2^{(i)}>0$, let N_i $(1\leqq i\leqq 4)$ be the 2-cycle on $X_n\setminus D$ defined by

$$N_i=\{(z_1^{(i)}, z_2^{(i)})\,\big|\,|z_1^{(i)}|=r_1^{(i)},\ |z_2^{(i)}|=r_2^{(i)}\} \tag{3.5}$$

with orientation $(\arg z_1^{(i)}, \arg z_2^{(i)})$.

If M_i is the chain

$$M_i=\{(z_1^{(i)}, z_2^{(i)})\,\big|\,|z_1^{(i)}|=r_1^{(i)},\ |z_2^{(i)}|\leqq r_2^{(i)}\}, \qquad r_1^{(i)}, r_2^{(i)}>0 \tag{3.6}$$

with orientation $(\arg z_1^{(i)}, \Re z_2^{(i)}, \Im z_2^{(i)})$ ($\Re z$ and $\Im z$ denote the real part and the imaginary part of z respectively), then the boundary of M_i is N_i.

$$\partial M_i=N_i \tag{3.7}$$

We have

Lemma 3.2.

(1) *Let ν_i be the homology class of N_i in $H_2(X_n\setminus D; \boldsymbol{Z})$ and μ_i the homology class of M_i in $H_3(X_n, X_n\setminus D; \boldsymbol{Z})$. Then we have*

(i) $\nu_i=\partial_*(\mu_i)$,

(ii) $\nu_i=-\nu_{i+1}$, $\mu_i=-\mu_{i+1}$ $(1\leqq i\leqq 4,\ \nu_5=\nu_1,\ \mu_5=\mu_1)$.

(2) *$H_3(X_n, X_n\setminus D; \boldsymbol{Z})\cong\boldsymbol{Z}$, generated by μ_i.*

Proof. By (2.5), (3.6), and (3.7), we have (1). It follows from the duality theorem that

$$\begin{aligned} H_3(X_n, X_n\setminus D; \boldsymbol{Z}) &\cong H^1(D; \boldsymbol{Z}) \\ &\cong H_1(D; \boldsymbol{Z})^* \\ &\cong \boldsymbol{Z}\,. \end{aligned} \tag{3.8}$$

Let γ_i be an injective path in D_i from $D_{i-1}\cap D_i$ to $D_i\cap D_{i+1}$ $(D_0=D_4)$ and $\gamma=$

$\gamma_1+\cdots+\gamma_4$. Then $H_1(D; \boldsymbol{Z})$ is generated by the homology class of γ. Since the intersection number $M_i\cdot\gamma=(-1)^i$, μ_i is a generator of $H_3(X_n, X_n\setminus D; \boldsymbol{Z})$ by Poincaré duality. ■

The Poincaré duality yields a canonical isomorphism

$$H_c^2(X_n\setminus D; \boldsymbol{Z})\cong H_2(X_n\setminus D; \boldsymbol{Z}),$$

where $H_c^2(X_n\setminus D; \boldsymbol{Z})$ is the second cohomology group of $X_n\setminus D$ with compact supports. This duality induces the intersection product on $H_2(X_n\setminus D; \boldsymbol{Z})$

Proposition 3.3. *Let $\nu=\partial_*(\mu)$ be the image of a generator μ of $H_3(X_n, X_n\setminus D; \boldsymbol{Z})$ and α_i, $1\leqq i\leqq n-1$, the class of $\Gamma_{i,i+1}$ in $H_2(X_n\setminus D; \boldsymbol{Z})$. Then $H_2(X_n\setminus D; \boldsymbol{Z})$ is generated by $\nu, \alpha_1, \cdots, \alpha_{n-1}$ and the intersection pairing is given by*

$$\nu^2=0, \qquad \nu\cdot\alpha_i=0 \ (1\leqq i\leqq n-1)$$

$$\alpha_i\cdot\alpha_j=\begin{cases} -2, & \text{if } i=j, \\ 1, & \text{if } |i-j|=1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since

$$\Pi=\{i_*(\alpha_i)=e_i-e_{i+1}\,|\,1\leqq i\leqq n-1\}$$

is a basis of the lattice Q (Proposition 3.1), we have, by (3.3),

$$H_2(X_n\setminus D; \boldsymbol{Z})\cong \boldsymbol{Z}\nu\oplus\boldsymbol{Z}\alpha_1\oplus\cdots\oplus\boldsymbol{Z}\alpha_{n-1}.$$

The intersection numbers of these generators are given as follows: since $i_*(\nu)=0$, we have

$$\nu\cdot\nu=i_*(\nu)\cdot i_*(\nu)=0,$$

$$\nu\cdot\alpha_i=i_*(\nu)\cdot i_*(\alpha_i)=0,$$

$$\begin{aligned}\alpha_i\cdot\alpha_j&=i_*(\alpha_i)\cdot i_*(\alpha_j)\\ &=(e_i-e_{i+1})\cdot(e_j-e_{j+1})\\ &=\begin{cases} -2, & \text{if } i=j, \\ 1, & \text{if } |i-j|=1, \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

§ 4. Torelli Theorem for the Pairs (X_n, D)

We now prove a Torelli theorem for the pairs (X_n, D), where X_n $(n\geqq 2)$ is the blowing up of Σ_n at n points $P_1, \cdots, P_n$ in general position and $D\in\mathfrak{d}(X_n)$

is a marked anticanonical divisor on X_n. The formulation is similar to that of [7].

Let μ be a generator of $H_3(X_n, X_n \setminus D; \boldsymbol{Z})$ and

$$\nu=\partial_*(\mu)\,.$$

There exists a unique meromorphic 2-form ω_ν on X_n such that ω_ν has poles only along D and $\omega_\nu(\nu)=1$. By Lemma 3.2, ν is the homology class of N_1 or N_2 (see (3.5)). If ν is the homology class of N_1, then

$$\omega_\nu=\frac{1}{(2\pi\sqrt{-1})^2}\,\frac{dz_1^{(i)}\wedge dz_2^{(i)}}{z_1^{(i)}z_2^{(i)}} \qquad \text{for } i=1, 2 \tag{4.1}$$

in a neighbourhood of $D_i \cap D_{i+1}$, where $D_1=\bar{F}_1$, $D_2=\bar{C}$, and $D_3=\bar{F}_2$. The residue map gives the 1-form

$$\operatorname{Res}_{\bar{C}}\omega_\nu=\frac{1}{(2\pi\sqrt{-1})^2}\,\frac{dz_1^{(i)}}{z_1^{(i)}} \tag{4.2}$$

on $\bar{C}$.

Lemma 4.1. *Let $\Gamma_{i,j}$ be the 2-cycle defined by* (3.4). *Let $\nu=\partial_*(\mu)\in H_2(X_n \setminus D; \boldsymbol{Z})$ is the homology class of N_r ($r=1$ or 2), then*

$$\exp\left(2\pi\sqrt{-1}\int_{\Gamma_{i,j}}\omega_\nu\right)=\begin{cases}[\bar{F}_1\cap\bar{C},\ \bar{F}_2\cap\bar{C}\,;\ E_j\cap\bar{C},\ E_i\cap\bar{C}], & \text{if } r=1,\\ [\bar{F}_2\cap\bar{C},\ \bar{F}_1\cap\bar{C}\,;\ E_j\cap\bar{C},\ E_i\cap\bar{C}], & \text{if } r=2,\end{cases}$$

where $[Q_1, Q_2\,; Q_3, Q_4]$ denotes the cross ratio of the points Q_1, Q_2, Q_3, and Q_4 on $\boldsymbol{P}^1$.

Proof. Since E_i and E_j are the inverse image of the points P_i and P_j respectively, we have

$$\int_{E_i\setminus(E_i\cap T)}\omega_\nu=\int_{E_j\setminus(E_j\cap T)}\omega_\nu=0\,.$$

Therefore

$$\int_{\Gamma_{i,j}}\omega_\nu=\int_{\partial T|_\gamma}\omega_\nu\,.$$

By the residue formula, we have

$$\begin{aligned}\int_{\partial T|_\gamma}\omega_\nu&=2\pi\sqrt{-1}\int_\gamma\operatorname{Res}_{\bar{C}}\omega_\nu\\ &=\frac{1}{2\pi\sqrt{-1}}\int_\gamma\frac{dz_1^{(r)}}{z_1^{(r)}}\\ &=\frac{1}{2\pi\sqrt{-1}}\int_{t_i}^{t_j}\frac{dz_1^{(r)}}{z_1^{(r)}}\\ &=\frac{1}{2\pi\sqrt{-1}}\log\frac{t_j}{t_i} \pmod{\boldsymbol{Z}},\end{aligned}$$

where t_i and t_j are the affine coordinates of the points $E_i\cap\bar{C}$ and $E_j\cap\bar{C}$ respectively ($\bar{F}_r\cap\bar{C}=0$). Then we have

$$\exp\left(2\pi\sqrt{-1}\int_{\Gamma_{i,j}}\omega_\nu\right)=\frac{t_j}{t_i}$$

$$=[(1:0),\ (0:1)\,;\ (1:t_j),\ (1:t_i)].$$

Thus lemma follows. ∎

We shall define a character $\chi_\nu: Q\to\boldsymbol{C}^*$ by

$$\chi_\nu(i_*[\Gamma])=\exp\left(2\pi\sqrt{-1}\int_\Gamma\omega_\nu\right),\qquad \Gamma\in H_2(X_n\setminus D\,;\,\boldsymbol{Z})\,. \tag{4.3}$$

$$\begin{array}{ccc} H_2(X_n\setminus D\,;\,\boldsymbol{Z}) & \xrightarrow{\ i_*\ } & Q \\ {\scriptstyle \exp 2\pi\sqrt{-1}\int_{[\cdot]}\omega_\nu}\downarrow & & \downarrow{\scriptstyle \chi_\nu} \\ \boldsymbol{C}^* & = & \boldsymbol{C}^* \end{array}$$

Notation 4.2. Let ν be the homology class of N_r ($r=1, 2$). We shall introduce the marking of $\bar{F}_1$, $\bar{F}_2$ in the following way:

$$\begin{cases} \bar{F}_0=\bar{F}_1,\quad \bar{F}_\infty=\bar{F}_2, & \text{in case } \nu=[N_1]\,, \\ \bar{F}_0=\bar{F}_2,\quad \bar{F}_\infty=\bar{F}_1, & \text{if case } \nu=[N_2]\,. \end{cases}$$

Then Lemma 4.1 says that

$$\exp\left(2\pi\sqrt{-1}\int_{\Gamma_{i,j}}\omega_\nu\right)=[\bar{F}_0\cap\bar{C},\ \bar{F}_\infty\cap\bar{C}\,;\ E_j\cap\bar{C},\ E_i\cap\bar{C}]\,. \tag{4.4}$$

We now have the Torelli theorem for the pairs (X_n, D).

Theorem 4.3. *Let $p: X_n\to\Sigma_n$ (resp. $p': X'_n\to\Sigma_n$) be the morphism obtained by blowing up n points $P_1, \cdots, P_n$ (resp. $P'_1, \cdots, P'_n$) in general position and $D\in\mathfrak{d}(X_n)$ (resp. $D'\in\mathfrak{d}(X'_n)$) with the marking $D=\bar{F}_0+\bar{F}_\infty+\bar{S}+\bar{C}$ (resp. $D'=\bar{F}'_0+\bar{F}'_\infty+\bar{S}'+\bar{C}'$) associated with the homology class $\nu\in H_2(X_n\setminus D\,;\,\boldsymbol{Z})$ (resp. $\nu'\in H_2(X'_n\setminus D'\,;\,\boldsymbol{Z})$) (see Notation 4.2). Let χ_ν (resp. $\chi'_{\nu'}$) be the character of the root lattice defined by (4.3). If $\varphi: H_2(X_n\,;\,\boldsymbol{Z})\to H_2(X'_n\,;\,\boldsymbol{Z})$ is an isometry such that*

(1) $\varphi([\bar{F}_0])=[\bar{F}'_0]$, $\varphi([\bar{F}_\infty])=[\bar{F}'_\infty]$, $\varphi([\bar{C}])=[\bar{C}']$, $\varphi([\bar{S}])=[\bar{S}']$,

(2) $\varphi^*(\chi'_{\nu'})=\chi_\nu$,

then there exists an isomorphism $\Phi: X_n\to X'_n$ which induces φ and maps $\bar{F}_0$ to $\bar{F}'_0$, $\bar{F}_\infty$ to $\bar{F}'_\infty$, $\bar{C}$ to $\bar{C}$, and $\bar{S}$ to $\bar{S}'$.

Proof. Let Q (resp. Q') and R (resp. R') be the root lattice and the root system defined in Section 3. It follows from the condition (1) and the Proposition 3.1 that

$$\varphi(Q)=Q'\,,\qquad \varphi(R)=R'\,.$$

Let $E_i=p^{-1}(P_i)$ and e_i its class in $H_2(X_n; \boldsymbol{Z})$. By (1), $e'_i=\varphi(e_i)$ is the class of the exceptional curve on X'_n and let E'_i be the exceptional curve for p' corresponding to the class e'_i. We change the suffixes of $P'_1, \cdots, P'_n$ in such a way that $E'_i=(p')^{-1}(P'_i)$. It follows from the condition (2) and (4.4) that

$$[F_0\cap C, F_\infty\cap C; P_j\cap C, P_i\cap C]=[F'_0\cap C', F'_\infty\cap C'; P'_j\cap C', P'_i\cap C'].$$

Thus the theorem follows from the Proposition 2.6.

§5. A Family of the Hirzebruch Surfaces with n Points Blown up

Let T be a maximal torus of $SL(n, \boldsymbol{C})$ $(n\geqq 2)$ and W the Weyl group of $SL(n, \boldsymbol{C})$, which is isomorphic to the symmetric group $\mathfrak{S}_n$ of degree n. The quotient space $S=T/W$ is isomorphic to $\boldsymbol{C}^{n-1}$. In this section, we construct a family of Hirzebruch surfaces with n points blown up over S. For $t=(a_1, \cdots, a_{n-1})\in S=\boldsymbol{C}^{n-1}$, put

$$f_t(x)=x^n+a_1x^{n-1}+\cdots+a_{n-1}x+1$$

$$(5.1)\quad \begin{cases} \mathcal{U}_1=\{(x_1, y_1)(\lambda:\mu)(a_1, \cdots, a_{n-1})\in \boldsymbol{C}^2\times \boldsymbol{P}^1\times S \mid \lambda y_1+\mu f_t(x_1)=0\} \\ \mathcal{U}_2=\{(x_2, y_2)(a_1, \cdots, a_{n-1})\in \boldsymbol{C}^2\times S \mid x_2^n f_t(x_2^{-1})\neq 0\} \\ \mathcal{U}_3=\{(x_3, y_3)(a_1, \cdots, a_{n-1})\in \boldsymbol{C}^2\times S\} \\ \mathcal{U}_4=\{(x_4, y_4)(a_1, \cdots, a_{n-1})\in \boldsymbol{C}^2\times S\}. \end{cases}$$

The manifold $\mathfrak{X}$ is obtained by glueing $\mathcal{U}_i$, $1\leqq i\leqq 4$, as follows:

$$(5.2)\quad \begin{cases} x_1x_2=x_2x_4=1, \quad x_2=x_3, \\ y_2=x_1^{-n}y_1, \quad y_1y_4=1, \quad y_2y_3=1. \end{cases}$$

Let $\varphi: \mathfrak{X}\to S$ be the projection to S.

Remark 5.1. The open sets $\mathcal{U}_i$ are glued in the same way as the open sets U_i of Hirzebruch surface Σ_n (see (2.4), (2.5)).

Proposition 5.2. *Let $\mathfrak{X}$ and S be as above, then $\mathfrak{X}$ is nonsingular. Put*

$$\Delta=\{t\in S \mid \text{the equation } f_t(x)=0 \text{ has a multiple root}\}.$$

Then we have

(1) *If $t\in S\setminus\Delta$, the fiber $\mathfrak{X}_t=\varphi^{-1}(t)$ is a nonsingular surface. In this case, if $t_1, \cdots, t_n$ be the roots of the equation $f_t(x)=0$, then $\mathfrak{X}_t$ is isomorphic to the surface obtained by blowing up*

$$\Sigma_n=\{(\zeta_0:\zeta_1:\zeta_2)(s:t)\in \boldsymbol{P}^2\times \boldsymbol{P}^1 \mid s^n\zeta_0=t^n\zeta_1\}$$

at the points $(1:t_i^n:0)(t_i:1)$, $1\leqq i\leqq n$, on the n-section C_0 (see (2.3)).

(2) *If* $t\in\Delta$, *the fiber* $\mathfrak{X}_t$ *has singularities. Put*

$$f_t(x)=(x-t_1)^{k_1}\cdots(x-t_r)^{k_r}, \qquad t_i\neq t_j\ (i\neq j).$$

Then $\mathfrak{X}_t$ *has simple singularities of type* A_{k_i-1}, $1\leqq i\leqq r$.

Proof. For $i\neq 1$, $\mathcal{V}_i$ has no singular point. Let $\mathcal{W}_1$ (resp. $\mathcal{W}_2$) be the open set defined by $\lambda\neq 0$ (resp. $\mu\neq 0$) in $\mathcal{V}_1$.

$$\mathcal{V}_1=\mathcal{W}_1\cup\mathcal{W}_2 .$$

Then

$$(5.3)\qquad \begin{cases} \mathcal{W}_1\cong\{(x, y, z)(a_1, \cdots, a_{n-1})\in \boldsymbol{C}^3\times S\,|\, y+zf_t(x)=0\} \\ \mathcal{W}_2\cong\{(x, y, z)(a_1, \cdots, a_{n-1})\in \boldsymbol{C}^3\times S\,|\, yz+f_t(x)=0\}. \end{cases}$$

Put

$$g_1(x, y, z, a_1, \cdots, a_{n-1})=y+zf_t(x),$$

$$g_2(x, y, z, a_1, \cdots, a_{n-1})=yz+f_t(x),$$

then the rank of the Jacobian matrix of g_i is not 0. Thus $\mathfrak{X}$ is nonsingular. (1) Let $t=(a_1, \cdots, a_{n-1})\in S\setminus\Delta$ and $\tilde{U}_1=\mathcal{V}_1\cap\mathfrak{X}_t$, then the first projection $\tilde{U}_1\to V_1=\boldsymbol{C}^2$ is nothing but the blowing up of $\boldsymbol{C}^2$ at the points $P_i=(t_i, 0)$, $1\leqq i\leqq n$, where $t_1, \cdots, t_n$ are the roots of $f_t(x)=0$. If we identify V_1 with the open set U_1 of Σ_n (see (2.4)), the point $P_i=(t_i, 0)\in V_1$ corresponds to the point $(1:t_i^n:0)(t_i:1)\in\Sigma_n$. Thus we have (1) by Remark 5.1. (2) Let $t\in\Delta$ and put

$$\tilde{U}_1=\mathcal{V}_1\cap\mathfrak{X}_t ,$$

$$W_1=\mathcal{W}_1\cap\mathfrak{X}_t ,$$

$$W_2=\mathcal{W}_2\cap\mathfrak{X}_t .$$

Then $\tilde{U}_1=W_1\cup W_2$. The open set W_1 has no singularity. Since

$$yz+(x-t_1)^{k_1}\cdots(x-t_r)^{k_r}=0$$

is the defining equation of W_2 and $t_i\neq t_j$ for $i\neq j$, we can take a neighbourhood around $x=t_i$ such that

$$\prod_{\substack{j=1\\ j\neq i}}^{r}(x-t_j)^{k_j}\neq 0 .$$

Thus we can choose a local coordinate

$$x'=(x-t_i)\Big(\prod_{\substack{j=1\\ j\neq i}}^{r}(x-t_j)^{k_j}\Big)^{1/k_i}.$$

Then we have

$$yz-x'^{k_i}=0 .$$

This has the simple singularity of type A_{k_i-1} at the origin. ■

We next consider a meromorphic 2-form ω on $\mathfrak{X}$ defined by

$$\omega=\frac{(-1)^{i-1}}{(2\pi\sqrt{-1})^2}\frac{dx_i\wedge dy_i}{x_iy_i} \qquad \text{on } \mathcal{U}_i,\ 1\leqq i\leqq 4\,.$$

Let $\mathfrak{D}$ be the pole divisor of ω and $\mathfrak{D}_t=\mathfrak{D}\cap\mathfrak{X}_t$ for $t\in S$.

Proposition 5.3. *For* $t\in S\setminus\Delta$, *we have* $\mathfrak{D}_t=\bar{F}+\bar{F}'+\bar{S}+\bar{C}_0$, *where* $\bar{F}$, $\bar{F}'$, $\bar{S}$, *and* $\bar{C}_0$ *are the strict transforms of the divisors* F, F', S, *and* C_0 *on* Σ_n *defined in* (2.2) *and* (2.3).

Proof. Let $t=(a_1,\cdots,a_{n-1})\in S\setminus\Delta$ and $U_i=\mathcal{U}_i\cap\mathfrak{X}_t$. Let V_i $(i=1, 2)$ be the open subspace of U_1 defined by

$$V_1=\{(x_1, y_1)(\lambda:\mu)\in \boldsymbol{C}^2\times\boldsymbol{P}^1\,|\,\lambda y_1+\mu f_t(x_1)=0,\ \mu\neq 0\}\,,$$

$$V_2=\{(x_1, y_1)(\lambda:\mu)\in \boldsymbol{C}^2\times\boldsymbol{P}^1\,|\,\lambda y_1+\mu f_t(x_1)=0,\ \lambda\neq 0\}\,,$$

where $f_t(x)=x^n+a_1x^{n-1}+\cdots+a_{n-1}x+1$. Then

$$U_1=V_1\cup V_2\,.$$

Put $x=x_1$, $y=y_1$ and $z=\lambda/\mu$, then

$$V_1\cong\{(x, y, z)\in\boldsymbol{C}^3\,|\,yz+f_t(x)=0\}\,.$$

Let ω_t be the restriction of ω to the fiber $\mathfrak{X}_t$. On V_1,

$$\omega_t=\frac{1}{(2\pi\sqrt{-1})^2}\frac{dx\wedge dy}{xy}\,.$$

It follows from $yz+f_t(x)=0$ that

$$\frac{df_t}{dx}(x)dx+zdy+ydz=0\,.$$

Thus we have

$$\frac{dx\wedge dy}{y}=-\frac{dx\wedge dz}{z}=\frac{dy\wedge dz}{\dfrac{df_t}{dx}(x)}\,. \tag{5.4}$$

In V_1, the poles of this form lie on the set

$$W=\left\{(x, y, z)\in\boldsymbol{C}^3\,|\,yz+f_t(x)=0,\ y=z=\frac{df_t}{dx}(x)=0\right\}.$$

If $(x, y, z)\in W$, then $f_t(x)=0$ by $y=z=0$. Since $t\in S\setminus\Delta$, $f_t(x)=0$ has no multiple root. Thus there is no common root of $f_t(x)=0$ and $(df_t/dx)(x)=0$ and we have $W=\phi$. Hence the 2-form $(dx\wedge dy)/y$ is holomorphic on V_1 and ω_t has poles only along the divisor defined by $x=0$ on V_1, which is $\bar{F}\cap V_1$.

We next put $z=\mu/\lambda$, then

$$V_2\cong\{(x, y, z)\in\boldsymbol{C}^3\,|\,y+zf_t(x)=0\}$$

and

$$\omega_t = \frac{1}{(2\pi\sqrt{-1})^2}\frac{dx\wedge dy}{xy}$$

on V_1. We have

$$z\frac{df_t}{dx}(x)dx + dy + f_t(x)dz = 0\,.$$

Thus

$$\frac{dx\wedge dy}{xy} = \frac{dx\wedge dz}{xz} = -\frac{dy\wedge dz}{xz^2\frac{df_t}{dx}(x)}\,.$$

Since $z=0$ implies $y=0$ and $\bar{F}$, $\bar{C}_0$ are defined by $x=0$, $z=y=0$ respectively, this form has poles along the divisors $\bar{F}\cap V_2$ and $\bar{C}_0\cap V_2$ on V_2.

By (5.1)

$$U_2 \cong \{(x_2, y_2)\in \boldsymbol{C}^2 \mid x_2^n f_t(x_2^{-1})\neq 0\}$$

and

$$\omega_t = -\frac{1}{(2\pi\sqrt{-1})^2}\frac{dx_2\wedge dy_2}{x_2 y_2}$$

on U_2. Since $x_2=0$ and $y_2=0$ define the divisors $\bar{F}'$ and $\bar{C}_0$ respectively, ω_t has poles along $\bar{F}'\cap U_2$ and $\bar{C}_0\cap U_2$ on U_2.

Similarly ω_t has poles along $(\bar{F}'\cup\bar{S})\cap U_3$ on U_3 and $(\bar{F}\cup\bar{S})\cap U_4$ on U_4. ∎

Remark 5.4. (1) For a semi-universal deformation $\mathfrak{Y}\to S$ of simple surface singularity of type E_l ($l=6, 7, 8$), there is a family $\bar{\mathfrak{Y}}\to S$ whose general fibers are Del Pezzo surfaces and regarded as the compactifications of general fibers of $\mathfrak{Y}\to S$ ([12]).

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & \bar{\mathfrak{Y}} \\ \downarrow & & \downarrow \\ S & = & S \end{array}$$

The fiber $\bar{\mathfrak{Y}}_s$ is l points blowing up of $\boldsymbol{P}^2$ and $C=\bar{\mathfrak{Y}}_s\setminus\mathfrak{Y}_s$ is an anticanonical divisor of $\bar{\mathfrak{Y}}_s$ which is a rational curve with a cusp. It is well known ([5][8]) that

$$R = \{\alpha\in H_2(\bar{\mathfrak{Y}}_s;\boldsymbol{Z}) \mid \alpha\cdot[C]=0,\ \alpha\cdot\alpha=-2\}$$

is a root system of type E_l in $H_2(\bar{\mathfrak{Y}}_s;\boldsymbol{Z})^{\perp}\otimes_{\boldsymbol{Z}}\boldsymbol{R}$, where $[C]$ is the class of the curve C and $H_2(\bar{\mathfrak{Y}}_s;\boldsymbol{Z})^{\perp}$ is the orthogonal complement of C in $H_2(\bar{\mathfrak{Y}}_s;\boldsymbol{Z})$.

On the other hand, for a semi-universal deformation $\mathfrak{Z}\to S$ of simple surface singularity of type A_{n-1}, we can construct a similar family $\bar{\mathfrak{Z}}\to S$ as follows: Put $S=\boldsymbol{C}^{n-1}$ and for $t=(a_1, \cdots, a_{n-1})\in S$, put

$$g_t(x) = x^n + a_1 x^{n-2} + \cdots + a_{n-1}\,.$$

Let

$$\begin{cases} \mathcal{V}_1=\{(x_1, y_1)(\lambda:\mu)(a_1, \cdots, a_{n-1})\in C^2\times P^1\times S \mid \lambda y_1+\mu g_t(x_1)=0\} \\ \mathcal{V}_2=\{(x_2, y_2)(a_1, \cdots, a_{n-1})\in C^2\times S \mid x_2^n g_t(x_2^{-1})\neq 0\} \\ \mathcal{V}_3=\{(x_3, y_3)(a_1, \cdots, a_{n-1})\in C^2\times S\} \\ \mathcal{V}_4=\{(x_4, y_4)(a_1, \cdots, a_{n-1})\in C^2\times S\}. \end{cases}$$

Let $\bar{\mathfrak{Z}}$ be the manifold obtained by glueing $\mathcal{V}_i$, $1\leqq i\leqq 4$, in the same way as $\mathcal{U}_i$ (see (5.2)). Let

$$S'=\{s\in S \mid \text{the fiber } \bar{\mathfrak{Z}}_s \text{ is nonsingular}\}.$$

The fiber $\bar{\mathfrak{Z}}_s$, $s\in S'$, is isomorphic to the n points blowing up of the Hirzebruch surface. Let $\mathfrak{Z}$ be the open subspace of $\mathcal{V}_1$ defined by $\mu\neq 0$, then $\mathfrak{Z}\to S$ can be regarded as a semi-universal deformation of simple surface singularity of type A_{n-1}. The fiber $\bar{\mathfrak{Z}}_s$ is a compactification of $\mathfrak{Z}_s$. If $s\in S'$, the complement $\bar{\mathfrak{Z}}_s\setminus\mathfrak{Z}_s$ has three components $\bar{F}'$, $\bar{S}$ and $\bar{C}_0$, which are the divisors in Proposition 5.3. The classes of $\bar{F}'$, $\bar{S}$ and $\bar{C}_0$ are f, s and $nf+s-e_1-\cdots-e_n$ (see Notation 2.4). For $s\in S'$, let

$$R=\{\alpha\in H_2(\bar{\mathfrak{Z}}_s; Z) \mid \alpha\cdot f=\alpha\cdot s=\alpha\cdot[\bar{C}_0]=0,\ \alpha\cdot\alpha=-2\}.$$

then R is a root system of type A_{n-1} in $H_2(\bar{\mathfrak{Z}}_s; Z)^{\perp}\otimes_Z R$ by Proposition 3.1, where $H_2(\bar{\mathfrak{Z}}_s, Z)^{\perp}$ is the orthogonal complement of f, s, and $[\bar{C}_0]$ in $H_2(\bar{\mathfrak{Z}}_s, Z)$ with respect to the intersection pairing.

(2) For $s=(a_1, \cdots, a_{n-1})\in S'$ with $a_{n-1}\neq 0$, let

$$\omega_s^{(1)}=\frac{1}{(2\pi\sqrt{-1})^2}\frac{dx_1\wedge dy_1}{x_1 y_1}$$

be a 2-form on $\bar{\mathfrak{Z}}_s\cap\mathcal{V}_1$, which has poles along $\bar{F}'\cup\bar{S}$. Extending $\omega_s^{(1)}$ to $\bar{\mathfrak{Z}}_s$ by the transition functions (5.2), we obtain the 2-form ω_s on $\bar{\mathfrak{Z}}_s$. This is the same 2-form as in the proof of Proposition 5.3.

§ 6. Monodromy Representation of $\pi_1(S\setminus\Delta)$ on $H_2(\mathfrak{X}_t\setminus\mathfrak{D}_t; Z)$

Let $\varphi:\mathfrak{X}\to S$, $\mathfrak{D}$ and Δ be as in Section 5. Let

$$S'=S\setminus\Delta$$

$$\mathfrak{X}'=\mathfrak{X}\setminus(\mathfrak{D}\cup\varphi^{-1}(\Delta)).$$

Then $\varphi:\mathfrak{X}'\to S'$ is a locally trivial fiber bundle whose fibers are open surfaces $\mathfrak{X}_t\setminus\mathfrak{D}_t$, $t\in S'$. The fundamental group $\pi_1(S', t)$ acts on $H_2(\mathfrak{X}_t\setminus\mathfrak{D}_t; Z)$ as the monodromy of $\varphi:\mathfrak{X}'\to S'$.

Let

$$T=\{(t_1, \cdots, t_n)\in(C^*)^n \mid t_1\cdots t_n=1\}$$

$$\tilde{\Delta}=\{(t_1, \cdots, t_n)\in T \mid \prod_{1\leq i<j\leq n}(t_i-t_j)=0\}.$$

The symmetric group $\mathfrak{S}_n$ of degree n acts on T by permutations of coordinates and its quotient space is isomorphic to S:

$$S\cong T/\mathfrak{S}_n .$$

By the definition of Δ, we have

$$S'\cong(T\setminus\tilde{\Delta})/\mathfrak{S}_n . \tag{6.1}$$

Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{t}_C$ its complexification

$$\mathfrak{t}=\left\{(x_1, \cdots, x_n)\in \boldsymbol{R}^n \,\middle|\, \sum_{i=1}^{n} x_i=0\right\}$$

$$\mathfrak{t}_C=\mathfrak{t}\otimes_{\boldsymbol{R}}\boldsymbol{C} .$$

Let

$$Q=\{(x_1, \cdots, x_n)\in\mathfrak{t} \mid x_i\in\boldsymbol{Z}\}.$$

Then the kernel of exponential mapping $\exp:\mathfrak{t}_C\to T$ is $2\pi\sqrt{-1}Q$, where

$$\exp((x_1, \cdots, x_n))=(e^{x_1}, \cdots, e^{x_n}).$$

$$0\longrightarrow 2\pi\sqrt{-1}Q\longrightarrow \mathfrak{t}_C \xrightarrow{\exp} T\longrightarrow 1 . \tag{6.2}$$

Therefore we have

$$S\cong T/\mathfrak{S}_n\cong\mathfrak{t}_C/(Q\rtimes\mathfrak{S}_n),$$

where $\mathfrak{S}_n$ acts on $\mathfrak{t}_C$ by permutations of coordinates and Q acts on $\mathfrak{t}_C$ by translations:

$$\alpha\cdot x=x+2\pi\sqrt{-1}\alpha, \qquad (\alpha\in Q,\ x\in\mathfrak{t}_C).$$

We denote by $\tilde{W}$ the group $Q\rtimes\mathfrak{S}_n$. It is isomorphic to the affine Weyl group of the root system of type A_{n-1} (see [2]). Let

$$H_{i,j}=\{(x_1, \cdots, x_n)\in\mathfrak{t} \mid x_i=x_j\},$$

$$L_{i,j}^{k}=\{(x_1, \cdots, x_n)\in\mathfrak{t} \mid x_i-x_j=k\}.$$

where $1\leqq i<j\leqq n$, $k\in\boldsymbol{Z}$. A fundamental region of $\tilde{W}$ in $\mathfrak{t}_C$ is given by

$$\mathfrak{t}+2\pi\sqrt{-1}\,\bar{A} ,$$

where

$$\bar{A}=\{(x_1, \cdots, x_n)\in\mathfrak{t} \mid x_i\geqq x_j \text{ for } i<j,\ x_1-x_n\leqq 1\}. \tag{6.3}$$

Thus each element of $\mathfrak{t}_C$ is equivalent under $\tilde{W}$ to just one element of $\mathfrak{t}+2\pi\sqrt{-1}\,\bar{A}$.

Let

$$\mathfrak{t}_C'=\mathfrak{t}_C\setminus \bigcup_{\substack{i,j,k\\ 1\leq i<j\leq n\\ k\in \boldsymbol{Z}}} (H_{i,j}+2\pi\sqrt{-1}L_{i,j}^k).$$

It follows from (6.1) and (6.2) that

$$S'=\mathfrak{t}_C'/\widetilde{W}. \tag{6.4}$$

A fundamental region of $\widetilde{W}$ in $\mathfrak{t}_C'$ is given by

$$\mathfrak{t}+2\pi\sqrt{-1}A,$$

where A is the interior of $\bar{A}$. By (6.3), the hyperplanes $L_{1,n}^1, H_{1,2}, \cdots, H_{n-1,n}$ are the walls of A. Let us denote $L_{1,n}^1$ by H_0 and $H_{i,i+1}$ by H_i. Let $w_0, w_1, \cdots, w_{n-1}$ be the orthogonal reflections fixing these hyperplanes with respect to the bilinear form on $\mathfrak{t}$ given by

$$\langle (x_i), (y_i)\rangle = \sum_{i=1}^n x_i y_i .$$

Let m_{ij} be the order of $w_i w_j$. Then

$$\begin{cases} m_{ij}=1, & \text{if } i=j, \\ m_{ij}=3, & \text{if } |i-j|=1, \\ m_{ij}=3, & \text{if } (i, j)=(0, n) \text{ or } (n, 0), \\ m_{ij}=2, & \text{otherwise.} \end{cases}$$

The following result has been proved by Nguyễn Viêt Dũng.

Theorem 6.1 ([11]). *The fundamental group $\pi_1(S')$ has a presentation with generators $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}$ and relations*:

$$\underbrace{\sigma_i\sigma_j\sigma_i\cdots}_{m_{ij}\text{ times}}=\underbrace{\sigma_j\sigma_i\sigma_j\cdots}_{m_{ij}\text{ times}}$$

The loop corresponding to the generator σ_i can be given as follows (see [11]). Let

$$\omega_i=\frac{1}{n}(n-i, n-i, \cdots, \overset{i}{\smile}{n-i}, -i, \cdots, -i)\in\mathfrak{t}.$$

Then $\omega_1, \cdots, \omega_{n-1}$ and O are the vertices of the convex polyhedron $\bar{A}$. Let us denote by $\tilde{\omega}$ the bary-center of $\bar{A}$:

$$\begin{aligned} \tilde{\omega}&=\frac{1}{n}(\omega_1+\cdots+\omega_{n-1}) \\ &=\frac{1}{2n}(n-1, n-3, \cdots, -(n-3), -(n-1)). \end{aligned} \tag{6.5}$$

Let

$$\alpha'_i=(0,\cdots,0,\overset{i}{\overbrace{1}},-1,0,\cdots,0),\qquad 1\leqq i\leqq n-1$$

$$\alpha'_0=(-1,0,\cdots,0,1),$$

$$\tilde{\omega}^{(i)}=\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n-1}\omega_j,\qquad 0\leqq i\leqq n-1\,.$$

Then $\tilde{\omega}^{(i)}$ is the bary-center of the face $\bar{A}\cap H_i$, $0\leqq i\leqq n-1$. Let

$$\gamma_i^{(1)}:[0,1]\longrightarrow \mathfrak{t}_C$$

be the path from $2\pi\sqrt{-1}\tilde{\omega}$ to $\alpha'_i+2\pi\sqrt{-1}\tilde{\omega}^{(i)}$ defined by

$$\gamma_i^{(1)}=(1-s)(2\pi\sqrt{-1}\tilde{\omega})+s(\alpha'_i+2\pi\sqrt{-1}\tilde{\omega}^{(i)})\,.$$

Let

$$\gamma_i^{(2)}:[0,1]\longrightarrow \mathfrak{t}_C$$

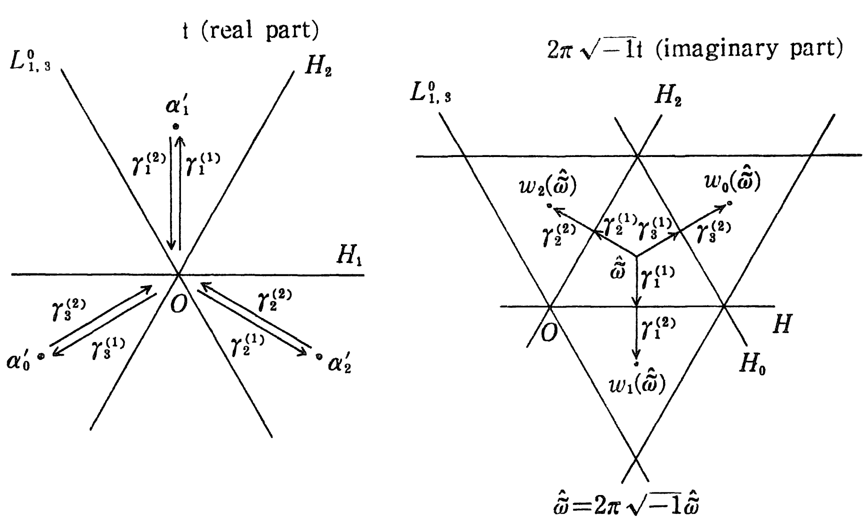
be the path from $\alpha'_i+2\pi\sqrt{-1}\tilde{\omega}^{(i)}$ to $w_i(2\pi\sqrt{-1}\tilde{\omega})$ defined by

$$\gamma_i^{(2)}=(1-s)(\alpha'_i+2\pi\sqrt{-1}\tilde{\omega}^{(i)})+s(w_i(2\pi\sqrt{-1}\tilde{\omega}))\,.$$

Let us denote by γ_i the product of $\gamma_i^{(1)}$ and $\gamma_i^{(2)}$ beginning at $2\pi\sqrt{-1}\tilde{\omega}$ and ending at $w_i(2\pi\sqrt{-1}\tilde{\omega})$.

$$\gamma_i=\gamma_i^{(1)}\gamma_i^{(2)} \tag{6.6}$$

The path γ_i is in $\mathfrak{t}_C'$. The image $\bar{\gamma}_i$ of γ_i under the projection $\mathfrak{t}_C'\to S'=\mathfrak{t}_C'/\widetilde{W}$ gives the generator σ_i of $\pi_1(S')$. When $n=3$, the path γ_i is given as in the following figure: $\mathfrak{t}_C=\mathfrak{t}+2\pi\sqrt{-1}\mathfrak{t}$



Let $\zeta=e^{2\pi\sqrt{-1}/2n}$ and

$$\hat{t}_0=(\zeta^{n-1}, \zeta^{n-3}, \cdots, \zeta^{-(n-1)})\in T\setminus\tilde{\Delta}.$$

Then $\hat{t}_0=\exp(2\pi\sqrt{-1}\tilde{\omega})$. Let t_0 be the image of $\hat{t}_0$ under the quotient mapping $T\to T/\mathfrak{S}_n$. Before describing the monodromy, we shall give 2-cycles of $\mathfrak{X}_{t_0}$ whose homology classes are generators of $H_2(\mathfrak{X}_{t_0}\setminus\mathfrak{D}_{t_0}; \boldsymbol{Z})$. It follows from Proposition 5.2 that the fiber $\mathfrak{X}_{t_0}$ is the surface obtained by blowing up Σ_n at the points $(1:\zeta^{n(n-2i+1)}:0)(\zeta^{n-2i+1}:1)$, $1\leqq i\leqq n$. By the arguement in the proof of Proposition 5.2,

$$U_1=\mathcal{U}_1\cap\mathfrak{X}_{t_0}$$

is the blowing up of $\boldsymbol{C}^2$ at the points $P_i=(\zeta^{n-2i+1}, 0)$, $1\leqq i\leqq n$,

$$U_1\cong\left\{(x_1, y_1)(\lambda:\mu)\in\boldsymbol{C}^2\times\boldsymbol{P}^1 \,|\, \lambda y_1+\mu\prod_{i=1}^{n}(x_1-\zeta^{n-2i+1})=0\right\}.$$

For $\varepsilon>0$, let N be the torus in U_1 defined by $|x_1|=|y_1|=\varepsilon$ with orientation $(\arg x_1, \arg y_1)$. Let E_i be the exceptional curve defined by $x_1=\zeta^{n-2i+1}$, $y_1=0$ in U_1. Let $\mathcal{V}$ be the closed tubular neighborhood of the x_1-axis in U_1 such that $\mathcal{V}\cap E_i$ is a fiber. Let τ_i be the path from $(\zeta^{n-2i+1}, 0)(1:0)$ to $(\zeta^{n-2(i+1)+1}, 0)(1:0)$ in U_1 defined by

$$\tau_i(s)=(e^{-(2\pi\sqrt{-1}/n)s}\zeta^{n-2i+1}, 0)(1:0), \qquad s\in[0, 1].$$

Then we can construct a 2-cycle $\Gamma_{i,i+1}$ as in Section 3 (see (3.4)):

$$\Gamma_{i,i+1}=(E_i\setminus(E_i\cap\mathcal{V}))\cup\partial\mathcal{V}|_{\tau_i}\cup(E_{i+1}\setminus(E_{i+1}\cap\mathcal{V})), \tag{6.7}$$

with orientation $(\arg x_1, \arg y_1)$.

Let ν and α_i are the homology classes of N and $\Gamma_{i,i+1}$ respectively. It follows from Proposition 3.3 that $H_2(\mathfrak{X}_{t_0}\setminus\mathfrak{D}_{t_0}; \boldsymbol{Z})$ is generated by $\nu, \alpha_1, \cdots, \alpha_{n-1}$.

Theorem 6.2. *Let $t_0\in S'$ be the point corresponding to $\tilde{\omega}$. Let*

$$\rho:\pi_1(S', t_0)\longrightarrow \mathrm{Aut}(H_2(\mathfrak{X}_{t_0}\setminus\mathfrak{D}_{t_0}; \boldsymbol{Z}))$$

be the monodromy of the fibration $\varphi:\mathfrak{X}'\to S'$. Let $\nu, \alpha_1, \cdots, \alpha_{n-1}$ be the generators of $H_2(\mathfrak{X}_{t_0}\setminus\mathfrak{D}_{t_0}; \boldsymbol{Z})$) and $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}$ the generators of $\pi_1(S', t_0)$ as above. Let $\alpha_0=-\nu-\alpha_1-\cdots-\alpha_{n-1}$, then σ_i acts on $H_2(\mathfrak{X}_{t_0}\setminus\mathfrak{D}_{t_0}; \boldsymbol{Z})$ by

$$\rho(\sigma_i)(x)=x-\frac{2x\cdot\alpha_i}{\alpha_i\cdot\alpha_i}\alpha_i,$$

$$\rho(\sigma_i\sigma_j)(x)=\rho(\sigma_j)\rho(\sigma_i)(x),$$

where the dot $\cdot$ denote the intersection pairing given in Proposition 3.3. The monodromy group, the image $\rho(\pi_1(S', t_0))$, is isomorphic to the affine Weyl group of the root system of type A_{n-1}.

Proof. In order to describe the action of σ_i, we consider the parallel displacement of the cycles $N, \Gamma_{1,2}, \cdots, \Gamma_{n-1,n}$ along the path $\tilde{\gamma}_i$ in $S'=(T\setminus\tilde{\Delta})/\mathfrak{S}_n$.

First we consider the action of σ_i for $i=1, \cdots, n-1$. The path γ_i, $1 \leqq i \leqq n-1$, gives the path in $T \setminus \tilde{\Delta}$ from $(\zeta^{n-1}, \zeta^{n-3}, \cdots, \zeta^{-(n-1)})$ to

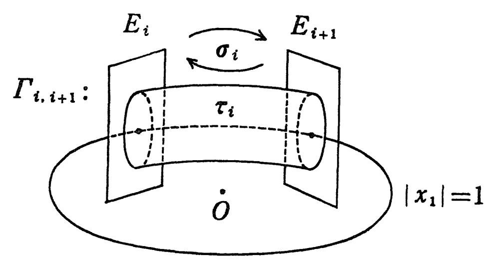
$$(\zeta^{n-1}, \zeta^{n-3}, \cdots, \zeta^{n-2(i-1)+1}, \zeta^{n-2(i+1)+1}, \zeta^{n-2i+1}, \zeta^{n-2(i+2)+1}, \cdots, \zeta^{-(n-1)}).$$

By the construction of the cycles N and $\Gamma_{i,i+1}$, the resulting cycles N' and $\Gamma'_{i,i+1}$ of parallel displacement of N and $\Gamma_{i,i+1}$ along the path $\bar{\gamma}_i$ are given by (up to homology):

$$\begin{cases} [N']=[N]=\nu\,, \\ [\Gamma'_{i-1,i}]=[\Gamma_{i-1,i}]+[\Gamma_{i,i+1}]=\alpha_{i-1}+\alpha_i\,, \\ [\Gamma'_{i,i+1}]=-[\Gamma_{i,i+1}]=-\alpha_i\,, \\ [\Gamma'_{i+1,i+2}]=[\Gamma_{i,i+1}]+[\Gamma_{i+1,i+2}]=\alpha_i+\alpha_{i+1}\,, \\ [\Gamma'_{j,j+1}]=[\Gamma_{j,j+1}]=\alpha_j \qquad \text{if } j \neq i-1,\, i,\, i+1\,. \end{cases}$$

Thus we have

$$\rho(\sigma_i)(x)=x-\frac{2x\cdot\alpha_i}{\alpha_i\cdot\alpha_i}\alpha_i\,, \qquad 1 \leqq i \leqq n-1\,.$$



We next describe the action of σ_0. The path γ_0 gives the path in $T \setminus \tilde{\Delta}$ from $(\zeta^{n-1}, \zeta^{n-3}, \cdots, \zeta^{-(n-1)})$ to $(\zeta^{-(n-1)}, \zeta^{n-3}, \cdots, \zeta^{n-1})$. On the homology we have

$$[N']=[N]=\nu\,,$$

$$[\Gamma'_{i,i+1}]=[\Gamma_{i,i+1}]=\alpha_i \qquad \text{if } i=2, 3, \cdots, n-2\,.$$

The classes $[\Gamma'_{1,2}]$ and $[\Gamma'_{n-1,n}]$ can be calculated as follows. Let τ'_1 be the path from $(\zeta^{-(n-1)}, 0)(1:0)$ to $(\zeta^{n-1}, 0)(1:0)$ in U_1 defined by

$$\tau'_1(s)=(e^{-(2\pi\sqrt{-1}/n)s}\zeta^{-(n-1)}, 0)(1:0)\,, \qquad s \in [0, 1]\,.$$

Then the cycle $\Gamma'_{1,2}$ is homologous to the cycle

$$\Gamma'=(E_n \setminus (E_n \cap \mathcal{V})) \cup \partial\mathcal{V}|_{\tau'_1\tau_1} \cup (E_2 \setminus (E_2 \cap \mathcal{V}))\,.$$

The cycle $\Gamma'+\Gamma_{2,3}+\cdots+\Gamma_{n-1,n}$ is homologous to $-N$. Hence we have

$$\Gamma'_{1,2} = -[N] - [\Gamma_{2,3}] - \cdots - [\Gamma_{n-1,n}]$$
$$= -(\nu + \alpha_2 + \cdots + \alpha_{n-1}) .$$

Similarly we have

$$\Gamma'_{n-1,n} = -[N] - [\Gamma_{1,2}] - \cdots - [\Gamma_{n-2,n-1}]$$
$$= -(\nu + \alpha_1 + \cdots + \alpha_{n-2}) .$$

Thus we have

$$\rho(\sigma_0)(x) = x - \frac{2x \cdot \alpha_0}{\alpha_0 \cdot \alpha_0} \alpha_0 ,$$

where $\alpha_0 = -(\nu + \alpha_1 + \cdots + \alpha_{n-1})$.

The elements $\rho(\sigma_i)$, $0 \leqq i \leqq n-1$, have the same relations as w_i, $0 \leqq i \leqq n-1$. Thus $\rho(\pi_1(S', t_0))$ is isomorphic to the affine Weyl group of the root system of type A_{n-1}. ■

The fundamental group $\pi_1(S', t_0)$ acts on $\mathrm{Hom}_{\boldsymbol{Z}}(H_2(\mathfrak{X}_{t_0} \setminus \mathfrak{D}_{t_0}; \boldsymbol{Z}), \boldsymbol{C})$ by

$$(\sigma \cdot f)(x) = f(\rho(\sigma)^{-1}(x)) \qquad \sigma \in \pi_1(S', t_0) .$$

Let

(6.8) $$\Omega = \{ f \in \mathrm{Hom}_{\boldsymbol{Z}}(H_2(\mathfrak{X}_{t_0} \setminus \mathfrak{D}_{t_0}; \boldsymbol{Z}), \boldsymbol{C}) \mid f(\nu) = 1 \} .$$

Since ν is fixed by the action of $\pi_1(S', t_0)$, Ω is stable under the action of $\pi_1(S', t_0)$. Let

$$\rho^* : \pi_1(S', t_0) \longrightarrow \mathrm{Aut}\,(\Omega)$$

be the homomorphism defined by this action.

Let ν^* and α_i^*, $1 \leqq i \leqq n-1$, be the elements of $\mathrm{Hom}_{\boldsymbol{Z}}(H_2(\mathfrak{X}_{t_0} \setminus \mathfrak{D}_{t_0}; \boldsymbol{Z}), \boldsymbol{C})$ defined by

$$\nu^*(x) = \begin{cases} 1 , & \text{if } x = \nu , \\ 0 , & \text{otherwise,} \end{cases}$$

$$\alpha_i^*(x) = \alpha_i \cdot x .$$

Let $f \in \Omega$. Since $f(\nu) = 1$, f has the form $f = \nu^* + \sum_{i=1}^{n-1} a_i \alpha_i^*$ $(a_i \in \boldsymbol{C})$. Let V^* be the vector space $\sum_{i=1}^{n-1} \boldsymbol{R} \alpha_i^*$, then

$$\Omega = \{ \nu^* + v_1^* + \sqrt{-1} v_2^* \mid v_i^* \in V^* \} .$$

We shall define a non-degenerate bilinear form on V^* by

$$\langle x^*, y^* \rangle = \Bigl(\sum_{i=1}^{n-1} x_i \alpha_i \Bigr) \cdot \Bigl(\sum_{i=1}^{n-1} y_i \alpha_i \Bigr) ,$$

where $x^* = (\sum_{i=1}^{n-1} x_i \alpha_i^*)$ and $y^* = (\sum_{i=1}^{n-1} y_i \alpha_i^*)$. Let us write $f = \nu^* + v_{f,R}^* + \sqrt{-1} v_{f,I}^* \in \Omega$ $(v_{f,R}^*, v_{f,I}^* \in V^*)$.

Let $w_{\alpha_i^*}$, $1 \leqq i \leqq n-1$, be the reflection in the hyperplane orthogonal to α_i^* and W^* the group generated by $w_{\alpha_1^*}, \cdots, w_{\alpha_{n-1}^*}$. Let $R^*=W^*(\{\alpha_1^*, \cdots, \alpha_{n-1}^*\})$. Then R^* forms a root system of type A_{n-1} in V^*. For $\alpha^* \in R^*$ and $k \in \boldsymbol{Z}$, let

$$L_{\alpha^*, k}^*=\{v^* \in V^* \mid \langle\alpha^*, v^*\rangle=k\}. \tag{6.9}$$

The group $\widetilde{W}^*$ generated by the reflections $w_{\alpha^*, k}$ in the hyperplanes $L_{\alpha^*, k}^*(\alpha^* \in R^*,\ k \in \boldsymbol{Z})$ is isomorphic to the affine Weyl group of the root system of type A_{n-1}.

Corollary 6.3. *The fundamental group $\pi_1(S', t_0)$ acts on Ω as affine transformations*:

$$\rho^*(\sigma_i)(f)=\begin{cases} \nu^*+w_{\alpha_i^*}(v_{f, R}^*)+\sqrt{-1} w_{\alpha_i^*}(v_{f, I}^*), & \text{if } 1 \leqq i \leqq n-1, \\ \nu^*+w_{\beta^*, 1}(v_{f, R}^*)+\sqrt{-1} w_{\beta^*, 0}(v_{f, I}^*), & \text{if } i=0, \end{cases}$$

where $f=\nu^*+v_{f, R}^*+\sqrt{-1} v_{f, I}^* \in \Omega$ $(v_{f, R}^*, v_{f, I}^* \in V^*)$, $\beta^*=-(\alpha_1^*+\cdots+\alpha_{n-1}^*)$. The image of ρ^* is isomorphic to the affine Weyl group of the root system of type A_{n-1}.

Proof. For $i \geqq 1$ and $f \in \Omega$,

$$\begin{aligned}(\rho^*(\sigma_i) f)(x) & =f(\rho^*(\sigma_i)^{-1}(x)) \\ & =f(x+(\alpha_i \cdot x) \alpha_i) \quad \text{(by Theorem 6.2)} \\ & =f(x)+f(\alpha_i)(\alpha_i \cdot x) \\ & =f(x)+(\langle v_{f, R}^*, \alpha_i^*\rangle+\sqrt{-1}\langle v_{f, I}^*, \alpha_i^*\rangle) \alpha_i^*(x).\end{aligned}$$

Thus we have

$$\rho^*(\sigma_i) f=\nu^*+(v_{f, R}^*+\langle v_{f, R}^*, \alpha_i^*\rangle \alpha_i^*)+\sqrt{-1}(v_{f, I}^*+\langle v_{f, I}^*, \alpha_i^*\rangle \alpha_i^*).$$

This implies that σ_i acts on V^* and $\sqrt{-1} V^*$ as the reflection in the hyperplane L_i^* orthogonal to α_i^* $(1 \leqq i \leqq n-1)$. For $i=0$, let $\beta=-(\alpha_1+\cdots+\alpha_{n-1})$ and $\beta^*=-(\alpha_1^*+\cdots+\alpha_{n-1}^*)$. Since

$$\begin{aligned}f(\alpha_0) & =(\nu^*+v_{f, R}^*+\sqrt{-1} v_{f, I}^*)(-\nu+\beta) \\ & =-1+\langle v_{f, R}^*, \beta^*\rangle+\sqrt{-1}\langle v_{f, I}^*, \beta^*\rangle\end{aligned}$$

and

$$\alpha_0 \cdot x=\beta \cdot x=\beta^*(x),$$

we have

$$\begin{aligned}(\rho^*(\sigma_0) f)(x) & =f(x)+f(\alpha_0)(\alpha_0 \cdot x) \\ & =f(x)+(-1+\langle v_{f, R}^*, \beta^*\rangle+\sqrt{-1}\langle v_{f, I}^*, \beta^*\rangle) \beta^*(x).\end{aligned}$$

Thus we have

$$\rho^*(\sigma_0)f=\nu^*+(v^*_{f,R}+(\langle v^*_{f,R},\ \beta^*\rangle-1)\beta^*)+\sqrt{-1}(v^*_{f,I}+\langle v^*_{f,I},\ \beta^*\rangle\beta^*)\,.$$

This implies that σ_0 acts on V^* as the reflection in the hyperplane $L^*_{\beta^*,1}$ in V^* and on $\sqrt{-1}V^*$ as the reflection in the hyperplane orthogonal to β^*.

The orbit of the set $\{\alpha^*_1, \cdots, \alpha^*_{n-1}\}$ under the action of the group generated by $\rho^*(\sigma_1), \cdots, \rho^*(\sigma_{n-1})$ forms a root system R^* of type A_{n-1} in V^* and $\{\alpha^*_1, \cdots, \alpha^*_{n-1}\}$ is a basis of R^*. The element $-\beta^*\in R^*$ is the highest root. Thus the generators $\rho^*(\sigma_0), \cdots, \rho^*(\sigma_{n-1})$ of the image $\rho^*(\pi_1(S', t_0))$ have the same relations as w_i, $0\leqq i\leqq n-1$. Therefore $\rho^*(\pi_1(S', t_0))$ is isomorphic to the affine Weyl group of the root system of type A_{n-1}.

§ 7. Period Mapping for the Fibration $\varphi: \mathfrak{X}'\to S'$

In this final section, we shall define a period mapping for the family $\varphi: \mathfrak{X}'\to S'$. Let $\tilde{S}'$ be the covering $\pi: \tilde{S}'\to S'$ of S' which is the quotient of the universal covering of S' by the kernel of the monodromy representation $\rho: \pi_1(S', t_0)\to \mathrm{Aut}\,(H_2(\mathfrak{X}_{t_0}\setminus\mathfrak{D}_{t_0}; \boldsymbol{Z}))$. Thus $\tilde{S}'$ is the regular covering of S' with the monodromy group $G=\rho(\pi_1(S', t_0))$ as covering transformation group.

Let $t_0\in S'$ be the base point defined in Section 6 and write $\tilde{t}_0=(t_0, [e])\in\tilde{S}'$, where $[e]$ is the unit of $\pi_1(S', t_0)/\ker\rho$. By the parallel displacement of a 2-cycle γ on $\mathfrak{X}_{t_0}\setminus\mathfrak{D}_{t_0}$ along paths in S' from t_0 to t, we have a horizontal family of homology classes $\gamma(\tilde{t})\in H_2(\mathfrak{X}_t\setminus\mathfrak{D}_t; \boldsymbol{Z})$, $(\tilde{t}\in\tilde{S}',\ t=\pi(\tilde{t}))$.

We associate a point $\tilde{t}\in\tilde{S}'$ with an element $f_{\tilde{t}}\in\mathrm{Hom}_{\boldsymbol{Z}}(H_2(\mathfrak{X}_{t_0}\setminus\mathfrak{D}_{t_0}; \boldsymbol{Z}, \boldsymbol{C})$ as follows: let

$$f_{\tilde{t}}([\gamma])=\int_{\gamma(\tilde{t})}\omega_t\,. \qquad t=\pi(\tilde{t})\,.$$

$$\begin{array}{ccc} H_2(\mathfrak{X}_{t_0}\setminus\mathfrak{D}_{t_0}; \boldsymbol{Z}) & \xrightarrow{(\tilde{t})_*} & H_2(\mathfrak{X}_t\setminus\mathfrak{D}_t; \boldsymbol{Z}) \\ f_{\tilde{t}}\downarrow & & \downarrow\int_{[\cdot]}\omega_t \\ \boldsymbol{C} & = & \boldsymbol{C} \end{array}$$

where $[\gamma]$ is the homology class of γ and $(\tilde{t})_*$ is the mapping induced by the parallel displacement along the path from t_0 to t which represents $\tilde{t}$. Thus we have a mapping

$$\mathcal{P}: \tilde{S}'\longrightarrow \mathrm{Hom}_{\boldsymbol{Z}}(H_2(\mathfrak{X}_{t_0}\setminus\mathfrak{D}_{t_0}; \boldsymbol{Z}), \boldsymbol{C})\,.$$

We call $\mathcal{P}$ the period mapping for $\varphi: \mathfrak{X}'\to S'$.

Let Ω be the affine subspace of $\mathrm{Hom}\,(H_2(\mathfrak{X}_{t_0}\setminus\mathfrak{D}_{t_0}; \boldsymbol{Z}), \boldsymbol{C})$ defined by (6.8). The fundamental group $\pi_1(S', t_0)$ acts on Ω as affine transformations as in Corollary 6.3. Let $L^*_{\alpha^*,k}$ be the hyperplane in V^* defined by (6.9) and put

$$\Omega'=\Omega\setminus\bigcup_{\substack{\alpha^*\in R^*\\ k\in \boldsymbol{Z}}}(\nu^*+L^*_{\alpha^*,k}+\sqrt{-1}L^*_{\alpha^*,0})\,.$$

Theorem 7.1. *The image of the period mapping* $\mathcal{P}$ *is in* Ω' *and the mapping*

$$\mathcal{P} : \tilde{S}' \longrightarrow \Omega'$$

is biholomorphic. The monodromy group G, *the image of the monodromy representation* $\rho : \pi_1(S', t_0) \to \mathrm{Aut}(H_2(\mathfrak{X}_{t_0} \setminus \mathfrak{D}_{t_0}; \boldsymbol{Z}))$, *acts on* $\tilde{S}'$ *as covering transformation group and on* Ω' *as affine transformations through the representation* ρ^*. *The period mapping* $\mathcal{P}$ *is equivariant with these actions. Thus we have an isomorphism*

$$S' = \tilde{S}'/G \cong \Omega'/G .$$

Proof. As in the discussion of Section 6, an element of S' can be represented by an element $x \in \mathfrak{t} + 2\pi\sqrt{-1}A \subset \mathfrak{t}'_C$ and an element $w \in \tilde{W}$ as follows. By Theorem 6.2, the monodromy group $G = \rho(\pi_1(S', t_0))$ is isomorphic to the affine Weyl group $\tilde{W}$ and $\mathfrak{t} + 2\pi\sqrt{-1}A$ is a fundamental region of $\tilde{W}$ in $\mathfrak{t}'_C$. Thus, by (6.4), an element $\tilde{t} \in \tilde{S}'$ can be represented by an element $x_{\tilde{t}} \in \mathfrak{t} + 2\pi\sqrt{-1}A$ and $w_{\tilde{t}} \in \tilde{W}$ uniquely.

$$\tilde{t} = (x_{\tilde{t}}, w_{\tilde{t}}), \qquad x_{\tilde{t}} \in \mathfrak{t} + 2\pi\sqrt{-1}A, \qquad w_{\tilde{t}} \in \tilde{W} .$$

Let $x_{\tilde{t}} = (x_1, \cdots, x_n) \in \mathfrak{t} + 2\pi\sqrt{-1}A$ and $\exp(x_{\tilde{t}}) = (t_1, \cdots, t_n)$, where $t_i = \exp(x_i)$. By (6.3),

$$\arg t_1 > \cdots > \arg t_n, \qquad \arg t_1 - \arg t_n \leqq 2\pi, \qquad \sum_{i=1}^{n} \arg t_i = 0. \tag{7.1}$$

Let $U_1 = \mathcal{U}_1 \cap \mathfrak{X}_t$ (see (5.1)), $t = \pi(\tilde{t})$. Then

$$U_1 \cong \left\{ (x_1, y_1)(\lambda : \mu) \in \boldsymbol{C}^2 \times \boldsymbol{P}^1 \,\middle|\, \lambda y_1 + \mu \prod_{i=1}^{n} (x_1 - t_i) = 0 \right\} .$$

Let $N(t)$ be the torus in U_1 defined as in Section 6. Let $\tau_i(t)$ be the path from $(t_i, 0)(1:0)$ to $(t_{i+1}, 0)(1:0)$ in U_1 defined by

$$\tau_i(t)(s) = (((1-s)|t_i| + s|t_{i+1}|) e^{\sqrt{-1}((1-s)\arg t_i + s \arg t_{i+1})}, 0)(1:0), \quad 0 \leqq s \leqq 1 .$$

This path τ_i gives a 2-cycle $\Gamma_{i,i+1}(t)$ as in (6.7). Let $\nu(t)$ and $\alpha_i(t)$ be the homology classes of $N(t)$ and $\Gamma_{i,i+1}(t)$ respectively. Then these classes are the generators of $H_2(\mathfrak{X}_t \setminus \mathfrak{D}_t; \boldsymbol{Z})$ and the horizontal families of homology classes are given as follows:

$$\nu(\tilde{t}) = w_{\tilde{t}}^{-1}(\nu(t)) ,$$

$$\alpha_i(\tilde{t}) = w_{\tilde{t}}^{-1}(\alpha_i(t)) ,$$

where the action of $\tilde{W} \cong G$ on $H_2(\mathfrak{X}_t \setminus \mathfrak{D}_t; \boldsymbol{Z})$ is given by the same formula in Theorem 6.2.

Let ν, α_i be the homology classes of 2-cycles N, $\Gamma_{i,i+1}$ on $\mathfrak{X}_{t_0} \setminus \mathfrak{D}_{t_0}$ defined in Section 6. Then

(7.2)
$$\mathcal{P}(\tilde{t})(\nu)=\int_{\nu(\tilde{t})}\omega_t$$
$$=\int_{w_{\tilde{t}}^{-1}(\nu(t))}\omega_t$$
$$=\int_{\nu(t)}\omega_t$$
$$=1 .$$

Hence $\mathcal{P}(\tilde{t})\in\Omega$.

(7.3)
$$\mathcal{P}(\tilde{t})(\alpha_i)=\int_{\alpha_i(\tilde{t})}\omega_t$$
$$=\int_{w_{\tilde{t}}^{-1}(\alpha_i(t))}\omega_t .$$

If we write $w_{\tilde{t}}^{-1}(\alpha_i(t))=c_0\nu(t)+\sum_{i=1}^{n-1}c_i\alpha_i(t)$, then
$$\mathcal{P}(\tilde{t})(\alpha_i)=c_0+\sum_{i=1}^{n-1}c_i\int_{\alpha_i(t)}\omega_t .$$

By the same arguement as in Section 4, we have

(7.4)
$$\int_{\alpha_i(t)}\omega_t=\frac{1}{2\pi\sqrt{-1}}\left\{\log\left|\frac{t_{i+1}}{t_i}\right|+\sqrt{-1}(\arg(t_{i+1})-\arg(t_i))\right\}$$
$$=\frac{1}{2\pi}(\arg(t_{i+1})-\arg(t_i))-\frac{\sqrt{-1}}{2\pi}\log\left|\frac{t_{i+1}}{t_i}\right|.$$

Then we have

(7.5)
$$\mathcal{P}(\tilde{t})(\alpha_i)=c_0+\sum_{i=1}^{n-1}c_i\left\{\frac{1}{2\pi}(\arg(t_{i+1})-\arg(t_i))+\frac{\sqrt{-1}}{2\pi}\log\left|\frac{t_i}{t_{i+1}}\right|\right\}.$$

If $w_{\tilde{t}}=1$, then

(7.6)
$$\mathcal{P}(\tilde{t})(\alpha_i)=\frac{1}{2\pi}(\arg(t_{i+1})-\arg(t_i))+\frac{\sqrt{-1}}{2\pi}\log\left|\frac{t_i}{t_{i+1}}\right|.$$

Let
$$A^*=\{v^*\in V^*\mid v^*(\alpha_i)<0,\ v^*(\alpha_1+\cdots+\alpha_{n-1})>-1\},$$
then $\nu^*+A^*+\sqrt{-1}V^*$ is a fundamental region of $\rho^*(\pi_1(S',t_0))$ in Ω'. Let
$$x_{\tilde{t}}=(y_1,\cdots,y_n)+2\pi\sqrt{-1}(z_1,\cdots,z_n)\qquad(y_i)\in\mathfrak{t},\ (z_i)\in A .$$
Then $t_i=e^{y_i+2\pi\sqrt{-1}z_i}$ and
$$\mathcal{P}(\tilde{t})(\alpha_i)=(z_{i+1}-z_i)+\frac{\sqrt{-1}}{2\pi}\log e^{y_i-y_{i+1}}$$
$$=(z_{i+1}-z_i)+\frac{\sqrt{-1}}{2\pi}(y_i-y_{i+1}) .$$

Therefore it follows from (7.1) and (7.6) that $\mathcal{P}$ gives a bijective mapping from

$$\{\tilde{t}=(x_{\tilde{t}},\ 1)\in\tilde{S}' \mid x_{\tilde{t}}\in \mathfrak{t}+2\pi\sqrt{-1}A\}$$

to the space

$$\{f\in\Omega' \mid f\in\nu^*+A^*+\sqrt{-1}V^*\}.$$

By (7.2) and (7.3), the action of $\pi_1(S',\ t_0)$ on $\tilde{S}'$ is compatible with that on Ω'. Therefore the period mapping $\mathcal{P}$ is bijective.

We next show that $\mathcal{P}$ is biholomorphic. Put

$$x_i'=\frac{1}{2\pi}\left\{(\arg(t_{i+1})-\arg(t_i))+\sqrt{-1}\log\left|\frac{t_i}{t_{i+1}}\right|\right\}.$$

Let $(a_1,\ \cdots,\ a_{n-1})$ be a coordinate system of S as in (5.1). By (7.5) and (7.6), it suffice to show that the Jacobian of the mapping $(a_1,\ \cdots,\ a_{n-1})\to(x_1',\ \cdots,\ x_{n-1}')$ does not vanish, where

$$x^n+a_1x^{n-1}+\cdots+a_{n-1}x+1=\prod_{i=1}^{n}(x-t_i).$$

Since

$$e^{2\pi\sqrt{-1}x_i'}=\frac{t_{i+1}}{t_i}\qquad\left(\prod_{i=1}^{n}t_i=1\right), \tag{7.7}$$

we shall calculate the Jacobians of the mappings

$$(t_1,\ \cdots,\ t_{n-1})\longrightarrow(t_2/t_1,\ \cdots,\ t_n/t_{n-1})$$

and

$$(t_1,\ \cdots,\ t_{n-1})\longrightarrow(a_1,\ \cdots,\ a_{n-1}).$$

Since

$$\frac{t_n}{t_{n-1}}=\frac{1}{t_1\cdots t_{n-2}t_{n-1}^2},$$

$$\frac{\partial(t_n/t_{n-1})}{\partial t_i}=\begin{cases}-\dfrac{1}{t_i}\dfrac{1}{t_1\cdots t_{n-2}t_{n-1}^2}, & \text{if } i\leqq n-2,\\[2ex] -\dfrac{2}{t_{n-1}}\dfrac{1}{t_1\cdots t_{n-2}t_{n-1}^2} & \text{if } i=n-1,\end{cases}$$

we have

$$J_1=\frac{\partial(t_2/t_1,\ \cdots,\ t_n/t_{n-1})}{\partial(t_1,\ \cdots,\ t_{n-1})}$$

$$=\frac{-1}{t_1\cdots t_{n-2}t_{n-1}^2}\begin{vmatrix}-t_2/t_1^2 & 1/t_1 & 0 & & \cdots & 0\\ 0 & -t_3/t_2^2 & 1/t_2 & 0 & \cdots & 0\\ \vdots & & & \ddots & & \vdots\\ 0 & & 0 & -t_{n-1}/t_{n-2}^2 & & 1/t_{n-2}\\ 1/t_1 & & \cdots & 1/t_{n-2} & & 2/t_{n-1}\end{vmatrix}$$

From the $(n-1)$-th column to the second column, multiply the i-th column by t_i/t_{i-1} and add it to the $(i-1)$-th column, then we have

$$J_1=\frac{-1}{t_1\cdots t_{n-2}t_{n-1}^2}\begin{vmatrix} 0 & 1/t_1 & 0 & & \cdots & 0 \\ 0 & 0 & 1/t_2 & 0 & \cdots & 0 \\ & & & \ddots & & \vdots \\ & & & \ddots & 1/t_{n-3} & 0 \\ \vdots & & & \cdots & 0 & 1/t_{n-2} \\ n/t_1 & & & \cdots & 3/t_{n-2} & 2/t_{n-1} \end{vmatrix}$$

$$=(-1)^{n-1}\frac{n}{t_1}\frac{1}{(t_1\cdots t_{n-1})^2}.$$

Thus $J_1\neq 0$. We next show that $J_2=(\partial(a_1, \cdots, a_{n-1})/\partial(t_1, \cdots, t_{n-1}))\neq 0$. Let I_i, $1\leqq i\leqq n$, be the i-th elementary symmetric polynomial in variables $x_1, \cdots, x_n$. It is well known that

$$\frac{\partial(I_1, \cdots, I_n)}{\partial(x_1, \cdots, x_n)}=\prod_{1\leqq i<j\leqq n}(x_i-x_j).$$

Thus the mapping $(x_1, \cdots, x_{n-1})\to(I_1, \cdots, I_{n-1})$ is locally biholomorphic on the submanifold defined by $x_i\neq x_j$ (if $i\neq j$) and $x_1\cdots x_n=1$, which is isomorphic to S'. Therefore we have $J_2\neq 0$.

It follows from $J_1\neq 0$, $J_2\neq 0$, and (7.7) that the mapping $(a_1, \cdots, a_{n-1})\to(x_1', \cdots, x_{n-1}')$ is locally biholomorphic. It follows from the bijectivity of $\mathcal{P}$ that $\mathcal{P}$ is biholomorphic. The action of the monodromy group G on $\tilde{S}'$ and Ω' are equivariant and G acts on these spaces freely. Therefore $\mathcal{P}$ induces an isomorphism

$$S'\cong\Omega'/G$$

and the theorem is proved.

References

[1] Barth, W., Peters, C. and Van de Ven, A., *Compact Complex Surfaces*, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1984.

[2] Bourbaki, N., *Groupes et algèbres de Lie*, Chap. 4, 5, 6, Hermann, Paris, 1968.

[3] Cayley, A., On the triple tangent planes of surfaces of the third order, *Camb. and Dublin Math. Journal.*, **IV** (1849), 118-132; Collected papers **I**, Cambridge Univ. Press (1889), 445-456.

[4] Cartan, É., Sur la réduction à sa forme canonique de la structure d'un groupe de transformations fini et continu, *Amer. J. Math.*, **18** (1896), 1-61.

[5] Demazure, M., Surfaces de Del Pezzo II, III, IV, V, Séminaire sur Les Singularités des Surfaces, *Lecture Notes in Math.*, **777** (1980), Springer-Verlag.

[6] Hartshorne, R., *Algebraic Geometry*, 3rd edition, Springer-Verlag, 1983.

[7] Looijenga, E., Rational surfaces with an anti-canonical cycle, *Ann. of Math.*, **114** (1981), 267-322.

[8] Manin, Yu. I., *Cubic Forms*, 2nd edition, North-Holland, Amsterdam, 1986.

[9] Matsuzawa, J., Monoidal transformations of Hirzebruch surfaces and Weyl groups of type C, *J. Fac. Sci. Univ. Tokyo*, **35** (1988), 425–429; Correction, *J. Fac. Sci. Univ. Tokyo*, **36** (1989), 827.

[10] Naruki, I., Cross ratio variety as a moduli space of cubic surfaces, with Appendix by E. Looijenga, *Proc. London Math. Soc. (3)*, **45** (1982), 1–30.

[11] Nguyễn Viêt Dũng, The fundamental groups of the spaces of regular orbits of the affine Weyl groups, *Topology*, **22** (1983), 425–435.

[12] Pinkham, H., Résolution simultanée de points doubles rationnels, Séminaire sur Les Singularités des Surfaces, *Lecture Notes in Math.*, **777** (1980), Springer-Verlag.