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Purely Inseparable Extensions of Complete Intersections

By

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§ 0. Introduction

Let R be a graded unique factorization domain and $h \in R$ a product of q distinct homogeneous factors in R. Let $S=R[z]/(z^m-h)$. If S is a Krull domain, then what is the divisor class group of S? In several cases the answer is that the divisor class group is a direct sum of $q-1$ copies of Z/mZ. For example, this is true if R is a polynomial ring over a field and h is a product of two variables ([7], page 58); also, if R is a polynomial ring in two or more variables of characteristic $p \neq 0$, the factors of h are homogeneous in R and m is a p-th power ([6], Proposition 3.11, page 627). In this paper the same phenomenon is verified in the following case.

Let k be an algebraically closed field of characteristic $p \neq 0$, $Y \subset A_k^n$ a complete intersection of dimension greater than one, and assume that the ideal that defines Y is homogeneous. The above question is considered when R is the coordinate ring of Y and $h \in R$ is the image of a homogeneous element of $k^{[n]}$ of degree not divisible by p. We prove that if m is a p-th power, then $Cl(S)$ is a direct sum of $q-1$ copies of Z/mZ. This substantially improves a theorem obtained in a previous article ([5], page 569, Theorem 5.7).

§ 1. Preliminaries

This paper assumes familiarity with the subject of divisor class group of a Krull domain. Two excellent references are P. Samuel's 1964 Tata notes [7] and R. Fossum's, "The Divisor Class Group of a Krull Domain" [2]. If A is a Krull domain, we denote the divisor class group of A by $Cl(A)$. If $Y=\operatorname{Spec} A$, we will refer to "the divisor class group of Y", denoted $Cl(Y)$; by this we mean $Cl(A)$. In this section we recall results from [2], [4] and [7].

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Theorem 1. *Let $S \subset T$ be Krull domains with T integral over S. Then there is a well defined group homomorphism $\phi: Cl(S) \to Cl(T)$.* ([7], theorem (6.2), page 20).

Theorem 2. *Let T be a Krull domain of characteristic $p \neq 0$ and $\mathcal{G}$ a finite group of derivation of T. Let $S \subset T$ be the fixed subring of $\mathcal{G}$ and let L denote the quotient field of T. Let $\Delta_1, \cdots, \Delta_m$ be a basis of $\mathcal{G}$ over Z/pZ. Then S is a Krull domain, T is integral over S, and the kernel of the homomorphism $\phi: Cl(S) \to Cl(T)$ described in* (1) *is isomorphic to a subgroup of V_0/V_0' where V_0 and V_0' are the following additive subgroups of L^m: $V_0 = \{(t^{-1}\Delta_1 t, \cdots, t^{-1}\Delta_m t) : t \in L$ and $t^{-1}\Delta_i t \in T$ for $1 \leqq i \leqq m\}$ and $V_0' = \{(u^{-1}\Delta_1 u, \cdots, u^{-1}\Delta_m u) : u$ is a unit in $T\}$.* ([2], Corollary 17.3, page 92).

Remark 3. If I is a divisorial ideal of S whose class group is in the kernel of $Cl(S) \to Cl(T)$, then $T:(T:IT)$ is a principal ideal, say tT, for some $t \in T$. The injection $\ker\phi \to V_0/V_0'$ maps I to $(t^{-1}\Delta_1 t, \cdots, t^{-1}\Delta_m t)$.

Theorem 4. *Let $T[\mathcal{G}]$ denote the S-subalgebra of $\mathrm{End}_s(T)$ generated by T and $\mathcal{G}$. If $T[\mathcal{G}] = \mathrm{End}_s(T)$, then $\ker\phi \to V_0/V_0'$ is an isomorphism.* ([2], theorem 17.4, page 93).

Proposition 5. *Let L' be the quotient field of S. If $[L:L'] = p$ and $D(T)$ is contained in no height one prime ideal of T, then $\ker\phi \to V_0/V_0'$ is an isomorphism.* ([7], Theorem 2.1, page 62).

Proposition 6. *Let A be a graded Krull domain. Let $\mathrm{Div}_h(A)$ denote the subgroup of $\mathrm{Div}(A)$ generated by the homogeneous divisorial (prime) ideals and set $\mathrm{Prin}_h(A) = \mathrm{Prin}(A) \cap \mathrm{Div}_h(A)$. Then the inclusion $\mathrm{Div}_h(A) \to \mathrm{Div}(A)$ induces a bijection*

$$\mathrm{Div}_h A/\mathrm{Prin}_h(A) \longrightarrow Cl(A).$$

([2], proposition 10.2, page 42).

Proposition 7. *In* (2) *assume $T = T_0 \oplus T_1 \oplus T_2 \oplus \cdots$ is a graded ring and Δ_i, $1 \leqq i \leqq m$, are graded derivations of degree e_i (i.e. $\Delta_i(T_j) \subset T_{j+e_i}$ for each j). For each $j = 0, 1, \cdots$, let $S_j = T_j \cap (\bigcap_{\mathcal{G}} \Delta^{-1}(0))$. Assume $S_0 = T_0$. Then $V_0' = 0$ and S is the graded subring of T, $S = S_0 \oplus S_1 \oplus \cdots$. If, in addition, T is a unique factorization domain, then $Cl(S) = \ker\phi$ and the image of the mapping $\ker\phi \to V_0$ is spanned by m-tuples $(t^{-1}\Delta_1 t, \cdots, t^{-1}\Delta_m t) \in T^m$, where t is homogeneous irreducible in T.*

Proof. If u is a unit in T, $u \in T_0$. Since $S_0 = T_0$, $V_0' = 0$. Given $j = 0, 1, \cdots, T_j \cap (\bigcap_{\mathcal{G}} \Delta^{-1}(0)) \subset T_j \cap \left(\bigcap_{i=1}^{m} \Delta_i^{-1}(0)\right)$. The reverse inclusion holds since the Δ_i span $\mathcal{G}$. Therefore $S_j = T_j \cap \left(\bigcap_{i=1}^{m} \Delta_i^{-1}(0)\right)$. Since the Δ_i are homogeneous, an element

$t\in\bigcap_{i=1}^{m}\Delta_i^{-1}(0)$ if and only if its homogeneous parts belong to $\bigcap_{i=1}^{m}\Delta_i^{-1}(0)$. Thus $S=S_0\oplus S_1\oplus\cdots$.

If T is a unique factorization domain, then $Cl(T)=0$ and obviously $Cl(S)=\ker\phi$. By (2), (3) and (6), the image of $\ker\phi$ in V_0 will be generated by m-tuples $(t^{-1}\Delta_1 t, \cdots, t^{-1}\Delta_m t)\in T^m$ where t is a homogeneous element of T. t can be factored as a product $t=w_1^{n_1}\cdots w_s^{n_s}$, where the w_i are irreducible homogeneous elements in T and the n_i are positive integers. If one of the $n_i=0 \pmod p$), then t can be replaced by $w_i^{-n_i}t$; so we may assume no $n_i=0 \pmod p$).

If for some j, $t^{-1}\Delta_j t\neq 0$, then $t^{-1}\Delta_j t=\sum_i n_i w_i^{-1}\Delta_j(w_i)$. $t^{-1}\Delta_j t\in T$ implies $\sum_i n_i\dfrac{w_1\cdots w_s}{w_i}\Delta_j(w_i)\in w_1\cdots w_s T$. Since the w_i are pairwise coprime, w_i must divide $\Delta_j(w_i)$ in T. Thus $w_i^{-1}\Delta_j(w_i)\in T$ for each i and j and $(t^{-1}\Delta_1 t, \cdots, t^{-1}\Delta_m t)=\sum_i n_i(w_i^{-1}\Delta_1 w_i, \cdots, w_i^{-1}\Delta_m w_i)$. □

Definition 8. Let K be a field of characteristic $p\neq 0$. A set $\mathcal{D}$ of derivations of K is called a restricted K-Lie algebra of derivations of K if: (1) $\mathcal{D}$ is closed under addition: (2) $\mathcal{D}$ is closed under bracket product (3) $\mathcal{D}$ is closed under p-th powers; (4) $\mathcal{D}$ is closed under multiplication by elements of K.

Theorem 9 (Jacobson). *Let K be a field of characteristic $p\neq 0$. Let $\mathcal{D}$ be a restricted K-Lie algebra of derivations of K such that $[\mathcal{D}: K]=m<\infty$. Then*: (1) *If K' is the subfield of $\mathcal{D}$ constants, then K is purely inseparable of exponent $\leqq 1$ over K' and $[K: K']=p^m$*; (2) *if $\mathcal{D}$ is any derivation of K over K', then $D\in\mathcal{D}$*; (3) *if $(D_1, \cdots, D_m)$ is any basis for $\mathcal{D}$ over K, then the set of monomials*

$$D_1^{k_1}\cdots D_m^{k_m};\ 0\leqq k_i<p, \qquad (D_i^0=1)$$

is a basis for the ring $\mathrm{End}_{K'}(K)$ considered as a vector space over K. ([4], theorem 19, page 186).

Lemma 10. *Let K be a perfect field of characteristic $p\neq 0$. Let B be a finitely generated K-integral domain of dimension d. Let $C=B^p$, the ring of p-th powers of elements of B. Then the degree of B over C is p^d.*

Proof. $B=K[w_1, \cdots, w_t]$ for some $w_i\in B$. Then $C=K[w_1^p, \cdots, w_t^p]$. By Noether's normalization theorem, there exists $y_1, \cdots, y_d\in B$ such that B is separable algebraic over $K[y_1, \cdots, y_d]$ and $y_1, \cdots, y_d$ are algebraically independent over K. Let L_B, L_C be the fields of quotients of B, C, respectively. Clearly, $[L_B: K(y_1, \cdots, y_d)]=[L_C: K(y_1^p, \cdots, y_d^p)]$ and the result follows. □

§2. Purely Inseparable Extensions of Complete Intersections

1. Let k be an algebraically closed field of characteristic $p\neq 0$ and let $k^{[n]}$ denote the polynomial ring in n variables over k. Assume $h_1, \cdots, h_r$ $(r\leqq n-2)$

are homogeneous polynomials in $k^{[n]}$ such that the ideal P they generate is a height r prime ideal in $k^{[n]}$. Let $R=k^{[n]}/P$ and for $f\in k^{[n]}$ denote its image in R by $\bar{f}$. For each integer $m\geqq 0$, let $R_m=k[\bar{x}_1^{p^m}, \cdots, \bar{x}_n^{p^m}]\subset R$. R_m is a subring of R and is ring isomorphic (not k-isomorphic) to R since k is perfect.

Throughout assume R is a unique factorization domain. Also assume $h_{r+1}\in k^{[n]}$ is homogeneous such that $\deg(h_{r+1})\neq 0 \pmod p$ and h_{r+1} factors as a product $\bar{h}_{r+1}=\bar{u}_1\bar{u}_2\cdots\bar{u}_q$ of q distinct irreducible elements in R. Note that this last assumption implies $\bar{h}_{r+1}\notin R_1$.

For each integer $m\geqq 0$, let $S_m=R_m[\bar{h}_{r+1}]$ and $X_m\subset A_k^{n+1}$ be the variety defined by the equations $h_1=\cdots=h_r=x_{n+1}^{p^m}-h_{r+1}=0$.

Lemma 2. *For each m, the coordinate ring of X_m is isomorphic S_m.*

Proof. Let $\phi: k[x_1, \cdots, x_{n+1}]\to S_m$ be the surjection that sends x_i to $\bar{x}_i^{p^m}$ for $1\leqq i\leqq n$, α to α^{p^m} for $\alpha\in k$ and x_{n+1} to $\bar{h}_{r+1}$. Let $Q\subset k^{[n+1]}$ be the ideal generated by $h_1, \cdots, h_r, x_{n+1}^{p^m}-h_{r+1}$. Since $\bar{h}_{r+1}\notin R_1$, Q is a prime ideal of height $r+1$ contained in $\ker\phi$. We have $\dim(R_m[\bar{h}_{r+1}])=\dim(R)=n-r$. Thus $\ker\phi$ is a height $r+1$ prime ideal and $\ker\phi=Q$. Therefore $k^{[n+1]}/Q$ is isomorphic to S_m. □

3. For each integer $m\geqq 0$, let S'_m be the ring of p-th powers of elements of S_m. S'_m and S_m are isomorphic and $S'_m\subset S_{m+1}\subset S_m$. Denote by E_m, F_m and F'_m the quotient fields of R_m, S_m and S'_m, respectively.

Lemma 4. *For each $m\geqq 0$:* (i) $[E_m : E_{m+1}]=[F_m : F'_m]=p^{n-r}$; (ii) $[F_{m+1} : F'_m]=p$; (iii) $[F_m : F_{m+1}]=p^{n-r-1}$; (iv) $[F_m : E_m]=p^{m+1}$.

Proof. (i) is an immediate consequence of Lemma (1.10). (ii) is obvious. (iii) follows from (i) and (ii). To prove (iv), note that $F_m=E_m(\bar{h}_{r+1})$. Then $[F_m : E_m]=\prod_{i=0}^{m-1}[E_m(\bar{h}_{r+1}^{p^i}) : E_m(\bar{h}_{r+1}^{p^{i+1}})]$. $E_m(\bar{h}_{r+1}^{p^i})\cong E_{m-i}(\bar{h}_{r+1})=F_{m-i}$, and $E_m(\bar{h}_{r+1}^{p^{i+1}})\cong E_{m-i}(\bar{h}_{r+1}^{p})=F'_{m-i-1}$, where the two isomorphisms are nothing other than the operation of taking p^i-th roots. Thus $[F_m : E_m]=\prod_{i=0}^{m-1}[F_{m-i} : F'_{m-i-1}]=p^m$ by (ii).

□

5. From here on assume that X_1 is regular in codimension one. Then X_m is regular in codimension one for each m and S_m is noetherian integrally closed; hence S_m is a Krull domain. By (1.1) and (1.3) we obtain well defined group homomorphisms $\phi'_m: Cl(S'_m)\to Cl(S_{m+1})$ and $\phi_m: Cl(S_{m+1})\to Cl(S_m)$, which we'll study via (1.3).

Definition 6. If A is a ring and $f_1, \cdots, f_s\in A[x_1, \cdots, x_s]$, the polynomial ring in s variables over A, let $\partial(f_1, \cdots, f_s)/\partial(x_1, \cdots, x_s)$ denote the determinant of the Jacobian matrix $[\partial f_i/\partial x_j]$.

§ 3. The Calculation of $\ker\phi'_m$

1. For each $(r+1)$-tuple $I=(i_1, \cdots, i_{r+1})$ of integers with $1\leqq i_j\leqq n$, let D_I be the derivation on $k(x_1, \cdots, x_n)$ defined by $D_I=\partial(\ , h_1, \cdots, h_r)/\partial(x_{i_1}, \cdots, x_{i_{r+1}})$. Since $D_I(h_i)=0$ for each $i=1, \cdots, r$; D_I induces a derivation on $E_0=k(\bar{x}_1, \cdots, \bar{x}_m)$. We will also denote this derivation by D_I; it should be clear from the context which one is meant.

Lemma 2. $R\cap(\bigcap_I D_I^{-1}(0))=R_1$.

Proof. Since R is factorial, R is regular in codimension one. Thus the maximal minors of the Jacobian matrix $[\partial h_i/\partial x_j]_{1\leqq i\leqq r}$ have greatest common divisor 1 in R. In particular, at least one maximal minor (actually at least two) has nonzero image in R. Without loss of generality we may assume $\partial(h_1, \cdots, h_r)/\partial(x_1, \cdots, x_r)$ has nonzero image in R.

For each $s=r+1, \cdots, n$, let $D_s=D_{(1,\cdots,r,s)}$. Then $E_0\supset E_0\cap D_{r+1}^{-1}(0)\supset E_0\cap\left(\bigcap_{s=r+1}^{r+2} D_s^{-1}(0)\right)\supset\cdots\supset E_0\cap\left(\bigcap_{s=r+1}^{n} D_s^{-1}(0)\right)$. Each containment is proper since $x_t\in\bigcap_{s<t} D_s^{-1}(0)$ and $D_t(\bar{x}_t)\neq 0$. Also, $E_1\subset E_0\cap(\bigcap_I D_I^{-1}(0))\subset E_0\cap\left(\bigcap_{s=r+1}^{n} D_s^{-1}(0)\right)$. By (2.4), $[E_0: E_1]=p^{n-r}$, which forces $E_1=E_0\cap(\bigcap_I D_I^{-1}(0))$. Thus R_1 and $R\cap(\bigcap_I D_I^{-1}(0))$ have the same quotient field. Since R_1 is integrally closed, $R_1=R\cap(\bigcap_I D_I^{-1}(0))$. □

Remark 3. By (1.1), $\bar{h}_{r+1}\notin R_1$. By (3.2), there exists an $(r+1)$-tuple I_0 such that $D_{I_0}(\bar{h}_{r+1})\neq 0$. Let $\beta=D_{I_0}(\bar{h}_{r+1})$. For each integer $m\geqq 0$, let Δ_m be the restriction of the derivation $\beta^{-1}D_{I_0}$ on E_0 to S_m.

Lemma 4. *Δ_m maps S_m into S_m and has kernel S'_{m-1}.*

Proof. Let $\alpha\in S_m$. By (2.4), $\alpha=\sum_{i=0}^{p-1}\alpha_i\bar{h}_{r+1}^i$ for unique $\alpha_i\in S'_{m-1}$. Then $\Delta_m(\alpha)=\sum_{i=0}^{p-1} i\alpha_i\bar{h}_{r+1}^{i-1}$. Thus $\Delta_m(\alpha)\in S_m$ and $\Delta_m(\alpha)=0$ if and only if $\alpha_i=0$ for $1\leqq i\leqq p-1$; that is, if and only if $\alpha\in S'_{m-1}$. □

Proposition 5. *The mapping $\phi'_m: Cl(S'_m)\rightarrow Cl(S_{m+1})$ described in (2.5) is an injection.*

Proof. $\Delta_m(\bar{h}_{r+1})=1$ and $[F_{m+1}: F'_m]=p$. By (2.5), (1.7), and (3.4), $\ker\phi'_m$ is isomorphic to V_0, where V_0 is spanned by the logarithmic derivatives $t^{-1}\Delta_m t\in S_{m+1}$, where $t\in S_{m+1}$ is homogeneous with respect to the grading S_{m+1} inherits from R. t homogeneous and $\deg(\bar{h}_{r+1})\neq 0 \pmod p$ implies $t=\alpha\bar{h}_{r+1}^j$ for some

$\alpha \in S'_m$ and integer j, $0 \leqq j \leqq p-1$. If $t^{-1}\Delta_m t \in S_{m+1}$, then either $j=0$ or $\bar{h}_{r+1}^{-1} \in S_{m+1}$. Thus $t^{-1}\Delta_m t=0$ and $V_0=0$. □

Proposition 6. *Let $\mathcal{G}$ denote the vector space over E_0 generated by the D_I defined in* (3.1). *Then if D is a derivation on E_0, $D \in \mathcal{G}$.*

Proof. Let $\mathcal{D}$ denote the vector space of derivations on E_0. The map $D \to (D\bar{x}_1, \cdots, D\bar{x}_n)$ is a E_0-vector space monomorphism from $\mathcal{D}$ to E_0^n. If $D \in \mathcal{D}$, $0=D(\bar{h}_i)=\sum_i \frac{\overline{\partial h_i}}{\partial x_j} D\bar{x}_j$. Thus $D \in \mathcal{D}$ implies $\left[\frac{\overline{\partial h_i}}{\partial x_j}\right] D(\bar{x})=0$. Since R is factorial, R is regular in codimension 1, which implies the rank of $\left[\frac{\overline{\partial h_i}}{\partial x_j}\right]$ is r. Therefore $\mathcal{D}$ is of dimension at most $n-r$ over E_0.

Assuming again that $\partial(h_1, \cdots, h_r)/\partial(x_1, \cdots, x_r)$ has nonzero image in R, we get that the column matrix $[D_s]_{r+1 \leq s \leq n}$ (where D_s is defined in the proof of (3.2)) is mapped to an $(n-r)\times n$ matrix under $\mathcal{D} \to E_0^n$ that contains an $(n-r)\times(n-r)$ nonzero scalar submatrix $\overline{\partial(h_1, \cdots, h_r)/\partial(x_1, \cdots, x_r)} \cdot I_{n-r}$. Thus $\mathcal{G}$ has dimension at least $n-r$ over E_0, which shows that $\mathcal{D}=\mathcal{G}$. □

Corollary 7. *Let $\mathcal{G}$ denote the vector space over E_0 generated by the D_I. Then $\mathcal{G}$ is a restricted Lie algebra of derivations over E_0. Furthermore, $R[\mathcal{G}]=\mathrm{End}_{R_1}(R)$.*

Proof. $\mathcal{D}$, the space of derivations on E_0, is a restricted Lie algebra of derivations over E_0. By (3.6), $\mathcal{D}=\mathcal{G}$. Both $R[\mathcal{G}]$ and $\mathrm{End}_{R_1}(R)$ are locally free R_1 modules ([1], page 86, exercise 16 and page 99, exercise 5). By (1.9) and a rank argument $R[\mathcal{G}]=\mathrm{End}_{R_1}(R)$. □

Corollary 8. *Let $t \in E_0$. If $t^{-1}D_I t \in R$ for all I, then $D_I(t)=0$ for all I.*

Proof. Let $W=\{(t^{-1}D_I t): t \in E_0$ and $t^{-1}D_I t \in R$ for all $I\}$. Since the units of R are the nonzero elements of k, by (1.2), (1.4), (3.2) and (3.7), $W=0$. □

§4. The Calculation of $Cl(S_1)$

1. For each $(r+2)$-tuple $J=(i_1, \cdots, i_{r+2})$ of integers with $1 \leqq i_j \leqq n$, let D_J be the derivation on $k(x_1, \cdots, x_n)$ defined by $D_J=\partial(\ , h_1, \cdots, h_{r+1})/\partial(x_{i_1}, \cdots, x_{i_{r+2}})$. Since $D_J(h_i)=0$ for $1 \leqq i \leqq r$, D_J induces a derivation on E_0, which we will also denote by D_J.

Lemma 2. $R \cap (\bigcap_J D_J^{-1}(0))=S_1$.

Proof. Similar to (3.2). □

Proposition 3. *Let W_0 be the additive group generated by $\{(t^{-1}D_J t): t \in R$ is* *irreducible and homogeneous and* $t^{-1}D_J t \in R$ *for each* $J\}$. *Then* $Cl(S_1)$ *is isomorphic to a subgroup of* W_0.

Proof. Follows from (1.7) and (4.2). □

4. Recall that $\bar{h}_{r+1}=\bar{u}_1\cdots\bar{u}_q$ is a product of q distinct irreducible elements in R and $\deg(h_{r+1})\neq 0 \pmod p$.

Lemma 5. *For each* $i=1, \cdots, q-1$, *let* Q_i *be the height one prime ideal in* S_1 *generated by* x_{n+1} *and* u_i. *Then the classes of the* Q_i *generate a subgroup of* $Cl(S_1)$ *of order* p^{q-1}.

Proof. By (4.3), $Cl(S_1)\to W_0$ is an injection. By (1.3), Q_i maps to $(\bar{u}_i^{-1}D_J(\bar{u}_i))$ in W_0.

Claim. The elements $(\bar{u}_i^{-1}D_J(\bar{u}_i))$, $1\leqq i\leqq q-1$, are F_p-independent; F_p the prime subfield of k.

Suppose $e_i\in F_p$, $1\leqq i\leqq q-1$, such that $\sum_i e_i(\bar{u}_i^{-1}D_J\bar{u}_i)=0$. Let $H=\prod_i u_i^{e_i}$. Then $D_J(\bar{H})=0$ for each J, which implies by (4.2) that $E_1\subset E_1(\bar{H})\subset F_1=E_1(\bar{h}_{r+1})$. If $\bar{H}\notin E_1$, then $E_1(\bar{H})=E_1(\bar{h}_{r+1})$, which implies there exists $\alpha_j\in k^{[n]}$ such that $\bar{\alpha}_p^p\bar{H}=\sum_{j=0}^{p-1}\bar{\alpha}_j^p\bar{h}_{r+1}^j$. Since H and h_{r+1} are homogeneous elements and P a homogeneous ideal in $k^{[n]}$, we may assume that the α_j are homogeneous polynomials as well. Since $\deg(\alpha_j^p h_{r+1}^j)=j(\deg(h_{r+1})) \pmod p$ and $\deg(h_{r+1})\neq 0 \pmod p$, it follows $\bar{\alpha}_0^p\bar{H}=\bar{\alpha}_{j_0}^p\bar{h}_{r+1}^{j_0}$ for some $j_0=0, 1, \cdots, p-1$. If $j_0\neq 0$, then this implies $\bar{u}_q\in E_1$, which contradicts the irreducibility of $\bar{u}_q$ in R. If $j_0=0$, then $\bar{H}$ must in fact belong to E_1. But if $\bar{H}\in E_1$, then each $e_i=0 \pmod p$. This proves the claim and hence the lemma. □

Theorem 6. $Cl(S_1)$ *is a direct sum of* $q-1$ *copies of* Z/pZ.

Proof. By (4.3) and (4.5) it is enough to show that W_0 has order at most p^{q-1}. Let $\bar{u}\in R$ be irreducible homogeneous such that $\bar{w}_J=\bar{u}^{-1}D_J(\bar{u})\in R$ for each J. Then $\bar{u}$ divides the images in R of all of the maximal minors of the matrix

$$M=\begin{pmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} \\ \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_{r+1}}{\partial x_1} & \frac{\partial h_{r+1}}{\partial x_2} & \cdots & \frac{\partial h_{r+1}}{\partial x_n} \end{pmatrix}.$$

By Euler's formula, $\bar{u}$ divides in R the images of the maximal minors of the matrix obtained from M by replacing the j-th column by the $(r+2)\times 1$ vector

$$\begin{pmatrix} d\cdot u \\ 0 \\ \vdots \\ 0 \\ d'\cdot h_{r+1} \end{pmatrix}$$

where $d=\deg(u)$ and $d'=\deg(h_{r+1})$. Therefore $\bar{u}$ is a factor of $\bar{h}_{r+1}$ or $\bar{u}$ is a common factor of the images in R of each of the maximal minors of the matrix

$$\begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} \\ \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_r}{\partial x_1} & \frac{\partial h_r}{\partial x_2} & \cdots & \frac{\partial h_r}{\partial x_n} \end{bmatrix}.$$

The latter implies $\bar{u}^{-1}D_I\bar{u}\in R$ for each of the derivations D_I described in (3.1), which by (3.2) and (3.8) implies $\bar{u}\in R_1$. Since $\bar{u}$ is irreducible in R, it must be that $\bar{u}$ is a k^* multiple of one of the $\bar{u}_i$. By (4.3) we get W_0 is generated by $(\bar{u}_i^{-1}D_J\bar{u}_i)$, $1\leqq i\leqq q$. We have for each J, $\sum_{i=1}^{q}\bar{u}_i^{-1}D_J\bar{u}_i=\bar{h}_{r+1}^{-1}D_J(\bar{h}_{r+1})=0$. Therefore W_0 is generated by $(\bar{u}_i^{-1}D_J\bar{u}_i)$, $1\leqq i\leqq q-1$. Since W_0 is a p-group, the order of W_0 is at most p^{q-1}. □

Corollary 7. *$Cl(X_1)$ is generated by the codimension one cycles on X_1 defined by $u_i=0$, $1\leqq i\leqq q-1$.*

Proof. By the proof of (4.5) and by theorem (4.6). □

§ 5. The Calculation of $Cl(S_m)$

1. Given $\alpha\in F_m$, by (2.4) there exist unique $\alpha_i\in E_0$, such that $\alpha=\sum_{i=0}^{p^m-1}\alpha_i^{p^m}\bar{h}_{r+1}^i$. For each derivation D_J defined in (4.1) and each integer $m\geqq 0$, define a mapping $D_J^{(m)}$ on F_m by the formula

$$D_J^{(m)}(\alpha)=\sum_{i=0}^{p^m-1}(D_J(\alpha_i))^{p^m}\bar{h}_{r+1}^i.$$

Lemma 2. *$D_J^{(m)}$ is a derivation on F_m.*

Proof. $D_J^{(m)}$ is clearly additive. Given $\alpha=\sum_{i=0}^{p^m-1}\alpha_i^{p^m}\bar{h}_{r+1}^i$ and $\beta=\sum_{i=0}^{p^m-1}\beta_i^{p^m}\bar{h}_{r+1}^i$

in F_m, we must show that $D_J^{(m)}(\alpha\beta)=\alpha D_J^{(m)}\beta+\beta D_J^{(m)}\alpha$. We argue by induction on the number of nonzero coefficients appearing in α plus the number of nonzero coefficients appearing in β.

Suppose this sum is 2. Then $\alpha=\alpha_i^{p^m}\bar{h}_{r+1}^i$ and $\beta=\beta_j^{p^m}\bar{h}_{r+1}^j$ for some α_i, $\beta_j \in E_0$ and nonnegative integers i and j. Then $D_J^{(m)}(\alpha\beta)=D_J^{(m)}(\alpha_i\beta_j)^{p^m}h_{r+1}^{i+j}=(D_J(\alpha_i\beta_j))^{p^m}\bar{h}_{r+1}^{i+j}=(\alpha_i D_J\beta_j+\beta_j D_J\alpha_i)^{p^m}\bar{h}_{r+1}^{i+j}=\alpha_i^{p^m}\bar{h}_{r+1}^i D_J^{(m)}(\beta_j^{p^m}\bar{h}_{r+1}^j)+\beta_j^{p^m}\bar{h}_{r+1}^j D_J^{(m)}(\alpha_i^{p^m}\bar{h}_{r+1}^i)=\alpha D_J^{(m)}\beta+\beta D_J^{(m)}\alpha$.

Now assume that the total number of nonzero coefficients appearing in α and β is greater than 2. Let j_0 be the highest power of $\bar{h}_{r+1}$ with nonzero coefficient in β. Then $D_J^{(m)}(\alpha\beta)=D_J^{(m)}(\alpha(\beta-\beta_{j_0}^{p^m}\bar{h}_{r+1}^{j_0}))+D_J^{(m)}(\alpha\beta_{j_0}^{p^m}\bar{h}_{r+1}^{j_0})$, which by the induction hypothesis and the additivity of $D_J^{(m)}$, equals to $\alpha D_J^{(m)}\beta+\beta D_J^{(m)}\alpha$. □

Lemma 3. *S_{m+1} is the fixed subring of the derivations $D_J^{(m)}$ acting on S_m.*

Proof. Given $\alpha\in S_m$, $\alpha=\sum\alpha_i^{p^m}\bar{h}_{r+1}^i$, $\alpha_i\in R$. $D_J^{(m)}(\alpha)=0$ for each J if and only if $D_J(\alpha_i)=0$ for each J and $0\leqq i\leqq p^m-1$. By (4.2) we get $D_J^{(m)}(\alpha)=0$ for each J implies each $\alpha_i\in S_1$ and hence $\alpha\in S_{m+1}$. □

Definition 4. For each integer $m\geqq 1$, let W_m be the additive group generated by $\{(t^{-1}D_J^{(m)}t): t$ is a homogeneous element in S_m and $t^{-1}D_J^{(m)}t\in S_m$ for each $J\}$.

Lemma 5. *If $(u_J)\in W_0$ then $(u_J^{p^m})\in W_m$ and the mapping $(u_J)\to(u_J^{p^m})$ from W_0 to W_m is an isomorphism.*

Proof. If $t\in E_0$ and $t^{-1}D_Jt=u_J\in R$, then $t^{p^m}\in F_m$ and $t^{-p^m}D_J^{(m)}(t^{p^m})=(t^{-1}D_Jt)^{p^m}=u_J^{p^m}\in R_m\subset S_m$. This also shows that $(u_J)\to(u_J^{p^m})$ defines an injection from W_0 to W_m.

Suppose $t'\in S_m$ is homogeneous. Since $\deg(h_{r+1})\neq 0 \pmod p$, $t'=\bar{\alpha}^{p^m}\bar{h}_{r+1}^i$, where $\alpha\in k^{[n]}$ is homogeneous and i is a nonnegative integer. If $(t')^{-1}D_J^{(m)}(t')$ is a nonzero element of S_m, then $(t')^{-1}D_J^{(m)}(t')=(\alpha^{-1}D_J\alpha)^{p^m}\in S_m\cap E_m=R_m$. Thus $\alpha^{-1}D_J\alpha\in R$. Therefore $W_0\to W_m$ is also a surjection. □

Theorem 6. *$Cl(S_m)$ is a direct sum of $q-1$ copies of Z/p^mZ, generated by the height one primes $Q_i^{(m)}=\bar{h}_{r+1}S_m+\bar{u}_i^{p^m}S_m$ in S_m, $1\leqq i\leqq q-1$.*

Proof. For each $i=1,\cdots,q-1$ and positive integer m, let $P_i^{(m)}=\bar{h}_{r+1}^pS'_m+\bar{u}_i^{p^{m+1}}S'_m$. Then $Q_i^{(m)}\cap S_{m+1}=Q_i^{(m+1)}$ and $Q_i^{(m+1)}\cap S'_m=P_i^{(m)}$. Also, the ramification index of $Q_i^{(m)}$ over $Q_i^{(m+1)}$ is 1 and the ramification index of $Q_i^{(m+1)}$ over $P_i^{(m)}$ is p. Thus $\phi_m: Cl(S_{m+1})\to Cl(S_m)$ sends $Q_i^{(m+1)}$ to $Q_i^{(m)}$ and $\phi'_m: Cl(S'_m)\to Cl(S_{m+1})$ sends $P_i^{(m)}$ to $pQ_i^{(m+1)}$.

By (1.2), (1.3) and (1.6) it follows that the $\ker\phi_m$ is isomorphic to a subgroup of W_m. By (5.5) and the proof of (4.6), we have the $\ker\phi_m$ has order at most p^{q-1}.

Proceeding by induction we have that the primes $Q_i^{(m)}$ generate $Cl(S_m)$ and are each of order p^m, hence the same is true of the primes $P_i^{(m)}$ in S'_m. Since $\phi'_m: Cl(S'_m)\to Cl(S_{m+1})$ is injective by (3.5), we see that the elements $p^m Q_i^{(m+1)}$ are a Z/pZ-basis for $\ker\phi_m$. Since the ramification index of $Q_i^{(m)}$ over $Q_i^{(m+1)}$ is 1, ϕ_m is surjective, and the conclusion of the theorem follows. □

Corollary 7. *$Cl(X_m)$ is generated by the codimension one cycles on X_m defined by $u_i=0$, $1\leqq i\leqq q-1$.*

Proof. By (2.2) and (5.6). □

8. Let A be a Krull ring of characteristic $p\neq 0$ with quotient field k. Let $h_1, \cdots, h_{r+1}\in A[x_1, \cdots, x_n]$ satisfy all of the conditions of (2.1) and (2.5) as elements of $k[x_1, \cdots, x_n]$. Let $T=A[x_1, \cdots, x_{n+1}]/(h_1, \cdots, h_r, x_{n+1}^p-h_{r+1})$. Assume T is a Krull ring.

Corollary 9. *Let T be as in* (5.8), *$Cl(T)$ is isomorphic to $Cl(A)\oplus\left(\bigoplus_{i=1}^{q-1} Z/pZ\right)$.*

Proof. Let U be the multiplicative set of nonzero elements of A. Since A is a Krull ring, so is $S=U^{-1}T$. By a theorem of Nagata ([7], theorem 6.3, page 21), we obtain an exact sequence

$$0 \longrightarrow Cl(A) \longrightarrow Cl(T) \longrightarrow Cl(S) \longrightarrow 0.$$

The surjection $Cl(T)\to Cl(S)$ is a split surjection. Thus $Cl(T)\cong Cl(A)\oplus Cl(S)$. □

Corollary 10. *Let R be as in* (2.1) *and assume $g\in k^{[n]}$ is homogeneous such that $\bar{g}$ is irreducible in R. Let $Y\subset A_k^m$ be defined by the equations $h_1=\cdots=h_r=x_{n+1}^{p^m}-g=0$, where m is a positive integer. Then the coordinate ring of Y is factorial if and only if Y is regular in codimension one.*

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