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Note on the Equivariant K-Theory Spectrum

To the memory of Professor Masahisa Adachi

By

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Introduction

Let G be a finite group. For any finite G-CW complex X, denote by $KO_G(X)$ the equivariant real K-theory of X and $\mathrm{Sph}_G(X)$ the group of stable equivalence classes of spherical G-fibrations over X.

In [9] and [10], we showed that there are connective G-spectra kO_G and kF_G representing the theories $KO_G(-)$ and $\mathrm{Sph}_G(-)$ respectively. and a map of G-spectra $kO_G \to kF_G$ inducing the equivariant J-homomorphism

$$J_G : KO_G(X) \longrightarrow \mathrm{Sph}_G(X) .$$

(Cf. Theorem 4.3 of [10].) It follows, by [8], that $KO_G(-)$ and $\mathrm{Sph}_G(-)$ are equipped with transfers for G-equivariant bundles which are compatible with J_G.

In this note, we will show that kO_G is, in fact, a (-1)-connected cover of the G-spectrum representing the periodic KO_G-theory; that is, the equivariant infinite loop structures defined by kO_G and by the Bott periodicity are equivalent to each other. Thus, the transfer associated with kO_G coincides with the transfer coming from the Bott periodicity.

We also apply our results to deduce the equivariant Adams conjecture for arbitrary finite groups from one and two dimensional cases proved by Hauschild-Waner [4]. (Compare [2] and [7].)

§ 1. Construction of the Periodic KO_G-Theory Spectrum

This section is devoted to the proof of the following theorem. (For the definition of *G-spectrum*, see [6].)

Theorem 1. *There are a G-spectrum KO_G representing the periodic KO_G-theory and a map of G-spectra $l_G : kO_G \to KO_G$, defined in the stable category, inducing an equivalence between kO_G and the (-1)-connected cover of KO_G.*

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We begin by recalling from [9] and [11] the construction of kO_G. For each $n\geqq 0$, let O_n be the real orthogonal group regarded as a G-category with unique object and with trivial G-action. Then the union $\coprod_{n\geqq 0}O_n$ can be regarded as a bipermutative G-category with respect to Whitney sum and tensor product operations. By Theorem A of [11], we obtain an E_∞ ring G-spectrum

$$kO_G=E_G\Big(\coprod_{n\geqq 0}O_n\Big)$$

indexed on $\mathcal{A}=\{V^n; n\geqq 0\}$, where V is a real Spin G-module of dimension $8n$ such that every irreducible representation of G can be embedded in V.

For brevity, write $kO_G=\{E_n; n\geqq 0\}$, so that $E_0=\Omega^\infty kO_G$. We showed in [9, § 3] that E_0 is G-equivalent to the homology theoretic group completion of the G-monoid $\coprod_{n\geqq 0} BO_n(G)$ where $BO_n(G)$ is a classifying space for n-dimensional real G-vector bundles. Thus we have

$$KO_G(X)=[X^+, E_0]^G=[\Sigma^\infty X^+, kO_G]^G$$

for every finite G-CW complex X.

As in [11], let $\mathcal{C}$ be an E_∞ G-operad with $\mathcal{C}_j=|\mathrm{Cat}(EG, E\Sigma_j)|$ where EG denotes the translation category of G, that is, $\mathrm{ob}\,EG=G$ and $\mathrm{mor}\,EG=G\times G$ (and similarly for $E\Sigma_j$). Then kO_G comes equipped with an E_∞ ring structure defined by external $\mathcal{C}$-action

$$\xi_j: \mathcal{C}_j^+\wedge E_{n_1}\wedge\cdots\wedge E_{n_j}\to E_{n_1+\cdots+n_j};\quad j\geqq 0,\ n_1,\cdots, n_j\geqq 0\,.$$

Let ι_2 be the element of $\mathcal{C}_2$ represented by the constant $EG\to E\Sigma_2$ with value 1, and let $\mu: E_0\wedge E_0\to E_0$ be a multiplication defined by

$$\mu(x\wedge y)=\xi_2(\iota_2\wedge x\wedge y)\,.$$

One easily verifies that μ is strictly associative and homotopy commutative.

Let SV be the onepoint compactification of V and let $b: SV\to E_0$ be a G-map representing the Bott class in $\widetilde{KO}_G(SV)$. We define a G-prespectrum $D=\{D_n; n\geqq 0\}$ as follows. For each $n\geqq 0$, put $D_n=E_0$ and define its structure map $\delta: D_n\wedge SV\to D_{n+1}$ to be the composite

$$D_n\wedge SV\xrightarrow{1\wedge b} E_0\wedge E_0\xrightarrow{\mu} E_0=D_{n+1}\,.$$

By the Bott periodicity theorem, the homomorphism $\widetilde{KO}_G(X)\to\widetilde{KO}_G(X\wedge SV)$ induced by the adjoint $E_0\to\Omega^V E_0$ to δ is an isomorphism (cf. [1]). Hence D is an Ω-G-prespectrum and represents the periodic KO_G-theory.

Let KO_G denote the G-spectrum associated to D. We construct a map of G-spectra $kO_G\to KO_G$ by using the equivariant version of the up and across theorem. (Compare the proof of theorem 2.4 in [11].)

Let $\mathcal{A}\oplus\mathcal{A}$ denote the indexing set $\{V^m\oplus V^n; m, n\geqq 0\}$ in the universe

$V^\infty \oplus V^\infty$, and define a G-prespectrum $X=\{X_{m,n}\}$ indexed on $\mathcal{A}\oplus\mathcal{A}$ as follows. For $m, n\geqq 0$, put

$$X_{m,n}=E_n$$

and define G-maps $\chi' : X_{m,n}\wedge SV\to X_{m+1,n}$, $\chi'' : X_{m,n}\wedge SV\to X_{m,n+1}$ by

$$\chi'(x\wedge v)=\xi_2(\iota_2\wedge x\wedge b(v)), \qquad \chi''(x\wedge v)=\varepsilon(x\wedge v)$$

where ε denotes the structure map of kO_G.

By the property of external $\mathcal{C}$-action (cf. Definition 1 2 (1.3) of [11]), the following diagram is commutative:

$$\begin{tikzcd} X_{m,n}\wedge SV\wedge SV \arrow[r, "\chi'\wedge 1"] \arrow[d, "(\chi''\wedge 1)(1\wedge T)"'] & X_{m+1,n}\wedge SV \arrow[d, "\chi''"] \\ X_{m,n+1}\wedge SV \arrow[r, "\chi'"'] & X_{m+1,n+1} \end{tikzcd}$$

where T denotes the transposition $u\wedge v\to v\wedge u$. It follows that $X_{m,n}$ together with χ' and χ'' determine a G-prespectrum $X\in G\mathcal{P}(\mathcal{A}\oplus\mathcal{A})$.

Let i^* and j^* be the functors $G\mathcal{P}(\mathcal{A}\oplus\mathcal{A})\to G\mathcal{P}\mathcal{A}$ induced by the embeddings of V^∞ into the first and the second factor of $V^\infty\oplus V^\infty$ respectively, and let L be the functor which takes G-prespectra to their associated G-spectra. Then X satisfies the following properties:

(1) $Li^*X=KO_G$, $Lj^*X=kO_G$.

(2) The natural map $Li^*X\to i^*LX$ is a G-equivalence.

Moreover, there is a G-equivalence $j^*LX\cong i^*LX$ uniquely determined in the stable category. (See Theorem 1.7 of [6, Chapter II].) Thus we get a map of G-spectra $l_G: kO_G\to KO_G$ defined by the composite

$$kO_G=Lj^*X\to j^*LX\cong i^*LX\cong Li^*X=KO_G\,.$$

It is easily verified that l_G induces isomorphisms $\pi_n^H kO_G\cong\pi_n^H KO_G$ for all subgroups H and $n\geqq 0$. Therefore, kO_G is equivalent to the (-1)-connected cover of KO_G.

Note. The arguments above remain true if KO_G is replaced by K_G or KR_G. Thus we have complex and Real analogs of Theorem 1. Moreover, l_G can be taken to be a map of H_∞ ring spectra, which we hope to discuss elsewhere.

§2. The Equivariant Adams Conjecture

We now give a proof of the equivariant Adams conjecture formulated by McClure [7]. (For a slightly different formulation, see Fiedorowicz-Hauschild-May [2].)

For any finite G-CW complex X and every prime p, let $\mathrm{Sph}_G^{(p)}(X)$ denote the quotient of $\mathrm{Sph}_G(X)_{(p)}$ under the equivalence relation induced by *stable p-equivalence* (cf. [7, § 1]). Let $J_G^{(p)}$ be the composite

$$KO_G(X)_{(p)} \xrightarrow{J_G} \mathrm{Sph}_G(X)_{(p)} \longrightarrow \mathrm{Sph}_G^{(p)}(X)\,.$$

Theorem 2. *Let X be a finite G-CW complex, and $x \in KO_G(X)$. If k is prime to p and the order of G then $J_G^{(p)}(\psi^k x - x) = 0$.*

We prove this theorem by generalizing the arguments of Hashimoto [3]. (See also Nishida [8] and Kono [5].)

Observe first that it suffices to prove the case where x is the class of an odd dimensional G-vector bundle. Let ξ be a $(2m+1)$-dimensional real G-vector bundle over X and E the total space of the associated principal G-O_{2m+1} bundle. For a closed subgroup K of O_{2m+1}, let $\alpha: RO(K) \to KO_G(E/K)$ be a homomorphism which takes K-vector spaces V to the G-vector bundles $E \times_K V \to E/K$, and let

$$\pi_!: KO_G(E/K) \to KO_G(X)\,, \qquad \pi_!: \mathrm{Sph}_G(E/K) \to \mathrm{Sph}_G(X)$$

be the transfers, in the sense of [8], for the projection $\pi: E/K \to X$. Theorem 1 implies that the first $\pi_!$ is identical with the transfer coming from the Bott periodicity.

By the argument of [8, Proposition 3] and the fact that J_G is induced by a map of G-spectra, we obtain

Proposition 3. *The following diagram is commutative*

$$\begin{array}{ccccc}
RO(K)_{(p)} & \xrightarrow{\ \alpha\ } & KO_G(E/K)_{(p)} & \xrightarrow{\ J_G^{(p)}\ } & \mathrm{Sph}_G^{(p)}(E/K) \\
\Big\downarrow{\scriptstyle \mathrm{ind}_K^L} & & \Big\downarrow{\scriptstyle \pi_!} & & \Big\downarrow{\scriptstyle \pi_!^{(p)}} \\
RO(L)_{(p)} & \xrightarrow[\ \alpha\ ]{} & KO_G(X)_{(p)} & \xrightarrow[\ J_G^{(p)}\ ]{} & \mathrm{Sph}_G^{(p)}(X)\,,
\end{array}$$

where $L = O_{2m+1}$ and $\pi_!^{(p)}$ is induced from the transfer by passing to quotient.

We are now ready to prove Theorem 2. Put $K = O_2 \times O_{2m-1}$ and let ι be the identity representation of $L = O_{2m+1}$, so that $\xi = \alpha(\iota)$. By [3, Proposition 5], there exist a one dimensional representation ν of L and a two dimensional representation μ of K such that

$$\iota = \mathrm{ind}_K^L \mu + \nu\,.$$

Since Theorem 2 is true for one and two dimensional G-vector bundles by Hauschild-Waner [4], we have

$$\begin{aligned} J_G^{(p)}(\psi^k-1)(\xi) &= J_G^{(p)}(\psi^k-1)\alpha(\iota) \\ &= J_G^{(p)}(\psi^k-1)\alpha(\mathrm{ind}_K^L\mu)+J_G^{(p)}(\psi^k-1)\alpha(\nu) \\ &= J_G^{(p)}(\psi^k-1)\pi_!\alpha(\mu)+J_G^{(p)}(\psi^k-1)\alpha(\nu) \\ &= \pi_! J_G^{(p)}(\psi^k-1)\alpha(\mu)+J_G^{(p)}(\psi^k-1)\alpha(\nu) \\ &= 0\,. \end{aligned}$$

This completes the proof of Theorem 2.

References

[1] Atiyah, M.F., Bott periodicity and the index of elliptic operators, *Quart. J. Math. Oxford Ser.* (2), **19** (1968), 113-140.

[2] Fiedorowicz, Z., Hauschild, H. and May, J.P., Equivariant algebraic K-theory, *Lecture Notes in Math.*, **967** (1982), 23-80.

[3] Hashimoto, S., The transfer map in the KR_G-theory, *Osaka J. Math.*, **18** (1981), 501-507.

[4] Hauschild, H. and Waner, S., The equivariant Dold theorem mod k and the Adams conjecture, *Illinois J. Math.*, **27** (1983), 52-66.

[5] Kono, A., On the order of certain elements of $J(X)$ and the Adams conjecture, *Publ. RIMS, Kyoto Univ.*, **17** (1981), 557-564.

[6] Lewis, G., May, J.P. and Steinberger, M., Equivariant stable homotopy theory, *Lecture Notes in Math.*, **1213** (1986).

[7] McClure, J.E., On the groups JO_GX. I, *Math. Z.*, **183** (1983), 229-253.

[8] Nishida, G., The transfer homomorphism in equivariant generalized cohomology theories, *J. Math. Kyoto Univ.*, **18** (1978), 435-451.

[9] Shimakawa, K., Infinite loop G-spaces associated to monoidal G-graded categories, *Publ. RIMS, Kyoto Univ.*, **25** (1989), 239-262.

[10] ———, Mackey systems of monoidal categories and G-spectra, *K-Theory*, **5** (1992), 373-394.

[11] ———, An E_∞ ring G-spectrum representing the equivariant algebraic K-theory of a bipermutative G-category, *Quart. J. Math. Oxford Ser.* (2), to appear.