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Self-avoiding Paths on the Three Dimensional Sierpinski Gasket

Dedicated to Professor H. Ezawa on his 60th birthday

By

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Abstract

We study self-avoiding paths on the three-dimensional pre-Sierpinski gasket. We prove the existence of the limit distribution of the scaled path length, the exponent for the mean square displacement, and the continuum limit. We also prove that the continuum-limit process is a self-avoiding process on the three-dimensional Sierpinski gasket, and that a path almost surely has infinitely fine creases.

§ 1. Introduction

The three-dimensional pre-Sierpinski gasket is a pre-fractal which we introduce as a three-dimensional analog of the pre-Sierpinski gasket. Let $O=(0, 0, 0)$, $a_0=(1/2, \sqrt{3}/6, \sqrt{6}/3)$, $b_0=(1/2, \sqrt{3}/2, 0)$, $c_0=(1, 0, 0)$, and let F_0 be the set of vertices and edges of the tetrahedron $Oa_0b_0c_0$. We define a sequence of graphs $F_0, F_1, F_2, \cdots$, inductively by

$$F_{n+1}=F_n\cup(F_n+2^n a_0)\cup(F_n+2^n b_0)\cup(F_n+2^n c_0), \qquad n=0, 1, 2, \cdots,$$

where, $A+a=\{x+a \mid x\in A\}$, and $kA=\{kx \mid x\in A\}$. We call $F=\bigcup_{n=0}^{\infty} F_n$ the three-dimensional pre-Sierpinski gasket. We denote the set of vertices in F by G, and put $a_n=2^n a_0$, $b_n=2^n b_0$, $c_n=2^n c_0$.

Let $\boldsymbol{Z}_+=\{0, 1, 2, \cdots\}$ and define the set of self-avoiding paths W_0 on G to be the set of mappings $w: \boldsymbol{Z}_+\to G$ such that there exists $L(w)\in\boldsymbol{Z}_+\cup\{\infty\}$ for which

$$w(i)=w(L(w)), \qquad i\geqq L(w),$$

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$$w(i_1)\neq w(i_2), \qquad 0\leqq i_1<i_2\leqq L(w),$$

$$|w(i)-w(i+1)|=1, \qquad 0\leqq i\leqq L(w)-1, \quad \text{and}$$

$$\overline{w(i)w(i+1)}\subset F, \qquad 0\leqq i\leqq L(w)-1.$$

We call $L(w)$ the length of the path w.

In previous papers [2, 3, 4], we studied the self-avoiding paths on the (two-dimensional) Sierpinski gasket. Here, we study the self-avoiding paths on the three-dimensional Sierpinski gasket. In the case of the two-dimensional Sierpinski gasket, a self-avoiding path is allowed to pass through a unit triangle at most once, while in the case of the three-dimensional Sierpinski gasket, it is allowed to pass through a unit tetrahedron more than once. One might suspect that this fact affects properties that enabled the detailed analyses in the two-dimensional case. We will show in this paper that despite such complexties, we can carry on our analyses in the three-dimensional case.

Define $W^{*(n)}\subset W_0$ by

$$W^{*(n)}=\{w\in W_0 \mid w(0)=O,\ w(L(w))=a_n,\ w(\boldsymbol{Z}_+)\subset F_n\},$$

and let

$$Z_n^*(\beta)=\sum_{w\in W^{*(n)}}\exp(-\beta L(w)), \qquad \beta\in\boldsymbol{R},\ n\in\boldsymbol{Z}_+.$$

In Section 3 we prove the following.

Theorem 1.1. *There exists a constant β_c such that*

$$\lim_{n\to\infty} Z_n^*(\beta)=0, \qquad \beta>\beta_c,$$

$$\lim_{n\to\infty} Z_n^*(\beta)=x_c, \qquad \beta=\beta_c,$$

$$\lim_{n\to\infty} Z_n^*(\beta)=\infty, \qquad \beta<\beta_c.$$

This theorem suggests that in terms of statistical mechanics, $\beta>\beta_c$ is the low temperature regime, $\beta<\beta_c$ is the high temperature regime, and β_c is the critical point.

The asymptotic behavior of the partition function Z_n^* is related to the asymptotic distribution of the path length L. In fact, in Section 4 we prove the following. Let μ_n^* be a probability measure on $W^{*(n)}$ defined by

$$\mu_n^*[w]=Z_n^*(\beta_c)^{-1}\exp(-\beta_c L(w)), \qquad w\in W^{*(n)}.$$

Theorem 1.2. *There exists a constant λ satisfying $2<\lambda<3$ such that the distribution of 'scaled path length' $\lambda^{-n}L$ under μ_n^* converges to a probability measure p^* on $\boldsymbol{R}$ as $n\to\infty$. The measure p^* has a C^∞ density ρ, which satisfies $\rho(x)=0$, $x\leqq 0$, and $\rho(x)>0$, $x>0$.*

Numerically, $\lambda=2.7965\cdots$, to be compared with the corresponding constant λ_{SG_2} for the (two-dimensional) Sierpinski gasket, $\lambda_{SG_2}=(7-\sqrt{5})/2=2.381966\cdots$.

In Section 5, we study the continuum limit construction of self-avoiding process.

Let $\tilde{F}_n=2^{-n}F_n$, $n=0, 1, 2, \cdots$, and define the finite three-dimensional Sierpinski gasket $\tilde{F}$ by $\tilde{F}=\overline{\bigcup_{n=0}^{\infty}\tilde{F}_n}$. $\tilde{F}$ is a graph obtained by giving a substructure to a unit tetrahedron $Oa_0b_0c_0$. Let

$$C=\{w\in C([0, \infty)\to\tilde{F})\mid w(0)=O, \lim_{t\to\infty}w(t)=a_0\}.$$

C is a complete separable metric space with the metric

$$d(u, v)=\sup_{t\in[0,\infty)}|u(t)-v(t)|, \qquad u, v\in C.$$

Define $\gamma:\bigcup_n W^{*(n)}\to C$ as follows: For $u\in W^{*(n)}$, $\gamma u(j)=2^{-n}u(j)$ for $j\in\boldsymbol{Z}_+$, and for $t\notin\boldsymbol{Z}_+$, $u(t)$ is defined by linear interpolation. Also define time-scale transformation $U_n(\lambda): C\to C$, $n\in\boldsymbol{N}$, by $(U_n(\lambda)w)(t)=w(\lambda^n t)$, $w\in C$.

Denote by P_n the image measure of μ_n^* induced by $U_n(\lambda)\circ\gamma$.

Theorem 1.3. *P_n converges to a probability measure P on C weakly as $n\to\infty$. The stochastic process defined by P is almost surely self-avoiding, and the Hausdorff dimension of the trajectory, $\{w(t)\mid 0\leq t<\infty\}$, is almost surely greater than one.*

See Theorem 5.5, Theorem 5.15, and Theorem 5.16 in Section 5 for the proof.

Since P_n is supported on piecewise linear curves, the Hausdorff dimension of a curve is almost surely 1 with respect to P_n. The statement on the Hausdorff dimension in Theorem 1.3 implies that the continuum limit $n\to\infty$, is a nontrivial limit, and that with P-probability one, we have self-avoiding paths with infinitely fine creases.

The two ingredients for the proof of the results are the convergence of the distribution of crossing times of tetrahedrons, obtained from the results in Section 4, and the considerations about the distribution of the shape of the paths. See Proposition 5.2 for more properties on the crossing time distributions, and Theorem 5.11 and Proposition 5.14 for more on the distribution of the shape of the paths.

In Section 6 we consider a set of paths on the pre-Sierpinski gasket with a fixed length, instead of paths with fixed end points. Let $W^{(0)}=\{w\in W_0\mid w(0)=O\}$, and for each $k\in\boldsymbol{Z}_+$, let $N(k)$ be the number of elements in $\{w\in W^{(0)}\mid L(w)=k\}$. Let $\tilde{P}_k$, $k\in\boldsymbol{Z}_+$, be probability measures on $W^{(0)}$ defined by,

$$\tilde{P}_k(A)=N(k)^{-1}\#\{w\in A\mid L(w)=k\}, \qquad A\subset W^{(0)}.$$

For $w\in W^{(0)}$, let $\|w\|=\max\{|w(k)|\,;\ k=1, 2, \cdots, L(w)\}$, where $|\cdot|$ is the (Euclidean) length in $\boldsymbol{R}^3$. Put $\kappa=\log\lambda/\log 2$, where λ is as in Theorem 1.2.

Theorem 1.4. (1) $\lim_{k\to\infty} k^{-1}\log N(k)=\beta_c$.

(2) *There exists a positive constant* α *such that*

$$\lim_{k\to\infty}\tilde{P}_k[\|w\|<(\log k)^{-\alpha}k^{1/\kappa} \text{ or } \|w\|>(\log k)^{\alpha}k^{1/\kappa}]\exp((\log k)^2)=0.$$

(3) $$\lim_{k\to\infty}(\log k)^{-1}\log E^{\tilde{P}_k}[|w(k)|^{s\kappa}]=s, \qquad s>0.$$

This theorem says that the exponent for the mean square displacement of self-avoiding random walk on the three-dimensional Sierpinski gasket is $\log 2/\log\lambda$.

The starting point for the analyses in this paper is the study of renormalization group, which is a dynamical system in a certain parameter space which specifies the path ensemble. Such dynamical system is derived as the response in the parameter space to the change in n. In Section 2, we study the behaviors of this dynamical system. A certain graphical property of the three-dimensional Sierpinski gasket implies that the renormalization group is a finite dimensional dynamical system. (We prefer to call this property the finite ramification of the fractal.)

We would like to mention some previous works in the physics literature. The two-dimensional mapping defined by eq. (2.3) and eq. (2.4) in Proposition 2.1 is given in [1, 7]. In [1], a set of self-avoiding paths on a slightly different fractal is considered, where this mapping is an exact renormalization group recursion relation. In [7], self-avoiding paths on the three-dimensional Sierpinski gasket is studied, where the authors state (without explicit discussion or proof,) that the same mapping becomes 'relevant' (i. e. determines the asymptotic behavior) of the present problem. Numerical estimates of the fixed point ((x_c, y_c) in Proposition 3.1) and the derivative (p, q, and r in Proposition 3.7) of this mapping together with the exponent for mean square displacement are also given in these references. However, their results are based on the assumption (based on numerical studies) of the existence of these quantities and the good limiting behaviors. We believe that this is the first time to give mathematically rigorous proofs. A well-defined statement on the exponent (Theorem 1.4) seems also to have been lacking. We also believe that the limit theorems for path length distribution (Theorem 1.2) and the continuum limit (Theorem 1.3) are new.

We would also like to take this opportunity to note that the reference [1] should also have been included in the reference of [3].

We would like to thank Professor N. Asano and Professor H. Nakajima for bringing our attention to the references on dynamical systems.

§ 2. Recursion Relations for the Generating Functions

Let T denote the set of subgraphs in F which are the translations of F_0. T is a set of the unit tetrahedrons that compose F. For each $w\in W_0$, let $\hat{w}\subset F$ be the curve defined by joining each pair of points $w(i)$ and $w(i+1)$ with an edge of F, for all $i=0, 1, 2, \cdots, L(w)-1$. For each tetrahedron $\Delta\in T$, the intersection $\hat{w}\cap\Delta$ is either empty or one of the following four possibilities: case 1) one edge, case 2) two disjoint edges, case 3) two edges connected at a vertex, and case 4) three edges that constitute a chain. Correspondingly, define $S_i(w)$, $i=1, 2, 3, 4$, $w\in W_0$, as: $S_i(w)=\{\Delta\in T \mid \hat{w}\cap\Delta$ is of case i).$\}$. For each $i=1, 2, 3, 4$, let $s_i(w)$ be the number of the elements of $S_i(w)$. Note that $s_1+2s_2+2s_3+3s_4=L$.

For $n\in \boldsymbol{Z}_+$ and $p\in G$, $q\in G$, define $W^{(n,p,q)}\subset W_0$ by

$$W^{(n,p,q)}=\{w\in W_0 \mid w(0)=p,\ w(L(w))=q,\ w(\boldsymbol{Z}_+)\subset F_n\},$$

and let $W_i^{(n)}$, $i=1, 2, 3, 4$, $n\in\boldsymbol{Z}_+$, be as follows:

$$W_1^{(n)}=\{w\in W^{(n,O,a_n)} \mid w(\boldsymbol{Z}_+)\cap\{b_n, c_n\}=\emptyset\},$$

$$W_2^{(n)}=\{(w_1, w_2)\in W^{(n,O,a_n)}\times W^{(n,b_n,c_n)} \mid w_1(\boldsymbol{Z}_+)\cap w_2(\boldsymbol{Z}_+)=\emptyset\},$$

$$W_3^{(n)}=\{w\in W^{(n,O,a_n)} \mid w(\boldsymbol{Z}_+)\cap\{b_n, c_n\}=\{b_n\}\},$$

$$W_4^{(n)}=\{w\in W^{(n,O,a_n)} \mid \text{there exist two positive integers } i \text{ and } j \text{ such that } i<j,\ w(i)=b_n, \text{ and } w(j)=c_n\}.$$

For a subset W of W_0, let $X(W)$ be the generating function for W defined by

$$X(W)(\vec{x})=\sum_{w\in W}\prod_{i=1}^{4} x_i^{s_i(w)}, \qquad \vec{x}=(x_1, x_2, x_3, x_4)\in \boldsymbol{C}^4. \tag{2.1}$$

To extend the definition of s_i to $W_2^{(n)}$, note that if $w=(w', w'')\in W_2^{(n)}$ and $\Delta\in T$, $(\hat{w}'\cup\hat{w}'')\cap\Delta$ is either empty or one of the four possibilities mentioned above. Therefore, the definition of $S_i(w)$ given above has a natural extension to $w\in W_2^{(n)}\cdot s_i(w)$, $w\in W_2^{(n)}$, is again defined to be the number of the elements of $S_i(w)$.

Let $X_{i,n}(\vec{x})=X(W_i^{(n)})(\vec{x})$, $i=1, 2, 3, 4$.

Proposition 2.1. *The following recursion relation holds:*

$$\vec{X}_{n+1}(\vec{x})=\vec{\Phi}(\vec{X}_n(\vec{x})), \qquad n\in\boldsymbol{Z}_+, \tag{2.2}$$

where $\vec{X}_n(\vec{x})=(X_{1,n}(\vec{x}), X_{2,n}(\vec{x}), X_{3,n}(\vec{x}), X_{4,n}(\vec{x}))$, *and* $\vec{X}_0(\vec{x})=\vec{x}$. $\vec{\Phi}=(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$ *satisfies the following*:

(1) *Each function* Φ_i, $i=1, 2, 3, 4$, *is a polynomial of degree* 4 *in* 4 *variables with positive coefficients. Each term in* Φ_i, $i=1, 2, 3, 4$, *is at least of degree* 2, 4, 3, 4, *respectively.*

(2) *Let* $\Xi_0=\{\vec{x}\in\boldsymbol{R}^4 \mid x_i\geqq 0,\ i=1, 2, 3, 4,\ x_1^2\geqq x_2\}$ *and* $\Xi=\{\vec{x}\in\boldsymbol{R}^4 \mid x_i>0,\ i=1, 2, 3, 4,\ x_1^2\geqq x_2\}$. *Then* $\vec{\Phi}(\Xi_0)\subset\Xi_0$ *and* $\vec{\Phi}(\Xi)\subset\Xi$.

(3)

$$\Phi_1(x, y, 0, 0)=x^2+2x^3+2x^4+4x^3y+6x^2y^2, \tag{2.3}$$

$$\Phi_2(x, y, 0, 0)=x^4+4x^3y+22y^4. \tag{2.4}$$

(4) *There exist polynomials* $\Phi_{4,1}$, $\Phi_{4,2}$, *and* $\Phi_{3,0}$, *with positive coefficients, satisfying the following*:

$$\Phi_4(\vec{x})=\Phi_{4,1}(\vec{x})x_3+\Phi_{4,2}(\vec{x})x_4, \tag{2.5}$$

$$\Phi_3(\vec{x})=2(\Phi_{4,1}(\vec{x})x_1+\frac{1}{2}\Phi_{4,2}(\vec{x})x_3)+\Phi_{3,0}(\vec{x}). \tag{2.6}$$

Moreover, $\Phi_{3,0}(\vec{x})$ *has terms* $x_1^2x_3+2x_1^3x_3$.

(5) *There exist polynomials* $\Phi_{3,1}$, $\Phi_{3,2}$, *and* $\Phi_{1,0}$, *with positive coefficients, satisfying the following*:

$$\Phi_3(\vec{x})=\Phi_{3,1}(\vec{x})x_3+\Phi_{3,2}(\vec{x})x_4, \tag{2.7}$$

$$\Phi_1(\vec{x})=\Phi_{3,1}(\vec{x})x_1+\frac{1}{2}\Phi_{3,2}(\vec{x})x_3+\Phi_{1,0}(\vec{x}). \tag{2.8}$$

Moreover, $\Phi_{1,0}(\vec{x})$ *has terms* x_1^2, x_3^2, *and* x_4^2.

(6) $\Phi_1(\vec{x})$ *has terms* x_1x_3 *and* x_1x_4, $\Phi_2(\vec{x})$ *has terms* $x_1^3x_3$ *and* $x_1^3x_4$, *and* $\Phi_4(\vec{x})$ *has a term* $x_1^2x_3^2$. *Furthermore, all the terms in* $\Phi_4(\vec{x})$ *are at least of degree* 2 *in* x_3 *and* x_4.

Proof. The methods of obtaining the recursion relations eq. (2.2) are quite similar to the case of the (two-dimensional) Sierpinski gasket [3], so that the explanations will be brief. Note that F_{n+1} is composed of 4 tetrahedrons f^k, $k=1, 2, 3, 4$, say, each congruent to F_n. These 4 tetrahedrons assemble to form F_{n+1}, in the same way as 4 unit tetrahedrons, congruent to F_0, assemble to form F_1. This similarity of the composition leads to a natural mapping π: $W_i^{(n+1)}\to W_i^{(1)}$. For each $k=1, 2, 3, 4$, and for each $w\in W_i^{(n+1)}$, consider the intersection $w^k \overset{\text{def}}{=} f^k\cap w(\boldsymbol{Z}_+)$. Under the identification of f^k and F_n, $w^k\in\bigcup_{j=1}^4 W_j^{(n)}$. If one classifies the summation of w in eq. (2.1) for $X_{i,n+1}(\vec{x})$ by $\pi(w)$, one finds that eq. (2.2) holds with $\vec{\Phi}=\vec{X}_1$. Since there are 4 unit tetrahedrons in F_1, $\Phi_i(\vec{x})=X_{i,1}(\vec{x})$, $i=1, 2, 3, 4$, are polynomials of degree 4. Since $w\in W_1^{(1)}$ passes O and a_1, each term in $\Phi_1(\vec{x})=X_{1,1}(\vec{x})$ is at least of degree 2. Similar arguments hold for $\Phi_i(\vec{x})$, $i=2, 3, 4$, by which the assertion (1) is proved.

Let $\vec{x}\in\Xi$ ($\vec{x}\in\Xi_0$, respectively). From the assertion (1) it is clear that $\Phi_i(\vec{x})>0$ ($\Phi_i(\vec{x})\geqq 0$, respectively). Let $\vec{x}'=\vec{\Phi}(\vec{x})$.

$$
\begin{aligned}
x'^2_1 &= X_{1,1}(\vec{x})^2 \\
&= \sum_{w' \in W_1^{(1)}} \sum_{w'' \in W_1^{(1)}} \prod_{i=1}^{4} x_i^{s_i(w')+s_i(w'')} \\
&\geqq \sum_{w=(w',w'') \in W_2^{(1)}} \prod_{i=1}^{4} x_i^{s_i(w')+s_i(w'')} \\
&\geqq X_{2,1}(\vec{x}) = x'_2,
\end{aligned}
$$

where the last inequality comes from $x_1^2 \geqq x_2$. Therefore the assertion (2) holds.

The two formulas eq. (2.3) and eq. (2.4) are obtained through the explicit calculations of $\vec{\Phi}(x, y, 0, 0) = \vec{X}_1(x, y, 0, 0)$. An example of the paths $w \in W_1^{(1)}$ which contribute to the x^2y^2 term in $\Phi_1(x, y, 0, 0) = X_{1,1}(x, y, 0, 0)$ is given in Figure 1(a) (the slim lines in the figure represent $F_1 = Oa_1b_1c_1$ projected onto the Oa_1b_1 plane, and $\hat{w}$ is represented by the bold lines), and a path $w \in W_2^{(1)}$ for the y^4 term in $\Phi_2(x, y, 0, 0) = X_{2,1}(x, y, 0, 0)$ is given in Figure 1(b).

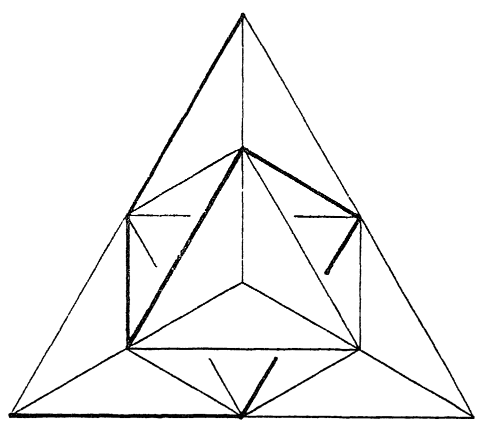

Fig. 1a.

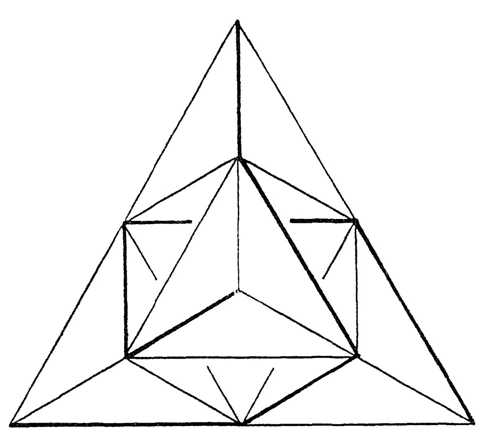

Fig. 1b.

Denote by Δ_c, the tetrahedron F_0+c_0, which is the unit tetrahedron in F_1 that has c_1 as one of its vertices. To obtain eq. (2.5), classify the summation over w in eq. (2.1) for $\Phi_4(\vec{x})=X(W_4^{(1)})(\vec{x})$ by the shape of the intersection $w(\boldsymbol{Z}_+)\cap\Delta_c$. Only case 3) or case 4) is possible for the intersection, which gives the contribution of factor x_3 or x_4 to each term in $\Phi_4(\vec{x})$, respectively, which in turn gives contribution of the first and the second term in the right hand side of eq. (2.5), respectively.

Consider now the relation between Φ_4 and Φ_3. Denote the four vertices of Δ_c by p, q, r, c_1. Note that the paths contributing to $\Phi_4(\vec{x})=X_{4,1}(\vec{x})$ must pass the vertex c_1. One can obtain a path contributing to Φ_3 from such a path, by short cutting c_1 such as $(\cdots, p, q, \cdots)$ instead of $(\cdots, p, c_1, q, \cdots)$. Denote the mapping induced by such operation by $\rho: W_4^{(1)}\to W_3^{(1)}$. For the paths contibuting to $\Phi_{4,2}(\vec{x})x_4$ in $\Phi_4(\vec{x})$, ρ is a two-to-one mapping, since $\rho((\cdots, p, c_1, r, q, \cdots))=(\cdots, p, r, q, \cdots)$ and $\rho((\cdots, p, r, c_1, q, \cdots))=(\cdots, p, r, q, \cdots)$. Therefore such paths, mapped by ρ, give contribution of $(1/2)\Phi_{4,2}(\vec{x})x_3$ to $\Phi_3(\vec{x})$. Similarly, the paths contributing to $\Phi_{4,1}(\vec{x})x_3$ are mapped one-to-one onto the paths contributing to the term $\Phi_{4,1}(\vec{x})x_1$ in $\Phi_3(\vec{x})$. Let

$$W_4'^{(n)}=\{w\in W^{(n,O,a_n)}\mid \text{there exist two positive integers } i \text{ and } j \text{ such that } i<j,\ w(i)=c_n, \text{ and } w(j)=b_n\}.$$

Then it is easy to see that $W_3^{(1)}\supset\rho(W_4'^{(1)})$ and $\rho(W_4^{(1)})\cap\rho(W_4'^{(1)})=\emptyset$. The paths in $\rho(W_4'^{(1)})$ give the same contribution to $\Phi_3(\vec{x})$ as the paths in $\rho(W_4^{(1)})$. Hence the factor 2 in the right hand side of eq. (2.6). There are paths $w\in W_3^{(1)}$ which are not in the image of ρ, namely, those paths that do not pass through Δ_c. Such contribution are denoted by $\Phi_{3,0}(\vec{x})$. In particular, there is a path in $W_3^{(1)}$ which starts at O, moves in a straight line to b_1, and then moves in a straight line to a_1, which gives a contribution $x_1^2x_2$ to $\Phi_{3,0}(\vec{x})$. This proves eq. (2.6). The formulas eq. (2.7) and eq. (2.8) are derived by similar arguments. The assertion (6) is straightforward. This completes the proof. □

Remark. By aid of computers, it is not difficult to obtain the full forms of the recursion relations $\vec{\Phi}$. The full explicit forms of $\vec{\Phi}$ are given in Appendix A. They are, howerver, unnecessary for the analyses of the limit $n\to\infty$ in this paper.

Let Ξ_0 be as defined in Proposition 2.1 (2). For $A\subset\Xi_0$, let $\bar{A}$ be the closure of A in Ξ_0, $A^c=\Xi_0\setminus A$, $\partial A=\bar{A}\cap\bar{A^c}$, and $A^o=A\setminus\partial A$. Let

$$D=\{\vec{x}\in\Xi_0\mid \sup_{n\in\boldsymbol{Z}_+}(X_{1,n}(\vec{x})+X_{2,n}(\vec{x}))<\infty\},$$

$$D'=\{\vec{x}\in\Xi_0\mid \max_{i\in\{1,2,3,4\}}\sup_{n\in\boldsymbol{Z}_+}X_{i,n}(\vec{x})\leqq 1\},$$

$$\tilde{D}=\{\vec{x}\in\Xi_0 \mid \lim_{n\to\infty}\max_{i\in\{1,2,3,4\}} X_{i,n}(\vec{x})=0\}.$$

Proposition 2.2. (1) *$D=D'$. In particular, D is a closed set in Ξ_0.*

(2) *Let $\vec{x}\in D$ and $\vec{x}'\in\Xi_0$. If $x'_i<x_i$ for all i with $x_i>0$, and $x'_i=0$ for all i with $x_i=0$, then $\vec{x}'\in\tilde{D}$.*

(3) *$\tilde{D}=D^o$.*

(4)

$$\vec{\Phi}(D^o)\subset D^o,\quad \vec{\Phi}(\partial D)\subset\partial D,\quad \text{and}\quad \vec{\Phi}(D^c)\subset D^c. \tag{2.9}$$

Proof. Note that by Proposition 2.1 (5) and eq. (2.2), $X_{1,n+1}(\vec{x})\geqq X_{1,n}(\vec{x})^2$, $X_{1,n+1}(\vec{x})\geqq X_{3,n}(\vec{x})^2$, and $X_{1,n+1}(\vec{x})\geqq X_{4,n}(\vec{x})^2$ hold. Also by eq. (2.4), $X_{2,n+1}(\vec{x})\geqq X_{2,n}(\vec{x})^4$ follows. Therefore, if $\vec{x}\in D'^c$, then $\lim_{n\to\infty}X_{1,n}(\vec{x})=\infty$ or $\lim_{n\to\infty}X_{2,n}(\vec{x})=\infty$ hold, which proves that $D'^c\subset D^c$. By definition, $D'\subset D$, hence $D'=D$. Since by definition D' is a closed set in Ξ_0, D is a closed set in Ξ_0.

Let $\vec{x}\in D$, $\vec{x}'\in\Xi_0$, and $x'_i<x_i$ for all i with $x_i>0$, and $x'_i=0$ for all i with $x_i=0$. If $\vec{x}'=\vec{0}$, then $\vec{x}'\in\tilde{D}$. Assume $\vec{x}'\neq\vec{0}$. Put $r=\max_{i;\,x_i\neq 0}(x'_i/x_i)$. By assumption, $0<r<1$, and $0\leqq x'_i\leqq rx_i$, $i=1, 2, 3, 4$. By Proposition 2.1 (1), $\Phi_i(\vec{x}')\leqq r^2\Phi_i(\vec{x})$, $i=1, 2, 3, 4$, hence by induction $X_{i,n}(\vec{x}')\leqq r^{2^n}X_{i,n}(\vec{x})$, and $0\leqq\limsup_{n\to\infty}\max_{i\in\{1,2,3,4\}}X_{i,n}(\vec{x}')\leqq(\lim_{n\to\infty}r^{2^n})\sup_{n\in\mathbf{Z}_+}\max_{i\in\{1,2,3,4\}}X_{i,n}(\vec{x})=0$, which proves the assertion (2).

For $\varepsilon>0$, define D_ε by $D_\varepsilon=\{\vec{x}\in\Xi_0 \mid \max_{i\in\{1,2,3,4\}}x_i<\varepsilon\}$. By Proposition 2.1 (1) there exists a positive constant M such that if $0<\varepsilon<1$ and $\vec{x}\in D_\varepsilon$, then $\Phi_i(\vec{x})\leqq M\varepsilon(x_1+x_2+x_3+x_4)/4$, $i\in\{1, 2, 3, 4\}$. Put $\varepsilon=\min\{1/2M, 1/2\}$. Then for all $\vec{x}\in D_\varepsilon$ and for all i, $\Phi_i(\vec{x})\leqq\varepsilon/2$ holds, which further implies $\Phi(\vec{x})\in D_\varepsilon$ and $\Phi_1(\vec{x})+\Phi_2(\vec{x})+\Phi_3(\vec{x})+\Phi_4(\vec{x})\leqq(x_1+x_2+x_3+x_4)/2$. By induction, $X_{1,n}(\vec{x})+X_{2,n}(\vec{x})+X_{3,n}(\vec{x})+X_{4,n}(\vec{x})\leqq 2^{-n}(x_1+x_2+x_3+x_4)$. Hence, there exists a positive constant ε such that

$$D_\varepsilon\subset\tilde{D}. \tag{2.10}$$

Fix such an ε and let $\vec{x}\in\tilde{D}$. By definition, there exists a positive integer n such that $\vec{X}_n(\vec{x})\in D_\varepsilon$ holds. Since $\vec{X}_n(\vec{x})$ is continuous with respect to $\vec{x}$, there exists an open set U in Ξ_0 such that for all $\vec{x}'\in U$, $\vec{X}_n(\vec{x}')\in D_\varepsilon$ holds. Then by eq. (2.10), $\lim_{n\to\infty}\max_{i\in\{1,2,3,4\}}X_{i,n}(\vec{x}')=0$. Therefore, $U\subset\tilde{D}$. This proves that $\tilde{D}$ is an open set in Ξ_0. By definition, $\tilde{D}\subset D$ and also D^o is the largest open set that is included in D. Therefore, $\tilde{D}\subset D^o$.

Now let $\vec{x}'\in D^o$. Since D^o is an open set in Ξ_0, there exists a $\vec{x}\in D^o$ such that $x'_i<x_i$, $i=1, 2, 3, 4$. From the assertion (2), $\vec{x}'\in\tilde{D}$. Hence, $D^o\subset\tilde{D}$, and the assertion (3) is proved.

By definition, $\vec{\Phi}(D)\subset D$, $\vec{\Phi}(D^c)\subset D^c$, and $\vec{\Phi}(\tilde{D})\subset\tilde{D}$. From the assertion (3), the assertion (4) follows. This completes the proof. □

Monotonicity properties similar to but slightly different from the Proposition 2.2 (2) hold, which shall be listed here.

Proposition 2.3. (1) *If* $\vec{x}\in D$, $\vec{x}'\in\Xi_0$ *and* $x_i'\leqq x_i$, $i=1, 2, 3, 4$, *then* $\vec{x}'\in D$.

(2) *If* $\vec{x}\in\partial D$, $\vec{x}'\in\Xi_0$, *and* $x_i'>x_i$, $i=1, 2, 3, 4$, *then* $\vec{x}'\in D^c$.

(3) *Let* Ξ *be as defined in Proposition* 2.1 (2). *If* $\vec{x}\in\partial D\cap\Xi$, $\vec{x}'\in\Xi_0$, $\vec{x}'\neq\vec{x}$, *and* $x_i'\leqq x_i$, $i=1, 2, 3, 4$, *then* $\vec{x}'\in D^o$.

(4) *If* $\vec{x}\in\partial D\cap\Xi$, $\vec{x}'\in\Xi_0$, $\vec{x}'\neq\vec{x}$, *and* $x_i'\geqq x_i$, $i=1, 2, 3, 4$, *then* $\vec{x}'\in D^c$.

Proof. Assume $\vec{x}\in D$, $\vec{x}'\in\Xi_0$, and $x_i'\leqq x_i$, $i=1, 2, 3, 4$. Since Φ_i, $i=1, 2, 3, 4$ are polynomials with positive coefficients, $X_{i,n}(\vec{x}')\leqq X_{i,n}(\vec{x})$, $n\in\boldsymbol{Z}_+$, $i=1, 2, 3, 4$. Therefore, $\vec{x}'\in D$ follows.

Next assume $\vec{x}\in\partial D$, $\vec{x}'\in\Xi_0$, and $x_i'>x_i$, $i=1, 2, 3, 4$. If $\vec{x}'\in D$ then by Proposition 2.2 (2) and Proposition 2.2 (3) $\vec{x}\in\tilde{D}=D^o$, which is a contradiction.

Next assume $\vec{x}\in\partial D\cap\Xi$, $\vec{x}'\in\Xi_0$, $\vec{x}'\neq\vec{x}$, and $x_i'\leqq x_i$, $i=1, 2, 3, 4$. Since $\vec{x}'\neq\vec{x}$, $x_i'<x_i$ holds for at least one i. If $x_4'<x_4$, then by eq. (2.7), $\Phi_3(\vec{x}')<\Phi_3(\vec{x})$. If $x_3'<x_3$, then by eq. (2.8), $\Phi_1(\vec{x}')<\Phi_1(\vec{x})$. If $x_2'<x_2$ or $x_1'<x_1$, then by eq. (2.3), $\Phi_1(\vec{x}')<\Phi_1(\vec{x})$. Therefore, for every case, $X_{1,n}(\vec{x}')<X_{1,n}(\vec{x})$, $n\geqq 2$, which, with eq. (2.3), eq. (2.4), eq. (2.6), eq. (2.5), implies that $X_{i,n}(\vec{x}')<X_{i,n}(\vec{x})$, $i=1, 2, 3, 4$, $n\geqq 4$. $\vec{x}\in\partial D$ and eq. (2.9) imply that $\vec{X}_n(\vec{x})\in\partial D\subset D$, $n\geqq 4$. From Proposition 2.2 (2) and Proposition 2.2 (3), $\vec{X}_n(\vec{x}')\in D^o$, hence $\vec{x}'\in D^o$.

Finally assume $\vec{x}\in\partial D\cap\Xi$, $\vec{x}'\in\Xi_0$, $\vec{x}'\neq\vec{x}$, and $x_i'\geqq x_i$, $i=1, 2, 3, 4$. By the same argument as above, it follows that $X_{i,n}(\vec{x}')>X_{i,n}(\vec{x})$, $i=1, 2, 3, 4$, $n\geqq 4$. By the assertion (2), $\vec{x}'\in D^c$ follows. This completes the proof. □

Define a function $R:\Xi\to\boldsymbol{R}$ by

$$R(\vec{x})=\max\left\{\frac{x_3}{x_1}, \frac{2x_4}{x_3}\right\},$$

and let

$$R_n(\vec{x})=R(\vec{X}_n(\vec{x})), \qquad \vec{x}\in\Xi,\ n\in\boldsymbol{Z}_+.$$

Here, Ξ is defined in Proposition 2.1 (2). Proposition 2.1 (2) implies that R_n is well-defined.

Proposition 2.4. (1) *For each* $\vec{x}\in\Xi$, $R_n(\vec{x})$ *is non-increasing in* n. *In particular,*

$$R_\infty(\vec{x})\overset{\text{def}}{=}\lim_{n\to\infty}R_n(\vec{x})$$

exists and is non-negative.

(2) *If* $\vec{x}\in D\cap\Xi$, *then* $R_\infty(\vec{x})=0$.

(3) *There exists a constant* γ *satisfying* $0<\gamma<1$ *such that for every* $\vec{x}\in D\cap\Xi$,

$$\limsup_{n\to\infty}\frac{R_{n+1}(\vec{x})}{R_n(\vec{x})}\leqq\gamma\,. \tag{2.11}$$

(4) *If* $\vec{x}\in D^{\circ}\cap\Xi$, *then* $\limsup_{n\to\infty}2^{-n}\log X_{j,n}(\vec{x})<0$, $j=1, 2$.

Proof. From Proposition 2.1 (4) and Proposition 2.1 (5) it follows that $\Phi_3(\vec{x})\leqq R(\vec{x})(\Phi_1(\vec{x})-\Phi_{1,0}(\vec{x}))$, $\vec{x}\in\Xi$, and $2\Phi_4(\vec{x})\leqq R(\vec{x})(\Phi_3(\vec{x})-\Phi_{3,0}(\vec{x}))$, $\vec{x}\in\Xi$, from which follows

$$\begin{aligned}R_{n+1}(\vec{x})&=\max\left\{\frac{\Phi_3(\vec{X}_n(\vec{x}))}{\Phi_1(\vec{X}_n(\vec{x}))},\ \frac{2\Phi_4(\vec{X}_n(\vec{x}))}{\Phi_3(\vec{X}_n(\vec{x}))}\right\}\\&\leqq R_n(\vec{x})\left\{1-\min\left\{\frac{\Phi_{1,0}(\vec{X}_n(\vec{x}))}{\Phi_1(\vec{X}_n(\vec{x}))},\ \frac{\Phi_{3,0}(\vec{X}_n(\vec{x}))}{\Phi_3(\vec{X}_n(\vec{x}))}\right\}\right\}.\end{aligned} \tag{2.12}$$

In particular, $R_n(\vec{x})$ is non-increasing in n.

Assume that $\vec{x}\in\Xi$. From Proposition 2.1 (5) and Proposition 2.1 (4), $\Phi_{1,0}(\vec{X}_n(\vec{x}))\geqq X_{1,n}(\vec{x})^2$ and $\Phi_{3,0}(\vec{X}_n(\vec{x}))\geqq X_{1,n}(\vec{x})^2X_{3,n}(\vec{x})$. Therefore eq. (2.12) implies

$$R_{n+1}(\vec{x})\leqq R_n(\vec{x})\left\{1-\min\left\{\frac{X_{1,n}(\vec{x})^2}{\Phi_1(\vec{X}_n(\vec{x}))},\ \frac{X_{1,n}(\vec{x})^2X_{3,n}(\vec{x})}{\Phi_3(\vec{X}_n(\vec{x}))}\right\}\right\}. \tag{2.13}$$

On the other hand, if $\vec{x}\in\Xi$, $\Phi_1(x)\leqq\Phi_1(x_1, x_1^2, x_3, x_4)$, so that with Proposition 2.1 (1) it follows that there exists a polynomial P_1 of 5 variables and with constant positive coefficients, such that $x_1^{-2}\Phi_1(\vec{x})\leqq P_1(x_1, x_3, x_4, x_3/x_1, x_4/x_1)$, and similarly, there exists a polynomial P_2 of 5 variables and with constant positive coefficients, such that $(x_1^2x_3)^{-1}\Phi_3(\vec{x})\leqq P_2(x_1, x_3, x_4, x_3/x_1, x_4/x_3)$. Note that if $\vec{x}\in D\cap\Xi$, then by Proposition 2.1 (2) and Proposition 2.2 (4) it follows that $\vec{X}_n(\vec{x})\in D\cap\Xi$, $n\in\mathbf{Z}_+$. Also, by Proposition 2.2 (1) it follows that if $\vec{x}\in D$, then $x_i\leqq 1$, $i=1, 2, 3, 4$. Therefore, from eq. (2.13) it follows that there exists a polynomial P of 1 variable and with constant positive coefficients, such that

$$R_{n+1}(\vec{x})\leqq R_n(\vec{x})(1-P(R_n(\vec{x}))^{-1}),\qquad \vec{x}\in D\cap\Xi\,. \tag{2.14}$$

From the assertion (1), $P(R_n(\vec{x}))\leqq P(R_0(\vec{x}))=P(R(\vec{x}))$, so that $1-P(R_n(\vec{x}))^{-1}$ is bounded from above by a constant $1-P(R(\vec{x}))^{-1}$ less than 1. The assertion (2) is thus proved.

From the assertion (2) it follows that there exists a positive constant $M>1$ independent of $\vec{x}$ such that

$$\limsup_{n\to\infty}P(R_n(\vec{x}))<M,\qquad \vec{x}\in D\cap\Xi\,.$$

The assertion (3) therefore follows from eq. (2.14).

Next let $\vec{x}\in D^{\circ}\cap\Xi$. From Proposition 2.1 (1) and Proposition 2.1 (3), there exists a polynomoial P_3 of three variables with positive coefficients and without constant terms (i. e. $P_3(0, 0, 0)=0$), satisfying

$$\Phi_1(\vec{x}) \leqq x_1^2(1+P_3(x_1, x_2, R(\vec{x}))).$$

This with eq. (2.2) implies

$$\text{(2.15)} \quad \log X_{1,n+1}(\vec{x}) \leqq 2\log X_{1,n}(\vec{x}) + \log(1+P_3(X_{1,n}(\vec{x}), X_{2,n}(\vec{x}), R_n(\vec{x}))).$$

This and Proposition 2.2 (3) and assertion (2) imply

$$\limsup_{n\to\infty}\Big(\log X_{1,n+1}(\vec{x}) - \frac{3}{2}\log X_{1,n}(\vec{x})\Big) = -\infty,$$

which, with Proposition 2.2 (3) implies

$$\limsup_{n\to\infty}\Big(\Big(\frac{2}{3}\Big)^n \log X_{1,n}(\vec{x})\Big) < 0.$$

This and eq. (2.15) with Proposition 2.2 (3) and assertion (2) imply

$$\limsup_{n\to\infty}\Big(\log X_{1,n+1}(\vec{x}) - \Big(2-\Big(\frac{2}{3}\Big)^n\Big)\log X_{1,n}(\vec{x})\Big) < 0.$$

Therefore for sufficiently large N,

$$\prod_{k=1}^{n}\Big(1-\frac{1}{2}\Big(\frac{2}{3}\Big)^{k+N-1}\Big)^{-1} 2^{-n}\log X_{1,n+N}(\vec{x}) \leqq \log X_{1,N}(\vec{x}).$$

This with Proposition 2.2 (3) implies assertion (4) for $j=1$. The case $j=2$ follows because $X_{2,n}(\vec{x}) \leqq X_{1,n}(\vec{x})^2$. This completes the proof. □

§ 3. Fixed Points of the Renormalization Group Flows

Let $\iota: \boldsymbol{R}^2 \to \boldsymbol{R}^4$ be a natural embedding of the x_1-x_2 plane: $\iota(x_1, x_2) = (x_1, x_2, 0, 0)$. From eq. (2.5) and eq. (2.7), x_1-x_2 plane is an invariant submanifold of the mapping $\vec{\Phi}$; $\vec{\Phi}(\iota(\boldsymbol{R}^2)) \subset \iota(\boldsymbol{R}^2)$. Therefore, the restriction of $\vec{\Phi}$ onto the x_1-x_2 plane,

$$\vec{\phi} \overset{\text{def}}{=} \iota^{-1}\circ\vec{\Phi}\circ\iota : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^2,$$

is well-defined. Let $\vec{\phi} = (\phi_1, \phi_2)$. From eq. (2.3) and eq. (2.4),

$$\text{(3.1)} \quad \phi_1(x, y) = x^2 + 2x^3 + 2x^4 + 4x^3y + 6x^2y^2,$$

$$\text{(3.2)} \quad \phi_2(x, y) = x^4 + 4x^3y + 22y^4.$$

Proposition 3.1. *The fixed points of the mapping $\vec{\phi}$ in the first quadrant $\boldsymbol{R}_+^2 \overset{\text{def}}{=} \{(x, y) \in \boldsymbol{R}^2 \mid x \geqq 0,\ y \geqq 0\}$ are $(0, 0)$, $(0, 22^{-1/3})$, $(1/3, 1/3)$, and (x_c, y_c), where x_c and y_c are positive constants. They satisfy $3/7 < x_c < 1/2$ and $0 < y_c < 9/49$, in particular, $x_c^2 > y_c$.*

Proof. Assume that (x, y) is a fixed point: $\phi_1(x, y) = x$ and $\phi_2(x, y) = y$.

If $x=0$ then $y=0$ or $y=22^{-1/3}$, and if $y=0$ then $x=0$. Assume in the following that $x>0$ and $y>0$. Then,

$$1=6xy^2+4x^2y+2x^3+2x^2+x\,, \tag{3.3}$$

$$x^4+4x^3y+22y^4-y=0\,. \tag{3.4}$$

From eq. (3.3) follows $1>2x^3+2x^2+x$. The right hand side of this inequality is increasing in x, and $2(1/2)^3+2(1/2)^2+1/2=5/4>1$. Therefore, $x<1/2$. From eq. (3.4) follows $y>22y^4$, from which follows $y<22^{-1/3}<2/5$. Then from eq. (3.3), $1<(49/25)x+(18/5)x^2+2x^3$. The right hand side of this inequality is increasing in x, and its value at $x=1/4$ is $597/800<1$. Therefore, $x>1/4$. Let $I=(1/4,\ 1/2)$. If $(x,\ y)$ is a fixed point, then $x\in I$.

Let

$$g_1(x,\ y)=y^4+22^{-1}(4x^3y-y+x^4)\,,$$

and

$$g_2(x,\ y)=y^2+\frac{2}{3}xy+(6x)^{-1}(2x^3+2x^2+x-1)\,.$$

Then the set of fixed point conditions eq. (3.3) and eq. (3.4) is, for $x>0$ and $y\geqq 0$, equivalent to a set of conditions $g_1(x,\ y)=g_2(x,\ y)=0$. Note that $g_1(x,\ y)=g_2(x,\ y)\times\{y^2-(2/3)xy+(4/9)x^2-(6x)^{-1}(2x^3+2x^2+x-1)\}+h(x,\ y)$, where,

$$h(x,\ y)=594^{-1}h_1(x)y+1188^{-1}x^{-2}h_2(x)\,,$$

$$h_1(x)=-159+132x+264x^2+196x^3\,,$$

$$h_2(x)=33-66x-99x^2+88x^3+176x^4+88x^5+10x^6\,.$$

$h_1(x)$ is increasing in x and $h_1(1/2)=-5/2$, so that $h_1(x)\neq 0$ for $x\in I=(1/4,\ 1/2)$. Therefore, the set of conditions $g_1(x,\ y)=g_2(x,\ y)=0$ is equivalent to a set of conditions, $g_2(x,\ y)=0$ and

$$y=-2^{-1}x^{-2}h_1(x)^{-1}h_2(x)\,. \tag{3.5}$$

Substituting eq. (3.5) into $g_2(x,\ y)=0$ and noting that $x>0$ and $h_1(x)\neq 0$ for the fixed points, one sees that the condition $g_2(x,\ y)=0$ is equivalent to

$$\left(x-\frac{1}{3}\right)f(x)=0\,,$$

where

$$\begin{aligned} f(x)=&\,19239-35211x-112167x^2-41179x^3+518440x^4+725492x^5\\ &+2124x^6-2096944x^7-3168164x^8-1048100x^9+3320820x^{10}\\ &+5564268x^{11}+4315264x^{12}+1787888x^{13}+353584x^{14}\,. \end{aligned}$$

A computer calculation was used here (and also in the following) to handle large coefficients. If $x=1/3$, then it follows from eq. (3.5) that $y=1/3$. Therefore $(1/3, 1/3)$ is a fixed point.

From the preceding arguments, a necessary and sufficient condition for (x, y) to be a fixed point satisfying $x>0$, $y>0$, and $x\neq 1/3$, is $f(x)=0$ and eq. (3.5). It has also been proved that if (x, y) is a fixed point satisfying $x>0$, $y>0$, then $x\in I=(1/4, 1/2)$, and that $h_1(x)\neq 0$. Assume that $x\in I$ in the following. Denote by $f^{(n)}$, the n-th derivative of f. Since

$$f^{(9)}(x)=-262025+8302050x+76508685x^2+237339520x^3 +319584980x^4+176968792x^5,$$

$f^{(9)}(x)$ is increasing for $x\in I$. This with $f^{(9)}(1/4)=1500788509/128>0$ implies $f^{(9)}(x)>0$, $x\in I$, hence $f^{(8)}(x)$ is increasing for $x\in I$.

Similarly, since $f^{(8)}(1/4)=51984265459/128>0$, and $f^{(7)}(1/4)=6064495535/512>0$, $f^{(6)}(x)$ is increasing for $x\in I$. $f^{(6)}(1/4)=-17104786191/16384<0$ and $f^{(6)}(1/2)=2781099317/64>0$. Therefore there exists an $x_1\in I$ such that $f^{(5)}(x)$ is decreasing for $x<x_1$ and increasing for $x>x_1$. $f^{(5)}(1/4)=-148288007935/32768<0$ and $f^{(5)}(1/2)=2507283275/32>0$. Therefore there exists an $x_2\in I$ such that $f^{(4)}(x)$ is decreasing for $x<x_2$ and increasing for $x>x_2$. $f^{(4)}(1/4)=-25860012481/262144<0$ and $f^{(4)}(1/2)=742821129/256>0$. Therefore there exists an $x_3\in I$ such that $f^{(3)}(x)$ is decreasing for $x<x_3$ and increasing for $x>x_3$. $f^{(3)}(1/4)=96806613501/32768>0$, $f^{(3)}(1/2)=81366081/32>0$, and $f^{(3)}(3/8)=-124154731777731/67108864<0$. Therefore there exist $x_4\in I$ and $x_5\in I$ such that $x_4<3/8<x_5$ and that $f^{(2)}(x)$ is increasing for $x<x_4$, decreasing for $x_4<x<x_5$. and increasing for $x>x_5$. $f^{(2)}(1/4)=106573809241/524288>0$ and $f^{(2)}(1/2)=-2264445/128<0$. Therefore there exists an $x_6\in I$ such that $f^{(1)}(x)$ is increasing for $x<x_6$ and decreasing for $x>x_6$. $f^{(1)}(1/4)=-120599934157/2097152<0$ and $f^{(1)}(1/2)=14625/256>0$. Therefore there exists an $x_7\in I$ such that $f(x)$ is decreasing for $x<x_7$ and increasing for $x>x_7$. Finally, $f(1/4)=89607188671/16777216>0$ and $f(1/2)=-125/1024<0$. Therefore there exists one and only one $x_c\in I$ such that $f(x)=0$.

From eq. (3.5), $y_c \stackrel{\text{def}}{=} -2^{-1}x_c^{-2}h_1(x_c)^{-1}h_2(x_c)$ is uniquely determined. Since $f(3/7)=216853862622/96889010407>0$, the above arguments imply $3/7<x_c<1/2$. From $3/7<x_c$ and eq. (3.3), it follows that

$$y_c^2+\frac{2}{7}y_c-\frac{8}{441}<0,$$

which imply $y_c<(\sqrt{17}-3)/21<(3/7)^2<x_c^2$. This completes the proof. $\square$

Remark.

(1) By the same arguments as the proof of $3/7<x_c$ and $y_c<(\sqrt{17}-3)/21$, it is not difficult to obtain (x_c, y_c) to an arbitrary precision. For example, one can prove $0.4294449<x_c<0.42944491$ and $0.0499839<y_c<0.049984$.

(2) x_c is not a rational number. This can be proved by standard arguments using the explicit form of $f(x)$.

Define $(x_n(x, y), y_n(x, y))$, $n=0, 1, 2, 3, \cdots$, inductively by $(x_0(x, y), y_0(x, y))=(x, y)$ and $(x_{n+1}(x, y), y_{n+1}(x, y))=\vec{\phi}(x_n(x, y), y_n(x, y))$. From eq. (2.2), it follows that $(x_n(x, y), y_n(x, y))$ is $\vec{X}_n(\vec{x})$ with $\vec{x}$ restricted to the x_1-x_2 plane: $(x_n(x, y), y_n(x, y))=(\iota^{-1}\circ\vec{X}_n\circ\iota)(x, y)$.

Let $D^{(2)}=\{(x, y)\in \boldsymbol{R}_+^2 \mid \sup_{n\in \boldsymbol{Z}_+}(x_n(x, y)+y_n(x, y))<\infty\}$. Define also $D^{(2)c}$, $D^{(2)o}$, and $\partial D^{(2)}$, to be the exterior, interior, and boundary, of $D^{(2)}$ in $\boldsymbol{R}_+^2$, respectively. Notice the slight difference in the previous definition of D. The condition $x^2\geqq y$ is dropped here and the whole first quadrant is considered. Let $\Xi_0^{(2)}=\{(x, y)\in\boldsymbol{R}_+^2 \mid x^2\geqq y\}$. Then from the definition of D, $D^{(2)}\cap\Xi_0^{(2)}$ is the restriction of D to the x_1-x_2 plane: $D^{(2)}\cap\Xi_0^{(2)}=\iota^{-1}(D)$. Note also that Proposition 2.1 (2) implies $\vec{\phi}(\Xi_0^{(2)})\subset\Xi_0^{(2)}$. Define also $D'^{(2)}=\{(x, y)\in\boldsymbol{R}_+^2 \mid \sup_{n\in\boldsymbol{Z}_+}\max\{x_n(x, y), y_n(x, y)\}\leqq 1\}$, and $\tilde{D}^{(2)}=\{(x, y)\in\boldsymbol{R}_+^2 \mid \lim_{n\to\infty}\max\{x_n(x, y), y_n(x, y)\}=0\}$.

Proposition 3.2. (1) *$D^{(2)}$ is a closed set in $\boldsymbol{R}_+^2$ satisfying $D^{(2)}=D'^{(2)}$. The interior of $D^{(2)}$ satisfies $D^{(2)o}=\tilde{D}^{(2)}$. They are invariant sets of $\vec{\phi}$:*

$$\vec{\phi}(D^{(2)o})\subset D^{(2)o},\qquad \vec{\phi}(\partial D^{(2)})\subset\partial D^{(2)},\quad \text{and}\quad \vec{\phi}(D^{(2)c})\subset D^{(2)c}.$$

(2) *There exist a positive constant c and a continuous strictly decreasing function p: $[0, c]\to\boldsymbol{R}$ such that $\partial D^{(2)}=\{(x, p(x)) \mid x\in[0, c]\}$. For $(x, y)\in\boldsymbol{R}_+^2$ it holds that $(x, y)\in D^{(2)}$ if and only if $y\leqq p(x)$.*

Proof. The assertion (1) may be proved in exactly the same way as Proposition 2.2, if one notes the explicit formula eq. (3.1) and eq. (3.2). Let $(x, y)\in\boldsymbol{R}_+^2$ in the following. From eq. (3.1) and eq. (3.2) it follows that $(0, y)\in\partial D^{(2)}$ if and only if $y=22^{-1/3}$.

Assume that $x>0$ and $(x, y)\in\partial D^{(2)}$. If $y'>y$, then eq. (3.1) and eq. (3.2) imply $\phi_1(x, y')>\phi_1(x, y)$ and $\phi_2(x, y')>\phi_2(x, y)$. Let

$$r=\min\left\{\frac{\phi_1(x, y')}{\phi_1(x, y)},\ \frac{\phi_2(x, y')}{\phi_2(x, y)}\right\}>1.$$

Then from eq. (3.1) and eq. (3.2), it follows by induction that

$$x_n(x, y')\geqq r^{2^{n-1}}x_n(x, y),\qquad n=1, 2, 3, \cdots,$$

and

$$y_n(x, y')\geqq r^{2^{n-1}}y_n(x, y),\qquad n=1, 2, 3, \cdots,$$

which imply $(x, y')\in D^{(2)c}$. If $y'<y$, then just in the same way as in the above argument, it follows that $(x, y')\in D^{(2)o}$. Therefore for each $x\geqq 0$, there exists at most one $y\geqq 0$ such that $(x, y)\in\partial D^{(2)}$.

Let $J=\{x\geqq 0 \mid$ there exists $y\geqq 0$ such that $(x, y)\in\partial D^{(2)}\}$, and define a function $p: J\to \boldsymbol{R}$ by $(x, p(x))\in\partial D^{(2)}$. The above arguments prove that p is well-defined, and that if $y<p(x)$ then $(x, y)\in D^{(2)o}$, while if $y>p(x)$ then $(x, y)\in D^{(2)c}$.

Let $x\in J$, $x'\in J$, and $x'>x$. Put $y=p(x)$ and $y'=p(x')$. If $y'\geqq y$, then from eq. (3.1) and eq. (3.2) $\phi_1(x', y')>\phi_1(x, y)$ and $\phi_2(x', y')>\phi_2(x, y)$, hence as in the previous arguments, $(x', y')\in D^{(2)c}$ follows. Therefore $y'<y$, which proves that the function p is strictly decreasing.

Now $D^{(2)}$ is a closed set in $\boldsymbol{R}_+^2$ and eq. (3.1) implies that if $x\geqq 1/2$ and $y\geqq 0$ then $(x, y)\in D^{(2)c}$, hence $\partial D^{(2)}$ intersects the x-axis. Therefore, there exists a positive constant c such that $p(c)=0$. If $x>c$ and $y\geqq 0$, then $\phi_1(x, y)>\phi_1(c, 0)$ and $\phi_2(x, y)>\phi_2(c, 0)$, so that $(x, y)\in D^{(2)c}$. Therefore if $x>c$ then $x\notin J$. If $x<c$ then the same argument as before implies $(x, 0)\in D^{(2)o}$, while $(x, 1)\in D^{(2)c}$. Therefore there exists a y such that $(x, y)\in\partial D^{(2)}$, hence $x\in J$. This proves that $J=[0, c]$. Since by construction $\partial D^{(2)}=\{(x, p(x)) \mid x\in[0, c]\}$, the continuity of p follows from the fact that $\partial D^{(2)}$ is the boundary of $D^{(2)}$. This completes the proof. □

A numerically obtained shape of $D^{(2)}$ together with the fixed points of $\vec{\phi}$ is given in Figure 2 (the black dots in the figure represent the fixed points).

If $(x, y)\in D^{(2)o}$, then Proposition 3.2 (1) implies $\lim_{n\to\infty} x_n(x, y)=0$ and $\lim_{n\to\infty} y_n(x, y)=0$. If $(x, y)\in D^{(2)c}$, then from Proposition 3.2 (1) and the explicit recursion formula eq. (3.1) and eq. (3.2) it easily follows that $\lim_{n\to\infty} y_n(x, y)=\infty$, and $\lim_{n\to\infty} x_n(x, y)=0$ if $x=0$ and $\lim_{n\to\infty} x_n(x, y)=\infty$ if $x\neq 0$. The convergence of the sequence $\{(x_n(x, y), y_n(x, y))\}$, $n=1, 2, 3, \cdots$, for the case $(x, y)\in\partial D^{(2)}$, is not obvious from previous arguments alone, because the sequence may either converge to one of the fixed points, or approach a non-

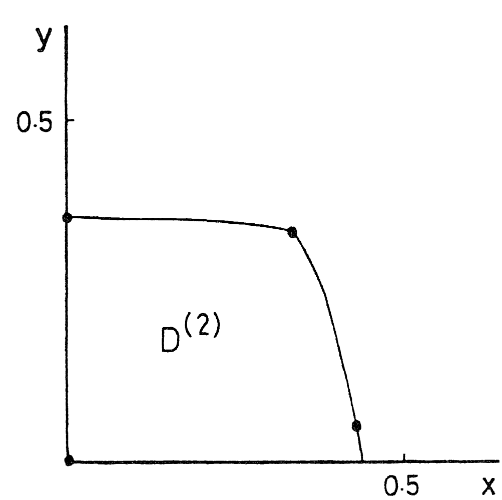

Fig. 2.

trivial attractor. However, the next two propositions show that the latter is not the case.

Proposition 3.3. *Let* $(x_1, y_1)\in\partial D^{(2)}$ *and* $(x_2, y_2)\in\partial D^{(2)}$. *If* $x_1<x_2$, *then* $\phi_1(x_1, y_1)<\phi_1(x_2, y_2)$.

Proof. The mapping $\vec{\phi}$ is a C^∞ mapping. Its differential $\mathcal{T}$ is, from eq. (3.1) and eq. (3.2),

$$\mathcal{T}(x, y)=\begin{pmatrix} \phi_{1,x}(x, y) & \phi_{1,y}(x, y) \\ \phi_{2,x}(x, y) & \phi_{2,y}(x, y) \end{pmatrix} \tag{3.6}$$

$$=\begin{pmatrix} 2x+6x^2+8x^3+12x^2y+12xy^2 & 4x^3+12x^2y \\ 4x^3+12x^2y & 4x^3+88y^3 \end{pmatrix}.$$

Therefore the Jacobian is,

$$\det\mathcal{T}(x, y)=8x(2x^5+3x^4+x^3-6x^4y-12x^3y^2$$
$$+132y^5+132xy^4+88x^2y^3+66xy^3+22y^3)$$
$$=8x(3x^4+x^3+132y^5+88x^2y^3+66xy^3+22y^3$$
$$+x^3(x-3y)^2+x\left(x^2-\frac{21}{2}y^2\right)^2+\frac{87}{4}xy^4\right).$$

Since $(0, 0)\in D^{(2)o}$ and $\partial D^{(2)}$ is a closed set, it follows that there exists a positive constant ε_1 such that

$$\det\mathcal{T}(x, y)>\varepsilon_1, \qquad (x, y)\in\partial D^{(2)}. \tag{3.7}$$

Denote by $\mathcal{U}_\varepsilon(x, y)$ the ε-neighborhood of (x, y) in $\boldsymbol{R}_+^2$. The mapping $\vec{\phi}$ is a C^∞ mapping on $\boldsymbol{R}_+^2$, and $\det\mathcal{T}(x, y)>0$ if $(x, y)\in\partial D^{(2)}$. Therefore for each $(x, y)\in\partial D^{(2)}$ there exists a positive number $\varepsilon(x, y)$ such that $\vec{\phi}$, when restricted to $\mathcal{U}_{\varepsilon(x,y)}(x, y)$, is a diffeomorphism of class C^∞. Let $P=(x_0, y_0)\in\partial D^{(2)}$ and $\varepsilon=\varepsilon(x_0, y_0)$. By Proposition 3.2 (2), $p(x_0)=y_0$. The line segment of the line $x=x_0$ inside $\mathcal{U}=\mathcal{U}_\varepsilon(x_0, y_0)$ is mapped onto a smooth curve which cuts the domain $\vec{\phi}(\mathcal{U})$ into two pieces. By Proposition 3.2 (1), this curve intersects $\partial D^{(2)}$ at one point $\vec{\phi}(P)$. Put $\mathcal{U}_1=\mathcal{U}\cap\{x<x_0\}$, which is the left half of $\mathcal{U}$, and $\mathcal{U}_2=\mathcal{U}\cap\{x>x_0\}$, the right half of $\mathcal{U}$. Likewise denote by $\mathcal{U}_1'$ the piece of $\vec{\phi}(\mathcal{U})$ which contains the points of $\partial D^{(2)}$ which satisfy $x<\phi_1(x_0, y_0)$, and the other piece by $\mathcal{U}_2'$. Then $\mathcal{U}_1$ is mapped by $\vec{\phi}$ onto either $\mathcal{U}_1'$ or $\mathcal{U}_2'$, and $\mathcal{U}_2$ is mapped onto the other piece.

Let $0<\varepsilon'<\varepsilon$ and put $Q=(x_0+\varepsilon' y_0)$, $R=(x_0, y_0+\varepsilon')$. Denote by P', Q', R', the images of P, Q, R, by $\vec{\phi}$, respectively. Put $\vec{e}_1=\overrightarrow{PQ}$, $\vec{e}_2=\overrightarrow{PR}$, $\vec{e}_1'=\overrightarrow{P'Q'}$, and $\vec{e}_2'=\overrightarrow{P'R'}$. Define $\vec{e}_1\times\vec{e}_2\overset{\text{def}}{=}e_{1,x}e_{2,y}-e_{1,y}e_{2,x}$. Proposition 3.2 (1) implies that if $(x_0, y_0)\in\partial D^{(2)}$, then $x_0\leqq 1$ and $y_0\leqq 1$. Therefore from eq. (3.1) and eq. (3.2), there exists a positive constant M independent of (x, y) and ε' such that

$$\begin{aligned}
&|\vec{e}_1'\times\vec{e}_2'-(\det\mathcal{T}(x_0, y_0))\vec{e}_1\times\vec{e}_2| \\
&=|(\phi_1(x_0+\varepsilon', y_0)-\phi_1(x_0, y_0))(\phi_2(x_0, y_0+\varepsilon')-\phi_2(x_0, y_0)) \\
&\quad -(\phi_2(x_0+\varepsilon', y_0)-\phi_2(x_0, y_0))(\phi_1(x_0, y_0+\varepsilon')-\phi_1(x_0, y_0)) \\
&\quad -(\det\mathcal{T}(x_0, y_0))\varepsilon'^2)| \\
&\leqq M\varepsilon'^3.
\end{aligned}$$

By eq. (3.7) $\det\mathcal{T}(x_0, y_0)>\varepsilon_1$. Therefore for sufficiently small ε_2 it follows that if $0<\varepsilon'<\varepsilon_2$ then

$$(\det\mathcal{T}(x_0, y_0))\vec{e}_1\times\vec{e}_2=(\det\mathcal{T}(x_0, y_0))\varepsilon'^2>M\varepsilon'^3,$$

which implies that if $\varepsilon'<\varepsilon_2$ then $\vec{e}_1'\times\vec{e}_2'$ and $\vec{e}_1\times\vec{e}_2$ have the same sign. On the other hand, by Proposition 3.2 (2) $R\in D^{(2)c}$, and by Proposition 3.2 (1) $R'\in D^{(2)c}$, so that R' is above the curve $y=p(x)$. Therefore Q' is contained in the piece that contains the points in $\partial D^{(2)}$ with $x>\phi_1(x_0, y_0)$, that is, the piece $\mathcal{U}_2'$. Therefore, $\vec{\phi}(\mathcal{U}_i)=\mathcal{U}_i'$, $i=1, 2$. In particular, for $(x, y)\in\partial D^{(2)}\cap\mathcal{U}$, if $x>x_0$ then $\phi_1(x, y)>\phi_1(x_0, y_0)$, and if $x<x_0$ then $\phi_1(x, y)<\phi_1(x_0, y_0)$.

Therefore, for each point $(x, y)\in\partial D^{(2)}$, there exists an ε-neighborhood $\mathcal{U}(x)$ such that if $(x', y')\in\partial D^{(2)}\cap\mathcal{U}(x)$, then the order of x and x' is conserved by the map $\vec{\phi}$.

Now assume that $(x_1, y_1)\in\partial D^{(2)}$, $(x_2, y_2)\in\partial D^{(2)}$, and $x_1<x_2$. Since $\{\mathcal{U}(x)| x_1\leqq x\leqq x_2\}$ is an open ball covering of the closed set $\partial D^{(2)}\cap\{x_1\leqq x\leqq x_2\}$, one can choose a finite number of $\mathcal{U}(x)$s', say $\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n$, that covers the set $\partial D^{(2)}\cap\{x_1\leqq x\leqq x_2\}$. It is further possible to choose them in such a way that none of the $\mathcal{U}_i$ is included in some other $\mathcal{U}_j$. Arrange the balls in a way that the x-coordinates $x_1, x_2, x_3, \cdots, x_n$ of the centers of the balls are in increasing order. For each $i=1, 2, \cdots, n-1$, take $x_i'\in\mathcal{U}_i\cap\mathcal{U}_{i+1}\cap\partial D^{(2)}$. Then the order of the points

$$x_1<x_1'<x_2<x_2'<\cdots<x_{n-1}'<x_n$$

is conserved by $\vec{\phi}$. Since $(x_1, y_1)\in\mathcal{U}_1$ and $(x_2, y_2)\in\mathcal{U}_n$, it follows that the order of x_1 and x_2 is conserved by $\vec{\phi}$, that is, $\phi_1(x_1, y_1)<\phi_1(x_2, y_2)$. This completes the proof. □

Proposition 3.4. *If $(x, y)\in\partial D^{(2)}$, then $(x_n(x, y), y_n(x, y))$ converges to one of the fixed points of the mapping $\vec{\phi}$ as $n\to\infty$. In particular, if $(x, y)\in\partial D^{(2)}\cap\Xi_0^{(2)}$, then $\lim_{n\to\infty}(x_n(x, y), y_n(x, y))=(x_c, y_c)$, where (x_c, y_c) is as given in Proposition* 3.1.

Proof. Assume that $(x, y)\in\partial D^{(2)}$ and that the sequence $\{(x_n(x, y), y_n(x,$

$y))\}$, $n=1, 2, 3, \cdots$, has an accumulation point (x_1, y_1) which is not a fixed point. Put $(x_2, y_2)=\vec{\phi}(x_1, y_1)$. Since (x_1, y_1) is not a fixed point, $x_1 \neq x_2$.

Assume first that $x_2 > x_1$. The continuity of $\vec{\phi}$ implies that for any positive number δ there exists a positive number ε such that the ε-neighborhood $\mathcal{U}_\varepsilon(x_1, y_1)$ is mapped into the $\mathcal{U}_\delta(x_2, y_2)$. In particular, there exists $\varepsilon>0$ such that if $(x, y)\in\mathcal{U}_\varepsilon(x_1, y_1)$, then $\phi_1(x, y)>x_1$. But if $(x, y)\in\partial D^{(2)}$ and $x>x_1$, then Proposition 3.3 implies $\phi_1(x, y)>x_2>x_1$. By induction, if $(x, y)\in\mathcal{U}_\varepsilon(x_1, y_1)$, then $\phi_1(\vec{\phi}^n(x, y))>x_2$, $n=1, 2, 3, \cdots$, where $\vec{\phi}^n$ is the n-th iteration of $\vec{\phi}$. By assumption, the sequence $\{(x_n(x, y), y_n(x, y))\}$, $n=1, 2, 3, \cdots$, accumulates at (x_1, y_1), therefore there exists an integer N such that $(x_N(x, y), y_N(x, y))\in\mathcal{U}_\varepsilon(x_1, y_1)$. Therefore if $n>N$ then $\phi_1(x_n(x, y), y_n(x, y))>x_2>x_1$, which says that (x_1, y_1) cannot be an accumulation point, which is a contradiction. The case $x_2<x_1$ can be handled in the same way. The conclusion is that if $(x, y)\in\partial D^{(2)}$ then every accumulation point of the sequence $\{(x_n(x, y), y_n(x, y))\}$, $n=1, 2, 3, \cdots$, is a fixed point.

If $(x, y)\in\partial D^{(2)}\cap\Xi_0^{(2)}$, then from $\vec{\phi}(\Xi_0^{(2)})\subset\Xi_0^{(2)}$ it follows that the sequence $(x_n(x, y), y_n(x, y))$ accumulates at a fixed point in $\Xi_0^{(2)}$. Proposition 3.1 therefore implies $\lim_{n\to\infty}(x_n(x, y), y_n(x, y))=(x_c, y_c)$, where (x_c, y_c) is as given in Proposition 3.1. This completes the proof. □

The original problem of four dimensional parameter space is now considered.

Theorem 3.5.

$$\lim_{n\to\infty}\vec{X}_n(\vec{x})=(0, 0, 0, 0), \qquad \vec{x}\in D^o\cap\Xi,$$

$$\lim_{n\to\infty}\vec{X}_n(\vec{x})=(\infty, \infty, \infty, \infty), \qquad \vec{x}\in D^c\cap\Xi,$$

$$\lim_{n\to\infty}\vec{X}_n(\vec{x})=(x_c, y_c, 0, 0), \qquad \vec{x}\in\partial D\cap\Xi.$$

Here x_c *and* y_c *are as given in Proposition* 3.1.

Proof. The first two cases are direct consequences of Proposition 2.2 and Proposition 2.1. Consider the case $\vec{x}\in\partial D\cap\Xi$. Since ∂D is a closed set, the sequence $\{\vec{X}_n(\vec{x})\}$ has an accumulation point in ∂D. From Proposition 2.4 (2) and Proposition 2.2 (1) follows $\lim_{n\to\infty}X_{i,n}(\vec{x})=0$, $i=3, 4$. Therefore every accumulation point of $\{\vec{X}_n(\vec{x})\}$ is in $\iota(\partial D^{(2)}\cap\Xi_0^{(2)})$. Denote an accumulation point of the sequence by $\iota(\vec{y})$, $\vec{y}\in\partial D^{(2)}\cap\Xi_0^{(2)}$. Let $\{\vec{X}_{k_1(n)}(\vec{x})\}$, $n=1, 2, 3, \cdots$, be a subsequence of $\{\vec{X}_n(\vec{x})\}$ that converges to $\iota(\vec{y})$. By the same reasoning, every accumulation point of $\{\vec{X}_{k_1(n)-1}(\vec{x})\}$ is in $\iota(\partial D^{(2)}\cap\Xi_0^{(2)})$. Denote one of the points by $\iota(\vec{z}_1)$, $\vec{z}_1\in\partial D^{(2)}\cap\Xi_0^{(2)}$, and let $\{\vec{X}_{k_1(k_2(n))-1}(\vec{x})\}$, $n=1, 2, 3, \cdots$, be a subsequence of $\{\vec{X}_{k_1(n)-1}(\vec{x})\}$ that converges to $\iota(\vec{z}_1)$. By definition $\vec{X}_{k_1(k_2(n))}(\vec{x})=\vec{\Phi}(\vec{X}_{k_1(k_2(n))-1}(\vec{x}))$, and $\vec{\Phi}$ is a continuous map, therefore it follows that $\vec{y}=\vec{\phi}(\vec{z}_1)$, where $\vec{\phi}$ is defined at the beginning of the section. By induction, one obtains a sequence of points $\{\vec{z}_l\}$, $l=1, 2, 3, \cdots$, such that $\vec{z}_l\in\partial D^{(2)}\cap\Xi_0^{(2)}$, $l=1, 2, 3, \cdots$, and $\vec{y}=\vec{\phi}(\vec{z}_1)$, $\vec{z}_l=\vec{\phi}(\vec{z}_{l+1})$, $l=1, 2, 3, \cdots$.

Since $\partial D^{(2)}\cap\Xi_0^{(2)}$ is a closed set, every accumulation point of the sequence $\{\vec{z}_l\}$, $l=1, 2, 3, \cdots$, is in $\partial D^{(2)}\cap\Xi_0^{(2)}$. Let $\vec{w}\in\partial D^{(2)}\cap\Xi_0^{(2)}$ be one of the accumulation points. Assume that $\vec{w}$ is not a fixed point of the map $\vec{\phi}$: $\vec{w}'\overset{\text{def}}{=}\vec{\phi}(\vec{w})\neq\vec{w}$. Since $\vec{\phi}$ is a continuous map, $\vec{w}'$ is an accumulation point of $\{\vec{\phi}(\vec{z}_l)\}=\{\vec{z}_{l-1}\}$, $l=2, 3, \cdots$. Assume that $w_1>w_1'$, where $\vec{w}=(w_1, w_2)$. Proposition 3.3 implies that if there exists an l such that $z_{l,1}>w_1'$, then $z_{l+1,1}=\phi_1^{-1}(\vec{z}_l)>w_1>w_1'$, and consequently, $z_{l+n,1}=\vec{\phi}^{-n}(\vec{z}_l)_{,1}>w_1>w_1'$, $n\in\boldsymbol{Z}_+$. This contradicts the fact that $\vec{w}'$ is an accumulation point. Thus $z_{l,1}\leqq w_1'$ for all $l\in\boldsymbol{Z}_+$. But this contradicts the fact that $\vec{w}$ is also an accumulation point. The case $w_1<w_1'$ may be handled in a similar manner and a contradiction occurs. Therefore $w_1=w_1'$, hence $\vec{w}=\vec{w}'$.

Therefore every accumulation point $\vec{w}$ of the sequence $\{\vec{z}_l\}$, $l=1, 2, 3, \cdots$, is a fixed point of the map $\vec{\phi}$ in $\partial D^{(2)}\cap\Xi_0^{(2)}$. By Proposition 3.1 the only fixed point in $\partial D^{(2)}\cap\Xi_0^{(2)}$ is (x_c, y_c). Hence the sequence $\{\vec{z}_l\}$, $l=1, 2, 3, \cdots$, converges to (x_c, y_c) as $l\to\infty$.

It is proved that there is a sequence $\vec{z}_l$, $l=1, 2, 3, \cdots$, satisfying $\vec{z}_l\in\partial D^{(2)}\cap\Xi_0^{(2)}$, $l=1, 2, 3, \cdots$, and $\vec{y}=\vec{\phi}(\vec{z}_1)$, $\vec{z}_l=\vec{\phi}(\vec{z}_{l+1})$, $l=1, 2, 3, \cdots$, and $\lim_{l\to\infty}\vec{z}_l=(x_c, y_c)$. On the other hand, Proposition 3.3 implies that for $(x, y)\in\partial D^{(2)}$, if $x>\phi_1(x, y)$ then $\phi_1(x, y)>\phi_1(\vec{\phi}(x, y))$, and if $x<\phi_1(x, y)$ then $\phi_1(x, y)<\phi_1(\vec{\phi}(x, y))$. This with Proposition 3.4 implies that $\phi_1(\vec{\phi}^n(x, y))$ approaches x_c monotonically as $n\to\infty$. If $\vec{y}\neq(x_c, y_c)$, this is a contradiction. Therefore $\vec{y}=(x_c, y_c)$, which implies that the only accumulation point of $\{\vec{X}_n(\vec{x})\}$ is (x_c, y_c). Therfore $\vec{X}_n(\vec{x})$ converges as $n\to\infty$ to (x_c, y_c). This completes the proof. □

Let

$$Z_{i,n}(\beta)=\sum_{w\in W_i^{(n)}}\exp(-\beta L(w)),\qquad \beta\in\boldsymbol{R},\ i=1, 2, 3, 4,\ n\in\boldsymbol{Z}_+. \tag{3.8}$$

Then $\vec{Z}_n(\beta)=\vec{X}_n(\exp(-\beta), \exp(-2\beta), \exp(-2\beta), \exp(-3\beta))$.

Corollary 3.6. *There exists a constant β_c such that*

$$\lim_{n\to\infty}\vec{Z}_n(\beta)=(0, 0, 0, 0),\qquad \beta>\beta_c,$$

$$\lim_{n\to\infty}\vec{Z}_n(\beta)=(x_c, y_c, 0, 0),\qquad \beta=\beta_c,$$

$$\lim_{n\to\infty}\vec{Z}_n(\beta)=(\infty, \infty, \infty, \infty),\qquad \beta<\beta_c.$$

The proof of Theorem 1.1 is as follows. Note that

$$W^{*(n)}=W_1^{(n)}\cup W_3^{(n)}\cup W_4^{(n)}\cup W_3'^{(n)}\cup W_4'^{(n)},$$

where $W_i^{(n)}$, $i=1, 2, 3, 4$, are defined at the beginning of Section 2, and $W_4'^{(n)}$ in the proof of Proposition 2.1, and

$$W'^{(n)}_3=\{w\in W^{(n,0,a_n)}\,|\,w(\boldsymbol{Z}_+)\cap\{b_n,c_n\}=\{c_n\}\}.$$

Therefore, $Z_n^*=Z_{1,n}+2Z_{3,n}+2Z_{4,n}$ follows, which, with Corollary 3.6, implies Theorem 1.1.

The first derivatives of $\vec{\Phi}$ are used to study the distributions of path lengths. Let $\vec{a}=(a_1, a_2, a_3, a_4)=(x_c, y_c, 0, 0)$ and

$$B=\left(\frac{\partial^t\vec{\Phi}}{\partial x_1}(\vec{a}), \cdots, \frac{\partial^t\vec{\Phi}}{\partial x_4}(\vec{a})\right). \tag{3.9}$$

Proposition 3.7. (1) *The matrix B has a form*

$$B=\begin{pmatrix} p & q & B_{13} & B_{14} \\ q & r & B_{23} & B_{24} \\ 0 & 0 & B_{33} & B_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$p=8x_c^3+6x_c^2+2x_c+12x_c^2y_c+12x_cy_c^2,$$

$$q=4x_c^3+12x_c^2y_c,$$

and

$$r=4x_c^3+88y_c^3.$$

Every element is non-negative, and the four elements B_{ij}, $i=1, 2$, $j=3, 4$, are positive.

(2) *Denote the four eigenvalues of B by λ_i, $i=1, 2, 3, 4$. Then one can arrange the order of the eigenvalues so that they satisfy*

$$\lambda_1>1>\lambda_2>\lambda_3>\lambda_4=0,$$

and $B_{33}=\lambda_2$. In particular, B is diagonalizable by an invertible matrix P: $P^{-1}BP=\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, 0)$.

Proof. The four elements B_{ij}, $i=1, 2$, $j=1, 2$, are obtained from Proposition 2.1 (3). Proposition 2.1 (6) implies $B_{1j}\geqq x_c>0$, $j=3, 4$, and $B_{2j}\geqq x_c^2>0$, $j=3, 4$, and $B_{4j}=0$, $j=1, 2, 3, 4$. The non-negativity of elements are obvious.

It is easy to see that the four eigenvalues are $1/2\{p+r+\sqrt{(p-r)^2+4q^2}\}=2.7965\cdots$, $1/2\{p+r-\sqrt{(p-r)^2+4q^2}\}=0.2537\cdots$, B_{33}, and 0. The numerical values are derived by the estimates for x_c and y_c given in the remark after Proposition 3.1.

From Proposition 2.1 (5) one has $B_{33}=\Phi_{3,1}(\vec{a})$. From Proposition 2.1 (5) and the fact that $\vec{a}$ is fixed point it follows that $\Phi_{3,1}(\vec{a})\leqq 1-x_c<1$. From Proposition 2.1 (4) one has $B_{33}\geqq x_c^2+2x_c^3=0.3428\cdots$. This completes the proof.

□

Remark. The explicit form of B obtained using the explicit form of the recursion relations, is shown in Appendix B.

Let

$$\Xi_1 \stackrel{\text{def}}{=} \Xi \cup \iota(\Xi_0^{(2)}) \subset \Xi_0 . \tag{3.10}$$

Proposition 3.4 and Theorem 3.5 imply that if $\vec{x} \in \partial D \cap \Xi_1$, then $\lim_{n\to\infty} \vec{X}_n(\vec{x}) = (x_c, y_c, 0, 0)$.

The following proposition states how $\vec{X}_n(\vec{x})$ converges to $(x_c, y_c, 0, 0)$ as $n \to \infty$. Though the proof is similar to that for the case of diffeomorphisms in [5], we give a proof here for readers' convenience.

Proposition 3.8. *Assume $\vec{x} \in \partial D \cap \Xi_1$. Then there are positive constants C and $\rho < 1$, such that*

$$|X_{i,n}(\vec{x}) - a_i| < C\rho^n, \qquad i=1, 2, 3, 4, \ n \in \boldsymbol{Z}_+ .$$

Proof. $\vec{\Phi}(\vec{x})$ can be expressed as

$${}^t\vec{\Phi}(\vec{x}) = {}^t\vec{a} + B\,{}^t(\vec{x} - \vec{a}) + {}^t\vec{\theta}(\vec{x} - \vec{a}),$$

where $\vec{\theta}: \boldsymbol{R}^4 \to \boldsymbol{R}^4$ satisfies, for $\vec{x} \in \boldsymbol{R}^4$ *and* $|\vec{x} - \vec{a}| \leq 1$,

$$|\vec{\theta}(\vec{x} - \vec{a})| < C_1 |\vec{x} - \vec{a}|^2, \tag{3.11}$$

with a positive constant C_1. $\boldsymbol{R}^4$ splits into $\vec{\Phi}$-invariant stable and unstable subspaces

$$\boldsymbol{R}^4 = V_s \oplus V_u ,$$

where V_s is spanned by the eigenvectors corresponding to λ_2, λ_3 and 0, and V_u by that corresponding to λ_1, where λ_i s' are as in Proposition 3.7. Denote the restrictions of B to V_s and V_u by B_s and B_u, respectively. For $\vec{x} \in \boldsymbol{R}^4$ define norms

$$|\vec{x}|^* \stackrel{\text{def}}{=} |P^{-1}\vec{x}| ,$$

$$|\vec{x}|_o \stackrel{\text{def}}{=} \max\{|\vec{x}_s|^*, |\vec{x}_u|^*\} ,$$

and for an 4×4 matrix A,

$$\|A\|^* \stackrel{\text{def}}{=} \sup_{|\vec{x}|^*=1} |A\vec{x}|^* .$$

Let

$$\alpha \stackrel{\text{def}}{=} \max\{\|B_s\|^*, \|B_u^{-1}\|^*\} < 1 .$$

Take $\kappa > 0$ such that $\alpha + \kappa < 1$. Then by eq. (3.11) there is a δ, $0 < \delta < 1$, such that

$$|\vec{\theta}(\vec{x}-\vec{a})|_o<\kappa|\vec{x}-\vec{a}|_o,$$

for

$$|\vec{x}-\vec{a}|_o<\delta.$$

Assume that $\vec{x}\in\partial D\cap\Xi_1$. Then there is an n_0 such that

$$|\vec{X}_n(\vec{x})-\vec{a}|_o<\delta,$$

for $n\geqq n_0$. Let $\vec{x}_0=\vec{X}_{n_0}(\vec{x})$.

Consider

$$\vec{\Phi}(\vec{x}_0)-\vec{a}=(\vec{\Phi}(\vec{x}_0)-\vec{a})_s+(\vec{\Phi}(\vec{x}_0)-\vec{a})_u.$$

Then

$$\begin{aligned}|(\vec{\Phi}(\vec{x}_0)-\vec{a})_s|^*&\leqq\|B_s\|^*|(\vec{x}_0-\vec{a})_s|^*+\kappa|\vec{x}_0-\vec{a}|_o\\&\leqq(\alpha+\kappa)|\vec{x}_0-\vec{a}|_o.\end{aligned}$$

Suppose that $|\vec{x}_0-\vec{a}|_o=|(\vec{x}^0-\vec{a})_u|^*$. Then

$$\begin{aligned}|(\vec{\Phi}(\vec{x}_0)-\vec{a})_u|^*&\geqq(\alpha^{-1}-\kappa)|\vec{x}_0-\vec{a}|_o\\&>(\alpha+\kappa)^{-1}|\vec{x}_0-\vec{a}|_o.\end{aligned}$$

By induction,

$$|\vec{X}_n(\vec{x}_0)-\vec{a}|_o\geqq(\alpha+\kappa)^{-n}|\vec{x}_0-\vec{a}|_o.$$

Since $(\alpha+\kappa)^{-n}\to\infty$ as $n\to\infty$, this leads to a contradiction. Thus, $|\vec{x}_0-\vec{a}|_o=|(\vec{x}_0-\vec{a})_s|^*$.

A similar argument shows that $|\vec{\Phi}(\vec{x}_0)-\vec{a}|_o=|(\vec{\Phi}(\vec{x}_0)-\vec{a})_s|^*$. Thus $|\vec{\Phi}(\vec{x}_0)-\vec{a}|_o\leqq(\alpha+\kappa)|\vec{x}_0-\vec{a}|_o$. By induction it follows that

$$|\vec{X}_n(\vec{x}_0)-\vec{a}|_o\leqq\rho^n|\vec{x}_0-\vec{a}|_o,$$

where $\rho=\alpha+\kappa$, $0<\rho<1$. Thus for each $\vec{x}\in\partial D\cap\Xi_1$ there is a positive constant C' such that

$$|\vec{X}_n(\vec{x})-\vec{a}|_o\leqq C'\rho^n,$$

for all $n\in Z_+$. This completes the proof. □

§ 4. Limit Theorem for Distribution of Path Lengths

First we define probability measures $\mu_n(\vec{x})$, $\mu_{1,n}(\vec{x})$, and $\nu_n(\vec{x})$ on $W^{*(n)}$, $W_1^{(n)}$, and $W_2^{(n)}$, respectively. $W^{*(n)}$ is defined in Section 1, and $W_1^{(n)}$ and $W_2^{(n)}$ in Section 2. Each measure is parametrized by $\vec{x}=(x_1, x_2, x_3, x_4)$ taking values in $\boldsymbol{R}_+^4\setminus\{(0,0,0,0)\}$. To each $w\in W^{*(n)}$, we assign the weight

$$\mu_n(\vec{x})[w]\overset{\text{def}}{=}\{X_{1,n}(\vec{x})+2X_{3,n}(\vec{x})+2X_{4,n}(\vec{x})\}^{-1}\prod_{i=1}^{4}x_i^{s_i(w)},$$

to each $w\in W_1^{(n)}$,

$$\mu_{1,n}(\vec{x})[w] \stackrel{\text{def}}{=} \{X_{1,n}(\vec{x})\}^{-1} \prod_{i=1}^{4} x_i^{s_i(w)},$$

and to each $w \in W_2^{(n)}$,

$$\nu_n(\vec{x})[w] \stackrel{\text{def}}{=} \{X_{2,n}(\vec{x})\}^{-1} \prod_{i=1}^{4} x_i^{s_i(w)},$$

where $s_i(w)$ and $X_{i,n}(\vec{x})$, $i=1, \cdots, 4$, $n \in \boldsymbol{Z}_+$ are defined in Section 2.

Our objective in this section is to study the asymptotic distribution of path lengths $L(w)$ under $\mu_n(\vec{x})$, $\mu_{1,n}(\vec{x})$ and $\nu_n(\vec{x})$, respectively, as n tends to infinity. Each element of $W_2^{(n)}$ consists of two path segments. Since we want to deal with these segments separately, we define, for $w=(w_1, w_2) \in W_2^{(n)}$,

$$\tilde{s}_9(w) = s_2(w) - s_2(w_1) - s_2(w_2),$$

$$\tilde{s}_1(w) = s_1(w_1) - \tilde{s}_9(w),$$

$$\tilde{s}_2(w) = s_2(w_1),$$

$$\tilde{s}_3(w) = s_3(w_1),$$

$$\tilde{s}_4(w) = s_4(w_1),$$

$$\tilde{s}_5(w) = s_1(w_2) - \tilde{s}_9(w),$$

$$\tilde{s}_6(w) = s_2(w_2),$$

$$\tilde{s}_7(w) = s_3(w_2),$$

$$\tilde{s}_8(w) = s_4(w_2).$$

Note that

(4.1)
$$L(w_1) = \tilde{s}_1(w) + 2\tilde{s}_2(w) + 2\tilde{s}_3(w) + 3\tilde{s}_4(w) + \tilde{s}_9(w),$$

(4.2)
$$L(w_2) = \tilde{s}_5(w) + 2\tilde{s}_6(w) + 2\tilde{s}_7(w) + 3\tilde{s}_8(w) + \tilde{s}_9(w).$$

Let Y_n be another generating function for $W_2^{(n)}$ defined by

$$Y_n(z) \stackrel{\text{def}}{=} \sum_{w \in W_2^{(n)}} \prod_{i=1}^{9} x_i^{\tilde{s}_i(w)},$$

where

$$z = (\vec{z}_1, \vec{z}_2, x_9) \in \boldsymbol{C}^9,$$

$$\vec{z}_1 = (x_1, x_2, x_3, x_4),$$

$$\vec{z}_2 = (x_5, x_6, x_7, x_8).$$

Y_n satisfies the following recursion relation.

$$Y_{n+1}(z) = Y_1((\vec{X}_n(\vec{z}_1), \vec{X}_n(\vec{z}_2), Y_n(z))).$$

We also have

(4.3) $$Y_1((\vec{z}_1, \vec{z}_1, x_2)) = \Phi_2(x_1, x_2, x_3, x_4).$$

We start out with

$$H^{(n)}(z) = (H_1^{(n)}(z), \cdots, H_9^{(n)}(z)) \overset{\text{def}}{=} (\vec{X}_n(\vec{z}_1), \vec{X}_n(\vec{z}_2), Y_n(z)).$$

Let $\partial H^{(n)}(z)$ be an 9×9 matrix defined by

$$\partial H^{(n)}(z) \overset{\text{def}}{=} \left(\frac{\partial}{\partial x_1}{}^t H^{(n)}(z), \cdots, \frac{\partial}{\partial x_9}{}^t H^{(n)}(z)\right).$$

Since the recursion relations imply,

(4.4) $$H^{(n)}(z) = H^{(1)}(H^{(n-1)}(z)),$$

we have

(4.5) $$\partial H^{(n)}(z) = \partial H^{(1)}(H^{(n-1)}(z))\partial H^{(1)}(H^{(n-2)}(z))\cdots\partial H^{(1)}(z).$$

Throughout this section, we write,

$$a = (a_1, \cdots, a_9) \overset{\text{def}}{=} (x_c, y_c, 0, 0, x_c, y_c, 0, 0, y_c).$$

We have, in particular,

$$H^{(n)}(a) = a,$$

$$\partial H^{(n)}(a) = (\partial H^{(1)}(a))^n.$$

Proposition 4.1. (1) *All the eigenvalues of $\partial H^{(1)}(a)$ are non-negative. The largest of them, λ, is a double eigenvalue with corresponding left eigenvectors* $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, 0, 0, 0, 0, 0)$ *and* $(0, 0, 0, 0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, 0)$, *satisfying* $\alpha_i > 0$, $i = 1, 2, 3, 4$. *Any other eigenvalue is less than* 1.

(2)

$$\lim_{n\to\infty} \lambda^{-n}(\partial H^{(1)}(a))^n = \Lambda(a)$$

exists. The (i, j)-element of $\Lambda(a)$, $\Lambda_{ij}(a)$, is non-negative for $i=1, \cdots, 9$, $j=1, \cdots, 9$. *In particular,* $\Lambda_{11}(a) > 0$ *and* $\Lambda_{91}(a) > 0$.

Proof. It is easy to see that $\partial H^{(1)}(a)$ has the form

$$
(4.6)\qquad \partial H^{(1)}(a)=\begin{pmatrix}
\begin{matrix} & \mathbf{B} & \end{matrix} & \begin{matrix} & \mathbf{0} & \end{matrix} & \begin{matrix}0\\0\\0\\0\end{matrix}\\
\begin{matrix} & \mathbf{0} & \end{matrix} & \begin{matrix} & \mathbf{B} & \end{matrix} & \begin{matrix}0\\0\\0\\0\end{matrix}\\
\frac{q}{2}\ \ C_1\ \ \frac{B_{23}}{2}\ \ \frac{B_{24}}{2} & \frac{q}{2}\ \ C_1\ \ \frac{B_{23}}{2}\ \ \frac{B_{24}}{2} & r-2C_1
\end{pmatrix},
$$

where C_1 is a constant satisfying $0\leqq C_1 \leqq r/2$. The 4×4 matrix B and the positive constant r are defined in Proposition 3.7. $\partial H^{(1)}(a)$ has four double eigenvalues λ_1, λ_2, λ_3, 0, and a single eigenvalue $r-2C_1$. Note that $\lambda_1 > r = 0.3277 \cdots \geqq r-2C_1$. It is also easy to show that the right and left eigenvectors of B corresponding to λ_1 can be chosen as ${}^t(\alpha_1, \alpha_2, 0, 0)$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, with $\alpha_i > 0$, $i=1, 2, 3, 4$, $\alpha_1^2+\alpha_2^2=1$. The assertion (1) follows with $\lambda=\lambda_1$.

There is an invertible matrix $\tilde{P}$ such that

$$\tilde{P}^{-1}\partial H^{(1)}(a)\tilde{P}=\mathrm{diag}(\lambda, \lambda_2, \lambda_3, 0, \lambda, \lambda_2, \lambda_3, 0, r-2C_1),$$

and that the first and the fifth columns of $\tilde{P}$ are ${}^t(\alpha_1, \alpha_2, 0, \cdots, 0, C_2)$ and ${}^t(0, 0, 0, 0, \alpha_1, \alpha_2, 0, 0, C_2)$, $C_2>0$, respectively, and that the first row of $\tilde{P}^{-1}$ is $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, 0, \cdots, 0)$. Combining these with

$$\lim_{n\to\infty}\tilde{P}^{-1}\lambda^{-n}(\partial H^{(1)}(a))^n\tilde{P}=\mathrm{diag}(1, 0, 0, 0, 1, 0, 0, 0, 0),$$

one has the assertion (2). In particular, $\Lambda_{11}(a)=\alpha_1^2$ and $\Lambda_{91}(a)=C_2\alpha_1$. This completes the proof. □

From the proof of Proposition 3.7, we have $\lambda=2.7965\cdots$.

In studying the limit of $\lambda^{-n}(\partial H^{(1)}(z))^n$ for more general z, we make use of the following lemma. It can be proved in a similar fashion to Lemma (3.1) in [3].

Lemma 4.2. *Let A, A_n, $n=1, 2, \cdots$, be $N\times N$ matrices. Assume that there is an invertible $N\times N$ matrix P such that $P^{-1}AP=\mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, $\lambda_i\geqq 0$, $i=1, \cdots, N$, $\lambda_{\max}=\max\lambda_i>0$. Assume further that*

$$\sum_{n=1}^{\infty}\|A_n-A\|<\infty .$$

Then

(4.7) $$\lim_{m\to\infty}\limsup_{n\to\infty}\|\lambda_{\max}^{-n}(A_{n+m}A_{n+m-1}\cdots A_{m+1})-Q\|=0,$$

where $P^{-1}QP=\mathrm{diag}(q_1, \cdots, q_N)$ *with* $q_i=1$ *if* $\lambda_i=\lambda_{\max}$ *and* $q_i=0$ *otherwise. Moreover,* $\lim_{n\to\infty}\lambda_{\max}^{-n}A_n\cdots A_1$ *exists.*

Let Ξ_1 be as in eq. (3.10) and

$$\Gamma=\{z=(x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, x_2)\,|\,(x_1, x_2, x_3, x_4)\in\partial D\cap\Xi_1\}.$$

Proposition 4.3. *Let* $z=(z_1, \cdots, z_9)\in\Gamma$. *Then* $\lim_{n\to\infty}\lambda^{-n}\partial H^{(n)}(z)=\Lambda(z)$ *exists, and* $\Lambda_{ij}(z)\geqq 0$ *for* $i=1, \cdots, 9$, $j=1, \cdots, 9$. *In particular,* $\Lambda_{11}(z)>0$ *and* $\Lambda_{91}(z)>0$.

Proof. By the mean-value theorem,

(4.8) $$\partial H_{ij}(z)-\partial H_{ij}(a)=\sum_{k=1}^{9}\Big(\frac{\partial}{\partial z_k}\partial H_{ij}\Big)(u)\cdot(z_k-a_k),$$

where

$$u=a+\theta(z-a), \qquad 0<\theta<1.$$

Since Γ is a bounded region in $\boldsymbol{R}^9$ and $(\partial/\partial z_k)\partial H_{ij}(z)$ is a polynomial in $z_1, \cdots, z_9$, there is a positive constant M such that

(4.9) $$\Big|\frac{\partial}{\partial z_k}\partial H_{ij}(u)\Big|<M$$

for all $u=a+\theta(z-a)$, $0<\theta<1$, $z\in\Gamma$, $i, j, k=1, \cdots, 9$. On the other hand, by Proposition 3.8, for each $\vec{x}\in\partial D\cap\Xi_1$, there are positive constants C and ρ, $\rho<1$, such that

(4.10) $$|X_{i,n}(\vec{x})-a_i|\leqq C\rho^n, \qquad i=1, \cdots, 4.$$

From eq. (4.8), eq. (4.9), and eq. (4.10), it follows that there is a constant C_1 such that

(4.11) $$\|\partial H(H^{(n)}(z))-\partial H(a)\|\leqq C_1\rho^n.$$

Now let $A_n=\partial H(H^{(n)}(z))$ and $A=\partial H(a)$. From eq. (4.11),

$$\sum_{n=1}^{\infty}\|A_n-A\|<\infty.$$

Lemma 4.2 implies the existence of $\lim_{n\to\infty}\lambda^{-n}\partial H^{(n)}(z)$. Since $Q=\Lambda(a)$ in this case, eq. (4.7) implies that for sufficiently large m, the (1, 1) and the (9, 1)-elements of $\lim_{n\to\infty}\lambda^{-n}\partial H(H^{(n+m)}(z))\cdots\partial H(H^{(m+1)}(z))$ are positive. From Proposition 2.1 (3), $X_{1,1}(\vec{x})$ includes x_1^2, and $X_{2,1}(\vec{x})$ includes x_1^4, which implies that $\partial H(H^{(k)}(z))$, $k=1, \cdots, m$, has positive (1, 1) and (9, 1)-elements for $\vec{x}\in\partial D\cap\Xi_1$. Therefore, $\Lambda_{ij}(z)>0$ for $(i, j)=(1, 1)$ and $(i, j)=(9, 1)$. This completes the proof. □

Proposition 4.4. *Assume* $z\in\Gamma$. *Let*

$$H_i^{(n)}(ze^{\lambda^{-n}t})\overset{\text{def}}{=}H_i^{(n)}(z_1e^{\lambda^{-n}t_1}, z_2e^{\lambda^{-n}t_2}, \cdots, z_9e^{\lambda^{-n}t_9}), \qquad i=1, \cdots, 9.$$

(1) *There are entire functions* $H_i^*: \boldsymbol{C}^9\to\boldsymbol{C}$, *such that* $H_i^{(n)}(ze^{\lambda^{-n}t})\to H_i^*(t)$, *as* $n\to\infty$ *uniformly in* $\{t=(t_1, \cdots, t_9)\in\boldsymbol{C}^9 \mid |t_i|\leqq R, i=1, \cdots, 9\}$ *for all* $R>0$. *In particular*, $H_i^*(t)\equiv 0$, *for* $i=3, 4, 7, 8$.

(2) *Let* $H^*(t)=(H_1^*(t), H_2^*(t), 0, 0, H_5^*(t), H_6^*(t), 0, 0, H_9^*(t))$. *Then* $H^*(t)$ *satisfies,*

$$H^*(\lambda t)=H^{(1)}(H^*(t)), \tag{4.12}$$

for any $t\in\boldsymbol{C}^9$. *Moreover,*

$$\frac{\partial}{\partial t_j}H_i^*(0)=z_jA_{ij}(z).$$

Proof. Let

$$|z|_*\overset{\text{def}}{=}\alpha_1\frac{|z_1|+|z_5|}{2}+\alpha_2\max\left(\frac{|z_2|+|z_6|}{2}, |z_9|\right)+\alpha_3\frac{|z_3|+|z_7|}{2}+\alpha_4\frac{|z_4|+|z_8|}{2},$$

for $z=(z_1, \cdots, z_9)\in\boldsymbol{C}^9$. $|\cdot|_*$ satisfies the conditions for a norm. From Proposition 4.1, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, 0, 0, 0, 0, 0)$ and $(0, 0, 0, 0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, 0)$ are the left eigenvectors of $\partial H^{(1)}(a)$ corresponding to λ. It follows that

$$|\partial H^{(1)}(a)^t z|_*\leqq\lambda|z|_*. \tag{4.13}$$

Now fix a $z\in\Gamma$ and let $w\in\boldsymbol{R}^9$. Put

$$H^{(n)}(z+w)=a+v_n+w_n, \tag{4.14}$$

$$v_n\overset{\text{def}}{=}H^{(n)}(z)-a,$$

$$w_n\overset{\text{def}}{=}H^{(n)}(z+w)-H^{(n)}(z).$$

From the mean-value theorem combined with eq. (4.13), it follows that there is a positive constant C_1 such that

$$|H^{(1)}(a+v_n+w)-H^{(1)}(a+v_n)|_*\leqq\lambda(1+C_1|v_n|_*)(1+C_1|w|_*)|w|_* \tag{4.15}$$

for $|w|_*\leqq 1$ and $n\in\boldsymbol{Z}_+$. By Proposition 3.8, there are positive constants C_2 and $\rho<1$ such that,

$$|v_n|_*\leqq C_2\rho^n. \tag{4.16}$$

Take a positive number b such that

$$b\cdot\prod_{k=0}^{\infty}(1+C_3\rho^k)(1+C_1\lambda^{-k})\leqq 1,$$

where $C_3=C_1C_2$. Then by eq. (4.15) and induction,

(4.17)
$$|w_k|_* \leqq \lambda^k |w|_* \prod_{j=0}^{k-1}(1+C_3\rho^j)(1+C_1\lambda^{-(n-j)})$$
$$\leqq \lambda^k |w|_*/b,$$

for $k \in \boldsymbol{Z}_+$, $0 \leqq k \leqq n$, and $w \in \boldsymbol{R}^9$, $|w|_* \leqq b\lambda^{-n}$. Thus the estimates eq. (4.16) and eq. (4.17) together with eq. (4.14) show that there is a $\delta > 0$ and a $C_4 > 0$ such that

(4.18)
$$|H_i^{(n)}(ze^{\lambda^{-n}t})|_* \leqq C_4,$$

for $t \in \boldsymbol{R}^9$, $|t|_* < \delta$. Since each $H_i^{(n)}$ is a polynomial with positive coefficients, we see that eq. (4.18) holds also in $\Omega \stackrel{\text{def}}{=} \{t \in \boldsymbol{C}^9 \mid |t|_* < \delta\}$. Therefore, for each i,

$$\{H_i^{(n)}(ze^{\lambda^{-n}t})\}, \qquad n=1, 2, \cdots$$

forms a normal family of holomorphic functions in Ω. Let

(4.19)
$$H_i^{(n)}(ze^{\lambda^{-n}t}) = \sum_k a_i^{(n)}(\vec{k}) t_1^{k_1} t_2^{k_2} \cdots t_9^{k_9},$$
$$\vec{k} = (k_1, \cdots, k_9) \in \boldsymbol{Z}_+^9.$$

By Theorem 3.5,

(4.20)
$$a_i^{(n)}(\vec{0}) \longrightarrow a_i, \qquad n \longrightarrow \infty.$$

Define $\vec{e}^{(i)} \in \boldsymbol{Z}_+^9$ by $e_j^{(i)} = \delta_{ij}$. By Proposition 4.3,

(4.21)
$$a_i^{(n)}(\vec{e}^{(j)}) \longrightarrow z_j \Lambda_{ij}(z), \qquad n \longrightarrow \infty.$$

Substitute eq. (4.19) into eq. (4.4) and let $n \to \infty$. By induction starting with eq. (4.20) and eq. (4.21), we see that there are $a_i^*(\vec{k})$'s such that

$$a_i^{(n)}(\vec{k}) \longrightarrow a_i^*(\vec{k}), \qquad n \longrightarrow \infty.$$

Therefore there are holomorphic functions $H_i^*: \Omega \to \boldsymbol{C}$, such that $H_i^{(n)}(ze^{\lambda^{-n}t}) \to H_i^*(t)$, as $n \to \infty$ uniformly in $\{t \in \boldsymbol{C}^9 \mid |t|_* \leqq \delta/2\}$, satisfying eq. (4.12). For any $R > 0$, take an $m \in \boldsymbol{N}$ such that $\lambda^{-m}R < \delta/2$. Then

$$H_i^{(n+m)}(ze^{\lambda^{-(n+m)}t}) = H_i^{(m)}(H^{(n)}(ze^{\lambda^{-n}(\lambda^{-m}t)}))$$
$$\longrightarrow H_i^{(m)}(H^*(\lambda^{-m}t)),$$

as $n \to \infty$ uniformly in $\{t \in \boldsymbol{C}^9 \mid |t|_* \leqq R\}$. This shows that H^* can be extended to an entire function in $\boldsymbol{C}^9$, satisfying eq. (4.12).

Let $ze^{\lambda^{-n}(it)} = (z_1 e^{\lambda^{-n}it_1}, \cdots, z_9 e^{\lambda^{-n}it_9})$, $z = (z_1, \cdots, z_9) = (\vec{x}, \vec{x}, x_2) \in \Gamma$, and $(t_1, \cdots, t_9) \in \boldsymbol{R}^9$. Since $X_3^{(n)}$ is a polynomial with positive coefficients,

$$|H_3^{(n)}(ze^{\lambda^{-n}(it)})| \leqq |H_3^{(n)}(z)| = |X_3^{(n)}(\vec{x})|.$$

Therefore, by Theorem 3.5,

$$H_3^{(n)}(ze^{\lambda^{-n}(it)}) \longrightarrow 0, \qquad n \longrightarrow \infty.$$

This and the fact that H_3^* is an entire function leads to

$$H_3^*(t)\equiv 0$$

on C^9. In the same way, we have

$$H_i^*(t)\equiv 0$$

on C^9, for $i=4, 7, 8$. This completes the proof. □

Note that H^* has an $\vec{x}$-dependence, though we do not write explicitly.

Let $p_n(\vec{x})$ and $p_n^*(\vec{x})$ denote the law of $\lambda^{-n}L(w)$ under $\mu_n(\vec{x})$ and under $\mu_{1,n}(\vec{x})$, respectively. Let $q_n(\vec{x})$, $q_n^+(\vec{x})$, and $q_n^*(\vec{x})$ be the law of $(\lambda^{-n}L(w_1), \lambda^{-n}L(w_2))$, $\lambda^{-n}(L(w_1)+L(w_2))$, and $\lambda^{-n}L(w_1)$, respectively, under $\nu_n(\vec{x})$. Note that the law of $\lambda^{-n}L(w_2)$ under $\nu_n(\vec{x})$ is also equal to $q_n^*(\vec{x})$. We often omit writing the dependence on $\vec{x}$, when no confusion occurs.

We define

$$g^{(n)}(t)\stackrel{\text{def}}{=}\int_0^\infty e^{t\xi}p_n(d\xi), \qquad t\in\boldsymbol{C}, \tag{4.22}$$

$$g_1^{(n)}(t)\stackrel{\text{def}}{=}\int_0^\infty e^{t\xi}p_n^*(d\xi), \qquad t\in\boldsymbol{C}, \tag{4.23}$$

and

$$h^{(n)}(t_1, t_2)\stackrel{\text{def}}{=}\int_{\boldsymbol{R}_+\times\boldsymbol{R}_+} e^{t_1\xi_1+t_2\xi_2}q_n(d\xi_1 d\xi_2), \qquad (t_1, t_2)\in\boldsymbol{C}^2. \tag{4.24}$$

Note that

$$h^{(n)}(t, t)=\int_0^\infty e^{t\xi}q_n^+(d\xi),$$

$$h^{(n)}(t, 0)=\int_0^\infty e^{t\xi}q_n^*(d\xi).$$

From the relation, $L(w)=s_1(w)+2s_2(w)+2s_3(w)+3s_4(w)$, $w\in W^{*(n)}$, and eq. (4.1) and eq. (4.2), it follows that

$$\begin{aligned} &g^{(n)}(t)\\ &=\{X_{1,n}(\vec{x})+2X_{3,n}(\vec{x})+2X_{4,n}(\vec{x})\}^{-1}\{X_{1,n}(\vec{x}_{n,t})+2X_{3,n}(\vec{x}_{n,t})+2X_{4,n}(\vec{x}_{n,t})\},\\ &g_1^{(n)}(t)=\{X_{1,n}(\vec{x})\}^{-1}X_{1,n}(\vec{x}_{n,t}), \end{aligned}$$

where

$$\vec{x}_{n,t}\stackrel{\text{def}}{=}(x_1e^{\lambda^{-n}t}, x_2e^{2\lambda^{-n}t}, x_3e^{2\lambda^{-n}t}, x_4e^{3\lambda^{-n}t}),$$

and

$$h^{(n)}(t_1, t_2)=\{X_{2,n}(\vec{x})\}^{-1}F_n(\vec{x}_{n,t_1}, \vec{x}_{n,t_2}, x_2e^{\lambda^{-n}(t_1+t_2)}).$$

Theorem 3.5 and Proposition 4.4 imply that for $\vec{x}\in\partial D\cap\Xi_1$,

$$g^{(n)}(t) \longrightarrow \frac{1}{x_c} H_1^*(t,\ 2t,\ 2t,\ 3t,\ 0,\ 0,\ 0,\ 0,\ 0),$$

$$g_1^{(n)}(t) \longrightarrow \frac{1}{x_c} H_1^*(t,\ 2t,\ 2t,\ 3t,\ 0,\ 0,\ 0,\ 0,\ 0),$$

$$h^{(n)}(t_1,\ t_2) \longrightarrow \frac{1}{y_c} H_9^*(t_1,\ 2t_1,\ 2t_1,\ 3t_1,\ t_2,\ 2t_2,\ 2t_2,\ 3t_2,\ t_1+t_2),$$

as $n\to\infty$ uniformly in $\{t\in \boldsymbol{C} \mid |t|\leqq R\}$ and $\{(t_1,\ t_2)\in \boldsymbol{C}^2 \mid |t_i|\leqq R,\ i=1,\ 2\}$, respectively, for all $R>0$. This leads to

Proposition 4.5. *Assume* $\vec{x}=(x_1,\ x_2,\ x_3,\ x_4)\in\partial D\cap\Xi_1$. *There are entire functions* $g:\ \boldsymbol{C}\to\boldsymbol{C}$ *and* $h:\ \boldsymbol{C}^2\to\boldsymbol{C}$ *such that*

$$g_1^{(n)}(t) \longrightarrow g(t),$$

$$g^{(n)}(t) \longrightarrow g(t),$$

as $n\to\infty$ *uniformly in* $\{t\in \boldsymbol{C} \mid |t|\leqq R\}$, *and*

$$h^{(n)}(t_1,\ t_2) \longrightarrow h(t_1,\ t_2),$$

as $n\to\infty$ *uniformly in* $\{(t_1,\ t_2)\in \boldsymbol{C}^2 \mid |t_i|\leqq R,\ i=1,\ 2\}$, *for all* $R>0$. $g(t)$ *and* $h(t_1,\ t_2)$ *are the unique solution to;*

$$x_c g(\lambda t)=\phi_1(x_c g(t),\ y_c h(t,\ t)), \tag{4.25}$$

$$y_c h(\lambda t_1,\ \lambda t_2)=f(x_c g(t_1),\ y_c h(t_1,\ t_1),\ x_c g(t_2),\ y_c h(t_2,\ t_2),\ y_c h(t_1,\ t_2)), \tag{4.26}$$

where ϕ_1 *is defined in eq.* (3.1) *and*

$$f(y_1,\ y_2,\ y_3,\ y_4,\ y_5)\overset{\text{def}}{=}Y_1(y_1,\ y_2,\ 0,\ 0,\ y_3,\ y_4,\ 0,\ 0,\ y_5)$$
$$=y_1^2y_3^2+2(y_1^2y_3+y_1y_3^2)y_5+6(y_2^2+y_4^2)y_5^2+4(y_2+y_4)y_5^3+2y_5^4.$$

$$\frac{\partial g(0)}{\partial t}=\frac{1}{x_c}(x_1\Lambda_{11}(z)+2x_2\Lambda_{12}(z)+2x_3\Lambda_{13}(z)+3x_4\Lambda_{14}(z)), \tag{4.27}$$

$$\frac{\partial h(0,\ 0)}{\partial t_1}=\frac{\partial h(0,\ 0)}{\partial t_2} \tag{4.28}$$
$$=\frac{1}{y_c}(x_1\Lambda_{91}(z)+2x_2\Lambda_{92}(z)+2x_3\Lambda_{93}(z)+3x_4\Lambda_{94}(z)+x_2\Lambda_{99}(z)),$$

where $z=(\vec{x},\ \vec{x},\ x_2)$.

Proposition 4.6. *Assume* $\vec{x}=(x_1,\ x_2,\ x_3,\ x_4)\in\partial D\cap\Xi_1$. *There are probability measures,* $p(\vec{x})$, $q^+(\vec{x})$ *and* $q^*(\vec{x})$ *on* $\boldsymbol{R}$, *and a probability measure* $q(\vec{x})$ *on* $\boldsymbol{R}^2$ *such that*

$$p_n(\vec{x}) \Longrightarrow p(\vec{x}),$$

$$p_n^*(\vec{x}) \Longrightarrow p(\vec{x}),$$

$$q_n^+(\vec{x}) \Longrightarrow q^+(\vec{x}),$$

$$q_n^*(\vec{x}) \Longrightarrow q^*(\vec{x}),$$

and

$$q_n(\vec{x}) \Longrightarrow q(\vec{x}),$$

as $n\to\infty$, *where* $\Rightarrow$ *denotes weak convergence.*

(1) *The Laplace transforms of the limit measures are given by,*

$$\int_{-\infty}^{\infty} e^{t\xi} p(\vec{x})(d\xi)=g(t),$$

$$\int_{-\infty}^{\infty} e^{t\xi} q^+(\vec{x})(d\xi)=h(t, t),$$

$$\int_{-\infty}^{\infty} e^{t\xi} q^*(\vec{x})(d\xi)=h(t, 0),$$

and

$$\int_{R\times R} e^{t_1\xi_1+t_2\xi_2} q(\vec{x})(d\xi_1 d\xi_2)=h(t_1, t_2).$$

(2)

$$\int_{-\infty}^{\infty} \xi p(\vec{x})(d\xi)>0,$$

$$\int_{-\infty}^{\infty} \xi q^+(\vec{x})(d\xi)>0,$$

$$\int_{-\infty}^{\infty} \xi q^*(\vec{x})(d\xi)>0.$$

(3) *None of* $p(\vec{x})$, $q^+(\vec{x})$ *and* $q^*(\vec{x})$ *is concentrated on a single point.*

The assertion (2) follows from Proposition 4.3, eq. (4.27), and ed. (4.28). The assertion (2) combined with eq. (4.25) and eq. (4.26) leads to assertion (3).

Proposition 4.7. *There are positive constants* C_1 *and* C_2 *such that*

$$|g(it)|<C_2 e^{-C_1|t|^\kappa},$$

$$|h(it, it)|<C_2 e^{-C_1|t|^\kappa},$$

$$|h(it, 0)|=|h(0, it)|<C_2 e^{-C_1|t|^\kappa},$$

for $t\in\boldsymbol{R}$, *where* $\kappa=\log 2/\log\lambda$.

Proof. By Proposition 4.5 and eq. (4.3),

$$g(i\lambda t)=\frac{1}{x_c}\phi_1(x_c g(it), y_c h(it, it)), \tag{4.29}$$

$$h(i\lambda t, i\lambda t)=\frac{1}{y_c}\phi_2(x_c g(it), y_c h(it, it)). \tag{4.30}$$

Define

$$G(t) \overset{\text{def}}{=} -|t|^{-\kappa} \log|g(it)|,$$

and

$$H(t) \overset{\text{def}}{=} -|t|^{-\kappa} \log|h(it, it)|.$$

Substituting these in eq. (4.29) and eq. (4.30), and using the fact that $|g(it)| \leqq 1$ and $|h(it, it)| \leqq 1$, we have, from eq. (3.1) and eq. (3.2),

$$\log|g(i\lambda t)| \leqq 2 \log|g(it)|,$$

$$\log|h(i\lambda t, i\lambda t)| \leqq 2(\log|g(it)| \vee |h(it, it)|).$$

From these it follows that .

(4.31) $$G(\lambda t) \geqq G(t),$$

(4.32) $$H(\lambda t) \geqq G(t) \wedge H(t).$$

By Proposition 4.6 (3), there is a constant $\delta > 0$ and a constant C, $0 < C < 1$ such that

$$0 < |g(it)| < C,$$

$$0 < |h(it, it)| < C,$$

for any $t \in \boldsymbol{R}$, $\lambda^{-1}\delta \leqq |t| < \delta$. Therefore, eq. (4.31) and eq. (4.32) lead to

$$G(t) > C_1,$$

$$H(t) > C_1,$$

for any $t \geqq \delta$ with $C_1 = -\delta^{-\kappa} \log C > 0$. This implies

$$|g(it)| < e^{-C_1|t|^\kappa},$$

$$|h(it, it)| < e^{-C_1|t|^\kappa},$$

for $t \geqq \delta$. Take $C_2 > e^{C_1\delta^\kappa}$. Then we have

$$|g(it)| < C_2 e^{-C_1|t|^\kappa},$$

$$|h(it, it)| < C_2 e^{-C_1|t|^\kappa},$$

for $t \in \boldsymbol{R}$. The estimate for $|h(it, 0)|$ is obtained similarly. This completes the proof. □

We use the following property in later sections.

Proposition 4.8. *p, q^+, and q^* have C^∞ densities. In particular, ρ, the density of p, satisfies*

$$\rho(\xi) = 0, \quad \xi \leqq 0, \quad \textit{and} \quad \rho(\xi) > 0, \quad \xi > 0.$$

Proof. Proposition 4.7 and the fact that g and h are entire functions imply that $g(it)$, $h(it, it)$, and $h(it, 0)$ with $t\in\boldsymbol{R}$ are rapidly decreasing functions. From this, the existence of the C^{∞} densities for p, q^{+}, and q^{*} follows. Let ρ^{+} be the density of q^{+}. Then eq. (4.25) and eq. (3.1) imply that

$$\lambda^{-1}\rho(\lambda^{-1}\xi)=x_c\rho*\rho(\xi)+2x_c^2\rho*\rho*\rho(\xi)+2x_c^3\rho*\rho*\rho*\rho(\xi) \\ +4x_c^2y_c\rho*\rho*\rho*\rho^{+}(\xi)+6x_cy_c^2\rho*\rho*\rho^{+}*\rho^{+}(\xi). \tag{4.33}$$

Let A be the support of ρ. It is clear that $A\subset[0, \infty)$. From eq. (4.33) it follows that if $x, y, z\in A$, then $\lambda^{-1}(x+y)$, $\lambda^{-1}(x+y+z)\in A$. Note that $2<\lambda<3$. By Proposition 4.6 (2), there is an $x_0\in A$ such that $x_0>0$. Then $(2\lambda^{-1})^n x_0\in A$, $n\geqq 1$. Since A is a closed set, this leads to $0\in A$. Therefore, 0, $\lambda^{-1}x_0$, $2\lambda^{-1}x_0$, $3\lambda^{-1}x_0\in A$, and by induction, it follows that $m\lambda^{-n}x_0\in A$, $m=0, 1, \cdots, 3^n$. This implies $A=[0, \infty)$. This completes the proof. □

Note that μ_n^* in Section 1 is equal to $\mu_n(\vec{x}_c)$ with

$$\vec{x}_c=(\exp(-\beta_c),\ \exp(-2\beta_c),\ \exp(-2\beta_c),\ \exp(-3\beta_c)).$$

The definition of β_c implies $\vec{x}_c\in\partial D\cap\Xi_1$. Therefore Theorem 1.2 in Section 1 follows from Proposition 4.6 and Proposition 4.8.

§ 5. Continuum Limit of Self-avoiding Paths

Let F_n, $n=0, 1, 2, \cdots$ be the graphs defined in Section 1. Let $\tilde{F}_n=2^{-n}F_n$, $n=0, 1, 2, \cdots$. Each $\tilde{F}_n$ is a finite graph obtained by giving a substructure to a unit tetrahedron $Oa_0b_0c_0$. Let us define the finite three-dimensional Sierpinski Gasket by

$$\tilde{F}=\overline{\bigcup_{n=0}^{\infty}\tilde{F}_n}.$$

We define $\tilde{G}_n$ to be the set of vertices in $\tilde{F}_n$, and T_n to be the set of closed tetrahedrons in $\boldsymbol{R}^3$ whose vertices belong to $\tilde{G}_n$ and whose edges are of length 2^{-n}.

Let

$$C=\{w\in C([0, \infty)\to\tilde{F})\mid w(0)=O,\ \lim_{t\to\infty}w(t)=a_0\}.$$

C is a complete separable metric space with the metric

$$d(u, v)=\sup_{t\in[0,\infty)}|u(t)-v(t)|,$$

$u, v\in C$.

We define a mapping $\gamma: \bigcup_n W^{*(n)}\to C$ as follows. For $u\in W^{*(n)}$,

(1) $\gamma u(j)\overset{\text{def}}{=}2^{-n}u(j)$, for $j\in\boldsymbol{Z}_+$

(2) $\gamma u(t) \overset{\text{def}}{=} (j+1-t)\gamma u(j)+(t-j)\gamma u(j+1)$, for $j \leqq t < j+1$, $j \in \boldsymbol{Z}_+$.

Note that γ is an injection. We denote

$$\widetilde{W}^{*(n)} \overset{\text{def}}{=} \gamma W^{*(n)} .$$

$w \in \widetilde{W}^{*(n)}$ is self-avoiding in the sense that $w(t_1) \neq w(t_2)$ if $0 \leqq t_1 < t_2 \leqq L(\gamma^{-1}w)$.

Let $\tilde{\mu}_n(\vec{x})$ be the image measures of $\mu_n(\vec{x})$ induced by γ. $\tilde{\mu}_n(\vec{x})$ is a probability measure on C supported on $\widetilde{W}^{*(n)}$. Throughout this section we consider the case $\vec{x} \in \partial D \cap \Xi_1$. Our objective in this section is to study the limit of $\tilde{\mu}_n(\vec{x})$ as n tends to infinity.

Let us begin with some definitions we use in this section. First we define "hitting times", $T_i^k : C \to \boldsymbol{R}_+$, $k, i \in \boldsymbol{Z}_+$. Let $T_0^k(w)=0$, and by induction, for $i \geqq 1$,

$$T_i^k(w) = \inf\{t > T_{i-1}^k(w) \mid w(t) \in \tilde{G}_k \setminus \{w(T_{i-1}^k(w))\}\},$$

if the right hand side is finite, otherwise, $T_i^k(w)=\infty$. T_i^k is the time when w hits the elements of $\tilde{G}_k$ for the i-th time on condition that if w hits the same element of $\tilde{G}_k$ more than once on end, we consider it "once". Writing $w(\infty) = a_0$, and noting that $w(t) \to a_0$ as $t \to \infty$, we obtain a finite sequence $\{T_i^k\}_{i=1,\dots,M}$ such that $w(T_M^k(w))=a_0$, $w(T_i^k(w)) \neq a_0$, $i=1, \cdots, M-1$.

Next we define the "exit times", $\{T_i^{*k}(w)\}_{i=0,\dots,N(w)}$, and the "$k$-skeletons", the sequence of tetrahedrons a path passes through, $\sigma_k(w)=(\Delta_1, \cdots, \Delta_{N(w)})$. Let $\{T_i^k(w)\}_{i=0,\dots,M}$ be the finite sequence obtained above. Let $T_0^{*k}(w)=T_0^k(w) =0$. Δ_1 is defined to be the element of T_k that contains $O=(0, 0, 0)$. For $i \geqq 1$ we proceed by induction. Define

$$\text{exit}(i) \overset{\text{def}}{=} \min\{j \in \boldsymbol{Z}_+ \mid j < M,\ T_j^k(w) > T_{i-1}^{*k}(w),\ w(T_{j+1}^k) \notin \Delta_i\}.$$

As long as the right-hand side exists, we define $T_i^{*k}(w)=T_{\text{exit}(i)}^k$ and Δ_{i+1} to be the element of T_k that contains both $w(T_i^{*k}(w))$ and $w(T_{\text{exit}(i)+1}^k(w))$. $N= N(w)$ denotes the number of the elements of $\sigma_k(w)$ defined in this way. Let $T_N^{*k}(w)=T_M^k(w)$. We write $S_i^k(w)=T_i^{*k}(w)-T_{i-1}^{*k}(w)$ and call it the crossing time of Δ_i. In the following we denote an ordered set of tetrahedrons like $(\Delta_1, \cdots, \Delta_N)$ and an unordered set like $\{\Delta_1, \cdots, \Delta_N\}$. Let $w \in C$, $k \in \boldsymbol{Z}_+$, and $\sigma_k(w)= (\Delta_1, \cdots, \Delta_N)$. The following properties are straightforward consequences of the definition.

1. $O \in \Delta_1$, $a_0 \in \Delta_N$.
2. $\Delta_i \cap \Delta_{i+1}$ is equal to neither $\emptyset$ nor Δ_i.

If $w \in \widetilde{W}^{*(n)}$, $n, k \in \boldsymbol{Z}_+$, $n \geqq k$, $\sigma_k(w)$ further satisfies,

3. Each element of T_k appears at most twice in $(\Delta_1, \cdots, \Delta_N)$.
4. $\{\Delta_i, \Delta_{i+1}\} \neq \{\Delta_j, \Delta_{j+1}\}$, $i \neq j$ as unordered sets.

Let us denote $\mathcal{T}_k \overset{\text{def}}{=} \{\boldsymbol{\Delta}=(\Delta_1, \cdots, \Delta_N) \mid \Delta_i \in T_k,\ i=1, \cdots, N,\ N=1, 2, \cdots, \boldsymbol{\Delta}$ satisfies 1. through 4.$\}$, and $\mathcal{T}_k^{1,2} \overset{\text{def}}{=} \{\boldsymbol{\Delta}=(\Delta_1, \cdots, \Delta_N) \mid \Delta_i \in T_k,\ i=1, \cdots, N,\ N=1,$

$2, \cdots, \Delta$ satisfies 1. and 2.}.

For $n \in \boldsymbol{Z}_+$, we define a "decimation" map $Q_n: C \to C$ by

$$(Q_n w)(i) = w(T_i^n(w)),$$

for $i=0, 1, 2, \cdots, M$, with $w(T_M^n(w)) = a_0$,

$$(Q_n w)(t) = (i+1-t)\ (Q_n w)(i) + (t-i)\ (Q_n w)(i+1),$$

for $i \leqq t < i+1$, $i=0, 1, 2, \cdots, M-1$, and

$$(Q_n w)(t) = a_0$$

for $t \geqq M$. Note that if $k \leqq n$, we have $Q_k \circ Q_n = Q_k$.

Let $m \leqq n$ and $Q_m \tilde{\mu}_n(\vec{x})$ be the image measure of $\tilde{\mu}_n(\vec{x})$ induced by Q_m.

Proposition 5.1. *For $w \in \widetilde{W}^{*(n)}$ and $m \leqq n$, $Q_m w \in \widetilde{W}^{*(m)}$.*

$$Q_m \tilde{\mu}_n(\vec{x}) = \tilde{\mu}_m(\vec{X}_{n-m}(\vec{x})).$$

In particular, for $\vec{a} \stackrel{\text{def}}{=} (x_c, y_c, 0, 0)$,

$$Q_m \tilde{\mu}_n(\vec{a}) = \tilde{\mu}_m(\vec{a}).$$

The statement on $Q_m \tilde{\mu}_n(\vec{x})$ is obtained directly from the recursion relations, eq. (2.2) in Proposition 2.1.

We introduce a time-scale transformation $U_n(\alpha): C \to C$, $\alpha \in (0, \infty)$, $n \in \boldsymbol{N}$. For $w \in C$, define

$$(U_n(\alpha) w)(t) \stackrel{\text{def}}{=} w(\alpha^n t).$$

Let us denote by $P_n(\vec{x})$ the image measure of $\tilde{\mu}_n(\vec{x})$ induced by $U_n(\lambda)$. We omit the $\vec{x}$ dependence of P_n when no confusion occurs.

We define

$$V^{(n)} \stackrel{\text{def}}{=} \{w \in \widetilde{W}^{*(n)} \mid s_3(\gamma^{-1} w) = s_4(\gamma^{-1} w) = 0\}.$$

Note that for $w \in V^{(n)}$,

$$T_i^{*k}(w) = T_i^k(w),$$

for $k \leqq n$, $0 \leqq i \leqq N(w)$.

In the following we write, for example, $P_n[Q_m w = v]$ instead of $P_n[\{w \in C \mid Q_m w = v\}]$.

We obtain the following proposition in a similar way to the case of the two-dimensional Sierpinski Gasket.

Proposition 5.2. *Assume $m \leqq n$, $v \in V^{(m)}$, and $\sigma_m(v) = (\Delta_1, \cdots, \Delta_N)$. Under the conditional probability $P_n[\cdot \mid Q_m w = v]$, we have the following.*

(1) *The set of S_i^m's with $i \in \{i_1, i_2, \cdots, i_K\} \subset \{1, \cdots, N\}$ are independent random variables, if $\Delta_{i_j} \neq \Delta_{i_k}$, for any $j \neq k$.*

(2) *For* $1 \leqq i \leqq N$, *if* Δ_i *appears only once in* $\sigma_m(v)$, *the law of* $\lambda^m S_i^m$ *is equal to* p_{n-m}^*, *thus converges weakly to* p *as* $n \to \infty$.

(3) *If* $\Delta_i = \Delta_j$, $1 \leqq i < j \leqq N$, *then the law of* $(\lambda^m S_i^m, \lambda^m S_j^m)$ *is equal to* q_{n-m}, *thus converges weakly to* q *as* $n \to \infty$. *In particular, the law of* $\lambda^m S_i^m$ *is equal to* q_{n-m}^*, *converging weakly to* q^* *as* $n \to \infty$.

By Proposition 5.1 combined with Theorem 3.5, we have

Proposition 5.3. *For any* $k \in \boldsymbol{Z}_+$,

$$\lim_{n \to \infty} P_n[Q_k w = v] = \begin{cases} x_c^{N_1 - 1} y_c^{N_2} & \text{if } v \in V^{(k)}, \\ 0 & \text{otherwise,} \end{cases}$$

where $N_1 = s_1(\gamma^{-1} v)$ *and* $N_2 = s_2(\gamma^{-1} v)$.

Proposition 5.4. *The family of measures* P_n, $n = 1, 2, \cdots$, *is tight.*

Proof. Since we already have

$$P_n[w(0) = O] = 1,$$

it suffices to show that for any ε, $\eta > 0$, there exist a positive integer n_0 and a positive number δ such that

$$P_n[\sup_{|s-t|<\delta} |w(s) - w(t)| > \varepsilon] \leqq \eta, \quad n \geqq n_0.$$

For an arbitrarily given ε, choose $k \in \boldsymbol{Z}_+$ satisfying

$$2 \cdot 2^{-k} < \varepsilon.$$

We have,

$$\begin{aligned} &P_n[\sup_{|s-t|<\delta} |w(s) - w(t)| > \varepsilon] \\ &\leqq P_n[S_i^k(w) < \delta \text{ for some } i = 1, \cdots, N(w)] \\ &\leqq \sum_{v \in V^{(k)}} \sum_{i=1}^{L(\gamma^{-1} v)} P_n[S_i^k(w) < \delta \mid Q_k w = v] \cdot P_n[Q_k w = v] \\ &\quad + P_n[Q_k w \in \widetilde{W}^{*(k)} \setminus V^{(k)}] \\ &\leqq 2 \cdot 4^k (p_{n-k}^*[s : s < \lambda^k \delta] + q_{n-k}^*[s : s < \lambda^k \delta]) \\ &\quad + P_n[Q_k w \in \widetilde{W}^{*(k)} \setminus V^{(k)}]. \end{aligned}$$

The last inequality is obtained from Proposition 5.2 and the fact that $L(\gamma^{-1} v) \leqq 2 \cdot 4^k$, a.s.. By Proposition 5.3, there is an $n_1 \in \boldsymbol{Z}_+$ such that

$$P_n[Q_k w \in \widetilde{W}^{*(k)} \setminus V^{(k)}] \leqq \frac{\eta}{2}, \qquad \text{for } n > n_1.$$

Take a $\delta > 0$ such that

$$p[s:s<\lambda^k\delta]+q^*[s:s<\lambda^k\delta]\leqq\frac{1}{2}\cdot 4^{-k-1}\eta\,.$$

Then by Proposition 4.6 and Proposition 4.8, there is an $n_0\geqq n_1$ such that

$$p^*_{n-k}[s:s<\lambda^k\delta]+q^*_{n-k}[s:s<\lambda^k\delta]<4^{-k-1}\eta\,,$$

for $n\geqq n_0$. This completes the proof. □

Now we will show the convergence of the finite dimensional distributions. For $w\in C$, $0\leqq t_1\leqq\cdots\leqq t_m$, $m=1, 2, \cdots$, let us define

$$h_m(w(t_1), \cdots, w(t_m))=e^{ix_1 w(t_1)+\cdots+ix_m w(t_m)},\ (x_1, \cdots, x_m)\in \boldsymbol{R}^m\,.$$

For a probability measure Q on C, define $F_m(Q)(t_1, \cdots, t_m): \boldsymbol{R}^m\to \boldsymbol{C}$ by,

$$F_m(Q)(t_1, \cdots, t_m)\stackrel{\text{def}}{=}E^Q[h_m(w(t_1), \cdots, w(t_m))]\,.$$

Fix an $m\in \boldsymbol{Z}_+$. For any $k\in\boldsymbol{Z}_+$ and $n\geqq k$,

$$F_m(P_n)(t_1, \cdots, t_m)=\sum_{v\in V^{(k)}}E^{P_n}[h_m\,|\,Q_k w=v]P_n[Q_k w=v]$$

$$+E^{P_n}[h_m\,|\,Q_k w\in\tilde{W}^{*(k)}\setminus V^{(k)}]P_n[Q_k w\in\tilde{W}^{*(k)}\setminus V^{(k)}].$$

$$\begin{aligned}F(P_n)&\stackrel{\text{def}}{=}\sum_{v\in V^{(k)}}E^{P_n}[h_m\,|\,Q_k w=v]P_n[Q_k w=v]\\&=\sum_{v\in V^{(k)}}\sum_{\{r_i\}}E^{P_n}[h_m\,|\,Q_k w=v,\ T^{*k}_{r_i}\leqq t_i<T^{*k}_{r_{i+1}},\ i=1, \cdots, m]\\&\quad\times P_n[T^{*k}_{r_i}\leqq t_i<T^{*k}_{r_{i+1}},\ i=1, \cdots, m\,|\,Q_k w=v]P_n[Q_k w=v],\end{aligned}$$

where $\sum_{\{r_i\}}$ is taken over $\{1, 2, \cdots, N(v)\}^m$ with $r_1\leqq r_2\leqq\cdots\leqq r_m$.

For simplicity write

$$E^{P_n}\stackrel{\text{def}}{=}E^{P_n}[h_m\,|\,Q_k w=v,\ T^{*k}_{r_i}\leqq t_i<T^{*k}_{r_{i+1}},\ i=1, \cdots, m]\,,$$

$$P^*_n\stackrel{\text{def}}{=}P_n[T^{*k}_{r_i}\leqq t_i<T^{*k}_{r_{i+1}},\ i=1, \cdots, m\,|\,Q_k w=v]\,,$$

$$\tilde{P}_n\stackrel{\text{def}}{=}P_n[Q_k w=v]\,,$$

and

$$R_n\stackrel{\text{def}}{=}P_n[Q_k w\in\tilde{W}^{*(k)}\setminus V^{(k)}]\,.$$

Then for $n, n'\geqq k$,

$$\begin{aligned}&|F(P_n)-F(P_{n'})|\\&\leqq\sum_{v\in V^{(k)}}\sum_{\{r_i\}}|E^{P_n}-E^{P_{n'}}|P^*_n\tilde{P}_n+\sum_{v\in V^{(k)}}\sum_{\{r_i\}}|E^{P_{n'}}||P^*_n-P^*_{n'}|\tilde{P}_n\\&\quad+\sum_{v\in V^{(k)}}\sum_{\{r_i\}}|E^{P_{n'}}|P^*_{n'}|\tilde{P}_n-\tilde{P}_{n'}|\,.\end{aligned}$$

Put

$$u_i = w(T_i^{*k}), \qquad i=1, \cdots, N(v).$$

Under the condition that $Q_k w = v$, $v \in V^{(k)}$, $T_{r_i}^{*k} \leqq t_i < T_{r_i+1}^{*k}$, $i=1, \cdots, m$, there are positive constants C_1 and C_2, independent of k, v and $\{r_i\}$ such that

$$|h_m(w(t_1), \cdots, w(t_m)) - h_m(u_{r_1}, \cdots, u_{r_m})| \leqq C_1 2^{-C_2 k}.$$

Thus the first term is bounded by $C_1 2^{-C_2 k+1}$. For an arbitrarily given $\varepsilon > 0$, choose a k such that

$$C_1 2^{-C_2 k+1} < \frac{\varepsilon}{4}.$$

Note that for a fixed k, the summation over v and $\{r_i\}$ is finite. By Proposition 5.2 and Proposition 5.3, for sufficiently large n and n', the second and the third sum are less than $\varepsilon/4$, respectively, and

$$R_n + R_{n'} < \frac{\varepsilon}{4}.$$

Thus we see that $\{F_m(P_n)(t_1, \cdots, t_m)\}_{n=1,2,\dots}$ is a Cauchy sequence in $C(\boldsymbol{R}^m \to \boldsymbol{C})$, and therefore converges uniformly as $n \to \infty$. We have shown that the distribution of $(w(t_1), \cdots, w(t_m))$ converges for any $0 \leqq t_1 \leqq \cdots \leqq t_m$, $m \in \boldsymbol{Z}_+$. This result combined with Proposition 5.4 leads to the following theorem.

Theorem 5.5. *P_n converges to a probability measure P on C weakly as $n \to \infty$.*

Now we will proceed to study the properties of P. For $\Delta \in T_k$, let us denote its neighbouring elements of T_k, by $\Delta^{(1)}, \cdots, \Delta^{(3)}$, if Δ contains any element of G_0, or by $\Delta^{(1)}, \cdots, \Delta^{(4)}$, otherwise. Let us denote $\left(\bigcup_i \Delta^{(i)} \cap G_k\right) \setminus (\Delta \cap G_k)$ by $\partial N(\Delta)$, and $\left(\bigcup_i \Delta^{(i)} \cup \Delta\right) \setminus \partial N(\Delta)$ by $N(\Delta)$.

Proposition 5.6. *Assume $\boldsymbol{\Delta} = (\Delta_1, \cdots, \Delta_N) \in \mathcal{T}_k$, $k \in \boldsymbol{Z}_+$. Let $A = \{w \in C \mid \boldsymbol{\sigma}_k(w) = \boldsymbol{\Delta}\}$. A is an open subset of C.*

Proof. Take any $w \in A$. From the definition of the skeleton, for $T_{i-1}^{*k} \leqq t < T_i^{*k}$, $i=1, \cdots, N$,

$$w(t) \cap G_k \subset \Delta_i \cap G_k,$$

that is, $w(t) \in N(\Delta_i)$. Let

$$r_i = \inf\{d(w(t), \partial N(\Delta_i)) \mid T_{i-1}^{*k} \leqq t < T_i^{*k}\},$$

and

$$t_i = \inf\{t > T_{i-1}^{*k} \mid w(t) \in \Delta_i,\ d(\Delta_{i-1} \cap \Delta_i, w(t)) = 2^{-k-1}\}.$$

Noting that $r_i > 0$, we can find an $\varepsilon > 0$ satisfying

$$\varepsilon < \min_{i=1,\cdots,N} r_i \wedge 2^{-k-2}.$$

Then it follows that for any w', $d(w, w') < \varepsilon$,

$$w'(t_i) \in \Delta_i \setminus G_k,$$

$$\{w'(t) \mid t_i < t \leqq T_i^{*k}\} \cap G_k \subset \Delta_i \cap G_k,$$

$$\{w'(t) \mid T_{i-1}^{*k} \leqq t < t_i\} \cap G_k = \Delta_{i-1} \cap \Delta_i,$$

for $i=1, \cdots, N$, and $\Delta_i \neq \Delta_{i-2}$ for $i=3, \cdots, N$. This means $w' \in A$. This completes the proof. □

For $\Delta \in \mathcal{T}_k^{1,2}$, define a subset of C as $u(\Delta) \overset{\text{def}}{=} \{w \in C \mid$ there exists a sequence $0 < s_1 < s_2 < \cdots < s_N < \infty$ such that $w(s_i) \in \Delta_i \setminus G_k$ and $w((s_i, s_{i+1})) \cap G_k \subset \Delta_i$ for all $i=1, 2, \cdots, N\}$, where $w((a, b)) \overset{\text{def}}{=} \{w(t) \mid a < t < b\}$.

By a similar argument to the proof of Proposition 5.6, we have,

Proposition 5.7. *$u(\Delta)$ is an open set.*

Proposition 5.8. *If $\Delta \in \mathcal{T}_k^{1,2}$, then*

$$\{w \in C \mid \sigma_k(w) = \Delta\} \subset u(\Delta).$$

Proof. Let t_i, $i=1, \cdots, N$ be as defined in the proof of Proposition 5.6. Since $T_{i-1}^{*k} < t_i < T_i^{*k}$, $\{w(t) \mid T_{i-1}^{*k} < t < t_i\} \cap G_k \subset \Delta_{i-1} \cap \Delta_i$, and $\{w(t) \mid t_i < t < T_i^{*k}\} \cap G_k \subset \Delta_i$. Take $s_i = t_i$, $i=1, 2, \cdots, N$. This completes the proof. □

From this proposition we have,

Proposition 5.9. *If $\Delta \in \mathcal{T}_k^{1,2}$ does not satisfy the condition 3. or 4., then $P[\sigma_k(w)=\Delta]=0$.*

Proof.

$$\begin{aligned} P[\sigma_k(w)=\Delta] &\leqq P[u(\Delta)] \\ &\leqq \liminf_{n\to\infty} P_n[u(\Delta)] \\ &= \liminf_{n\to\infty} P_n[\sigma_k(w)=\Delta] \\ &= 0. \end{aligned}$$

The first inequality comes from Proposition 5.8 and the second comes from the weak convergence of P_n to P. The probability vanishes because P_n is supported on a set of self-avoiding paths. This completes the proof. □

For each $\Delta = (\Delta_1, \cdots, \Delta_N) \in \mathcal{T}_k$, there is a unique element v_Δ of $V^{(k)}$ such that $\sigma_k(v_\Delta)=\Delta$. v_Δ is determined by $v_\Delta(i)\in\Delta_i\cap\Delta_{i+1}$, $i=1, 2, \cdots, N-1$, and $v_\Delta(N)=a_0$. On the other hand, for each $v\in V^{(k)}$ there is a unique element Δ of $\mathcal{T}_k$, such that $\sigma_k(v_\Delta)=\Delta$. This defines a one-to-one mapping from $\mathcal{T}_k$ to $V^{(k)}$.

Proposition 5.10. *For* $\Delta=(\Delta_1, \cdots, \Delta_N)\in\mathcal{T}_k$,

$$P[\sigma_k(w)=\Delta]\leqq x_c^{N-2N_2-1}y_c^{N_2},$$

where N_2 *denotes the number of distinct tetrahedrons that appear twice in* Δ.

Proof. By Theorem 5.5, Proposition 5.6, and Proposition 5.3, we have

$$\begin{aligned} P[\sigma_k(w)=\Delta] &\leqq \liminf_{n\to\infty} P_n[\sigma_k(w)=\Delta] \\ &= \liminf_{n\to\infty} P_n[Q_k w=v_\Delta]+\liminf_{n\to\infty} P_n[\sigma_k(w)=\Delta,\ Q_k w\neq v_\Delta] \\ &= x_c^{N_1-1}y_c^{N_2} \\ &= x_c^{N-2N_2-1}y_c^{N_2}. \end{aligned}$$

This completes the proof. □

Theorem 5.11. *For* $\Delta\in\mathcal{T}_k^{1,2}$,

$$P[\sigma_k(w)=\Delta]=\begin{cases} x_c^{N-2N_2-1}y_c^{N_2} & \text{if } \Delta\in\mathcal{T}_k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Proposition 5.9 implies that P is supported on $\{w\in C\,|\,\sigma_k(w)\in\mathcal{T}_k\}$. Assume that for some $\Delta'\in\mathcal{T}_k$,

$$P[\sigma_k(w)=\Delta']<x_c^{N-2N_2-1}y_c^{N_2}.$$

This assumption together with Proposition 5.10 leads to

$$\begin{aligned} 1 &= \sum_{\Delta\in\mathcal{T}_k} P[\sigma_k(w)=\Delta] \\ &< \sum_{\Delta\in\mathcal{T}_k} x_c^{N-2N_2-1}y_c^{N_2} \\ &= x_c^{-1}x_k(x_c, y_c) \\ &= 1. \end{aligned}$$

Here, x_k is defined in Section 3 and we used

$$\sum_{\Delta\in\mathcal{T}_k} x^{N-2N_2-1}y^{N_2}=x_k(x, y),$$

which follows from the correspondence between $\mathcal{T}_k$ and $V^{(k)}$. This is a contradiction. This completes the proof. □

Remark. Though P itself has an $\vec{x}$-dependence, the probability that a path's skeleton takes a certain form is independent of $\tilde{x}$. The dependence appears only in the crossing times of tetrahedrons. (See Proposition 4.5.)

Proposition 5.12. *Let* $\sigma_k(w)=(\Delta_1^{(k)}(w), \cdots, \Delta_{N_k}^{(k)}(w))$, *and denote by* $u_i^{(k)}$ *and* $v_i^{(k)}$ *the two vertices of* $\Delta_i^{(k)}(w)$ *that are not contained in* $\{w(T_i^{*k}(w)), w(T_{i-1}^{*k}(w))\}$, $i=1, \cdots, N_k$. *Then*

$$P[w(t)\in\Delta_i^{(k)}(w)\setminus\{u_i^{(k)}, v_i^{(k)}\} \text{ for all } t,\ T_{i-1}^{*k}(w)<t<T_i^{*k}(w),$$
$$i=1, \cdots, N_k,\ k\in\boldsymbol{Z}_+]=1\,.$$

Proof. Assume for some i and k there exist $\Delta'\in T_k$, $\Delta'\neq\Delta_i^{(k)}$, and t', $T_{i-1}^{*k}(w)<t'<T_i^{*k}(w)$, such that $w(t')\in\Delta'\setminus G_k$. The definition of the exit times implies that there are $t_1, \cdots, t_4$, $t_1<t_2<t'<t_3<t_4$ such that $\{w(t_2)\}=\{w(t_3)\}=\Delta'\cap\Delta_i^{(k)}$, and $w(t_1), w(t_4)\in\Delta_i^{(k)}\setminus G_k$. Let $r=d(w(t'), \Delta'\cap\Delta_i^{(k)})$ and $r_j=d(w(t_j), \Delta'\cap\Delta_i^{(k)})$, $j=1, 4$. Choose an $m\in\boldsymbol{Z}_+$ such that $2\cdot 2^{-m}<\min(r, r_1, r_4)$. Let $\Delta^{(m)}$ and $\tilde{\Delta}^{(m)}$ be the elements of T_m satisfying

$$\Delta^{(m)}\subset\Delta_i^{(k)},\ \tilde{\Delta}^{(m)}\subset\Delta',\ \Delta'\cap\Delta_i^{(k)}=\Delta^{(m)}\cap\tilde{\Delta}^{(m)}\,.$$

Then if follows that $\sigma_m(w)$ contains the subsequence $(\Delta^{(m)}, \tilde{\Delta}^{(m)})$ or $(\tilde{\Delta}^{(m)}, \Delta^{(m)})$ at least twice. Thus for a fixed $\boldsymbol{\Delta}=(\Delta_1, \cdots, \Delta_N)\in\mathcal{T}_k$,

$$\begin{aligned}
&P[\sigma_k(w)=\boldsymbol{\Delta} \text{ and } w(t')\in\tilde{F}\setminus\Delta_i, \text{ for some } i \text{ and } t',\ T_{i-1}^{*k}(w)<t'<T_i^{*k}(w)]\\
&\leqq\sum_{i=1}^{N(\boldsymbol{\Delta})}P[\sigma_k(w)=\boldsymbol{\Delta} \text{ and } w(t')\in\tilde{F}\setminus\Delta_i, \text{ for some } t',\ T_{i-1}^{*k}(w)<t'<T_i^{*k}(w)]\\
&\leqq\sum_{i=1}^{N(\boldsymbol{\Delta})}\sum_{m=k+1}^{\infty}P[\sigma_k(w)=\boldsymbol{\Delta},\ \sigma_m(w) \text{ contains some successive pair}\\
&\qquad\{\Delta^{(m)}, \tilde{\Delta}^{(m)}\} \text{ twice.}]\\
&=0\,.
\end{aligned}$$

To obtain the last equality, we used Proposition 5.9. Summing up over all elements of $\mathcal{T}_k$ and all $k\in\boldsymbol{Z}_+$ we obtain,

(5.1) $\quad P[w(t)\in\Delta_i^{(k)}(w) \text{ for all } T_{i-1}^{*k}(w)<t<T_i^{*k}(w),\ i=1, \cdots, N_k,\ k\in\boldsymbol{Z}_+]=1.$

Again, for a fixed $\boldsymbol{\Delta}=(\Delta_1, \cdots, \Delta_N)\in\mathcal{T}_k$, and a fixed i, $1\leqq i\leqq N$, we denote the two vertices of Δ_i that are not contained in $\Delta_i\cap\Delta_{i+1}$ or $\Delta_{i-1}\cap\Delta_i$, by u and v, and the element of T_m that is contained in Δ_i and containing u, by $\Delta_u^{(m)}$, for $m=k, k+1, \cdots$. If $w(t)=u$ for some t, $T_{i-1}^{*k}(w)<t<T_i^{*k}(w)$, then w must go inside $\Delta_u^{(m)}$, which, from what we just proved above, implies that there is a subsequence of $\sigma_m(w)$, $\{\Delta_r^{(m)}, \Delta_{r+1}^{(m)}, \cdots, \Delta_{r+r_1}^{(m)}\}$, $r, r_1\in\boldsymbol{Z}_+$, $r_1>0$, such that $\Delta_j^{(m)}\subset\Delta_i$, $r\leqq j\leqq r+r_1$, $w(T_{i-1}^{*k}(w))\in\Delta_r^{(m)}$, $w(T_i^{*k}(w))\in\Delta_{r+r_1}^{(m)}$, and $\Delta_s^{(m)}=\Delta_u^{(m)}$ for some s satisfying $r\leqq s\leqq r+r_1$. In terms of probability,

$$P[\text{there exists } t,\ T^{*k}_{i-1}(w)<t<T^{*k}_{i}(w),\ \text{such that } w(t)=u \mid \sigma_k(w)=\Delta]$$

$$\leqq P[\sigma_m(w) \text{ has a subsequence satisfying above conditions} \mid \sigma_k(w)=\Delta]$$

$$\leqq b^{m-k},$$

where

$$b \stackrel{\text{def}}{=} \max\{x_c^{-1}(\phi_1(x_c,\ y_c)-x_c^2),\ y_c^{-1}(\phi_2(x_c,\ y_c)-x_c^4)\}.$$

Since $b<1$ and m can be chosen arbitrarily large, the first probability vanishes. The same holds for v instead of u. Summing up over i, $\Delta\in\mathcal{T}_k$, and k, and combining with eq. (5.1) we have the statement. This completes the proof. □

We go on to prove that the stocastic process defined by P is almost surely self-avoiding, that is, $w(t_1)\neq w(t_2)$, for $0\leqq t_1<t_2\leqq T_{a_0}(w)$, where

$$T_{a_0}(w)=\inf\{t>0 \mid w(t)=a_0\}.$$

We classify possible self-intersections as follows.

(1) There are $t_1\geqq 0$ and $t_0>0$ such that

$$w(t)=w(t_1), \qquad \text{for } t_1\leqq t\leqq t_1+t_0<T_{a_0}(w).$$

(2) There are t_1, t_2, and t_3, $t_1<t_2<t_3$ such that

$$w(t_1)=w(t_3),$$

$$w(t_2)\neq w(t_1).$$

Type (2) can be further classified into two cases:

(2-1) $w(t_1)\in\bigcup_k G_k$,

(2-2) $w(t_0)\in F\setminus\bigcup_k G_k$.

We start with dealing with type (1) case.

Proposition 5.13.

$$P[\textit{there exist } t_1\geqq 0 \textit{ and } t_0>0 \textit{ such that } w(t)=w(t_1)\neq a_0,\ t_1\leqq t\leqq t_1+t_0]$$

$$=0.$$

Proof. Let A_{k,t_0} be the set of $w\in C$ such that there exist $t_1\geqq 0$, and two adjoining elements of T_k, Δ and Δ', satisfying $w(t)\in((\Delta\cup\Delta')\setminus G_k)\cup(\Delta\cap\Delta')$ for $t_1\leqq t\leqq t_1+t_0$, or such that there exist $t_1\geqq 0$ satisfying $w(t)\in(\Delta_0\setminus G_k)\cup\{O\}$ for $t_1\leqq t\leqq t_1+t_0$, where Δ_0 is the element of T_k containing O. It is straightforward to see that A_{k,t_0} is an open subset of C, and we have

$$P[A_{k,t_0}]\leqq \liminf_{n\to\infty} P_n[A_{k,t_0}].$$

$$P_n[A_{k,t_0}]\leqq P_n[\text{there exists } i,\ 2\leqq i\leqq N(w),\ \text{such that } S^k_{i-1}(w)+S^k_i(w)>t_0]$$

$$=\sum_{\Delta\in\mathcal{J}_k} P_n[\sigma_k=\Delta]\cdot\sum_{i=1}^{N(\Delta)} P_n[S^k_{i-1}(w)+S^k_i(w)>t_0\,|\,\sigma_k=\Delta]$$

$$\leqq 4^{k+1}\Big(p^*_{n-k}\Big[t: t>\frac{\lambda^k t_0}{2}\Big]+q^*_{n-k}\Big[t: t>\frac{\lambda^k t_0}{2}\Big]\Big).$$

In the last inequality we used

$$P_n[S^k_{i-1}(w)+S^k_i(w)>t_0\,|\,\sigma_k=\Delta]$$

$$\leqq P_n\Big[S^k_{i-1}(w)>\frac{t_0}{2} \text{ or } S^k_i(w)>\frac{t_0}{2}\,\Big|\,\sigma_k=\Delta\Big]$$

$$\leqq 2\Big(p^*_{n-k}\Big[t: t>\frac{\lambda^k t_0}{2}\Big]+q^*_{n-k}\Big[t: t>\frac{\lambda^k t_0}{2}\Big]\Big).$$

By Chebyshev's inequality and Proposition 4.5, for any $a>0$ and $s>0$

$$\lim_{n\to\infty} p^*_n[t: t>a]$$

$$\leqq e^{-sa}\lim_{n\to\infty}\int_0^\infty e^{st}p^*_n(dt)$$

$$=e^{-sa}g(s)<\infty\,.$$

A similar inequality holds for $q^*_n[t: t>a]$. Therefore

$$P[A_{k,t_0}]\leqq\liminf_{n\to\infty} P_n[A_{k,t_0}]\leqq 4^{k+1}Ce^{-s(\lambda^k t_0/2)}\,,$$

where s and C are positive constants. Thus by Proposition 5.12, for any $k\in\boldsymbol{Z}_+$ and $t_0>0$,

$$P(t_0)\overset{\text{def}}{=}P[\text{there exist } t_1\geqq 0 \text{ such that } w(t)=w(t_1)\neq a_0,\ t_1\leqq t\leqq t_1+t_0]$$

$$\leqq P[A_{k,t_0}]\,.$$

Letting $k\to\infty$, we see that $P(t_0)$ is equal to zero. Therefore,

$$P[\text{there exist } t_1\geqq 0 \text{ and } t_0>0 \text{ such that } w(t)=w(t_1)\neq a_0,\ t_1\leqq t\leqq t_1+t_0]$$

$$\leqq\sum_{m=1}^{\infty}P\Big(\frac{1}{m}\Big)$$

$$=0.$$

This completes the proof. □

Next we will rule out the possibility that (2-1) occurs. With Proposition 5.12 taken into consideration, it is sufficient to show that $\{w(t)\}\neq\Delta_i\cap\Delta_{i+1}$ for $T^{*k}_{i-1}<t<T^{*k}_i$ almost surely, where Δ_i is the i-th component of $\sigma_k(w)$. Note that $\{w(T^{*k}_i)\}=\Delta_i\cap\Delta_{i+1}$, and assume there is a t_1, $T^{*k}_{i-1}<t_1<T^{*k}_i$, such that $\{w(t_1)\}=\Delta_i\cap\Delta_{i+1}$. Proposition 5.13 implies that w cannot stay at $\Delta_i\cap\Delta_{i+1}$ for

a finite interval of time. It follows that there must be an integer $m>k$ and $\Delta\in T_m$, $\Delta_i\cap\Delta_{i+1}\subset\Delta\subset\Delta_i$, that appears in $\sigma_m(w)$ three times. By Theorem 5.11, this occurs with probability zero. We can show in a similar fashion that $\{w(t)\}\neq\Delta_{i-1}\cap\Delta_i$, for $T^{*k}_{i-1}<t<T^{*k}_i$, a. s.. Therefore, we have,

Proposition 5.14.

$$P[w(t)\in\Delta_i^{(k)}(w)\setminus G_k \text{ for all } T^{*k}_{i-1}(w)<t<T^{*k}_i(w),\ i=1,\cdots,N_k,\ k\in\boldsymbol{Z}_+]=1,$$

where $\sigma_k(w)=(\Delta_1^{(k)}(w),\cdots,\Delta_{N_k}^{(k)}(w))$. *In particular,*

$$P[T^{*k}_i(w)=T^k_i(w), \text{ for all } i=1,\cdots,N_k,\ k\in\boldsymbol{Z}_+]=1.$$

What is left is to show that the probability for the type (2-2) case is zero. For $x\in\tilde{F}\setminus\bigcup_k G_k$, there is a sequence of tetrahedrons $\Delta_x^{(0)}, \Delta_x^{(1)}, \cdots, \Delta_x^{(k)}, \cdots$ such that

$$\Delta_x^{(k)}\in T_k,\quad x\in\Delta_x^{(k)}\setminus G_k\subset\Delta_x^{(k-1)}\setminus G_{k-1},\quad k\in\boldsymbol{Z}_+.$$

In order that w hits x twice, there must be an integer K such that $\sigma_k(w)$ contains $\Delta_x^{(k)}$ twice for any $k\geqq K$, and $\sigma_{K-1}(w)$ contains $\Delta_x^{(K-1)}$ only once. For $w\in C$ let $K(w)$ be the minimum integer, if exists, such that there exists a sequence $\{\Delta^{(k)}\}$, $k=K(w), K(w)+1, \cdots$ satisfying

(1) $\Delta^{(k)}\in T_k$, $\Delta^{(k+1)}\subset\Delta^{(k)}$,

(2) $\sigma_k(w)$ contains $\Delta^{(k)}$ twice.

Put $q_k=P[K(w)=k]$. For any $\Delta^*\in T_m$, $m=1, 2, \cdots$, let q be the probability that there exists a sequence $\{\Delta^{(k)}\}$, $k=m, m+1, \cdots$, satisfying (1) and (2) above with $\Delta^{(m)}=\Delta^*$, under the condition that Δ^* is contained in $\sigma_m(w)$ twice. Note that by Theorem 5.11, q is independent of m and the choice of Δ^*.

Classifying according to the four possibilities of $\Delta^{(m+1)}$ and using inclusion-exclusion principle, we have

$$q=y_c^{-1}\{4x_c^3y_cq+22y_c^4(4q-6q^2+4q^3-q^4)\}.$$

The only solution to this equation found in $0\leqq q\leqq 1$ is $q=0$. By Theorem 5.11, q and q_k's are related as follows;

$$q_k=x_c^{-1}x_{k-1}(x_c, y_c(1-q))-x_c^{-1}x_{k-1}(x_c(1-q_1), y_c),$$

$$q_1=4x_c^2y_cq+6x_cy_c^2(2q-q^2).$$

This leads to $q_k=0$, for all $k=1, 2, \cdots$. We thus have

$$P[\text{type (2-2) occurs}]\leqq\sum_{k=1}^{\infty}q_k=0.$$

Theorem 5.15. *The stochastic process defined by P is almost surely self-avoiding, that is,*

$$P[w(t_1)\neq w(t_2),\ 0\leqq t_1<t_2\leqq T_{a_0}(w)]=1\,.$$

Let $w\in C$. The image of w, $G(w)\overset{\text{def}}{=}w([0,\infty))$, is a subset in three-dimensional Euclidean space. We next study the Hausdorff dimension of $G(w)$.

In the case of the self-avoiding paths on (two-dimensional) Sierpinski gasket, Theorem 1.1 of [6] was sufficient for the probability one determination of the Hausdorff dimension of curve $G(w)$ (Section 1.4 of [2]). Unfortunately it is not sufficient for the present case. The problem is as follows.

From Proposition 5.12, it follows that

$$G(w)=\bigcap_{k=0}^{\infty}\bigcup_{\Delta\in\sigma_k(w)}\Delta,\ P\text{— } a.s..$$

Each skeleton $\sigma_k(w)$ is a sequence of tetrahedrons Δ of side length 2^{-k}. Note that from Theorem 5.11, there are two types of tetrahedrons in $\sigma_k(w)$ for each k, namely those that appear just once in $\sigma_k(w)$ and those that appear twice, both type appearing with positive probability. The family of tetrahedrons $\bigcup_k \sigma_k(w)$ resembles the "random constructions" of Mauldin and Williams, but their theory can be applicable to the case when only one type of tetrahedrons appear.

Here we will state a weaker result, a lower bound of the Hausdorff dimension of $G(w)$. This can be derived by considering a following subgraph of $G(w)$:

$$G'(w)\overset{\text{def}}{=}\bigcap_{k=0}^{\infty}\bigcup_{\Delta\in\sigma'_k(w)}\Delta\,,$$

where

$$\sigma'_k(w)=\{\Delta\in\sigma_k(w)\,|\,\Delta \text{ appears just once in } \sigma_k(w)\}\,.$$

The Hausdorff dimension of $G'(w)$ can be derived from Theorem 1.1 of [6], in a similar way as in [2], and the value can be used as the lower bound to the Hausdorff dimension of $G(w)$.

Theorem 5.16.

$$P[\textit{Hausdorff dimension of } w([0,\infty))\geqq\log(8x_c^3+6x_c^2+2x_c)/\log 2]=1\,.$$

Remark.

(1) $8x_c^3+6x_c^2+2x_c=2.599\cdots>2$.

(2) We conjucture that with P-probability 1 the Hausdorff dimension of $w([0,\infty))$ is $\log\lambda/\log 2$, where λ is as in Proposition 4.1. This could be derived from an extension of the theory of [6].

§6. Mean Square Deviations of Self-avoiding Paths

In this section, we return to the self-avoiding paths on the three-dimensional pre-Sierpinski gasket, but instead of considering a set of paths with fixed end points, we now consider a set of paths with a fixed length. The arguments are similar to those in [4].

Let $W^{(0)}=\{w\in W_0 \mid w(0)=O\}$, and for each $k\in \boldsymbol{Z}_+$, let $N(k)$ be the number of elements in $\{w\in W^{(0)} \mid L(w)=k\}$. The first step is to bound $N(k)$ from above and below.

Proposition 6.1. *Let b be a positive constant, and for $n\in \boldsymbol{Z}_+$ and $\xi\in \boldsymbol{R}$, let $h_n=b\lambda^{-n}\sqrt{n}$, and $g_n(\xi)=(\sqrt{2\pi}h_n)^{-1}\exp(-\xi^2/(2h_n^2))$. If b is sufficiently large, then*

$$(p_n^*(\tilde{x}_c)*g_n)(\xi)\overset{\mathrm{def}}{=}\int_{\boldsymbol{R}} g_n(\xi-\eta)p_n^*(\tilde{x}_c)(d\eta)\longrightarrow \rho(\tilde{x}_c)(\xi)$$

uniformly in $\xi\in \boldsymbol{R}$ as $n\to\infty$. Here, λ is as in Proposition 4.1, p_n^* as in Proposition 4.6. ρ as in Proposition 4.8, $\tilde{x}_c=(\exp(-\beta_c), \exp(-2\beta_c), \exp(-2\beta_c), \exp(-3\beta_c))$, and β_c is as in Corollary 3.6.

Proof. Let

$$\phi_n(t)=\int_{\boldsymbol{R}} \exp(i\xi t)(p_n^*(\tilde{x}_c)*g_n)(\xi)d\xi\,, \qquad t\in \boldsymbol{R}\,.$$

Then

$$\begin{aligned}\phi_n(t)&=Z_{1,n}(\beta_c)^{-1}Z_{1,n}(\beta_c-i\lambda^{-n}t)\exp(-h_n^2t^2/2)\\ &=g_1^{(n)}(it)\exp(-h_n^2t^2/2)\,,\end{aligned}$$

where $Z_{1,n}$ is defined in eq. (3.8), and $g_1^{(n)}$ is as in eq. (4.23) with $\tilde{x}=\tilde{x}_0$. Note that $\tilde{x}_c\in\partial D\cap\Xi_1$. Also note that from eq. (4.24), $h^{(n)}(it, it)=Z_{2,n}(\beta_c)^{-1}Z_{2,n}(\beta_c-i\lambda^{-n}t)$.

Let

$$A=\{t\in \boldsymbol{C} \mid \Im t\geqq 0,\ \lambda^{-1}\leqq |t|\leqq \lambda^2\}\,.$$

Proposition 4.8 implies that $\sup_{t\in A}|g(it)|<1$ and $\sup_{t\in A}|h(it, it)|<1$. Therefore from Proposition 4.5 and Corollary 3.6 it follows that there exist a positive number ε and a positive integer n_1 such that for $n\geqq n_1$ and $t\in A$,

$$|Z_{1,n}(\beta_c-i\lambda^{-n}t)|=|Z_{1,n}(\beta_c)g_1^{(n)}(it)|<x_c-\varepsilon\,,$$

and

$$|Z_{2,n}(\beta_c-i\lambda^{-n}t)|=|Z_{2,n}(\beta_c)h^{(n)}(it, it)|<y_c-\varepsilon\,.$$

By Proposition 2.2, $(x_c-\varepsilon, y_c-\varepsilon, 0, 0)\in D^o$, and since D^o is an open set in Ξ_0, there exists a positive number δ such that $(x_c-\varepsilon, y_c-\varepsilon, \delta, \delta)\in D^o$. Note that $|Z_{j,n}(\beta_c-i\lambda^{-n}t)|\leqq|Z_{j,n}(\beta_c)|$, $j=1, 2, 3, 4$, $t\in A$. From Corollary 3.6, there exists an integer $n_0\geqq n_1$ such that $|Z_{j,n}(\beta_c-i\lambda^{-n}t)|\leqq\delta$, $j=3, 4$, $n\geqq n_0$. Therefore,

$$|Z_{j,n+m}(\beta_c-i\lambda^{-n}t)|\leqq X_{j,m}(x_c-\varepsilon,\ y_c-\varepsilon,\ \delta,\ \delta),$$

$$j=1,\ 2,\quad n\geqq n_0,\quad m\geqq 1,\quad t\in A.$$

This together with Proposition 2.4 implies that there exist positive constants C and γ such that for $n\geqq n_0$, $m\geqq 1$, $t\in A$,

(6.1) $$|Z_{j,n+m}(\beta_c-i\lambda^{-n}t)|\leqq C\exp(-\gamma 2^m),\qquad j=1,\ 2,$$

(6.2) $$Z_{j,n+m}(\beta_c+\lambda^{-n})\leqq C\exp(-\gamma 2^m),\qquad j=1,\ 2,\ 3,\ 4.$$

Now let n_0 be as above. Let n be a positive integer satisfying $n>n_0$, and assume that $t\in\boldsymbol{R}$ and $|t|\in[1,\ \lambda^{n-n_0-1}]$. Let m be the integer part of $(\log|t|/\log\lambda)+1$. Then m satisfies $n-m\geqq n_0$ and $\lambda^{-1}\leqq\lambda^{-m}|t|<1$. Then

$$\begin{aligned}|\phi_n(t)|&\leqq Z_{1,n}(\beta_c)^{-1}|Z_{1,n}(\beta_c-i\lambda^{-n}t)|\\&=Z_{1,n}(\beta_c)^{-1}|Z_{1,n-m+m}(\beta_c-i\lambda^{-(n-m)}(\lambda^{-m}t))|\\&\leqq Z_{1,n}(\beta_c)^{-1}C\exp(-\gamma 2^m)\\&\leqq Z_{1,n}(\beta_c)^{-1}C\exp(-\gamma|t|^{\log 2/\log\lambda}).\end{aligned}$$

Since $Z_{1,n}(\beta_c)\to x_c$ and $\phi_n(t)\to g(it)$, $n\to\infty$, Proposition 4.7 and the dominated convergence theorem implies,

$$\int_{\boldsymbol{R}}|\chi_{[0,\lambda^{n-n_0-1}]}(|t|)\phi_n(t)-g(it)|\,dt\longrightarrow 0,\qquad n\to\infty.$$

On the other hand,

$$\begin{aligned}&\int_{\boldsymbol{R}}|\chi_{[0,\lambda^{n-n_0-1}]}(|t|)\phi_n(t)-\phi_n(t)|\,dt\\&\leqq 2\int_{\lambda^{n-n_0-1}}^{\infty}\exp(-h_n^2t^2/2)dt\\&\leqq 2(h_n^2\lambda^{n-n_0-1})^{-1}\exp(-(h_n\lambda^{n-n_0-1})^2/2)\\&=2\lambda^{n_0-1}b^{-2}\lambda^n n^{-1}\exp(-\lambda^{-2n_0+2}b^2n/2)\longrightarrow 0,\qquad n\to\infty.\end{aligned}$$

if b is sufficiently large. Hence for sufficiently large b,

$$\int_{\boldsymbol{R}}|\phi_n(t)-g(it)|\,dt\longrightarrow 0,\qquad n\to\infty,$$

which implies the Proposition. This completes the proof. □

Proposition 6.2. *There exist positive constants C_1, C_2, and real constants γ_1, γ_2, such that*

$$C_1k^{\gamma_1}\exp(\beta_ck)\leqq N(k)\leqq C_2k^{\gamma_2}\exp(\beta_ck),\qquad k\geqq 1.$$

Proof. Let $D: W^{(0)}\to\boldsymbol{Z}_+$ be a map defined by

(6.3) $$D(w)=\min\{n\geqq 0\,;\ w(i)\in F_n \text{ for all } i\geqq 0.\}.$$

Let

$$M_n=\sum_{w\in W^{(0)},\,D(w)\leqq n}\exp(-\beta_c L(w)),\qquad n\in \boldsymbol{Z}_+ .$$

Classifying the summation in the definition of M_{n+1} in a similar way as in the proof of the recursion relations in eq. (2.2), it follows that there exists a polynomial f_1 of four variables with positive coefficients, such that

(6.4) $$M_{n+1}\leqq f_1(\vec{Z}_n(\beta_c))M_n ,\qquad n\in \boldsymbol{Z}_+ .$$

By Corollary 3.6, $\vec{Z}_n(\beta_c)$ converges as $n\to\infty$, hence there exist positive constants A_1 and $A_2>1$ such that

(6.5) $$M_n\leqq A_1 A_2^n ,\qquad n\in \boldsymbol{Z}_+ .$$

By definition, $2^{D(w)-1}\leqq L(w)$ follows. Therefore,

$$\exp(-\beta_c k)N(k)\leqq M_{[\log k/\log 2]+1}\leqq A_1 A_2^2 A_2^{\log k/\log 2} ,$$

which proves the upper bound in the Proposition.

To prove the lower bound, let b be a sufficiently large number satisfying Proposition 6.1. Note that

$$(p_n^*(x_c)*g_n)(\xi)=\int_{\boldsymbol{R}} g_n(\xi-\eta)\,p_n^*(x_c)(d\eta)\,.$$

Let $k_n=\sqrt{2\log\lambda}\,bn\lambda^{-n}$. Since

$$\int_{\boldsymbol{R}\setminus[\xi-k_n,\,\xi+k_n]} g_n(\xi-\eta)\,p_n^*(x_c)(d\eta)\leqq g_n(k_n)=(2\pi b^2 n)^{-1/2}\longrightarrow 0,\qquad n\to\infty.$$

Proposition 6.1 implies that

$$\sup\{\,|\,\rho(x_c)(\xi)-\int_{[\xi-k_n,\,\xi+k_n]} g_n(\xi-\eta)\,p_n^*(x_c)(d\eta)|\ ;\ \xi\in\boldsymbol{R}\}\longrightarrow 0,\qquad n\to\infty.$$

From Proposition 4.8, this implies that there exist an integer $n_2\geqq 1$ and a positive constant ε such that $h_n^{-1}p_n^*(x_c)([\xi-k_n,\ \xi+k_n])\geqq\varepsilon$, $n\geqq n_2$, $\xi\in[\lambda^{-1},\ \lambda^2]$.

Let $k\in\boldsymbol{Z}_+$. Let n be the integer satisfying $\lambda^{-n}k\in[1,\ \lambda]$. For sufficiently large k, $n\geqq n_2$ and $k_n\leqq 1-\lambda^{-1}$ follows, hence $p_n^*(x_c)([\lambda^{-n}k-2k_n,\ \lambda^{-n}k])\geqq h_n\varepsilon$. Therefore,

$$\begin{aligned} Z_{1,n}(\beta_c)h_n\varepsilon &\leqq \sum_{w\in W_1^{(n)},\,k-2k_n\lambda^n\leqq L(w)\leqq k}\exp(-\beta_c L(w)) \\ &\leqq \exp(\beta_c 2k_n\lambda^n)\exp(-\beta_c k)N(k), \end{aligned}$$

because $w\in W_1^{(n)}$ with $L(w)\leqq k$ can be extended to a path in $W^{(0)}$ with $L=k$. It follows that

$$N(k)\geqq Z_{1,n}(\beta_c)\varepsilon b\lambda^{-n}n^{1/2}\exp(-2\beta_c b(2\log\lambda)^{1/2}n)\exp(\beta_c k)\,.$$

Since $n \leqq \log k/\log \lambda$, this implies the lower bound in the Proposition. This completes the proof. □

The next step is to give bounds for the numbers of short paths and long paths. Let

$$U_{n,m}=\sum_{w\in W^{(0)},\, D(w)\leq n,\, L(w)\geq \lambda^{n+m/2}} \exp(-\beta_c L(w)),$$

and

$$V_{n,m}=\sum_{w\in W^{(0)},\, D(w)=n+1,\, L(w)\leq \lambda^{n-m}} \exp(-\beta_c L(w)),\quad n\in \boldsymbol{Z}_+,\ m\in \boldsymbol{Z}_+.$$

Proposition 6.3. *There exist positive constants A_2, C, and γ such that*

$$U_{n,m}\leqq CA_2^n \exp(-\gamma\lambda^{m/2}),$$

and

$$V_{n,m}\leqq CA_2^n \exp(-\gamma 2^m),\quad n\in \boldsymbol{Z}_+,\quad m\in \boldsymbol{Z}_+.$$

A_2 may be taken to be the same as in eq. (6.5).

Proof. Put $r=(\lambda-\sqrt{\lambda})/5$, and let

$$S_{j,n,m}=\sum_{w\in W_j^{(n)},\, L(w)\geq \lambda^{n+m/2}r} \exp(-\beta_c L(w)),\quad n\in \boldsymbol{Z}_+,\ m\in \boldsymbol{Z}_+,\ j=1, 2, 3, 4.$$

By a graphical consideration similar to that used to obtain eq. (6.4), one finds

$$U_{n+1,m}\leqq f_1(\vec{Z}_n(\beta_c))U_{n,m+1}+\Big(\sum_{j=1}^{4} S_{j,n,m}\frac{\partial f_1}{\partial x_j}(\vec{Z}_n(\beta_c))\Big)M_n ,$$

where $f_1(\vec{x})$ may be chosen to be the same as that in eq. (6.4), and $f_1(0,0,0,0)=1$. In particular, f_1 is a polynomial of four variables with positive coefficients. As in the derivation of eq. (6.5), there exists an integer n_1 such that $f_1(\vec{Z}_n(\beta_c))\leqq A_2$, $n\geqq n_1$, where A_2 is as in eq. (6.5). Corollary 3.6 and eq. (6.5) imply that there exists a positive constant C_1 such that

$$A_2^{-(n+1)}U_{n+1,m}\leqq A_2^{-n}U_{n,m+1}+C_1\sum_{j=1}^{4} S_{j,n,m},\quad n\geqq n_1,\quad m\geqq 0.$$

Note that since $L(w)\leqq 3\cdot 4^{D(w)}$, it follows that $U_{n,m}=0$ if $\lambda^{n+m/2}>3\cdot 4^n$, which holds if $m\geqq 3n$ and $n>4$. We may assume that $n_1>4$. Hence

$$A_2^{-n}U_{n,m}\leqq C_1\sum_{k=0}^{[(3/4)n]}\sum_{j=1}^{4} S_{j,n-k-1,m+k},\quad n\geqq 4n_1,\quad m\geqq 0.$$

On the other hand,

$$\begin{aligned} S_{j,n,m} &\leqq \exp(-r\lambda^{m/2})\sum_{w\in W_j^{(n)}} \exp(-(\beta_c-\lambda^{-n})L(w)) \\ &=\exp(-r\lambda^{m/2})Z_{j,n}(\beta_c-\lambda^{-n}). \end{aligned}$$

Corollary 3.6 and Proposition 4.5 imply that there exists a constant C_2 such that

$$Z_{1,n}(\beta_c-\lambda^{-n})=g_1^{(n)}(1)Z_{1,n}(\beta_c)<C_2, \quad n\in\boldsymbol{Z}_+,$$

and

$$Z_{2,n}(\beta_c-\lambda^{-n})=h^{(n)}(1)Z_{2,n}(\beta_c)<C_2, \quad n\in\boldsymbol{Z}_+.$$

Proposition 2.4 implies

$$\begin{aligned} Z_{3,n}(\beta_c-\lambda^{-n}) &\leqq R_n(\vec{x}_{c,n})Z_{1,n}(\beta_c-\lambda^{-n}) \\ &\leqq R_0(\vec{x}_{c,n})C_2 \\ &=2x_{c,n}C_2, \end{aligned}$$

where $\vec{x}_{c,n}=(x_{c,n}, x_{c,n}^2, x_{c,n}^2, x_{c,n}^3)$ with $x_{c,n}=\exp(-\beta_c+\lambda^{-n})$. Similar argument holds also for $Z_{4,n}(\beta_c-\lambda^{-n})$. Therefore $Z_{j,n}(\beta_c-\lambda^{-n})$, $j=1, 2, 3, 4$, are bounded, which, together with the above estimates on $A_2^{-n}U_{n,m}$ and $S_{j,n,m}$ implies the bound for $U_{n,m}$ in the Proposition.

To prove the bound for $V_{n,m}$, let

$$T_{j,n,m} \sum_{w\in W_j^{(n+1)},\ L(w)\leq\lambda^{n-m}} \exp(-\beta_c L(w)),$$
$$n\in\boldsymbol{Z}_+,\ m\in\boldsymbol{Z}_+,\quad m\leqq n,\quad j=1, 2, 3, 4,$$

and put $\vec{T}_{n,m}=(T_{1,n,m}, T_{2,n,m}, T_{3,n,m}, T_{4,n,m})$. By a graphical consideration similar to that used above, one finds

$$V_{n,m}\leqq f_1(\vec{T}_{n,m})M_n-M_n=(f_1(\vec{T}_{n,m})-f_1(0, 0, 0, 0))M_n.$$

Note that if $L(w)\leqq\lambda^{n-m}$ then $1-\lambda^{m-n}L(w)\geqq 0$. Therefore for $j=1, 2, 3, 4$,

$$T_{j,n,m}\leqq \sum_{w\in W_j^{(n)}} \exp(-(\beta_c+\lambda^{m-n})L(w)+1)=eZ_{j,n}(\beta_c+\lambda^{m-n}).$$

This with eq. (6.2) and eq. (6.5) implies the bound for $V_{n,m}$. This completes the proof. □

The proof of Theorem 1.4 in Section 1 is as follows.

The assertion (1) is a direct consequence of Proposition 6.2.

To prove assertion (2), let $K(k)=[\log k/\log\lambda]$, $k\in\boldsymbol{Z}_+$. Note that $\lambda^{K(k)}\leqq k<\lambda^{K(k)+1}$. Then Proposition 6.3 implies that for $m\in\boldsymbol{Z}_+$ and $k\in\boldsymbol{Z}_+$ with $m\leqq K(k)$,

$$\begin{aligned} &\#\{w\in W^{(0)}\mid L(w)=k,\ D(w)\leqq K(k)-m\} \\ &\leqq\exp(\beta_c k)U_{K(k)-m,2m} \\ &\leqq C\exp(\beta_c k+(K(k)-m)\log A_2-\gamma\lambda^m) \\ &\leqq C\exp(\beta_c k+(\log A_2/\log\lambda)\log k-\gamma\lambda^m). \end{aligned}$$

This and Proposition 6.2 imply that for sufficiently large α,

$$\tilde{P}_k[D(w)\leqq K(k)-\alpha \log\log k]$$
$$\leqq CC_1^{-1}\exp((\log A_2/\log\lambda-\gamma_1)\log k-\gamma(\log k)^{\alpha\log\lambda})$$
$$\leqq C'\exp(-(\log k)^3).$$

Note that $2^{D(w)-1}\leqq\|w\|\leqq 2^{D(w)}$. Therefore for sufficiently large α

$$\tilde{P}_k[\|w\|<(\log k)^{-\alpha}k^{1/\kappa}]\exp((\log k)^2)\longrightarrow 0,\qquad k\to\infty.$$

Next note that for $m\in \boldsymbol{Z}_+$ and $l\in\boldsymbol{Z}_+$, Proposition 6.3 implies

$$\#\{w\in W^{(0)}\mid L(w)=k,\ D(w)=K(k)+m+l+2\}$$
$$\leqq \exp(\beta_c k)V_{K(k)+m+l+1,\,m+l}$$
$$\leqq C\exp(\beta_c k)A_2^{K(k)+m+l+1}\exp(-\gamma 2^{m+l})$$
$$\leqq CA_2\exp(\beta_c k+K(k)\log A_2+m\log A_2-\gamma 2^{m-1})A_2^l\exp(-\gamma 2^{l-1}).$$

Therefore,

$$\tilde{P}_k[D(w)\geqq K(k)+(\alpha/\log 2)\log\log k]$$
$$=\sum_{l=0}^{\infty}\tilde{P}_k[D(w)=K(k)+l+(\alpha/\log 2)\log\log k]$$
$$\leqq C(C_1A_2)^{-1}\sum_{l=0}^{\infty}A_2^l\exp(-\gamma 2^{l-1})$$
$$\times\exp((\log A_2/\log\lambda-\gamma_1)\log k+(\alpha\log A_2/\log 2)\log\log k-(\gamma/8)(\log k)^{\alpha}).$$

Therefore, for sufficiently large α

$$\tilde{P}_k[\|w\|>(\log k)^{\alpha}k^{1/\kappa}]\exp((\log k)^2)\longrightarrow 0,\qquad k\to\infty.$$

This proves the assertion (2).

Let $k\in\boldsymbol{Z}_+$ and $s>0$. Note that the reflection principle similar to the one in the proof of the Lemma (4.2) in [4] holds also in the present case of three-dimensional pre-Sierpinski gasket, which implies

$$E^{\tilde{P}_k}[2^{(D(w)-1)s},\ |w(k)|\leqq 2^{D(w)-1}]\leqq E^{\tilde{P}_k}[2^{(D(w)-1)s},\ |w(k)|\geqq 2^{D(w)-1}].$$

This with $|w(L(w))|\leqq\|w\|\leqq 2^{D(w)}$ implies

$$2^{-s-1}E^{\tilde{P}_k}[2^{sD(w)}]\leqq E^{\tilde{P}_k}[|w(k)|^s]\leqq E^{\tilde{P}_k}[\|w\|^s]\leqq E^{\tilde{P}_k}[2^{sD(w)}]. \tag{6.6}$$

Note next that

$$E^{\tilde{P}_k}[\|w\|^s]\geqq((\log k)^{-\alpha}k^{1/\kappa})^s(1-\tilde{P}_k[\|w\|\leqq(\log k)^{-\alpha}k^{1/\kappa}]).$$

With assertion (2) this implies

$$\liminf_{k\to\infty}(\log k)^{s\alpha}k^{-s/\kappa}E^{\tilde{P}_k}[\|w\|^s]>0. \tag{6.7}$$

Similarly,

$$E^{\tilde{P}_k}[\|w\|^s] \leqq ((\log k)^\alpha k^{1/\kappa})^s + k^s \tilde{P}_k[\|w\| \geqq (\log k)^\alpha k^{1/\kappa}],$$

and assertion (2) imply

$$\limsup_{k\to\infty} (\log k)^{-s\alpha} k^{-s/\kappa} E^{\tilde{P}_k}[\|w\|^s] < \infty . \tag{6.8}$$

It is easy to see that eq. (6.6), eq. (6.7), and eq. (6.8) imply assertion (3). This completes the proof. □

§ A. Recursion Relations

In this Appendix, we give the complete form of the function $\vec{\Phi}$ defined in Proposition 2.1.

$$\begin{aligned}
\Phi_1(x, y, z, w) = {} & x^2 + 2x^3 + 2x^4 + 4x^3y + 6x^2y^2 \\
& + 4xz + 4xw + 10x^2z + 8x^2w + 12x^3z + 8x^3w + 16x^2yz \\
& + 8x^2yw + 12xy^2z + 4z^2 + 8zw + 4w^2 + 14xz^2 + 16xzw \\
& + 20x^2z^2 + 16x^2zw + 12xyz^2 + 6z^3 + 8z^2w + 8xz^3 ,
\end{aligned}$$

$$\begin{aligned}
\Phi_2(x, y, z, w) = {} & x^4 + 4x^3y + 22y^4 \\
& + 8x^3z + 8x^3w + 24x^2yz + 24x^2yw + 20x^2z^2 + 32x^2zw \\
& + 8x^2w^2 + 36xyz^2 + 48xyzw + 16xz^3 + 24xz^2w + 8yz^3 + 2z^4,
\end{aligned}$$

$$\begin{aligned}
\Phi_3(x, y, z, w) = {} & x^2z + 2x^2w + 2x^3z + 4x^3w + 4x^2yz + 8x^2yw \\
& + 6xy^2z + 12xy^2w + 4xz^2 + 12xzw + 8xw^2 + 10x^2z^2 \\
& + 24x^2zw + 8x^2w^2 + 6y^2z^2 + 24xyzw + 12xyz^2 \\
& + 8zw^2 + 10z^2w + 3z^3 + 4yz^3 + 24xz^2w + 12xz^3 + 2z^4,
\end{aligned}$$

$$\begin{aligned}
\Phi_4(x, y, z, w) = {} & x^2z^2 + 4x^2zw + 4x^2w^2 + 2xyz^2 + 8xyzw \\
& + 8xyw^2 + 3y^2z^2 + 12y^2zw + 4xz^3 + 16xz^2w \\
& + 16xzw^2 + 4yz^3 + 12yz^2w + 3z^4 + 8z^3w .
\end{aligned}$$

§ B. Derivative Matrices

We give the explicit forms of a 4×4 matrix B and a 9×9 matrix $\partial H^{(1)}(a)$ here. B is defined by

$$B = \left(\frac{\partial^t\vec{\Phi}}{\partial x_1}(\vec{a}), \cdots, \frac{\partial^t\vec{\Phi}}{\partial x_4}(\vec{a})\right),$$

Then an explicit calculation gives

$$B=\begin{pmatrix} p & q & p+s & 2s \\ q & r & 2q & 2q \\ 0 & 0 & \frac{1}{2}(p-s) & p-s \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$p=8x_c^3+6x_c^2+2x_c+12x_c^2y_c+12x_cy_c^2,$$

$$q=4x_c^3+12x_c^2y_c,$$

$$r=4x_c^3+88y_c^3,$$

and

$$s=2x_c+4x_c^2+4x_c^3+4x_c^2y_c.$$

From this it is straightforward to obtain the eigenvalues,

$$\lambda_1=\frac{1}{2}\{p+r+\sqrt{(p-r)^2+4q^2}\}=2.7965\cdots,$$

$$\lambda_2=\frac{1}{2}(p-s)=0.3861\cdots,$$

$$\lambda_3=\frac{1}{2}\{p+r-\sqrt{(p-r)^2+4q^2}\}=0.2537\cdots,$$

and

$$\lambda_4=0.$$

Let

$$\partial H^{(n)}(z)\overset{\text{def}}{=}\left(\frac{\partial}{\partial x_1}{}^tH^{(n)}(z),\ \cdots,\ \frac{\partial}{\partial x_9}{}^tH^{(n)}(z)\right).$$

Then

$$\partial H^{(1)}(a)=\begin{pmatrix} p & q & p+s & 2s & 0 & 0 & 0 & 0 & 0 \\ q & r & 2q & 2q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(p-s) & p-s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & q & p+s & 2s & 0 \\ 0 & 0 & 0 & 0 & q & r & 2q & 2q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(p-s) & p-s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{q}{2} & 16y_c^4 & q & q & \frac{q}{2} & 16y_c^4 & q & q & 4x_c^3+56y_c^3 \end{pmatrix}.$$

$\partial H^{(1)}(a)$ has double eigenvalues, $\lambda=\lambda_1$, λ_2, λ_3, 0 and a single eigenvalue, $4x_c^3+56y_c^3=0.3276\cdots$. The left eigenvectors corresponding to λ are (1, α, 2, 2, 0, 0, 0, 0, 0) and (0, 0, 0, 0, 1, α, 2, 2, 0), with $\alpha=(\lambda-p)/q=0.1731\cdots$.

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